XVI. *Of the Attraction of such Solids as are terminated by Planes; and of Solids of greatest Attraction. By* Thomas Knight, *Esq. Communicated by Sir* H. Davy, *LL.D. Sec. R. S.*

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Mathematicians, in treating of the attraction of bodies, have confined their attention, almost entirely, to those solids which are bounded by continuous curve surfaces; and Mr. Playfair, if I do not mistake, is the only writer, who has given any example of that kind of inquiry, which is the chief object of the present paper. This learned mathematician has found expressions* for the action of a parallelopiped; and of an isosceles pyramid, with a rectangular base, on a point at its vertex; and observes, on occasion of the first mentioned problem, that what he has there done, "gives some hopes of "being able to determine generally the attraction of solids "bounded by any planes whatever."

It is this general problem, that I venture to attempt the solution of, in what follows: viz. *any solid, regular or irregular, terminated by plane surfaces, being given, to find, both in quantity and direction, its action, on a point, given in position, either within or without it.*

* Ed. Trans. Vol. VI. p. 228 to 243. It is proper however to observe, that Mr. Playfair's expression, at p. 242, for the action of a parallelopiped, requires to have its sign changed; being, as it stands at present, negative, from the manner of correcting the fluent.

Nor has the matter any difficulty, as far as *theory** *only* is concerned; although, *actually* to find the attraction, of a body of very complicated figure, may, no doubt, be exceedingly laborious and troublesome: for no one, I suppose, will conceive, that it can be done in any other manner, than by a previous partition into more simple forms, each of which must have its action found separately.

Having completed this part of my subject in the three first sections, I next apply the formulas, given in §. 1, to find the attraction of certain complex bodies, which, though not bounded by planes, have yet a natural connexion with the preceding part of the paper. Finally, the fifth section treats, pretty fully, of solids of greatest attraction, under various circumstances; and I do not know, that any one of the problems there given has been before considered by mathematicians; whilst, on the other hand, the results of former writers are easily derived as corollaries.

For the sake of perspicuity, I have divided the paper into propositions, and shall terminate this short introduction by expressing a hope, that I may not be chargeable with unnecessary prolixity.

§. I.

Of the Attraction of Planes bounded by right Lines.

As all such figures may be divided into triangles, it seems natural to begin with these.

* It is usual, I think, with mathematicians, to consider a thing as done, when it can be pointed out how it *may be* done. Thus M. LAGRANGE, in his excellent work "*De la Résolution des Equations numériques,*" says (p. 43) "cette méthode ne laisse, "ce me semble, rien à desirer." where, of course, he can only mean, as far as relates to *theory*.

Prop. 1.

Let rvm, fig. 1, be a triangle, right angled at r, and pm a right line, perpendicular to the plane of the triangle, at the angular point m; it is required to find the attraction of the triangle, on the point p, both in quantity and direction.

Conceive a plane to pass through the point p, parallel to the plane of the triangle, and, in it, the lines pg, po, respectively parallel to rm, rv. The problem will be solved, if we find the actions of the triangle, in the directions of the three rectangular co-ordinates pm, pg, po.

Draw ks parallel to rv, and put $a =$ pm, $b =$ rm, $T =$ mk, $t =$ kq; then pq $= \sqrt{a^2 + T^2 + t^2}$. Let $r =$ tang. vmr, then ks $= r \times$ km.

The element of the plane at q is $\dot{T} \times \dot{t}$, and its action, on p, in the direction pq, is $\frac{\dot{T}\dot{t}}{a^2 + T^2 + t^2}$; by resolving which, and putting A, B, C for the actions of the triangle, in the directions pm, pg, po, we get

$$A = \iint \frac{a\dot{T}\dot{t}}{(a^2 + T^2 + t^2)^{\frac{3}{2}}}; \quad B = \iint \frac{T\dot{T}\dot{t}}{(a^2 + T^2 + t^2)^{\frac{3}{2}}}; \quad C = \iint \frac{\dot{T}t\dot{t}}{(a^2 + T^2 + t^2)^{\frac{3}{2}}};$$

in all which expressions, we must first take the fluent, with respect to t, from $t = o$, to $t = rT$; and afterwards, with respect to T, from $T = o$, to $T = b$. To begin with A,— a first operation gives

$$A = \int \frac{arT\dot{T}}{(a^2 + T^2)(a^2 + (1 + r^2)T^2)^{\frac{1}{2}}}$$

which, if we put $\beta^2 = 1 + r^2$, will be changed to

$$A = \int \frac{arT\dot{T}}{\beta(a^2 + T^2)\left(\frac{a^2}{\beta^2} + T^2\right)^{\frac{1}{2}}}.$$

Put $z^2 = \frac{a^2}{\beta^2} + T^2$, whence $T^2 = z^2 - \frac{a^2}{\beta^2}$, $T\dot{T} = z\dot{z}$: by substituting these values we get

$$A = \int \frac{a r \dot{z}}{\beta \left(a^2 - \frac{a^2}{\beta^2} - a^2 + z^2\right)} = (\text{because } \beta^2 - 1 = r^2) \int \frac{a r \dot{z}}{\beta \left(\frac{a^2}{\beta^2} r^2 + z^2\right)},$$

which, if we multiply both numerator and denominator by $\frac{\beta^2}{a^2 r^2}$, becomes $\int \frac{\frac{\beta}{ar}\dot{z}}{1 + \frac{\beta^2}{a^2 r^2} z^2} = \text{arc}\left(\text{tang.} = \frac{\beta}{ar} z\right)$, and, by putting for z and β their values, we have at last

$$*A = \text{arc}\left(\text{tang.} = \frac{1}{r}\sqrt{1 + \frac{1+r^2}{a^2} b^2}\right) - \text{arc}\left(\text{tang.} = \frac{1}{r}\right).$$

In like manner, a first integration of B gives

$$B = \int \frac{r T^2 \dot{T}}{(a^2+T^2)\,(a^2+(1+r^2)\,T^2)^{\frac{1}{2}}} = \int \frac{r T^2 \dot{T}}{\sqrt{1+r^2}\,(a^2+T^2)\left(\frac{a^2}{1+r^2} + T^2\right)^{\frac{1}{2}}}$$

$$= \frac{r}{\sqrt{1+r^2}} \int \left\{ \frac{\dot{T}}{\left(\frac{a^2}{1+r^2} + T^2\right)^{\frac{1}{2}}} - \frac{a^2 \dot{T}}{(a^2+T^2)\left(\frac{a^2}{1+r^2} + T^2\right)^{\frac{1}{2}}} \right\}.$$

Put $T = a$ tang. ϖ, then $\dot{T} = a$ sect. $^2\varpi\dot{\varpi}$, $a^2 + T^2 = a^2$ sect. $^2\varpi$; by this means the last term under the sign of the fluent is changed to $\frac{\dot{\varpi}}{\left(\frac{1}{1+r^2} + \text{tang.}\,^2\varpi\right)^{\frac{1}{2}}} = -\frac{\sqrt{1+r^2}}{r} \times \frac{r \sin \dot{\overline{\varpi}}}{(1+r^2 \sin.\,^2\varpi)^{\frac{1}{2}}}$;

wherefore, observing that tang. $\varpi = \frac{T}{a}$, and consequently sin. $\varpi = \frac{T}{\sqrt{a^2+T^2}}$, we find at last

* This quantity can be put under another form, which may be better in some cases. If we denote by b' the side rv of the triangle, $r = \frac{b'}{b}$, and

$$A = \text{arc}\left(\text{tang.} = \frac{b}{b'} \times \frac{\sqrt{a^2+b^2+b'^2}}{a}\right) - \text{arc}\left(\text{tang.} = \frac{b}{b'}\right).$$

$$B = \frac{r}{\sqrt{1+r^2}} L\left(\frac{b}{a} + \sqrt{\frac{1}{1+r^2} + \frac{b^2}{a^2}}\right) - L\left\{\frac{rb}{\sqrt{a^2+b^2}} + \sqrt{1 + \frac{r^2b^2}{a^2+b^2}}\right\}$$
$$- \frac{r}{\sqrt{1+r^2}} L . \frac{1}{\sqrt{1+r^2}}.$$

We have yet to find the value of C; and at first we get

$$C = \int \frac{\dot{T}}{(a^2+T^2)^{\frac{1}{2}}} - \int \frac{\dot{T}}{(a^2+(1+r^2)T^2)^{\frac{1}{2}}} = \int \frac{\frac{\dot{T}}{a}}{\left(1+\frac{T^2}{a^2}\right)^{\frac{1}{2}}} - \frac{\dot{T}}{\sqrt{1+r^2}}$$
$$\int \frac{\frac{\dot{T}}{a}}{\left(\frac{1}{1+r^2}+\frac{T^2}{a^2}\right)^{\frac{1}{2}}},$$

which again integrated, becomes

$$C = L\left(\frac{b}{a} + \sqrt{1+\frac{b^2}{a^2}}\right) - \frac{1}{\sqrt{1+r^2}} L\left(\frac{b}{a} + \sqrt{\frac{1}{1+r^2}+\frac{b^2}{a^2}}\right)$$
$$+ \frac{1}{\sqrt{1+r^2}} L . \frac{1}{\sqrt{1+r^2}}.$$

The expressions we have thus arrived at, for the action of a right angled triangle, are of such continual use in the following propositions, that it will be convenient to represent them by some concise symbol; and as they are functions of a, b, and r we may put

$$A = \text{arc}\left(\text{tang.} = \frac{1}{r}\sqrt{1+\frac{1+r^2}{a^2}b^2}\right) - \text{arc}\left(\text{tang.} = \frac{1}{r}\right) =$$
$$\varphi(a, b, r) - \text{arc}\left(\text{tang.} = \frac{1}{r}\right).$$

$$B = \frac{r}{\sqrt{1+r^2}} L\left(\frac{\sqrt{1+r^2}\, b}{a} + \sqrt{1+\frac{1+r^2}{a^2}b^2}\right) - L\left(\frac{rb+\sqrt{a^2+(1+r^2)b^2}}{\sqrt{a^2+b^2}}\right)$$
$$= \chi(a, b, r).$$

$$C = L\left(\frac{b}{a} + \sqrt{1+\frac{b^2}{a^2}}\right) - \frac{1}{\sqrt{1+r^2}} L\left(\frac{\sqrt{1+r^2}\, b}{a} + \sqrt{1+\frac{1+r^2\, b^2}{a^2}}\right)$$
$$= \psi(a, b, r).$$

Cor. 1. If, whilst r remains constant, b and a are supposed to vary, but so as to preserve the same ratio to each other, the

partial forces A, B, C will remain unchanged, as, consequently, will the total force, both in quantity and direction.

For, if we put $m \times a$ for b, the expressions of the forces become $\varphi(1, m, r) - \text{arc}\left(\text{tang.} = \frac{1}{r}\right)$, $\chi(1, m, r)$, $\psi(1, m, r)$; which are independent of the *absolute* values of a and b. It is scarcely necessary to observe, that arc $\left(\text{tang.} = \frac{1}{r}\right)$ is the arc, to the radius unity, corresponding to the angle rvm.

Cor. 2. When r becomes infinite, the triangle rmv is changed into a parallelogram, infinitely extended in the direction rv; in which case, the expressions of the forces become very simple, viz. $A = \text{arc}\left(\text{tang.} = \frac{b}{a}\right)$, $B = L.\frac{\sqrt{a^2+b^2}}{a}$, $C = L.\frac{b+\sqrt{a^2+b^2}}{a}$.

Prop. 2.

Let vmu, fig. 2, be any triangle whatever, pm a line perpendicular to the plane of the triangle, at the angular point m: from whence, let fall the perpendicular mr on the opposite side uv; moreover, let pg, po, be respectively parallel to mr, vu.

It is required to find the actions of the triangle vmu on the point p, in the directions pm, pg, po.

If we keep the same denominations as before, and put, besides, r′ = tang. umr, it is plain from the last proposition, and because the action of the whole must necessarily equal the sum of the actions of its parts, that

$$A = \varphi(a,b,r) + \varphi(a,b,r') - \text{arc}\left(\text{tang.} = \frac{1}{r}\right) - \text{arc}\left(\text{tang.} = \frac{1}{r'}\right);$$

$$B = \chi(a,b,r) + \chi(a,b,r');\ C = \psi(a,b,r) - (a,b,r').$$

When umv is a right angle, we shall evidently have arc

$\left(\text{tang.} = \frac{1}{r}\right) + \text{arc}\left(\text{tang.} = \frac{1}{r'}\right) = \frac{\pi}{2}$, π being the number 3,1415, &c.: this makes the expression for A somewhat simpler, in that case.

If it is the triangle vmu′ whose attraction we seek, we have, putting $r' =$ tang. u′mr,

$$A = \varphi(a,b,r) - \varphi(a,b,r') - \text{arc}\left(\text{tang.} = \frac{1}{r}\right) + \text{arc}\left(\text{tang.} = \frac{1}{r'}\right);$$

$$B = \chi(a,b,r) - \chi(a,b,r'); \quad C = \psi(a,b,r) - \psi(a,b,r').$$

Cor. 1. As a rhombus may be divided, from its centre, into four equal triangles, like that in fig. 2, but right angled at m, the angle lying at the centre; if b represent a perpendicular from the centre of a rhombus on one of its sides, and r and r' the tangents of the angles, that this perpendicular makes, at the centre, with the semi-diameters of the figure, we shall have for the action of the rhombus, on a point situated perpendicularly over its centre, at the distance a,

$$A = 4\varphi(a,b,r) + 4\varphi(a,b,r') - 2\pi.$$

Cor. 2. As any plane, terminated by right lines, may be divided into triangles from a point within it, we may find, by means of this proposition, the attraction of such a plane, on a point above it, both in quantity and direction. Let, for example, uvu′v′u, fig. 6, be the plane, p the attracted point; let fall the perpendicular pm on the plane, and from m draw right lines to the angles u, v, u′, v′; the plane will thus be divided into triangles, situated, with respect to the point p, like that in the proposition.

The attraction may still be found, if the perpendicular should fall without the figure; as in

Prop. 3.

To find the attraction of a triangle umv, fig. 3, on a point p any how situated.

Let fall from p the perpendicular pm′, on the plane of the triangle; join m′m, m′u, m′v. Find, by the last Prop. the attractions of the triangles m′uv, m′um, m′vm, on the point p; and resolve them into others in the directions of any three rectangular co-ordinates: and, when thus resolved, let the actions of

$$\left\{\begin{matrix} m'uv \\ m'um \\ m'vm \end{matrix}\right\} \text{ in the directions of these co-ordinates be } \left\{\begin{matrix} A\ , B\ , C \\ A'\ , B'\ , C' \\ A'', B'', C'' \end{matrix}\right\}.$$

It is plain, that the actions of the triangle umv, on p, in the directions of the same co-ordinates, will be

$$A - A' - A''; \; B - B' - B''; \; C - C' - C''.$$

There may be other cases of this proposition, in which the triangle and point are placed differently, with respect to each other, from what I have represented in fig. 3; but the reader, who understands the case that has been considered, will have no difficulty in any other that may occur.

Though the preceding propositions contain every thing that is necessary, for finding the attraction, both in quantity and direction, of any plane bounded by right lines; yet there are some cases worthy of a particular notice: as

Prop. 4.

To find the attraction of a rectangle mrvr′, fig. 4, on a point p situated perpendicularly over one of its corners as m.

Draw pg, po, parallel to mr, mr′, the sides of the rectangle; put $b =$ rm, $b' =$ r′m, $r =$ tang. rmv, $r' =$ tang. r′mv: then, if

A, B, C represent the actions of the rectangle, in the directions pm, pg, po, we find, by means of Proposition 1,

$$A = \varphi(a, b, r) + \varphi(a, b', r') - \frac{\pi}{2};$$

$$B = \chi(a, b, r) + \psi(a, b', r'); \quad C = \psi(a, b, r) + \chi(a, b', r').$$

We may eliminate r and r', from these expressions, by means of their values $\frac{b'}{b}$ and $\frac{b}{b'}$; and thus we may put A under a very simple form. It becomes, at first,

$$A = \text{arc}\left(\text{tang.} = \frac{b}{b'} \times \frac{\sqrt{a^2+b^2+b'^2}}{a}\right) + \text{arc}\left(\text{tang.} = \frac{b'}{b} \times \frac{\sqrt{a^2+b^2+b^2}}{a}\right) - \frac{\pi}{2}.$$

But, by trigonometry, α and β representing any angles whatever, tang. $(\alpha + \beta = \frac{\text{tang.}\,\alpha + \text{tang.}\,\beta}{1 - \text{tang.}\,\alpha \times \text{tang.}\,\beta}$: the application of which formula gives us, instead of the foregoing expression,

$$A = \text{arc}\left(\text{tang.} = \frac{-a}{bb'}\sqrt{a^2 + b^2 + b'^2}\right) - \frac{\pi}{2};$$

and this again is easily changed into the following form,

$$A = \text{arc tang.} = \frac{bb'}{a\sqrt{a^2+b^2+b^2}}\Big),$$ which is easily perceived to be the same as Mr. PLAYFAIR's expression.

In a similar manner, might the expressions for B and C be simplified: but it is perhaps easier to find new forms, *ab initio*. Thus we may get B immediately, from the double integral $B = \iint \frac{T\dot{T}\dot{t}}{(a^2+T^2+t^2)^{\frac{3}{2}}}$, if the fluent, with respect to t, be taken from $t = o$ to $t = b'$; and, with respect to T, from $T = o$ to $T = b$. The first integration gives $B = \int \frac{b'T\dot{T}}{(a^2+T^2)(a^2+b'^2+T^2)^{\frac{1}{2}}}$; and, by the second,

$$B = L.\frac{\sqrt{a^2+b^2}+b'}{a} - L.\frac{\sqrt{a^2+b^2+b^2}+b'}{\sqrt{a^2+b^2}}.$$

It is plain that, to find C, we need only change b into b', and the reverse, in the last expression, whence

$$C = L.\frac{\sqrt{a^2+b^2}+b}{a} - L.\frac{\sqrt{a^2+b^2+b'^2}+b}{\sqrt{a^2+b'^2}}.$$

Prop. 5.

To find the attraction of a regular polygon, on a point situated perpendicularly over its centre.

As this figure is composed of isosceles triangles, if we put b for the perpendicular from the centre on one of the sides, and r for the tangent of half the angle at the centre, subtended by one of those sides, we have, by Prop. 1, for a polygon of n sides,

$A = 2n \text{ arc}\left(\text{tang.} = \frac{1}{r}\sqrt{1+\frac{1+r^2}{a^2}b^2}\right) - 2n \text{ arc}\left(\text{tang.} = \frac{1}{r}\right)$, which, because the last term $= (n-2)\pi$, by Euclid 1.32. *Cor.* 1, is

$$A = 2n \text{ arc}\left(\text{tang.} = \frac{1}{r}\sqrt{1+\frac{1+r^2}{a^2}b^2}\right) - (n-2)\pi.$$

Prop. 6.

To find the attraction of a circle, on a point situated perpendicularly over its centre.

This is only a particular case of the last proposition, when n is infinitely great and r infinitely small.

It is easy to see, that the arc whose tangent is $\frac{1}{r}\sqrt{1+\frac{1+r^2}{a^2}b^2}$ will have for its cosine $\frac{ra}{\sqrt{a^2+b^2}}$, if we keep only the first power of r; consequently we may put it under the form $\frac{\pi}{2}$ — arc $\left(\text{sine} = \frac{ra}{\sqrt{a^2+b^2}}\right) = .\frac{\pi}{2} - \frac{ra}{\sqrt{a^2+b^2}}$, very nearly; this multi-

plied by $2n$ is $n\pi - \frac{2nra}{\sqrt{a^2+b^2}} = n\pi - \frac{2\pi a}{\sqrt{a^2+b^2}}$ by putting π for nr. If we substitute this value in the expression for a regular polygon, it becomes $A = 2\pi \left\{1 - \frac{a}{\sqrt{a^2+b^2}}\right\}$, the well known expression of NEWTON.

§. II.

Of the Attraction of Pyramids, and generally of any Solids whatever that are bounded by Planes.

The simplest case of the attraction of such bodies as we are to consider, is that of a pyramid with the attracted point at the vertex: and it fortunately happens, that on this simple case the action of any body whatever may be made to depend; which is the reason of my placing the general problem in this section, though I afterwards treat separately of prisms.

This facility, in the case of pyramids, results from what was shewn in *Cor.* 1. *Prop.* 1, viz. that if we put x for a and mx for b, in the functions $\varphi\,(a, b, r)$, $\chi\,(a, b, r)$, $\psi\,(a, b, r)$, they will become $\varphi\,(1, m, r)$, $\chi\,(1, m, r)$ $\psi\,(1, m, r)$, into which x does not enter.

Prop. 7.

Let figure 5 represent a pyramid with a triangular base umv, the vertex p of the solid being in a line pm, perpendicular to the triangle at the angular point m. It is required to find the action of the pyramid on a point at that vertex.

Draw the perpendicular mr, also pg, po parallel to mr, rv. Join pr, and let r′m′ be parallel to rm. Call pm′, x; r′m′, y; then $y = mx$, m being the tangent of the angle rpm. The attraction of a triangular section of the solid, made by a plane,

passing through r′m′, parallel to the base, will be found by *Prop.* 2, if we put x for a, and mx for b; and is in the respective directions pm, pg, po.

$A = \varphi(1, m, r) + \psi(1, m, r') - \text{arc}\left(\text{tang.} = \frac{1}{r}\right) - \text{arc}\left(\text{tang.} = \frac{1}{r'}\right)$;

$B = \chi(1, m, r) + \chi(1, m, r')$; $C = \psi(1, m, r) - \psi(1, m, r')$.

If we multiply these by $\dot{x}$ and take the fluents, the actions of the pyramid are found to be

$A = x\varphi(1, m, r) + x\varphi(1, m, r') - x\,\text{arc}\left(\text{tang.} = \frac{1}{r}\right) - x\,\text{arc}\left(\text{tang.} = \frac{1}{r'}\right)$;

$B = x\chi(1, m, r) + x\chi(1, m, r')$; $C = x\psi(1, m, r) - x\psi(1, m, r')$.

If the pyramid, whose action we were seeking, had been that whose is u′mv, we must have used the other values of A, B, C given in *Prop.* 2.

Prop. 8.

Let fig. 6 represent any pyramid whatever, whose base uvu′v′u is terminated by right lines; to find its attraction, on a point at the vertex p, both in quantity and direction.

Let a perpendicular from p meet the base at m, and draw lines from this point to all the angles u, v, &c. of the base. It is plain, that the solid will thus be divided into such pyramids as were considered in the last proposition; so that the problem is already solved.

Cor. We may apply this to the pyramid whose base is a rhombus, and the vertex placed perpendicularly over its centre. By proceeding as in the proposition, it will be divided

into four equal triangular pyramids; and, using for each of them the notation of *Prop.* 7, we have for the action of the whole rhomboidal pyramid, on a point at its vertex,

$$A = 4x\phi\,(1, m, r) + 4x\phi\,(1, m, r') - 2\pi x.$$

The other attractions evidently destroy each other.

It is not necessary, in the above proposition, that the perpendicular pm should fall within the base; if it falls without, we shall however have occasion for the following problem.

Prop. 9.

Let umv, fig. 3, be any triangle whatever, p a point any how situated with respect to it; join pm, pu, pv.* It is required to find the attraction of the oblique pyramid pumv, whose base is the triangle umv, on a point at the vertex p.

Let fall, from p, the perpendicular pm′ on the plane of the base umv, draw the lines m′m, m′u, m′v. Find, by *Prop.* 7, the attractions of the pyramids pm′uv, pm′um, pm′vm, whose bases are m′uv, m′um, m′vm, and their common vertex p.

Resolve these attractions into others in the directions of any rectangular co-ordinates, and when thus resolved let the actions of the pyramids

$$\left\{\begin{matrix} pm'uv \\ pm'um \\ pm'vm \end{matrix}\right\} \text{ in the directions of these co-ordinates be } \left\{\begin{matrix} A\,, B\,, C \\ A'\,, B'\,, C' \\ A'', B'', C'' \end{matrix}\right\}.$$

It is plain, that the actions of the pyramid pumv, on the point p, in the directions of the same co-ordinates, will be $A - A' - A''$; $B - B' - B''$; $C - C' - C''$.

* I have not actually drawn the lines, to avoid confusion in the figure.

Prop. 10.

Let fig. 4 represent, in every respect, the same as it did in *Prop.* 4; join pr, pv, pr′; it is required to find the attraction of the square pyramid pmrvr′, on the point p at its vertex.

If we proceed as in *Prop.* 7, but make use of the expressions, for the action of the rectangle mrvr′, found in *Prop.* 4, and put x for a, mx for b, $m'x$ for b', (where $m =$ tang. rpm, $m' =$ tang. r′pm) there will result

$$A = x\phi\,(1, m, r) + x\phi\,(1, m', r') - \frac{\pi x}{2};$$

$$B = x\chi\,(1, m, r) + x\psi\,(1, m', r'); \; C = x\psi\,(1, m, r) + x\chi\,(1, m', r').$$

But it will be better to make use of the more simple expressions, that were given in *Prop.* 4, by which means, we get

$$A = \text{arc}\left(\text{tang.} = \frac{mm'}{\sqrt{1+m^2+m'^2}}\right) x$$

$$B = \left\{L.\,(\sqrt{1+m'^2} + m') - L.\,\frac{\sqrt{1+m'^2+m^2} + m'}{\sqrt{1+m^2}}\right\} x$$

$$C = \left\{L.\,(\sqrt{1+m^2} + m) - L.\,\frac{\sqrt{1+m^2+m'^2} + m}{\sqrt{1+m'^2}}\right\} x.$$

Prop. 11.

Let the base of the pyramid be a regular polygon; the vertex situated perpendicularly over the centre of the base; the attracted point at the vertex.

By making use of the expression in Proposition 5, putting x for a, and mx for b (m being the tangent of the angle, at the vertex, formed by the axis of the pyramid and a line drawn from the vertex to the middle of one of the sides of the base, we get

$A = 2nx$ arc $\left(\text{tang.} = \frac{1}{r}\sqrt{1 + (1+r^2)m^2}\right) - (n-2)\,\pi x.$

Hitherto, I have considered the action of a pyramid, only on a point at its vertex; the case which next presents itself, is that of the attraction on a point p′ (fig. 5 and 6) any where in the produced axis mpp′. It would be easy to give a direct solution to this problem, but I choose rather to make it depend on the propositions that have been already established: and to shew that the functions $\varphi\,(a, b, r)$ $\chi\,(a, b, r)$ $\psi\,(a, b, r)$, of which so much use has been made, in the preceding investigations, are sufficient in all cases of the attraction of bodies bounded by planes.

Prop. 12.

Let pumv (fig. 5) represent the same pyramid as in *Prop.* 7; to find its attraction on a point p′ in the produced axis.

Suppose p′u, p′v joined, the attraction of the pyramid pumv, on the point p′, is the difference of the attractions of the pyramids p′umv, p′upv on the same point; which point being at the common vertex of these two pyramids, their attractions are found by Propositions 7 and 9; and the problem is solved.

Prop. 13.

Let it now be the action of the pyramid pumv, fig. 7, (where the plane of the base umv is not perpendicular to the line mpp′) on the point p′, that is required.

The attraction sought for will still be the difference of the actions of the two pyramids p′umv, p′upv, but these must now *both* be found by *Prop.* 9.

Prop. 14.

Figure 6 representing, in every respect, the same as in *Prop.* 8, it is required to find the action of the pyramid, on the point p′, in the produced axis.

As this solid may be divided into others, situated, with respect to p′, like that in *Prop.* 12, the problem is solved by what was shewn there.

If the attracted point was not in the line p′pm, perpendicular to the base, but in some other line $\pi p \mu$, passing through the vertex, and meeting the base in μ; draw lines from μ to all the angles of the base, and the solid will be divided in such pyramids as were treated of in *Prop.* 13.

Prop. 15.

Let pumv, fig. 7, be any triangular pyramid whatever, and let it be any how cut by a plane, whose intersection with the pyramid is the triangle $\alpha\beta\gamma$; it is required to determine the action of the portion $\varphi\beta\gamma$vum (which is cut off by the plane) on a point at p.

The attraction, of the solid in question, is the difference of the actions of the pyramids pumv and p$\alpha\beta\gamma$, which actions are found by *Prop.* 9.

Prop. 16, *or general Problem.*

To find the action of any solid, bounded by planes, on a point either within or without it.

It is plain, that by drawing lines, from the attracted point through the solid, this may always be divided, either into such pyramids (with the point at the vertex) as were considered in

Prop. 9, or into portions of pyramids, like that treated of in *Prop.* 15; and consequently the solution of the problem may be obtained by means of these propositions.

§. III.

Of the Attraction of Prisms.

Prop. 17.

To find the attraction of a right prism, whose base is a regular polygon, on a point in the produced axis.

We saw, in *Prop.* 5, that the action of a regular polygon, on a point situated perpendicularly over its centre is

$$A = 2n \text{ arc } \left(\text{tang.} = \frac{1}{r}\sqrt{1 + \frac{1+r^2}{a^2}b^2}\right) - (n-2)\,\pi.$$

To find the attraction of the prism, change a into x, multiply by $\dot{x}$, and take the fluent.

Now $\int \dot{x} \text{ arc } \left(\text{tang.} = \frac{1}{rx}\sqrt{x^2 + (1+r^2)\,b^2}\right) = \int \dot{x}\,\varphi\,(x, b, r)$ $= x \text{ arc } \left(\text{tang.} = \frac{1}{rx}\sqrt{x^2 + (1+r^2)\,b^2}\right) - \int x \,\dot{\overline{\text{arc}}}\left(\text{tang.} = \frac{1}{rx}\sqrt{x^2 + (1+r^2)\,b^2}\right) = x \text{ arc } \left(\text{tang.} = \frac{1}{rx}\sqrt{x^2 + (1+r^2)\,b^2}\right)$ $+ \int \frac{rb^2 x\dot{x}}{(b^2+x^2)\sqrt{(1+r^2)\,b^2+x^2}}$, because $\dot{\overline{\text{arc}}} = \frac{\dot{\overline{\text{tang.}}}}{1 + \text{tang.}^2}$; and, by taking the fluent,* it becomes $= x \text{ arc } \left(\text{tang.} = \frac{1}{rx}\sqrt{x^2 + (1+r^2)\,b^2}\right) - b\,\text{L}.\,\frac{rb + \sqrt{(1+r^2)\,b^2+x^2}}{\sqrt{b^2+x^2}}$.

* Put $x^2 + (1+r^2)\,b^2 = z^2$, $b^2 + x^2 = z^2 - r^2b^2$, $x\dot{x} = z\dot{z}$; then $\int \frac{rb^2 x\dot{x}}{(b^2+x^2)\sqrt{(1+r^2)\,b^2+x^2}} = \int \frac{rb^2\dot{z}}{z^2 - r^2b^2} = \frac{b}{2}\,\text{L}.\,\frac{z+rb}{z-rb}$ (SIMPSON's Fluxions, p. 140,) $= \frac{b}{2}\,\text{L}.\,\frac{(z+rb)^2}{z^2-r^2b^2} = b\text{L}.\,\frac{\sqrt{(1+r^2)\,b^2+x^2} + rb}{\sqrt{b^2+x^2}}$.

For the sake of brevity, call this quantity $F(b, r, x)$ and we have for the attraction of the prism,

$$A = 2nF(b, r, x) - (n-2)\pi x + \text{corr.}$$

The attraction of any other right prism, *in the direction of its length,* depends on the same function $F(b, r, x)$; as in

Prop. 18.

To find the attraction of a right prism, whose base is a rectangle, on a point in the produced axis.

We saw in *Prop.* 4, that the action of a rectangle, on a point situated perpendicularly over its centre is

$$A = 4\phi(a, b, r) + 4\phi(a, b', r') - 2\pi,$$

where b and b' are the halves of the sides of the rectangle, and r and r' the tangents of the angles, formed respectively by those sides and the diagonal. By changing a into x, and multiplying by $\dot{x}$, we have, for the prism,

$A = 4\int\dot{x}\phi(x, b, r) + 4\int\dot{x}\phi(x, b', r') - 2\pi x$; whence, by what was done in the last proposition,

$$A = 4F(b, r, x) + 4F(b', r', x) - 2\pi x + \text{corr.}$$

Prop. 19.

Let the base of the prism be a rhombus, the attracted point in the produced axis of the prism.

We found, in *Cor.* 1, *Prop.* 2, that the action of a rhombus on a point situated perpendicularly over its centre, is (keeping the notation there used)

$A = 4\phi(a, b, r) + 4\phi(a, b, r') - 2\pi$, therefore, proceeding as before, we have, for the prism,

$$A = 4F(b, r, x) + 4F(b, r', x) - 2\pi x + \text{corr.}$$

We will now consider the attractions of prisms more generally.

Prop. 20.

Let pr′v′vrm, fig. 8, be a right prism, whose base is the triangle vrm right angled at r. It is required to determine its attraction on a point at p, in the directions pm, pr′, po; po being parallel to r′v′.

If we wish to obtain a solution by means of what has been already done, we may conceive the solid under consideration to be divided into two pyramids; viz. the pyramid pmrv with the triangular base mrv; and the other pyramid pr′v′vr, whose base is the rectangle r′v′vr; the point p being at the common vertex of both.

Put pm $= x$, mr $=$ pr′ $= x'$; conceive the diagonal vr′ to be drawn, and put tang. rmv $=$ tang. r′pv′ $= r$; tang. v′rr $= r'$; tang. vr′v′ $= r''$; tang. rpm $= m$; tang. rpr′ $= m'$.

We get immediately, from propositions 7 and 10 (putting A, B, C for the respective actions in the directions pm, pr′, po)

$$A = x\varphi\,(1, m, r) - x \text{ arc}\left(\text{tang.} = \frac{1}{r}\right) + x'\chi\,(1, m', r') + x'\psi\,(1, r, r'');$$

$$B = x\chi\,(1, m, r) + x'\varphi\,(1, m', r') + x'\varphi\,(1, r, r'') - x'\,\frac{\pi}{2};$$

$$C = x\psi\,(1, m, r) + x'\psi\,(1, m', r') + x'\chi\,)1, r, r'').$$

In finding these values, the first expressions of *Prop.* 10 were made use of; if we take the others, there will result the simpler forms

$$A = x\varphi\,(1, m, r) - x \text{ arc}\left(\text{tang.} = \frac{1}{r}\right) + x'\left\{\text{L.}\left(\sqrt{1+r^2}+r\right) - \text{L.}\,\frac{\sqrt{1+r^2+m'^2}+r}{\sqrt{1+m'^2}}\right\};$$

$$B = x\chi\,(1, m, r) + x' \text{ arc}\left(\text{tang.} = \frac{rm'}{\sqrt{1+r^2+m'^2}}\right);$$

$C = x\psi\ (1,\ m,\ r) + x'\left\{L\ .\ (\sqrt{1+m'^2} + m') - L\ .\ \frac{\sqrt{1+m'^2+r^2} + m'}{\sqrt{1+r^2}}\right\}.$

These are, perhaps, as simple expressions as can be had in this case; we may however find others; nor is it necessary to conceive the solid divided into pyramids.

Thus we may obtain the value of the force A, by the method of the preceding propositions: a triangular section of the prism, parallel to the end r′pv′, and at a distance x from that end, will have its action on p expressed by $\varphi\ (x, x', r)$ — arc (tang. $= \frac{1}{r}$), using the same notation as before; therefore, the action of the pyramid, in the direction pm, is

$A = \int \dot{x}\varphi\ (x, x', r) - x$ arc (tang. $= \frac{1}{r}$) $= F\ (x', r, x) - x$ arc (tang. $= \frac{1}{r}$) + corr.

Another Way of finding the force B.

Draw ks parallel to r′v′; call pk, x'; then ks $= r \times x'$. Conceive a plane to pass through ks, parallel to the back rvv′r′ of the prism; the section made by it will be a rectangle whose sides are x and rx'. The action of this rectangle on p, in the direction pr′, is arc (tang. $= \frac{rx}{\sqrt{x'^2 + r^2x'^2 + x^2}}$) by *Prop.* 4, wherefore the action of the prism, in the same direction, is

$B = \int \dot{x}'$ arc (tang. $= \frac{rx}{\sqrt{x^2 + (1+r^2)\,x'^2}}$) where x' is the variable quantity, we have then

$B = x'$ arc (tang. $= \frac{rx}{\sqrt{x^2 + (1+r^2)\,x'^2}}$) $- \int x'\ \dot{\overline{\text{arc}}}$ (tang. $= \frac{rx}{\sqrt{x^2+(1+r^2)x'^2}}$).

Put $\frac{x^2}{1+r^2} = c^2$, and the last term becomes $-\int \dot{x}' \text{ arc (tang.} \frac{rc}{\sqrt{c^2+x^2}}) = + rc \int \left\{ \frac{\dot{x}'}{\sqrt{c^2+x^2}} - \frac{(1+r^2)c^2\dot{x}'}{\left\{(1+r^2)c^2+x'^2\right\}\sqrt{c^2+x'^2}} \right\} = rc$

$$\left\{ L\,(x'+\sqrt{c^2+x'^2}) - \frac{\sqrt{1+r^2}}{r} L\left(\frac{x'}{\sqrt{x^2+x'^2}} + \sqrt{\frac{1}{r^2}+\frac{x'^2}{x^2+x'^2}}\right)\right\};$$

therefore

$$B = x' \text{ arc (tang.} = \frac{rx}{\sqrt{x^2+(1+r^2)x'^2}}) + rc\, L\,(x'+\sqrt{c^2+x'^2})$$

$$- x\, L\left(\frac{x'}{\sqrt{x^2+x'^2}} + \sqrt{\frac{1}{r^2}+\frac{x'^2}{x^2+x'^2}}\right) + \text{corr.}$$

If the fluent is to begin when $x' = o$, the correction is

$$- rc\, L\,.\, c + x\, L\, \frac{1}{r}.$$

I have dwelt the longer on this proposition, because the attraction of right prisms, in all cases, may be made to depend on it.

Cor. 1. It is in the first place evident, that by means of this proposition, we may find, by parts, the force, both in quantity and direction, with which a point q, any where on the edge pm of the scalene prism represented by fig. 9, is attracted.

The same may be said of the action on p′ any where in the produced axis; this will be the difference of the actions of two prisms, like that in the figure.

Cor. 2. Moreover, if, instead of the prism in fig. 9, the point q was placed any where on the edge of a prism whose base is the triangle quv, fig. 10, the action may still be found; for it will now depend on the difference of the action of such prisms as were treated of in the Proposition; that is to say, the action of the prism whose base is quv as the difference of the actions of those whose bases are qvr, qur, qr being a perpendicular on vu produced.

Prop. 21.

To find the quantity and direction of the attraction of a right prism, whose base is any triangle whatever, on a point any where on its surface.

Let the triangle uvu′, fig. 11, be a section of the prism, parallel to its base, and through the attracted point p. Let fall the perpendiculars pr, pr′, on the opposite sides: and the solid may be divided into four prisms, whose bases are the right angled triangles pur, prv, pvr′, pr′u′, and the attraction of each of these, both in quantity and direction, is given by *Prop.* 20.

It is plain that there may be other cases of this problem, besides the one here considered; for instance, one of the perpendiculars may fall beyond the base; but it would be endless, in a subject of this kind, to consider every particular case, and in none can the intelligent reader find the smallest difficulty.

Prop. 22.

To find the attraction of any prism, fig. 12, whose base is a *convex* polygon uvu′v′, on a point q any where within it.

As such a solid may be divided into triangular prisms, like those in *Prop.* 20 and its corollaries, with the attracted point on the common edge pm, the problem is already solved.

If the point be at p′, in the line mp produced, the action on it may still be found, being the difference of the actions of two prisms, like that in the figure.

Prop. 23.

Let vuv′, fig. 13, be the section of an isosceles prism; pv a line passing through the vertex v perpendicularly on the

middle of the base. It is required to find the action of the prism on any point p in the line pv.

This action is equal to the action of the prism whose base is the triangle v′pu (given by *Prop*. 20), less the actions of the prisms whose bases, or sections, are the triangles v′vp, uvp, which are found by *Cor*. 2, *Prop*. 20.

Scholium.

I considered here only an isosceles prism; but the solution is just the same, if the section of the prism is any triangle whatever, as v′uv, fig. 14, and the action on a point p, (situated in the line uv produced) is required. For the attraction wanted will be the difference of attractions of the two prisms whose bases are the triangles v′up, v′vp, and these are given by *Cor*. 2, *Prop*. 20.

Suppose the base of the prism, whose attraction is required, to be the trapezium v′u$\alpha\beta$, fig. 14, the action of this on p, being the difference of the actions of the triangular prisms, whose bases are v′uv, $\beta\alpha$v, is found by the case just now considered.

In this manner, might cases be multiplied without end; but I think it is sufficiently plain, that by means of the preceding propositions and scholium, we may find the action of any prism whatever, on a point either within or without it.

§. IV.

Of the Attraction of certain Solids not terminated by Planes.

The expressions, arrived at in the first section, are useful in finding the attraction, not only of such solids as are bounded by planes, but of a great variety of others; viz. of such as

have their sections in one direction continuous curves, whilst, being cut in a different way, there results, from their intersection with a plane, a polygon, or rectangle, or some other right lined figure.

As no one, that I know of, has considered the actions of such bodies, I shall offer no apology for giving a few examples.

Let uvv′u′, fig. 15, represent any regular polygon, whose plane is perpendicular to the line pm, and its centre in that line: moreover, let this polygon be variable in magnitude, and move parallel to itself in the direction pm, in such a manner, that the middle point r of each of its sides uv, may describe a given curve pr.

Prop. 24.

Let it be required to find the attraction of the solid thus generated by the polygon, when the curve pr is a circle,* and the attracted point at the vertex p of the solid.

The attraction of a regular polygon was found in *Prop.* 5 and it will be adapted to our present purpose, by putting x^2 for a^2, and $2kx - x^2$ for b^2, where k is the radius of the circle pr: and we have, for the attraction of the solid,

$$A = 2nf\dot{x} \text{ arc } \left(\text{tang.} = \frac{1}{r}\sqrt{1 + \frac{1+r^2}{x^2}(2kx - x^2)}\right) - (n-2)\,\pi x$$

$$\text{or } A = 2nf\dot{x} \text{ arc } \left(\text{tang.} = \frac{1}{rx}\sqrt{2k(1+r^2)\,x - r^2x^2}\right) - (n-2)\,\pi x.$$

That part of A, under the sign of integration, equals

$$2nx \text{ arc } \left(\text{tang.} = \frac{1}{rx}\sqrt{2k(1+r^2)\,x - r^2x^2}\right) - 2nfx\, \dot{\overline{\text{arc}}} \left(\text{tang.} = \frac{1}{rx}\sqrt{2k(1+r^2)\,x - r^2x^2}\right),$$

* We may, not improperly, term this solid *a polygonal sphere.*

and, if we put $zz = x$, the last term becomes

$$-\ 2n\int z^2\ \overset{\cdot}{\overline{\text{arc}}}\ \left(\text{tang.} = \frac{1}{rz}\sqrt{2k\,(1+r^2) - r^2z^2}\right)$$

$$= 2n\int \frac{rz^2\dot{z}}{\sqrt{2k\,(1+r^2) - r^2z^2}} = 2n\int \frac{z^2\dot{z}}{\sqrt{\frac{2k\,(1+r^2)}{r^2} - z^2}};\ \text{this fluent is}$$

$$2n\left\{\frac{k\,(1+r^2)}{r^2}\ \text{arc}\ \left(\text{sine} = \frac{rz}{\sqrt{2k\,(1+r^2)}}\right) - \frac{1}{2}\,z\sqrt{\frac{2k\,(1+r^2)}{r^2} - z^2}\right\}$$

or by putting its value for z,

$$\frac{2nk\,(1+r^2)}{r^2}\ \text{arc}\ \left(\text{sine} = \frac{r\sqrt{x}}{\sqrt{2k\,(1+r^2)}}\right) - n\sqrt{\frac{2k\,(1+r^2)}{r^2}\,x - x^2}.$$

Collecting all the parts of A, we have at length

$A = 2nx$ arc $\left(\text{tang.} = \frac{1}{rx}\sqrt{2k\,(1+r^2)\,x - r^2x^2}\right) - \frac{n}{r}$ $\sqrt{2k\,(1+r^2)\,x - r^2x^2} + \frac{2nk\,(1+r^2)}{r^2}$ arc $\left(\text{sine} = \frac{r\sqrt{x}}{\sqrt{2k\,(1+r^2)}}\right) -$ $(n-2)\,\pi x +$ corr.

But it is easy to see that each of the arcs in this expression is the complement of the other; put then $A =$ arc $\left(\text{tang.} = \frac{1}{rx}\right.$ $\left.\sqrt{2k\,(1+r^2)\,x - r^2x^2}\right)$ and the expression becomes

$$A = \left\{\frac{nk}{r^2} + n\,(k-x)\right\}(\pi - 2A) + 2x\pi - nx\,.\,\text{tang.}\,A + \text{corr.}$$

When $x = 0$, $A = \frac{\pi}{2}$, so that, if the fluent is to begin when $x = 0$, no correction is necessary.

When $x = 2k$, $A =$ arc $\left(\text{tang.} = \frac{1}{r}\right) = \frac{(n-2)\,\pi}{2n}$, and $\sqrt{2k\,(1+r^2)\,x - r^2x^2} = 2k$, $\pi - 2A = \pi - \frac{(n-2)\,\pi}{n} = \frac{2\pi}{n}$: whence we have for *the attraction of the whole solid*

$$A = \frac{1+r^2}{r^2} \times 2k\pi - \frac{2nk}{r} \ldots\ldots\ldots\ (\alpha).$$

This will appear to be an expression of great simplicity, if we reflect what very different solids it belongs to, from that whose section is a triangle, to the sphere whose section is a circle.

In the latter case n is infinitely great and r infinitely small; r is also the tangent of $\frac{\pi}{n} = \frac{\pi}{n} + \frac{1}{3} \cdot \frac{\pi^3}{n^3}$, if we reject the other terms of the expansion, on account of their smallness.

The expression for A may then, in this case, be put into the form

$$A = 2k\pi + \frac{2k\pi}{r\left\{\frac{\pi}{n} + \frac{1}{3} \cdot \frac{\pi^3}{n^3}\right\}} - \frac{2nk}{r};$$

now $r\left\{\frac{\pi}{n} + \frac{1}{3} \cdot \frac{\pi^3}{n^3}\right\} = (q \cdot p)\, r\left\{\frac{\pi}{n} + \frac{1}{3} \cdot \frac{\pi}{n} r^2\right\} = r\left(1 + \frac{r^2}{3}\right)\frac{\pi}{n}$, whence the second term of the last number is changed into

$$\frac{2kn}{r\left(1 + \frac{r^2}{3}\right)} = \frac{2kn\left(1 - \frac{r^2}{3}\right)}{r} = \frac{2kn}{r} - \frac{2}{3} knr = \frac{2nk}{r} - \frac{2}{3} k\pi, \text{ by}$$

putting π for nr: and, by substituting this value, we have at last $A = \frac{4}{3} k\pi$; which is the well known expression for the attraction of a sphere on a point at its surface.

If the generating polygon is a square instead of a circle, $r =$ tang. $45^\circ = 1$, and equation (α) gives $A = 4k\,(\pi - 2)$ $= 4k \times 1{,}14159$, &c. which exceeds the attraction of the sphere by about one-tenth, if pr is the same circle in both.

Cor. If we would know the radius (k') of a sphere, which shall attract, a point at its surface, as much as a polygonal sphere, of the length $2k$, does a point at its vertex, we have only to put

$$\frac{4}{3}\pi k' = \frac{1+r^2}{r^2}\, 2\pi k - \frac{2nk}{r}, \text{ whence}$$

$$k' = \frac{3}{2} k\left\{\frac{1+r^2}{r^2} - \frac{n}{\pi r}\right\}.$$

Prop. 25.

Let the directing curve pr be a parabola; the attracted point at the vertex p of the solid.

We must here make use of the formula in *Prop.* 5, as we did in the last example; but as the equation of a parabola is $y^2 = \alpha x$, this latter quantity must be put for b^2. Thus we get, for the attraction of the solid,

$A = 2n\int \dot{x}$ arc $\left(\text{tang.} = \frac{1}{rx} \sqrt{x^2 + (1+r^2)\alpha x}\right) - (n-2)\pi x$.

The part, having the sign of integration, may be put under the form $2nx$ arc. $\left(\text{tang.} = \frac{1}{rx} \sqrt{x^2 + (1+r^2)\alpha x}\right)$ $2n\int x \,\dot{\overline{\text{arc}}}$ $\left(\text{tang.} = \frac{1}{rx} \sqrt{x^2 + (1+r^2)\alpha x}\right)$; in the last term of which put $z^2 = x$, and it will become

$$- 2n\int z^2 \,\dot{\overline{\text{arc}}} \left(\text{tang.} = \frac{1}{rz} \sqrt{z^2 + (1+r^2)\alpha}\right)$$
$$= 2n\alpha r \int \left\{ \frac{\dot{z}}{\sqrt{z^2 + (1+r^2)\alpha}} - \frac{\alpha \dot{z}}{(z^2+\alpha)\sqrt{z^2+(1+r^2)\alpha}} \right\}$$
$$= 2n\alpha r \left\{ \text{L}\left(z + \sqrt{z^2 + (1+r^2)\alpha}\right) - \frac{1}{r} \text{arc} \left(\text{sine} = \frac{r}{\sqrt{1+r^2}} \times \frac{z}{\sqrt{z^2+\alpha}}\right) \right\}.$$

Collecting all the terms, we have at length

$$A = 2nx \text{ arc} \left(\text{tang.} = \frac{1}{rx}\sqrt{x^2 + (1+r^2)\alpha x}\right) - 2n\alpha \text{ arc} \left(\text{sine} = \frac{r}{\sqrt{1+r^2}} \times \frac{\sqrt{x}}{\sqrt{x+\alpha}}\right) + 2n\alpha r \text{ L} \left(\sqrt{x} + \sqrt{x + (1+r^2)\alpha}\right) - (n-2)\pi x + \text{corr.}$$

It is observable here, as in the last proposition, that each of the arcs in this expression is the complement of the other; put $A =$ arc $\left(\text{tang.} = \frac{1}{rx}\sqrt{x^2 + (1+r^2)\alpha x}\right)$, and the attrac-

tion becomes $A = n(x + \alpha) 2A - \pi) + 2x\pi + 2n\alpha r L(\sqrt{x} + \sqrt{x + (1 + r^2)\alpha}) +$ corr.

When $x = o$, $A = \frac{\pi}{2}$; so that, if the fluent is to begin at that term, we have corr. $= - 2n\alpha r L \sqrt{(1 + r^2)\alpha}$.

If we would find, from this expression, the attraction of the limit of these solids (which is the parabolic conoid) we must observe, that the arc A may be put under the form $\frac{\pi}{2}$ — arc $\left(\text{sine} = \frac{r}{\sqrt{1+r^2}} \times \frac{\sqrt{x}}{\sqrt{x+\alpha}}\right)$, whence, because r is infinitely small, $2A - \pi = - 2r \cdot \frac{\sqrt{x}}{\sqrt{x+\alpha}} = - 2\frac{\pi}{n} \cdot \frac{\sqrt{x}}{\sqrt{x+\alpha}}$, qu. prox.; substituting this value, and π for nr, and neglecting r^2, we get

$$A = 2\pi \left\{x - \sqrt{x^2 + \alpha x} + \alpha L(\sqrt{x} + \sqrt{x + \alpha})\right\} + \text{corr.}$$

for the action of a parabolic conoid on a point at its vertex.

Prop. 26.

Let the curve pr be a parabola *convex* to the axis pm, in which case $y = \frac{x^2}{\alpha}$; and we have, by proceeding as before,

$A = 2n\int \dot{x}$ arc $\left(\text{tang.} = \frac{1}{r}\sqrt{1 + \frac{(1+r^2)}{\alpha^2} x^2}\right) - (n-2)\pi x$; or

$A = 2nx$ arc $\left(\text{tang.} = \frac{1}{r}\sqrt{1 + \frac{1+r^2}{\alpha^2} x^2}\right) 2n\int x\, \dot{\overline{\text{arc}}}\left(\text{tang.} = \frac{1}{r}\sqrt{1 + \frac{1+r^2}{\alpha^2} x^2}\right) - (n - 2)\pi x$; if we put $a^2 = \frac{\alpha^2}{1+r^2}$, the term, under the integral sign, becomes

$$- 2n\int x\, \dot{\overline{\text{arc}}}\left(\text{tang.} = \frac{1}{ar}\sqrt{a^2 + x^2}\right)$$

$$= - 2nar \int \left\{\frac{\dot{x}}{\sqrt{a^2+x^2}} - \frac{(1+r^2)\, a^2 \dot{x}}{\left\{(1+r^2)a^2+x^2\right\}\sqrt{a^2+x^2}}\right\}$$

$$= -2nar\left\{L\left(x+\sqrt{a^2+x^2}\right) - \frac{\sqrt{1+r^2}}{r} L\left(\frac{x}{\sqrt{x^2+\alpha^2}} + \sqrt{\frac{1}{r^2}+\frac{x^2}{x^2+\alpha^2}}\right)\right\};$$ so that

$$A = 2nx \text{ arc } (\text{tang.} = \frac{1}{r}\sqrt{1+\frac{1+r^2}{\alpha^2}x^2}) - (n-2)\pi x + \text{corr.}$$
$$-\ 2nr\frac{\alpha}{\sqrt{1+r^2}} L\left(x+\sqrt{\frac{\alpha^2}{1+r^2}+x^2}\right) + 2n\alpha L\left\{\frac{x}{\sqrt{x^2+a^2}} + \sqrt{\frac{1}{r^2}+\frac{x^2}{x^2+\alpha^2}}\right\}.$$

This is the attraction of a polygonal parabolic spike, on a particle at the point. When the polygon becomes a circle,

$$\text{Arc } (\text{tang.} = \frac{1}{r}\sqrt{1+\frac{1+r^2}{\alpha^2}x^2}) = \frac{\pi}{2} - \frac{r\alpha}{\sqrt{\alpha^2+x^2}},$$ and

$$L\left\{\frac{x}{\sqrt{x^2+\alpha^2}} + \sqrt{\frac{1}{r^2}+\frac{x^2}{x^2+\alpha^2}}\right\} = L\frac{1}{r} + \frac{rx}{\sqrt{\alpha^2+x^2}},$$ whence it will easily appear that, in the case under consideration,

$$A = 2\pi\left\{x - \alpha L\left(x+\sqrt{\alpha^2+x^2}\right)\right\} + \text{corr.}$$

We may conceive the plane uvv′u′, fig. 15, instead of a regular polygon to be a rectangle, moving along the line pm, as in the former case, with its centre in that line; and, with the middle points r and r′ of its sides, touching curves pr, pr′ either of the same or different kinds.

The section of the generated solid, or *groin*, perpendicular to its axis, will have its action on the point p (if we put $x = \text{pm}$, and b and b' for the sides of the rectangle) expressed by

$$4 \text{ arc } (\text{tang.} = \frac{bb'}{x\sqrt{x^2+b^2+b'^2}};$$

and if we multiply this by $\dot{x}$, and put for b and b' their values, given by the equations of the curves pr, pr′, the fluent will be the attraction of the generated solid.

Prop. 27.

Let pr be a circle, pr′ a parabola; to find the attraction of the solid on a point p at its vertex.

Let the radius of the circle be k, the parameter of the parabola α: then we have $b = \sqrt{2kx - x^2}$; $b' = \sqrt{\alpha x}$, and

$$A = 4\int \dot{x} \text{ arc } \left(\text{tang.} = \frac{\sqrt{\alpha x}\ \sqrt{2kx - x^2}}{x\ \sqrt{(2k+\alpha)\, x}}\right),$$

or $A = 4\int \dot{x}$ arc $\left(\text{tang.} = \frac{c\sqrt{2k-x}}{\sqrt{x}}\right)$, if we put $c = \frac{\sqrt{\alpha}}{\sqrt{2k+\alpha}}$, and taking the fluent by parts

$$A = 4x \text{ arc } \left(\text{tang.} = \frac{c\sqrt{2k-x}}{\sqrt{x}}\right) - 4\int x \dot{\overline{\text{arc}}} \left(\text{tang.} = \frac{c\sqrt{2k-x}}{\sqrt{x}}\right),$$

the last term of which, if we put z^2 for x, becomes

$-4\int z^2 \dot{\overline{\text{arc}}} \left(\text{tang.} = \frac{c\sqrt{2k-z^2}}{z}\right)$ and this, by actually taking the fluxion of the arc $= \int \frac{8kcz^2\dot{z}}{\left\{(1+c^2)\, z^2 + 2kc^2\right\} \sqrt{2k - z^2}}$, or,

by restoring the value of c, $= 4\sqrt{\alpha} \,.\, \sqrt{2k+\alpha} \,.\int \frac{z^2 \dot{z}}{(z^2+\alpha)\sqrt{2k-z^2}}$, or by division

$$= 4\sqrt{\alpha} \,.\, \sqrt{2k+\alpha} \,.\int \left\{\frac{\dot{z}}{\sqrt{2k-z^2}} - \frac{\alpha \dot{z}}{(z^2+\alpha)\sqrt{2k-z^2}}\right\}$$

$= 4\sqrt{\alpha}\ \sqrt{2k+\alpha}$ arc $\left(\text{sine} = \frac{z}{\sqrt{2k}}\right) - 4\alpha$ arc $\left(\text{sine} = \frac{\sqrt{2k+\alpha}}{\sqrt{2k}} \times \frac{z}{\sqrt{z^2+\alpha}}\right)$;

so that we have at last

$A = 4x$ arc $\left(\text{tang.} = \frac{\sqrt{\alpha}}{\sqrt{2k+\alpha}} \times \frac{\sqrt{2k-x}}{\sqrt{x}}\right) + 4\sqrt{\alpha}\ \sqrt{2k+\alpha}$ arc $\left(\text{sine} = \frac{\sqrt{x}}{\sqrt{2k}}\right) - 4\alpha$ arc $\left(\text{sine} = \frac{\sqrt{2k+\alpha}}{\sqrt{2k}} \times \frac{\sqrt{x}}{\sqrt{x+\alpha}}\right) +$ corr......(β).

As this expression vanishes when $x = o$, if the fluent is to

begin at that value of x, no correction is to be added. The last of the arcs, in the expression above, is the complement of the first; denote then the first by A, and we have

$$A = 4(x+\alpha)A - 2\alpha\pi + 4\sqrt{\alpha}\sqrt{2k+\alpha}\,\text{arc}\left(\text{sine} = \frac{\sqrt{x}}{\sqrt{2k}}\right);$$

lastly, if we want the whole fluent, when $x = 2k$, we get

$$A = 2\pi\left(\sqrt{2k\alpha + \alpha^2} - \alpha\right).$$

Cor. 1. If we make α infinite, in this last value of A, it becomes $A = 2k\pi$; which is the action of *an infinitely long circular cylinder*, on a point at its surface. This is the attraction of the whole cylinder when $x = 2k$; to find the same for *any* value of x, make α infinite in formula (β); this gives

$$A = 4x\,\text{arc}\left(\text{tang.} = \frac{\sqrt{2k-x}}{\sqrt{x}}\right) + 4(\alpha+k)\,\text{arc}\left(\text{sine} = \frac{\sqrt{x}}{\sqrt{2k}}\right)$$

$-4\alpha\,\text{arc}\left\{\text{sine} = \left(1 + \frac{2k-x}{2\alpha}\right)\frac{\sqrt{x}}{\sqrt{2k}}\right\}$; but (EULERI Calc. Diff. p. 376)

$$\text{Arc}\left\{\text{sine} = \left(1 + \frac{2k-x}{2\alpha}\right)\frac{\sqrt{x}}{\sqrt{2k}}\right\} = \text{arc}\left(\text{sine} = \frac{\sqrt{x}}{\sqrt{2k}}\right) + \frac{\sqrt{2kx - x^2}}{2\alpha}$$

qu. prox.; the substitution of this value gives

$$A = 4x\,\text{arc}\left(\text{tang.} = \frac{\sqrt{2k-x}}{\sqrt{x}}\right) + 4k\,\text{arc}\left(\text{sine} = \frac{\sqrt{x}}{\sqrt{2k}}\right) - 2\sqrt{2kx - x^2},$$

which, because the latter arc is the complement of the former, is changed to

$$A = 2k\pi - 4(k-x)\,\text{arc}\left(\text{tang.} = \frac{\sqrt{2k-x}}{\sqrt{x}}\right) - 2\sqrt{2kx - x^2}.$$

Cor. 2. In like manner we may find the *attraction of an infinitely long parabolic cylinder*, on a point in its surface, at the vertex of the parabola; this is effected by making k infinite in formula (β), whence there results

$$A = 4x\,\text{arc}\left(\text{tang.} = \frac{\sqrt{\alpha}}{\sqrt{x}}\right) - 4\alpha\,\text{arc}\left(\text{tang.} = \frac{\sqrt{x}}{\sqrt{\alpha}}\right) + 4\sqrt{\alpha x};\ \text{or}$$

$$A = 4(x + \alpha) \text{ arc } \left(\text{tang.} = \frac{\sqrt{\alpha}}{\sqrt{x}}\right) + 4\sqrt{\alpha x} - 2\alpha\pi.$$

Cor. 3. If $\alpha = \frac{2}{3}k$, the attraction of the solid in the proposition equals that of a sphere whose radius is k; for by substituting $\frac{2}{3}k$ for α, in the expression $2\pi\,(\sqrt{2k\alpha + \alpha^2} - \alpha)$, it becomes $\frac{4}{3}k\pi$, the action of a sphere whose radius is k on a point in its surface.

The attractions of cylinders of finite length, in directions perpendicular to their axes, are to be found after the manner of this last proposition; but there are not many cases in which they can be expressed by circular arcs and logarithms.

Prop. 28.

Let fig. 16 represent a circle, C the centre, ab, cd two parallel chords; conceive a right cylinder, whose section is the portion abcd of the circle, terminated by the chords ab, cd, to be extended to the distance d above and below the plane of the figure.

It is required to determine the action of this cylinder on a point at C.

Put k for the radius of the circle, and let x be the distance from C of a chord parallel to ab. Then, using the same formula as in the last *Prop.* we have $b = d$, $b' = \sqrt{k^2 - x^2}$, and for the action of the solid, $A = 4\int \dot{x} \text{ arc } \left(\text{tang.} = \frac{d\sqrt{k^2 - x^2}}{x\sqrt{d^2 + k^2}}\right)$ or

$$A = 4x \text{ arc } \left(\text{tang.} = \frac{d\sqrt{k^2 - x^2}}{x\sqrt{d^2 + k^2}}\right) - 4\int x \, \dot{\text{arc}} \left(\text{tang.} = \frac{d\sqrt{k^2 - x^2}}{x\sqrt{d^2 + k^2}}\right)$$

$$= 4x \text{ arc } \left(\text{tang.} = \frac{d\sqrt{k^2 - x^2}}{x\sqrt{d^2 + k^2}}\right) - 4dL \cdot \frac{\sqrt{d^2 + k^2} + \sqrt{k^2 - x^2}}{\sqrt{d^2 + x^2}} + \text{corr.}$$

If the fluent is to begin when $x = o$, the correction is

$4dL.\frac{\sqrt{d^2+k^2}+k}{d}$; and the fluent, taken from $x=o$ to $x=k$, is

$$A = 4dL\,\frac{\sqrt{d^2+k^2}+k}{d},$$

which agrees with what Mr. PLAYFAIR found in a different manner; except in this case, the object of the present proposition is different from that of his, which finds the action of such portions of the cylinder as have *sectors* for their bases.

Prop. 29.

Let the base of the cylinder be the figure parmnbp, fig. 17, the curves par, pbn being inverted parabolas; or in which $pm^2 = \alpha \times rm$. Let the attracted point be at p; and let the cylinder be extended to the distance d above and below the plane of the figure.

Using the same formula as before, and putting $pm = x$, we have $b = d$, $b' = \frac{x^2}{\alpha}$; and, for the action of the solid,

$$A = 4\int \dot{x} \text{ arc } \left(\text{tang.} = \frac{dx}{\sqrt{\alpha^2 d^2 + \alpha^2 x^2 + x^4}}\right), \text{ or}$$

$$A = 4x \text{ arc } \left(\text{tang.} = \frac{dx}{\sqrt{\alpha^2 d^2 + \alpha^2 x^2 + x^4}}\right) - 4\int x\, \dot{\overline{\text{arc}}}\ \left(\text{tang.} = \frac{dx}{\sqrt{\alpha^2 d^2 + \alpha^2 x^2 + x^4}}\right);$$

the last term of which becomes, by taking the fluxion of the arc,

$$-4\int \frac{\alpha^2 d^3 x\dot{x} - dx^5 \dot{x}}{\left\{\alpha^2 d^2 + (\alpha^2 + d^2)x^2 + x^4\right\} \sqrt{\alpha^2 d^2 + \alpha^2 x^2 + x^4}}$$

an expression integrable by circular arcs and logarithms.

When d is infinite, this fluent takes a very simple form, viz.

$$-4\int \frac{\alpha x\dot{x}}{\alpha^2 + x^2} = -2\alpha \int \frac{\frac{2x\dot{x}}{\alpha^2}}{1 + \frac{x^2}{\alpha^2}} = -2\alpha L\left(1 + \frac{x^2}{\alpha^2}\right), \text{ and, in}$$

this case

$$A = 4x \text{ arc } \left(\text{tang.} = \frac{x}{\alpha}\right) - 2\alpha L \left(1 + \frac{x^2}{\alpha^2}\right).$$

Cor. 1. Draw the right lines pr, pn; because $\frac{x}{\alpha} = \frac{\text{pm}}{\alpha} = \frac{\text{rm}}{\text{pm}}$ = tang, rpm, the last expression may be put into the form

$$A = 4x \text{ arc. rpm} - 4\alpha L \text{. sec. rpm.}$$

Cor. 2. If, in the last value of B in *Prop.* 20, we make x infinite, there results $B = x'$ arc (tang. $= r$); from whence it is plain that the first term of the expression for A in the last cor. viz. $4x$ arc rpm, expresses the action of an infinitely long prism, whose base is the triangle rpn, on the point p.

Consequently, the other term of A, or $4\alpha L$. sect. rpm, is the action of the infinitely long solid whose base consists of the parabolic segments parp, pbnp.

We may next consider the generating plane uvv′u′, fig. 15, to be a rhombus, given in *species*, and so varying in magnitude, as to touch four similar and equal curves, at those points where perpendiculars from the centre of the rhombus fall on its sides.

Prop. 30.

Let the guiding curves be semi-circles, to the radius k; the attracted point at the vertex p.

We saw, in *Prop.* 2, *Cor.* 1, that the action of a rhombus, on a point placed perpendicularly over its centre, is $A = 4$ arc $\left(\text{tang.} = \frac{1}{r}\sqrt{1 + \frac{1+r^2}{a^2} b^2}\right) + 4$ arc $\left(\text{tang.} = \frac{1}{r'}\sqrt{1 + \frac{1+r'^2}{a^2} b^2}\right)$ $- 2\pi$; in which we must put x^2 for a^2, $2kx - x^2$ for b^2, and we get, for the attraction of the solid,

$$A = 4\int\dot{x} \text{ arc } \left(\text{tang.} = \frac{1}{rx}\sqrt{2k(1+r^2)x - r^2x^2}\right) + 4\int\dot{x} \text{ arc } \left(\text{tang.} = \frac{1}{r'x}\sqrt{2k(1+r'^2)x - r'^2x^2}\right) - 2\pi x.$$

These fluents being exactly similar to the one in *Prop.* 24, if we put $A = \text{arc}\left(\text{tang.} = \frac{1}{rx}\sqrt{2k(1+r^2)x - r^2x^2}\right)$

$$A' = \text{arc}\left(\text{tang.} = \frac{1}{r'x}\sqrt{2k(1+r'^2)x - r'^2x^2}\right)$$

it is easy to see that

$$A = \left\{\frac{2k}{r^2} + 2(k-x)\right\}(\pi - 2A) - \frac{2}{r}\sqrt{2k(1+r^2)x - r^2x^2}$$
$$+ 2\pi x$$
$$+ \left\{\frac{2k}{r'^2} + 2(k-x)\right\}(\pi - 2A') - \frac{2}{r'}\sqrt{2k(1+r'^2)x - r'^2x^2}$$
$$+ \text{corr.}$$

If the fluent is to begin when $x = o$, no correction is necessary; for at that term $A = A' = \frac{\pi}{2}$.

When $x = 2k$, $A = \text{arc}\left(\text{tang.} = \frac{1}{r}\right)$, $A' = \text{arc}\left(\text{tang.} = \frac{1}{r'}\right)$ and

$$A = \left\{\frac{2k(1-r^2)}{r^2}\right\}\left\{\pi - 2\,\text{arc}\left(\text{tang.} = \frac{1}{r}\right)\right\} - \frac{4k}{r} + 4k\pi$$
$$+ \left\{\frac{2k(1-r'^2)}{r'^2}\right\}\left\{\pi - 2\,\text{arc}\left(\text{tang.} = \frac{1}{r'}\right)\right\} - \frac{4k}{r'}.$$

If we thought proper, this might still be put under a different form; for $r' = \frac{1}{r}$, and the arcs the complements of each other; and $\pi - 2\,\text{arc}\left(\text{tang.} = \frac{1}{r}\right) = 2\,\text{arc}(\text{tang.} = r)$; also $\pi - 2$ arc $\left(\text{tang.} = \frac{1}{r'}\right) = 2\,\text{arc}(\text{tang.} = r')$.

Cor. 1. When the rhombus is a square, $r = r' = 1$; and the action becomes $A = 4k\pi - 8k$, as we found in *Prop.* 24.

Cor. 2. Let r' be infinite, then $r = o$, and the solid becomes an infinitely long circular cylinder; and it is easy to see that the value of A is reduced to $2k\pi$, as we found before in a different manner.

The foregoing problems, which I have chosen from a great

variety that occurred to me, are sufficient to shew the use that may be made of the expressions given in the first section.

The attractions of certain infinitely long cylinders, which were derived, as corollaries, from some of the preceding propositions, present us with several curious relations; with these I shall terminate the present division of my subject.

Let pout, fig. 18, represent the base, or section, of a circular cylinder, infinitely extended both above and below the plane of the figure. Let p be an attracted point in the circumference of the section. Draw the diameter pu, and, at right angles to it, the diameter ot.

By *Cor.* 1, *Prop.* 27, the action of the whole cylinder on the point p is $2k\pi$ (k being the radius of the circular section); the action of that half of the cylinder, whose base is the semicircle opto, is $2k(\pi - 1)$; the action of the other half of the cylinder, which is furthest from p, is $2k$: therefore,

1. The attraction of a sphere is to that of an infinite circular cylinder of the same diameter (on a point at the surface of each) as $\frac{2}{3}$ to 1, which is the ratio of the solidity of a sphere to that of its circumscribing cylinder.

2. The attraction of the whole infinite cylinder, on p, is to the attraction of that half which is furthest from that point, as the circumference of a circle is to its diameter.

3. Consequently, the attraction of the nearest half, is to that of the furthest half, as the difference between the circumference and diameter of a circle is to the diameter; or *nearly* as 2 to 1.

4. In the circle optu, fig. 18, inscribe the parabola owpvt, whose equation is $kx = y^2$, so that its vertex may be at p, and its axis coincide with pu: this parabola will plainly cut the

circle at the quadrantal points o and t; and, I say, that the action, on the point p, of the infinitely long cylinder, whose base is the parabolic area owpvtco, is to the attraction of the furthest half of the infinite circular cylinder, *exactly* as 2 to 1. For the latter action has been shewn to be $2k$; and if in the expression, obtained in *Cor.* 2, *Prop.* 27, we make α and x both $= k$, it is reduced to $A = 4k$.

5. In fig. 18, draw fug perpendicular to pu at u; and from p, through o and t, the lines pof, ptg. The attraction, on the point p, of the infinitely long prism whose base is the triangle pfg, is equal to the attraction of the infinitely long circular cylinder. For the action of the prism is $4 \times \text{pu} \times \text{arc. fpu}$ (by *Prop.* 20) $= 8k \times \frac{\pi}{4} = 2k\pi$; and this has been already shewn to be the attraction of the circular solid.

§. V.

Of Solids of greatest Attraction.

The subject of this section has occupied the attention of Mr. PLAYFAIR,* in the same paper I have before noticed; it had previously been treated of by SILVABELLE. FRISI also, in the third volume of his works, gives a solution of the same problem as that which is first considered by Mr. PLAYFAIR, but his result is an erroneous one. None of these writers have pursued the matter any further than what relates to the figure of a homogeneous solid of revolution. My manner of treat-

* The problems which I investigate are similar to *the first* of Mr. PLAYFAIR's, where the *equation of a curve* is sought; nor do I at all meddle with that other class of problems which he treats of in the subsequent part of the paper.

ing the subject connects it intimately with the preceding parts of this paper; otherwise I should not have given the following problems a place here.

Prop. 31.

Suppose that a given quantity of matter is to be formed into a right cylinder of the length $2d$; what must be the figure of its base, so that it shall attract, with the greatest force possible, a point in its surface, and in the middle with respect to its two ends?

Let fig. 19 represent a section of the cylinder, at the attracted point p, parallel to its base. It is plain enough, that, whatever is the nature of the curve pab, we may draw a line pb from p, which shall divide the area into two equal and similar portions pabp, pcbp.

Put the absciss pd $= x$, the ordinate ad $= y$: the mass of the cylinder is $4d\int y\dot{x}$; and, by *Prop.* 4, its attraction on p is

$$4\int \dot{x} \text{ arc } \left(\text{tang.} = \frac{dy}{x\sqrt{d^2+x^2+y^2}}\right).$$

Let C be a constant quantity, and we have only to make the fluxion of the following expression, with respect to y, equal to nothing,* viz.

$$\text{Arc } \left(\text{tang.} = \frac{dy}{x\sqrt{d^2+x^2+y^2}}\right) + \text{C}dy:$$

this gives

$$\frac{x}{(x^2+y^2)\sqrt{d^2+x^2+y^2}} + \text{C} = o, \text{ or } x^2 - \text{C}^2)x^2+y^2)^2(d^2+x^2+y^2) = o,$$

for the equation of the curve pab. Make $y = o$, and let a be the corresponding value of x; the equation becomes

$1 - \text{C}^2a^2(d^2+a^2) = o$, whence $\text{C}^2 = \frac{1}{a(d^2+a^2)}$; by substituting

* EULERI "Methodus, &c." p. 42 and 185-6-7.

which value in the equation of the curve, it is ultimately

$$a^2(d^2+a^2)x^2-(x^2+y^2)^2(d^2+x^2+y^2)=o.$$

It is proper to remark, that though, in the enunciation, I spoke of the point as being in the surface of the cylinder, yet there is nothing in the above method of investigation that supposes it to be in contact with the solid: if there is to be a given distance between them, the nature of the curve will be the same.

Cor. 1. If we make d infinitely small, there results $a^4x^2-(x^2+y^2)^3=o$ for the equation of the curve bounding *the plane of greatest attraction;* and it is evident, that, by the revolution of this curve about its axis, will be generated *the solid of greatest attraction*, when it is sought for without any such conditions or restrictions as entered into the preceding problem.

This exactly agrees with the conclusion arrived at by SILVABELLE and Mr. PLAYFAIR.

Cor. 2. If, on the other hand, we make d infinitely great, the equation is reduced to $y^2=ax-x^2$, which is that of a circle whose diameter is a, the attracted point being in the circumference. Therefore,—*of all infinitely long cylinders, having the areas of their bases, or transverse sections, equal, that which has a circle for the circumference of the said base, shall exert the greatest action on a point at its surface.*

Prop. 32.

Let a given quantity of matter be fashioned into such a solid as was treated of at the beginning of the last section (in Propositions 24, 25, 26), viz. having its section perpendicular to the axis a regular polygon. The polygon being given in

species, it is required to determine the nature of the curve pr, fig. 15, so that the solid may have the greatest possible action on a point at its vertex p.

If we put x for the distance of the generating polygon from the vertex, and y for the perpendicular let fall from the centre of the polygon on one of its sides, the action of the solid is

$$2n\int\dot{x} \text{ arc } \left(\text{tang.} = \frac{1}{r}\sqrt{1 + \frac{1+r^2}{x^2}y^2}\right) - \int (n-2)\,\pi\dot{x}$$

by *Prop.* 5, and the mass of the solid is $nr\int y^2\dot{x}$; so that the quantity whose fluxion, with respect to y, must $= o$, is

$$2n \text{ arc } \left(\text{tang.} = \frac{1}{r}\sqrt{1 + \frac{1+r^2}{x^2}y^2}\right) + Cnry^2, \text{ and we get}$$

$$\frac{x}{(x^2+y^2)\sqrt{x^2+(1+r^2)y^2}} + C = o, \text{ or } x^2 - C^2(x^2+y^2)^2(x^2+(1+r^2)y^2) = o.$$

Let a be the value of x when $y = o$, then $C^2 = \frac{1}{a^4}$, and the equation of the curve becomes

$$a^4x^2 - (x^2+y^2)^2\left\{x^2 + (1+r^2)y^2\right\} = o.$$

When the polygon is a circle, $r = o$, and the equation is reduced to $a^4x^2 - (x^2+y^2)^3 = o$, the same as we found in *Cor.* 1, *Prop.* 31.

Lemma 1.

To find the attraction of the right prism whose base is the triangle mrv, fig. 1, and height $\dot{a}$, on the point p, in the direction pm; on the supposition, that the density at the ordinate ks is as any function of the absciss mk, and distance pm.

If we use the same notation as in *Prop.* 1, and put f (a, T) for the density of a particle any where at the line ks, we shall find, by proceeding, as we did there,

$$A = \dot{a}\int\frac{ar\,f(a,T)\,T\dot{T}}{(a^2+T^2)(a^2+(1+r^2)T^2)^{\frac{1}{2}}}.$$

Hence the attraction of a prism, whose height is $\dot{a}$, and base a regular polygon of n sides, composed of triangles having such a law of density as was supposed above, will be, on a point placed perpendicularly over its centre,

$$A = 2n\dot{a}\int \frac{ar\,\mathrm{f}\,(a,T)\,T\dot{T}}{(a^2+T^2)\,(a^2+(1+r^2)T^2)^{\frac{1}{2}}}.$$

This expression would be easily integrable on various suppositions. Thus we might conceive the density at the ordinate ks to vary as the line ps, drawn from the attracted point to its extremity s; this would be to make $\mathrm{f}\,(a,T) = \sqrt{a^2+(1+r^2)\,T^2}$; whence $A = 2n\dot{a}\int \frac{arT\dot{T}}{a^2+T^2} = rna\dot{a}\left\{\mathrm{L}\,.\,(a^2+T^2) - \mathrm{L}\,.\,a^2\right\}$.

Again, we might suppose $\mathrm{f}\,(a,T) = a^2 + T^2 = (\mathrm{pk})^2$; this would give $A = 2n\dot{a}\int \frac{arT\dot{T}}{\sqrt{a^2+(1+r^2)\,T^2}} = \frac{2nra\dot{a}}{1+r^2}\left\{\sqrt{a^2+(1+r^2)T^2} - a\right\}$.

But the kind of problems we are engaged about does not require us to know the value of A itself; its fluxional coefficient with respect to T being alone wanted, and this is *always*

$$\frac{2nar\,\mathrm{f}\,(a,T)\,T}{(a^2+T^2)\,\sqrt{a^2+(1+r^2)\,T^2}} \times \dot{a} \;\ldots\ldots\ldots\; (1).$$

For suppose we had actually found the fluent; when we make use of it in such a problem as the last, we must change T into y, and take the fluxion with respect to y, and the result must necessarily be $\frac{2nar\,\mathrm{f}\,(a,y)\,y}{(a^2+y^2)\,\sqrt{a^2+(1+r^2)\,y^2}} \times \dot{a}$; which we might have arrived at simply by changing T into y in the expression marked (1).

Lemma 2.

To find the quantity of matter in a right prism, whose base is the triangle rmv, fig. 1, and height $\dot{a}$; supposing the density at any ordinate ks to be $\mathrm{f}\,(a,T)$.

The *magnitude* of the element of the prism is $\dot{a}Tr\dot{T}$, and the *mass* of this element is $r\dot{a} \times \mathrm{f}\,(a,T)\; T\dot{T}$; whence the mass of the whole prism is $r\dot{a}\int \mathrm{f}\,(a,T)\; T\dot{T}$.

The mass of a prism whose height is $\dot{a}$ and base a regular polygon of n sides, formed of triangles having this law of density, is $2nr\dot{a}\int \mathrm{f}\,(a,T)\; T\dot{T}$ and its fluxional coefficient, with respect to T, is $2nr\,\mathrm{f}\,(a,T)\; T \times \dot{a}$.

Prop. 33.

Let the last proposition be again proposed, but with this difference, that the solid, instead of being homogeneous, is to be formed of polygonal prismatic elements, having such a law of density as in the preceding lemmas.

By proceeding as before, we shall have for the equation of the curve pr, fig. 15,

$$\frac{2nxr\,\mathrm{f}\,(x,y)\,y}{(x^2+y^2)\sqrt{x^2+(1+r^2)y^2}} + C2nr\,\mathrm{f}\,(x,y)\,y = o,$$

or $\frac{x}{(x^2+y^2)\sqrt{x^2+(1+r^2)y^2}} + C = o$; *which is exactly the same equation as when the solid was supposed to be homogeneous.*

When $r = o$, we have, as before, $a^4x^2 = (x^2+y^2)^2$; which shews that *the result of Mr.* PLAYFAIR *extends to an infinity of cases besides that of homogeneity.*

When, as in our last supposition, $r = o$, and the mass is a solid of revolution, the function $\mathrm{f}\,(a,T)$ expressing the density, is a function of the perpendicular let fall from any particle on the axis of the solid, and of the distance between the foot of that perpendicular and the attracted point.

Scholium.

If the preceding lemmas had been treated on the supposition that the density was variable *along* the line ks, fig. 1, (which is the same as making the density f (t), or more generally f (a, T, t) a function of a, T, and t) their application to the problem we have been considering, would give an indefinite number of different equations, for the curve pr, fig. 15, according to the nature of the assumed function f (a, T, t): every one of which equations will, however, have this peculiarity, that if we make $r = o$, it will become $a^4x^2 = (x^2 + y^2)^3$. For when $r = o$, $t = o$, and f (a, T, t) becomes a function of a and T only, and the case enters into *Prop.* 33 just now considered.

It may be worth while to see an example of this; we should have had, in general, for the action of the polygonal prismatic element of the solid, by *Prop.* 1,

$$A = 2n\dot{a}\iint \frac{a\,\mathrm{f}\,(a, T, t)\,\dot{T}\dot{t}}{(a^2 + T^2 + t^2)^{\frac{3}{2}}};$$

and the mass of the same element would have been

$$2n\dot{a}\iint \mathrm{f}\,(a, T, t)\,\dot{T}\dot{t}.$$

These must be integrated, with respect to t, from $t = o$ to $t = rT$: which cannot be done till we assign a form for the function f (a, T, t). Let this be $a^2 + T^2 + t^2$, that is to say, let the density at any point q, in the triangle vrm, be as the square of its distance pq from the attracted point p. This will give

$$A = 2n\dot{a}\iint \frac{a\dot{T}\dot{t}}{(a^2 + T^2 + t^2)^{\frac{1}{2}}} = 2na\dot{a}\int \left\{ \mathrm{L}\,(rT + \sqrt{a^2 + (1 + r^2)\,T^2}) - \mathrm{L}\,\sqrt{a^2 + T^2} \right\} \dot{T};$$ and for the mass

$$2nd\!f\!f\,(a^2 + T^2 + t^2)\,\dot{T}\dot{t} = 2nd\!f\,(a^2rT + rT^3 + \frac{r^3T^3}{3})\,\dot{T}.$$

If therefore we solve *Prop.* 32, on this supposition of density, we have for the equation of the curve pr, fig. 15, when the solid has the greatest attraction,

$$x\mathrm{L}\,(ry\sqrt{x^2 + (1 + r^2)\,y^2}) - x\mathrm{L}\sqrt{x^2 + y^2} + \mathrm{C}\,(x^2ry + ry^3 + \frac{r^3y^3}{3}) = o.$$

Now, when r is infinitely small, we shall have, by neglecting all the higher powers thereof,

$$\mathrm{L}\,(ry + \sqrt{x^2 + (1 + r^2)y^2}) = \mathrm{L}\sqrt{x^2 + y^2} + \frac{ry}{\sqrt{x^2+y^2}};$$

by substituting which our equation becomes

$\frac{x}{\sqrt{x^2+y^2}} + \mathrm{C}\,(x^2 + y^2)$, or $a^4x^2 - (x^2 + y^2)^3 = o$, as we shewed *a priori* must necessarily happen.

I shall just remark here, that, as the results of *Prop.* 32, are not altered by conceiving the density any function of a and T, such is also the case with respect to Problem 31, if T there represent the distance of any particle from a plane passing through the attracted point and the axis of the cylinder. This the reader may easily convince himself of.

The proposition just mentioned (31) is only a particular case of the following very general one.

Prop. 34.

Let uvv′u′, fig. 15, be a rectangle, whose plane is perpendicular to the line pm, and its centre in that line. Let this rectangle move parallel to itself, in the direction pm, and vary in such a manner, that the middle points r and r′ of its sides may continually touch two different curves.

The quantity of matter in the solid so generated being given, and the nature of one of the curves as pr′, to find what must be the other curve pr, so that the action of the solid, on a point at its vertex p, may be the greatest possible.

Put x for the absciss pm, y' and y for the ordinates mr′, mr; then by *Prop.* 4, the action of the solid will be

$4\int \dot{x}$ arc $\left(\text{tang.} = \frac{yy'}{x\sqrt{x^2+y^2+y'^2}}\right)$; and its mass is $4\int yy'\dot{x}$.

But y' is a given function of x, suppose $f(x)$. The quantity

$$\text{Arc}\left(\text{tang.} = \frac{yf(x)}{x\sqrt{x^2+y^2+f(x)^2}}\right) + Cyf(x)$$

is therefore to have its fluxion, with respect to y, made $= o$: and this gives, for the equation of the curve pr,

$\frac{x}{(x^2+y^2)\sqrt{x^2+y^2+f(x)^2}} + C = o$, or $x^2 - C^2(x^2+y^2)^2(x^2+y^2+f(x)^2) = o$.

Ex. 1. Let $f(x) = ax$, or pr′ be a straight line, the equation of pr must be $x^2 - C^2(x^2+y^2)^2\left\{(1+a^2)x^2+y^2\right\} = o$.

Ex. 2. Let pr′ be a circle, or $f(x)^2 = 2kx - x^2$, k being the radius, then $x^2 - C^2(x^2+y^2)^2(2kx+y^2) = o$, is the equation of the other curve.

Ex. 3. If pr′ is a parabola, or $f(x)^2 = ax$, the equation of pr is $x^2 - C^2(x^2+y^2)^2(ax+x^2+y^2) = o$.

Scholium.

In *Prop.* 27, after having found the action of the solid there treated of, we derived, as corollaries, the action of parabolic and circular cylinders of infinite length, by separately making infinite the diameter of the circle and the parameter of the parabola. Perhaps it might therefore be supposed, that if we made k infinite in the second of the preceding examples, or a

infinite in the third, the result would be the equation of the base of the infinitely long *cylinder* of greatest attraction; which however is by no means the case; for that was found to be a circle, whereas the equation we get here is

$$x - C'(x^2 + y^2) = o,$$

and if we make a infinitely great in the first example, the equation becomes $C' = x^2 + y^2$, or the line pr is a circle with its centre at the attracted point.

We might resolve this problem, on a variety of hypotheses respecting the density; or we might add other conditions of a different kind; for instance, not only the mass of the solid, but the area of the section, passing through the required curve pr and axis pm, might be supposed constant. But I pass on to other suppositions respecting the force of attraction; which will be treated with as much brevity as possible.

Lemma 3.

To find the attraction of the triangle vrm, fig. 1, on the point p, in the direction pm, supposing the force to be inversely as the mth power of the distance.

Keeping the same notation as in *Prop.* 1, we have, for the attraction of an element at q, $\frac{\dot{T}\dot{t}}{(a^2+T^2+t^2)^{\frac{m}{2}}}$; which being resolved, gives, for the force of the whole triangle, in the direction pm, $A = \int\int \frac{a\dot{T}\dot{t}}{(a^2+T^2+t^2)^{\frac{m+1}{2}}}$; the fluent is to be taken from $t = o$ to $t = rT$, and we have

$$A = \int \frac{ar\dot{T}}{(a^2+T^2)(a^2+(1+r^2)T^2)^{\frac{m-1}{2}}} \left\{ 1 - (2-m)\frac{r^2T^2}{3(a^2+T^2)} + (2-m)(4-m)\frac{r^4T^4}{3\cdot 5(a^2+T^2)^2} - \&c. \right\}.$$

The further integration, with respect to T, is not necessary for our purpose.

Cor. 1. If we multiply this by $2n$, it will be the attraction of a regular polygon of n sides; and making n infinitely great and r infinitely small, the attraction of a circle to the radius T is found to be

$$A = \int \frac{2nraT\dot{T}}{(a^2+T^2)^{\frac{m+1}{2}}} = \frac{-2nra}{(m-1)(a^2+T^2)^{\frac{m-1}{2}}} + \frac{2nra}{(m-1)\,a^{m-1}}$$

which, by putting π for nr, is

$$A = 2\pi \left\{ \frac{1}{(m-1)\,a^{m-2}} - \frac{a}{(m-1)(a^2+T^2)^{\frac{m-1}{2}}} \right\}$$

the same as is found differently by other writers.

Cor. 2. When r becomes infinite, the triangle vrm is changed into a parallelogram, infinitely extended in the direction rv; and we have

$$A = \int \frac{a\dot{T}}{(a^2+T^2)(rT)^{m-2}} \left\{ 1 - (2-m)\frac{r^2T^2}{3(a^2+T^2)} + (2-m)(4-m) \frac{r^4T^4}{3.5\,(a^2+T^2)^2} - \&c. \right\}$$

which, when m is an even positive whole number greater than 2, is reduced to $A = \frac{2.4.6\ldots\ldots(m-2)}{3.5.7\ldots\ldots(m-1)} \int \frac{a\dot{T}}{(a^2+T^2)^{\frac{m}{2}}}$.

Cor. 3. If instead of the action of the triangle vrm, that of a rectangle, whose sides are rm (y) and rv (y'), had been required, we must have proceeded exactly in the same manner, but the fluent, with respect to t, must have been taken from $t = o$, to $t = y'$; so that we have only to substitute y' for rT, in the value found by the lemma, and there results

$$A = \int \frac{ay\dot{T}}{(a^2+T^2)(a^2+T^2+y'^2)^{\frac{m-1}{2}}} \left\{ 1 - (2-m)\frac{y'^2}{3(a^2+T^2)} + (2-m)(4-m)\frac{y'^4}{3.5\,(a^2+T^2)^2} - \&c. \right\}$$

Another Method of finding the Action of the Triangle vrm.

The expressions we have found will terminate only when m is one of the series of numbers 2, 4, 6, &c. If m is among the odd numbers 1, 3, 5, &c. $\frac{m+1}{2}$ will be a whole positive number; and we have for the action of the triangle, or rectangle (accordingly as the fluent, with respect to t, is taken from $t = o$, to $t = rT$, or from $t = o$, to $t = y'$) provided m is greater than 3,

$$A = \int a\dot{T}\Big\{\frac{1}{(m-1)(a^2+T^2)} \times \frac{t}{(a^2+T^2+t^2)^{\frac{m-1}{2}}} + \frac{m-2}{(m-1)(m-3)(a^2+T^2)^2} \times \frac{t}{(a^2+T^2+t^2)^{\frac{m-3}{2}}} + \frac{(m-2)(m-4)}{(m-1)(m-3)(m-5)(a^2+T^2)^3} \times \frac{t}{(a^2+T^2+t^2)^{\frac{m-5}{2}}} + \ldots\ldots + \frac{(m-2)(m-4)\ldots\ldots 3}{(m-1)(m-3)(m-5)\ldots\ldots 2\,(a^2+T^2)^{\frac{m-1}{2}}} \times \frac{t}{a^2+T^2+t^2} + \frac{(m-2)(m-4)\ldots\ldots 3}{(m-1)(m-3)(m-5)\ldots\ldots 2\,(a^2+T^2)^{\frac{m-1}{2}}} \times \frac{1}{\sqrt{a^2+T^2}} \times \text{arc}\left(\text{tang.} = \frac{t}{\sqrt{a^2+T^2}}\right)\Big\}.$$

Let us, for brevity, denote this quantity by $\int a\dot{T}\varphi\,(a, m, T, t)$; then for the action of the triangle vrm we have $A = \int a\dot{T}\varphi\,(a, m, T, rT)$; and for the rectangle, whose sides are rm (y) and rv (y') $A = \int a\dot{T}\varphi\,(a, m, T, y')$. When $m = 1$, or 3, the above expression will not give the attraction; but we evidently have, in the case of $m = 1$,

$$A = \int \frac{a\dot{T}}{\sqrt{a^2+T^2}} \times \text{arc}\left(\text{tang.} = \frac{t}{\sqrt{a^2+T^2}}\right);$$ and when $m = 3$,

$$A = \int a\dot{T}\Big\{\frac{1}{2(a^2+T^2)} \times \frac{t}{(a^2+T^2+t^2)} + \frac{1}{2(a^2+T^2)^{\frac{3}{2}}} \times \text{arc}\left(\text{tang.} = \frac{t}{\sqrt{a^2+T^2}}\right)\Big\}.$$

Cor. 1. For a polygon of n sides, these expressions must be multiplied by $2n$ as usual; and when m is greater than 3, $A = 2nfa\dot{T}\phi\ (a, m, T, rT)$, if in this we make n infinitely great and r infinitely small, it ought to enter into the general case of the attraction of a circle given in *Cor.* 1, to the first part of the lemma: and in fact we get

$$A = \left\{\frac{1}{m-1} + \frac{m-2}{(m-1)(m-3)} + \frac{(m-2)(m-4)}{(m-1)(m-3)(m-5)} + \dots\dots\dots\dots + \frac{(m-2)(m-4)\dots\dots 3}{(m-1)(m-3)\dots\dots 2} + \frac{(m-2)(m-4)\dots\dots\dots 3}{(m-1)(m-3)\dots\dots\dots 2}\right\} \times \int \frac{2narT\dot{T}}{(a^2+T^2)^{\frac{m+1}{2}}},$$

or, because the quantity between the brackets is plainly equal to unity, becomes $A = \int \frac{2nraT\dot{T}}{(a^2+T^2)^{\frac{m+1}{2}}}$ which is the same form as was found before for the general case.

Cor. 2. When r becomes infinite, and the triangle rmv is changed into an infinitely extended rectangle, we have for its attraction

$$A = \frac{3.5.7\dots\dots\dots(m-2)}{2.4.6\dots\dots\dots(m-1)} \int \frac{a\pi\dot{T}}{2\,(a^2+T^2)^{\frac{m}{2}}},$$

except when $m = 1$, in which case, $A = \int \frac{a\pi\dot{T}}{2\sqrt{a^2+T^2}}$.

Scholium.

This lemma has been treated on the supposition that the density is the same at every part of the triangle rmv, fig. 1; but there are other hypotheses which render the solution easier: for instance, we may conceive the density of a particle at q to be as its distance (t) from the line rm, in which case

$$A = \iint \frac{a\dot{T}t\dot{t}}{(a^2+T^2+t^2)^{\frac{m+1}{2}}} = \int \left\{\frac{-a\dot{T}}{(m-1)(a^2+T^2+t^2)^{\frac{m-1}{2}}} + \frac{a\dot{T}}{(m-1)(a^2+T^2)^{\frac{m-1}{2}}}\right\}$$

where t must be made rT or y' accordingly as the action of a triangle or rectangle is required.

From this simple case, we may not only arrive at some curious results, connected with the particular hypothesis of density, but may find with equal ease the figure of a *homogeneous* solid of revolution of greatest attraction, as will just now be seen.

If the density was to be a function of a and T only, it would be sufficient to multiply the values found in the lemma, by that function, see lemma 1.

Prop. 35.

Let *Prop.* 31 be again proposed, but with this difference, that the force is inversely as the mth power of the distance, and that the density of any particle of the cylinder is as its distance (t) from that middle section (parallel to the ends of the cylinder) which passes through the attracted point.

In the expression we just now found, in the preceding scholium, put x for a, and d (half the length of the cylinder) for t. The action of the cylinder is

$$\mathrm{A} = 4\iint\left\{\frac{-x\dot{x}\dot{T}}{(m-1)(x^2+T^2+d^2)^{\frac{m-1}{2}}} + \frac{x\dot{x}\dot{T}}{(m-1)(x^2+T^2)^{\frac{m-1}{2}}}\right\};$$

its quantity of matter is $4\iiint\dot{x}\dot{T}\dot{t}t = 2\iint\dot{x}\dot{T}t^2$; so that we have, for the equation of the curve bounding the base,

$$\frac{x}{(x^2+y^2)^{\frac{m-1}{2}}} - \frac{x}{(x^2+y^2+d^2)^{\frac{m-1}{2}}} + \mathrm{C}d^2 = o.$$

Cor. 1. When d is infinitely small, this becomes

$$\frac{m-1}{2} \times \frac{xd^2}{(x^2+y^2)^{\frac{m+1}{2}}} + \mathrm{C}d^2, \text{ or } \frac{x}{(x^2+y^2)^{\frac{m+1}{2}}} + \mathrm{C}' = o.$$

Let a be the value of x when $y = o$, then $\mathrm{C}' = -\frac{1}{a^m}$; and

the equation of the curve, bounding *the plane of greatest attraction*, is

$$a^m x = (x^2 + y^2)^{\frac{m+1}{2}},$$

which is exactly the same result as that obtained by Mr. PLAYFAIR, p. 203, on the supposition of *homogeneity*; and this was to be expected; for, though a certain condition of the density of the cylinder entered into the foregoing problem, yet when d vanishes, and the solid becomes a plane, we must evidently obtain the same result as if it had been arrived at by supposing the cylinder homogeneous; which in fact it will be when the length is evanescent.

Nor is this observation to be confined to that particular case when the density is as t: if we had solved the problem on the supposition of any function of x, T, and t, for the density, it is easy to see that though different functions will give different results when d is finite, yet when the solid becomes *a plane*, and $d = o$, the equation will always be reduced to

$$a^m x = (x^2 + y^2)^{\frac{m+1}{2}}.$$

Hence we may conclude, that, *the solid of revolution which shall exercise the greatest attraction on a point in its axis, when the force is inversely as the mth power of the distance, and the density either uniform, or any function whatever of x and T (T being the perpendicular let fall from any particle to the axis of the solid, and x the distance between the foot of that perpendicular and the attracted point) will have, for the equation of its generating curve,* $a^m x = (x^2 + y^2)^{\frac{m+1}{2}}$.

Cor. 2. Nothing can be learned from the equation

$$\frac{x}{(x^2+y^2)^{\frac{m-1}{2}}} - \frac{x}{(x^2+y^2+d^2)^{\frac{m-1}{2}}} + Cd^2 = o,$$

when $m = 1$. The curve is then transcendent, and has for its equation $x\mathrm{L}.(x^2+y^2+d^2) - x\mathrm{L}.(x^2+y^2) + \mathrm{C}d^2 = 0$.

Cor. 3. If the cylinder becomes infinitely long, (m being positive and greater than unity) the equation of its base is

$$\frac{x}{(x^2+y^2)^{\frac{m-1}{2}}} + \mathrm{C}' = 0;$$

let a be the value of x when $y = 0$; then $\mathrm{C}' = \frac{-1}{a^{m-2}}$, and the equation becomes

$$\frac{x}{(x^2+y^2)^{\frac{m-1}{2}}} - \frac{1}{a^{m-2}} = 0.$$

If $m = 2$, as in the case of nature, this becomes $\frac{x^2}{x^2+y^2} = 1$, *so that the infinitely long cylinder of greatest attraction will be an infinitely long rectangle, with its edge turned to the attracted point.*

If $m = 3$, we have $ax = x^2 + y^2$, the equation of a circle with the attracted point in its circumference.

If $m = 4$, the equation is $a^2x = (x^2+y^2)^{\frac{3}{2}}$, which is Mr. PLAYFAIR's *curve of equal attraction.*

If we want the figure of the infinite cylinder of greatest attraction, when $m = 1$, we must have recourse to the last corollary; where we found

$$x\mathrm{L}(x^2+y^2+d^2) - x\mathrm{L}(x^2+y^2) = \mathrm{C}'.$$

This, when d is infinite gives $x\mathrm{L}.d^2 = \mathrm{C}'$, or, $x = \text{const.}$, *the equation of a plane perpendicular to the axis of x.*

Cor. 4. If we would solve *Proposition* 34, but with this difference, that the force is now inversely as the mth power of the distance, and the density, in the generating rectangle uvv′u′, fig. 15, is, at any point, as its distance from rm or y; we need only put f(x) (given by the nature of the curve pr) for d, in the equation here found, and we get that of pr, in the case of

greatest attraction; viz.

$$\frac{x}{(x^2+y^2)^{\frac{m-1}{2}}} - \frac{x}{(x^2+y^2+\mathrm{f}\,(x)^2)^{\frac{m-1}{2}}} + \mathrm{Cf}\,(x)^2 = o.$$

Prop. 36.

To solve *Prop.* 32, the force being supposed inversely as the mth power of the distance, and the generating polygon being composed of triangles having such a law of density as that in the scholium to lemma 3.

By using the value found in that scholium, and proceeding, in other respects, as in the similar propositions already given, we find, for the equation of the curve touching the sides of the polygon,

$$\frac{x}{(x^2+y^2)^{\frac{m-1}{2}}} - \frac{x}{(x^2+(1+r^2)\,y^2)^{\frac{m-1}{2}}} + \mathrm{C}r^2y^2 = o.$$

Prop. 37.

Let *Prop.* 32 be yet once more resolved, on the supposition that the force is inversely as the mth power of the distance; and the density, in the triangles forming the generating polygon, either uniform, or as any function of x and T.

If we make use of the first value of A in lemma 3, we get, for the equation of the curve touching the sides of the polygon,

$$\frac{x}{(x^2+y^2)\,(x^2+(1+r^2)\,y^2)^{\frac{m-1}{2}}}\left\{1-(2-m)\,\frac{r^2y^2}{3\,(x^2+y^2)} + (2-m)\,(4-m)\,\frac{r^4y^4}{3\cdot5\,(x^2+y^2)^2} - \&c.\right\} + \mathrm{C} = o.$$

When $r = o$, or the polygon becomes a circle, this equation is reduced to $\frac{x}{(x^2+y^2)^{\frac{m+1}{2}}} + \mathrm{C} = o$, as was found in another manner, in *Cor.* 1, *Prop.* 35.

If r is finite, the above expression will terminate when m is a whole positive even number; and consequently the guiding curve will then be algebraic. But, if m be amongst the numbers 5, 7, 9, 11, &c., we must use the other expression found in the lemma, and there arises, for the guiding curve, the transcendent equation

$$x\phi\,(x, m, y, ry) + Cry = o.$$

If $m = 1$, the equation is

$$\frac{x}{\sqrt{x^2+y^2}} \times \text{arc}\left(\text{tang.} = \frac{ry}{\sqrt{x^2+y^2}}\right) + Cry = o;$$ and, finally, when $m = 3$,

$$\frac{x}{x^2+y^2} \times \frac{ry}{x^2+(1+r^2)y^2} + \frac{x}{(x^2+y^2)^{\frac{3}{2}}}\,\text{arc}\left(\text{tang.} = \frac{ry}{\sqrt{x^2+y^2}}\right) + Cry = o.$$

In like manner, might be solved *Prop.* 31 and 34, the force and density being as in Lemma 3, but this I leave to the reader.

Prop. 38.

The force being inversely as the mth power of the distance (where m is any whole positive number), and the density either uniform or any function of x and T,* the base of the infinitely long cylinder of greatest attraction has, for its equation,

$$\frac{x}{(x^2+y^2)^{\frac{m}{2}}} + C = o;$$

for it will appear from lemma 3, and its corollaries, that, whether m be odd or even (that is to say when it is any number in the series 1, 2, 3, 4, 5, &c.), the attraction of an infinite cylinder will be of the form

* What this means with respect to a cylinder, is shewn at the end of the scholium to *Prop.* 33; and with respect to a solid of revolution in *Prop.* 33.

$$A = D\int\int \frac{x f(x, T)\,\dot{T}\dot{x}}{(x^2 + T^2)^{\frac{m}{2}}},$$ D being a function of m;

hence the truth of the proposition is manifest. And because the equation of the curve generating the solid of revolution of greatest attraction (on the same hypotheses of force and density) has been shewn to be $\frac{x}{(x^2+y^2)^{\frac{m+1}{2}}} + C = o$, we have the following remarkable

Theorem.

m being any whole positive number, and the density either uniform or as any function of x and T, the same curve which, by revolving, generates the solid of revolution of greatest attraction, when the force is inversely as the mth power, shall be the base of the infinitely long cylinder of greatest attraction, when the force is inversely as the (m + 1th) power.

Numberless other interesting questions might be proposed, relating to solids of greatest attraction; for instance, we may inquire what must be the curve bounding the base of a cylinder of given mass and length so that it shall exercise the greatest action in a direction *parallel* to its axis.

But as this kind of inquiry proceeds exactly in the same way as the other (only we must use the attraction B, instead of A, in *Prop.* 1), it is unnecessary to lengthen a paper which has already been extended too far.

APPENDIX TO §. III.

Of the Attraction of an infinitely long Prism, whose Base is any right lined Figure whatever.

Prop. A.

Let the rectangle bb′ c′c, fig. 20, be the section or base of a prism, infinitely extended on both sides of it, and let the line psu bisect the opposite sides bb′, cc′ of the rectangle.

It is required to find the attraction of the infinitely long solid, on the point p, in the direction psu.

Let C be the centre of the rectangle, put $k = s\mathrm{C}$, $a = \mathrm{bs}$, $u = \mathrm{pC}$; draw rm perpendicular to $s\mathrm{Cu}$, and put $x = \mathrm{Cm}$. Now it appears, from *Cor.* 2, *Prop.* 1 of the paper (putting A for the required attraction) that

$\mathrm{A} = 4\int \dot{\overline{\mathrm{pm}}} \times \text{arc}\left(\text{tang.} = \frac{\mathrm{rm}}{\mathrm{pm}}\right) = 4\int \dot{x}\ \text{arc}\left(\text{tang.} = \frac{a}{u+x}\right)$ $= 4x\ \text{arc}\left(\text{tang.} = \frac{a}{u+x}\right) - 4\int x\ \dot{\overline{\text{arc}}}\left(\text{tang.} = \frac{a}{u+x}\right)$ the last term of which is $4\int \frac{a x \dot{x}}{(u+x)^2 + a^2}$; put $u + x = z$, $\dot{x} = \dot{z}$, $x = z - u$, and it becomes $4\int \frac{(az - au)\dot{z}}{a^2 + z^2}$, which is $4a\mathrm{L}.(a^2 + z^2)^{\frac{1}{2}} - 4u\ \text{arc}\left(\text{tang.} = \frac{z}{a}\right)$ so that

$\mathrm{A} = 4x\ \text{arc}\left(\text{tang.} = \frac{a}{u+x}\right) - 4u\ \text{arc}\left(\text{tang.} = \frac{u+x}{a}\right) + 4a\mathrm{L}.(a^2 + (u+x)^2)^{\frac{1}{2}}$, or $\mathrm{A} = 4(x+u)\ \text{arc}\left(\text{tang.} = \frac{a}{u+x}\right) - 2u\pi + 4a\mathrm{L}.(a^2 + (u+x)^2)^{\frac{1}{2}}$,

which fluent being taken from $x = -k$ to $x = k$ gives

$\mathrm{A} = 4(u+k)\ \text{arc}\left(\text{tang.} = \frac{a}{u+k}\right) - 4(u-k)\ \text{arc}\left(\text{tang.} = \frac{a}{u-k}\right) + 4a\mathrm{L}.(a^2 + (u+k^2)^{\frac{1}{2}} - 4a\mathrm{L}.(a^2 + (u-k^2)^{\frac{1}{2}}$.

If we choose to express this by the lines and angles of the figure (20), it is

$A = 4 \times \text{pu} \times \text{arc}, c\text{pu} - 4 \times \text{ps} \times \text{arc}, \text{bps} + 4 \times \text{bs} \times L\frac{\text{pc}}{\text{pb}}$.

Prop. B.

Let the section of the prism be an isosceles triangle; the attracted point p being in the line psm (fig. 21), which passes through the vertex s to the middle of the base r′p′.

Draw rm parallel to the base, and put $r =$ tang. rsm; call ps, u; sm, x; then rm $= rx$; and we have for the attraction of the infinite solid

$A = 4\int \dot{x}$ arc $\left(\text{tang.} = \frac{rx}{u+x}\right) = 4x$ arc $\left(\text{tang.} = \frac{rx}{u+x}\right) - 4\int \frac{x\left(\frac{rx}{u+x}\right)^{\cdot}}{1+\left(\frac{rx}{u+x}\right)^2}$; the last term is $-4\int \frac{r\dot{x}(u+x) - rx^2\dot{x}}{(u+x)^2 + r^2x^2} = - 4\int \frac{rux\dot{x}}{(u+x)^2 + r^2x^2} = -4\int \frac{rux\dot{x}}{u^2 + 2ux + (1+r^2)x^2} = -\frac{4}{1+r^2}\int \frac{rux\dot{x}}{\frac{u^2}{1+r^2} + \frac{2u}{1+r^2}x + x^2} = -\frac{4}{1+r^2}\int \frac{rux\dot{x}}{\left(x+\frac{u}{1+r^2}\right)^2 + \left(\frac{ur}{1+r^2}\right)^2} = -\frac{4}{1+r^2}\int \frac{rux\dot{x}}{(x+\alpha)^2 + r^2\alpha^2}$, if we put $\alpha = \frac{u}{1+r^2}$. Make, moreover $x + \alpha = z$, $x = z - \alpha$, $\dot{x} = \dot{z}$, and it becomes $-\frac{4}{1+r^2}\int \frac{(ruz - ru\alpha)\dot{z}}{z^2 + r^2\alpha^2}$, which fluent is

$$-\frac{4}{1+r^2}\left\{ruL.\sqrt{z^2 + r^2\alpha^2} - u \text{ arc } \left(\text{tang.} = \frac{z}{r\alpha}\right)\right\},$$

we have then at length

$A = 4x$ arc $\left(\text{tang.} = \frac{rx}{u+x}\right) - \frac{4}{1+r^2}\left\{ruL\sqrt{(x+\alpha)^2 + r^2\alpha^2} - u\right.$ arc $\left.\left(\text{tang.} = \frac{x+\alpha}{r\alpha}\right)\right\}$ + cor.

Cor. If the position of the attracting solid be reversed, as in fig. 22, call ps, u, and the attraction will be given by the same formula; only the fluent (if it begin at the point) must now

be taken from o to $-x$, instead of from o to x. A being a function of r, x, and u, may be represented by $2\varphi\,(r, x, u)$. To correct the fluent, let sm (fig. 23) $=$ X, sm$'=x$, then, the attraction of the solid, whose base is the quadrilateral figure ρrr$'\rho'$, will be $2\varphi\,(r, x, u) - 2\varphi\,(r, \mathrm{X}, u)$.

In figure 24, call ps, u; sm, X; sm$'$, x. The action of the solid, whose base is ρrr$'\rho'$, is expressed by $2\varphi\,(r, -x, u) - 2\varphi\,(r, -\mathrm{X}, u)$.

In fig. 25, put ps $= u$, ps$' = u'$, sm $=$ s$'$m $= x$, tang. of rsm $= r$: the attraction of the solid, whose base is the rhombus srs$'\rho$, on a point p in the produced diameter of the section, is $2\varphi\,(r, x, u) - 2\varphi\,(r, o, u) + 2\varphi\,(r, -x, u') - 2\varphi\,(r, -o, u')$.

Prop. C.

Let fig. 26 represent the base or section of an infinitely long prism, and let this base be any right lined figure whatever, regular or irregular: from p, a point in the same plane, draw *any line* pq, cutting the base at s and m$'''$. It is required to find the action of the solid on the point p, in the direction pq.

From the angles r, r$'$, r$''$, r$'''$, &c. of the base, let fall the perpendiculars rm, r$'$m$'$, r$''$m$''$, r$'''$m$'''$, &c. on the line pq. Prolong the sides of the polygon till they meet pq at the points s, s$'$, s$''$, s$'''$, &c.

Put $u =$ ps, $u' =$ ps$'$, $u'' =$ ps$''$, $u''' =$ ps$'''$, &c.; and

$$x = \mathrm{sm}, \left\{\begin{matrix} x' = \mathrm{s'm'} \\ \mathrm{X}' = \mathrm{s'm} \end{matrix}\right\}, \left\{\begin{matrix} x'' = \mathrm{s''m''} \\ \mathrm{X}'' = \mathrm{s''m'} \end{matrix}\right\}, \left\{\begin{matrix} x''' = \mathrm{s'''m''} \\ \mathrm{X}''' = \mathrm{s''m'''} \end{matrix}\right\}, \text{\&c.}$$

Also, let $r =$ tang. rsm, r$' =$ tang. r$'$s$'$m$'$, r$'' =$ tang. r$''$s$''$m$''$, $r''' =$ tang. r$'''$s$'''$m$'''$, &c. then it appears from the last proposition, that the attraction, of the upper half of the solid, is

expressed by

$$\begin{array}{rlcl} & \varphi\,(r\ ,\ \ x\ ,u\ ) & - & \varphi\,(r\ ,\ \ o\ ,u\ ) \\ + & \varphi\,(r'\ ,\ \ x'\ ,u'\ ) & - & \varphi\,(r'\ ,\ \ X'\ ,u'\ ) \\ + & \varphi\,(r''\ ,\ \ x''\ ,u''\ ) & - & \varphi\,(r''\ ,\ \ X''\ ,u''\ ) \\ + & \varphi\,(r''',-x''',u''') & - & \varphi\,(r''',-X''',u''') \\ + & \&c. & - & \&c. \end{array}$$

And in the same manner is found the attraction of the lower portion. If any part of the polygon, as $\rho\rho'$, is parallel to pq, the attraction of that portion of the solid may be found by *Prop.* A.

Scholium to Prop. 25, page 273.

The following expression includes the attraction (on a point at the pole or vertex) of all this class of solids, where the generating plane is a regular polygon, and guiding curve a conic section: or where $y^2 = \alpha^2\,(\beta x + \gamma x^2)$.

$$A = 2n\left(x + \frac{\beta\alpha^2}{1+\gamma\alpha^2}\right)\text{arc}\left(\text{tang.} = \frac{1}{rx}\sqrt{\mu x + \nu x^2}\right) - \left\{(n-2)\,x + \frac{n\beta\alpha^2}{1+\gamma\alpha^2}\right\}\pi + \varphi$$

in which $\mu = \beta\alpha^2\,(1 + r^2)$, $\nu = 1 + \gamma\alpha^2\,(1 + r^2)$, and

$$\varphi = \frac{2nr\beta\alpha^2}{\sqrt{\nu}\,(1+\gamma\alpha^2)}\,L\left(\frac{\sqrt{\nu x}}{\sqrt{\mu}} + \sqrt{\frac{\nu x}{\mu} + 1}\right),\ \text{or} = \frac{2nr\beta\alpha^2}{\sqrt{-\nu}\,(1+\gamma\alpha^2)}\ \text{arc}\left(\text{sine} = \frac{\sqrt{-\nu x}}{\sqrt{\mu}}\right),$$

accordingly as ν is positive or negative.

Ex. 1. Let $\gamma = 0$, $\alpha = 1$, $y = \beta x$; in which case the solid is the polygonal parabolic conoid treated of in the proposition; and we have $\mu = \beta\,(1 + r^2)$, $\nu = 1$, whence

$A = 2n(x+\beta)$ arc (tang. $= \frac{1}{rx}\sqrt{\beta(1+r^2)x+x^2}) - \{(n-2)$
$x + n\beta\}\pi + 2nr\beta L\left(\frac{\sqrt{x}}{\sqrt{\beta(1+r^2)}} + \sqrt{\frac{x}{\beta(1+r^2)}+1}\right)$,
the same as was found before.

Ex. 2. Let $\alpha^2 = \frac{b^2}{a^2}$, $\beta = a$, $\gamma = -1$, $y^2 = \frac{b^2}{a^2}(ax - x^2)$: here the curve pr, fig. 15, is an ellipsis whose diameters are a and b, a being that which coincides with the axis pm. We have, in this case, $\mu = \frac{b^2}{a}(1+r^2)$, $\nu = 1 - \frac{b^2}{a^2}(1+r^2)$, and the attraction of a *polygonal spheroid*, on a point at its pole, is

$$A = 2n\left(x + \frac{b^2a}{a^2-b^2}\right) \text{arc}\left(\text{tang.} = \frac{1}{rx}\sqrt{\frac{b^2}{a}(1+r^2)x + \left(1-\frac{b^2}{a^2}(1+r^2)\right)x^2}\right)$$
$$- \left\{(n-2)x + \frac{nb^2a}{a^2-b^2}\right\}\pi + \varphi,$$

where $\varphi = \frac{2nra^2b^2}{(a^2-b^2)\sqrt{a^2-(1+r^2)b^2}} L\left\{\frac{\sqrt{\{a^2-(1+r^2)b^2\}x}}{\sqrt{b^2a(1+r^2)}} + \sqrt{\frac{\{a^2-(1+r^2)b^2\}x}{b^2a(1+r^2)}+1}\right\}$, or

$$= \frac{2nra^2b^2}{(a^2-b^2)\sqrt{(1+r^2)b^2-a^2}} \text{arc}\left(\text{sine} = \sqrt{\frac{\{(1+r^2)b^2-a^2\}x}{b^2a(1+r^2)}}\right),$$

accordingly as $\frac{a^2}{b^2}$ is greater or less than $1+r^2$, or as $\frac{a}{b}$ is greater or less than the secant of half the angle formed at the centre of the generating polygon by one of its sides.

When $x = a$, the first arc in the above expression becomes simply arc $\left(\text{tang.} = \frac{1}{r}\right) = \frac{(n-2)\pi}{2n}$, and we have for the action of the *whole solid*, $A = \psi - \frac{2ab^2}{a^2-b^2}\pi$, ψ representing φ after a has been put for x.

In like manner, may the action of the solid be found when

the guiding curve is an hyperbola; the only difference between that case, and the one we have just considered, being in the value of γ, which must be taken $+1$ instead of -1.

Scholium to Cor. 3, *Prop.* 27, page 278.

If the variable rectangle is given *in species*, and the touching curves are conic sections; that is, if

$$y^2 = \alpha^2 (\beta x + \gamma x^2), y'^2 = \alpha'^2 (\beta x + \gamma x^2),$$

we shall have, for the action of the generated solid, on a point at its vertex by *Prop.* 4,

$$A = 4 \int \dot{x} \text{ arc } \left(\text{tang.} = \frac{1}{rx} \sqrt{x^2 + (1 + r^2) \alpha^2 (\beta x + \gamma x^2)}\right) +$$

$$4 \int \dot{x} \text{ arc } \left(\text{tang.} = \frac{1}{r'x} \sqrt{x^2 + (1 + r'^2) \alpha'^2 (\beta x + \gamma x^2)}\right) - 2\pi x,$$

where $r = \frac{\alpha'}{\alpha}$, $r' = \frac{\alpha}{\alpha'}$; and by actually taking the fluent,

$$A = 4 \left(x + \frac{\beta\alpha^2}{1 + \gamma\alpha^2}\right) \text{arc} \left(\text{tang.} = \frac{1}{rx} \sqrt{\mu x + \nu x^2}\right) - \frac{2\beta\alpha^2}{1 + \gamma\alpha^2} \pi + \phi$$

$$+ 4 \left(x + \frac{\beta\alpha'^2}{1 + \gamma\alpha'^2}\right) \text{arc} \left(\text{tang.} = \frac{1}{r'x} \sqrt{\mu' x + \nu' x^2}\right) - \frac{2\beta\alpha'^2}{1 + \gamma\alpha'^2} \pi + \phi'$$

$-2\pi x$, where $\mu = \beta\alpha^2 (1 + r^2)$, $\nu = 1 + \gamma\alpha^2 (1 + r^2)$, $\mu' = \beta\alpha'^2 (1 + r'^2)$, $\nu' = 1 + \gamma\alpha'^2 (1 + r'^2)$,

$$\phi = \frac{4r\beta\alpha^2}{\sqrt{\nu}\,(1 + \gamma\alpha^2)} \text{L} \left(\frac{\sqrt{\nu x}}{\sqrt{\mu}} + \sqrt{\frac{\nu x}{\mu} + 1}\right), \text{ or } = \frac{4r\beta\alpha^2}{\sqrt{-\nu}\,(1 + \gamma\alpha^2)} \text{ arc } \left(\text{sine} = \frac{\sqrt{-\nu x}}{\sqrt{\mu}}\right)$$

according as ν is positive or negative,

$$\phi' = \frac{4r'\beta\alpha'^2}{\sqrt{\nu'}\,(1 + \gamma\alpha'^2)} \text{L} \left(\frac{\sqrt{\nu' x}}{\sqrt{\mu'}} + \sqrt{\frac{\nu' x}{\mu'} + 1}\right), \text{ or } = \frac{4r'\beta\alpha'^2}{\sqrt{-\nu'}\,(1 + \gamma\alpha'^2)} \text{ arc } \left(\text{sine} = \frac{\sqrt{-\nu' x}}{\sqrt{\mu'}}\right)$$

as ν' is positive or negative.

If, in the preceding expression, we make r and α' infinite, and $r' = 0$, it is reduced to

$A = 4\left(x + \frac{\beta\alpha^2}{1+\gamma\alpha^2}\right) \text{arc}\left(\text{tang.} = \frac{\alpha}{x}\sqrt{\beta x + \gamma x^2}\right) - \frac{2\beta\alpha^2}{1+\gamma\alpha^2}\pi + \varphi$

where $\varphi = \frac{4\beta\alpha}{\sqrt{\gamma}\,(1+\gamma\alpha^2)}\,\mathrm{L}\,\frac{\sqrt{\gamma x}}{\sqrt{\beta}}\Big) + \sqrt{\frac{\gamma x}{\beta} + 1}\Big)$, or $= \frac{4\beta\alpha}{\sqrt{-\gamma}\,(1+\gamma\alpha^2)}$

arc $\left(\text{sine} = \frac{\sqrt{-\gamma x}}{\sqrt{\beta}}\right)$ as ν or γ is positive or negative.

This is *the action of an infinitely long cylinder on a point at the vertex of its transverse section, the equation of the said section being* $y^2 = \alpha^2(\beta x + \gamma x^2)$.

Ex. If the base, or transverse section, is an ellipsis, or if $y^2 = \frac{b^2}{a^2}(ax - x^2)$, we have $\alpha^2 = \frac{b^2}{a^2}$, $\beta = a$, $\gamma = -1$; and $A = 4\left(x + \frac{ab^2}{a^2 - b^2}\right)$ arc $\left(\text{tang.} = \frac{b}{ax}\sqrt{ax - x^2}\right) + \frac{4a^2b}{a^2-b^2}$ arc $\left(\text{sine} = \frac{\sqrt{x}}{\sqrt{a}}\right) - \frac{2ab^2}{a^2-b^2}\pi$. When $x = a$, this expression is reduced to

$$A = \frac{2ab}{a+b}\pi.$$

Scholium to Cor. 2, *Prop.* 30, page 281.

If we would have a general expression for the attraction of such solids as the one we considered in the proposition, when the guiding curve is any conic section, or when $y^2 = \alpha^2(\beta x + \gamma x^2)$, there arises at first (from the formula for the action of a rhombus)

$A = 4\int\dot{x}$ arc $\left(\text{tang.} = \frac{1}{rx}\sqrt{x^2 + (1+r^2)\,\alpha^2(\beta x + \gamma x^2)}\right) +$
$4\int\dot{x}$ arc $\left(\text{tang.} = \frac{1}{r'x}\sqrt{x^2 + (1+r'^2)\,\alpha^2(\beta x + \gamma x^2)}\right) - 2\pi x$,

and by actually taking the fluent

$A = 4\left(x + \frac{\beta\alpha^2}{1+\gamma\alpha^2}\right)\Big\{$arc $\left(\text{tang.} = \frac{1}{rx}\sqrt{\mu x + \nu x^2}\right) +$ arc $\left(\text{tang.} = \frac{1}{r'x}\sqrt{\mu' x + \nu' x^2}\right)\Big\} - \left(\frac{4\beta\alpha^2}{1+\gamma\alpha^2} + 2x\right)\pi + \varphi + \varphi'$,

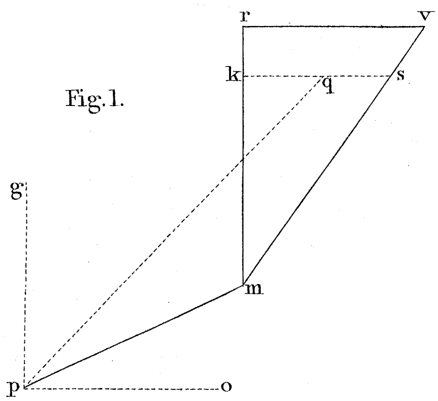


Fig. 1.

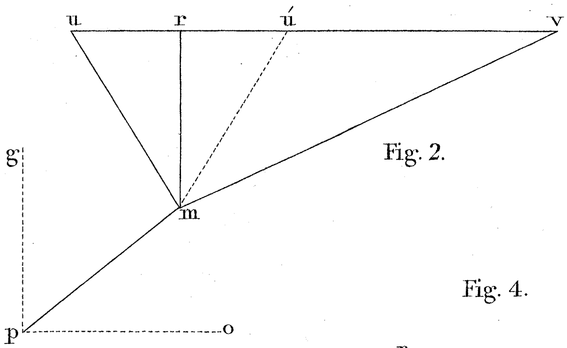


Fig. 2.

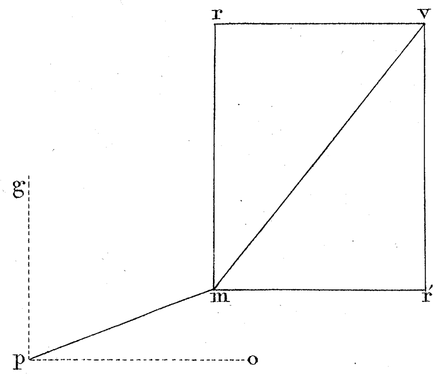


Fig. 4.

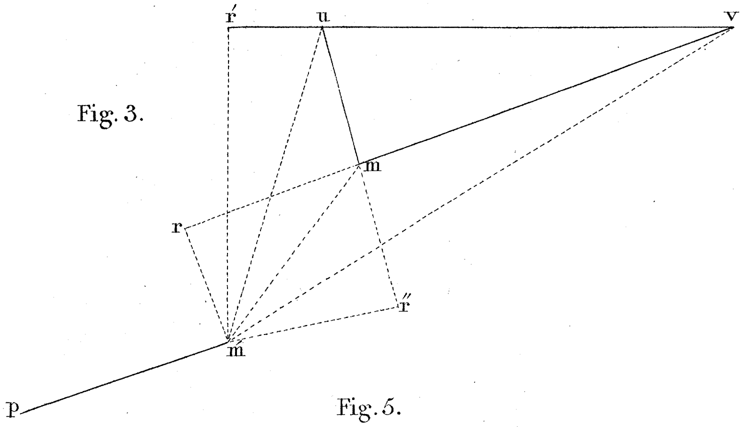


Fig. 3.

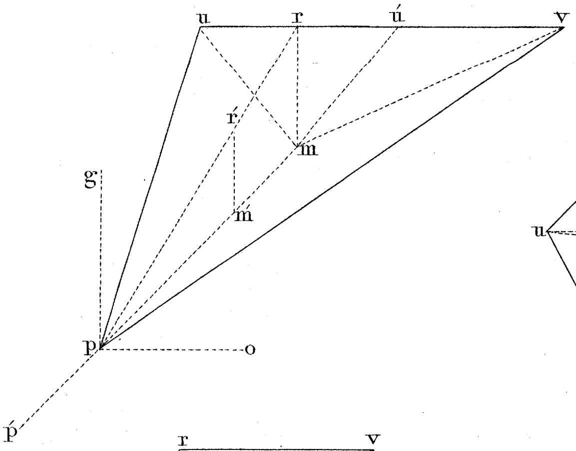


Fig. 5.

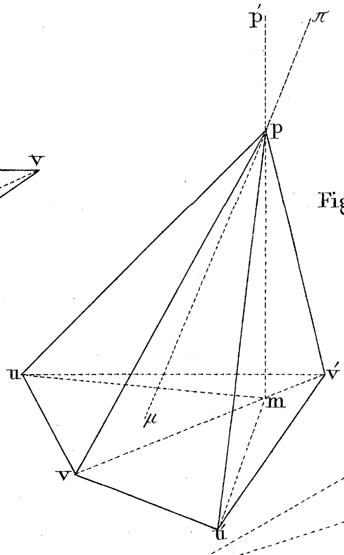
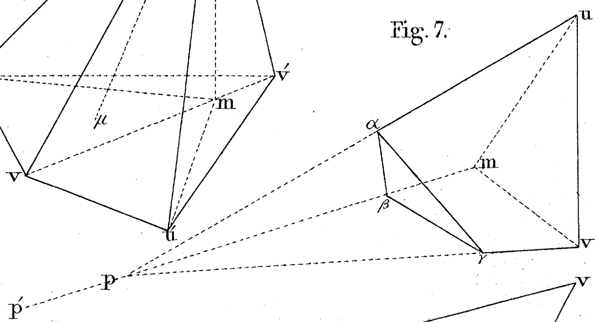


Fig. 6.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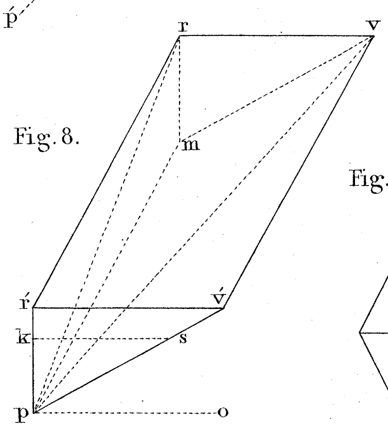
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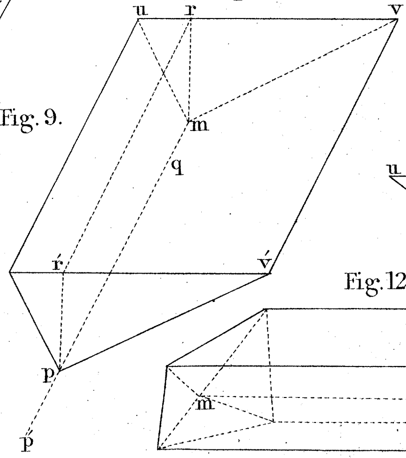
Fig. 9.

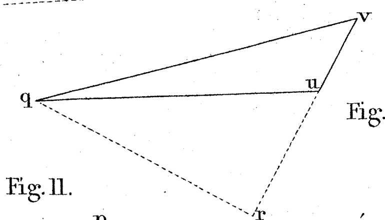
Fig. 10.

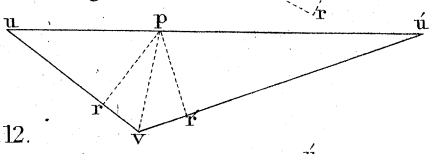
Fig. 11.

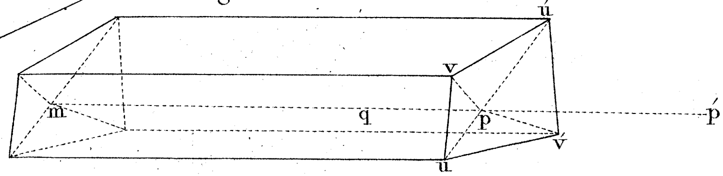
Fig. 12.

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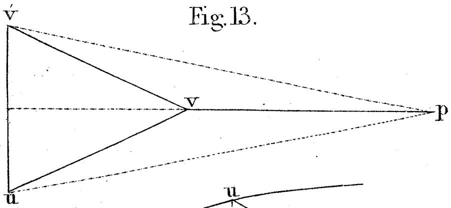

Fig. 13.

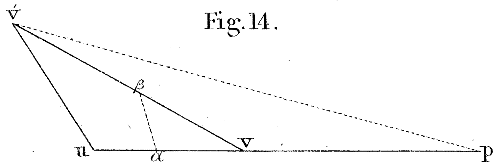

Fig. 14.

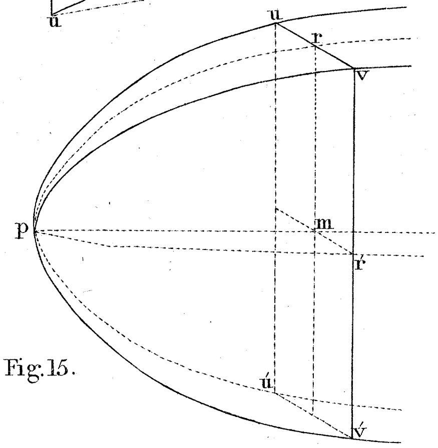

Fig. 15.

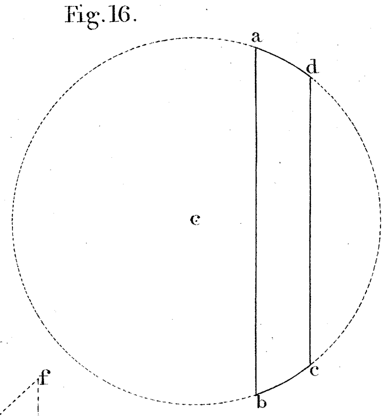

Fig. 16.

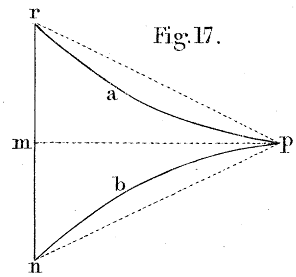

Fig. 17.

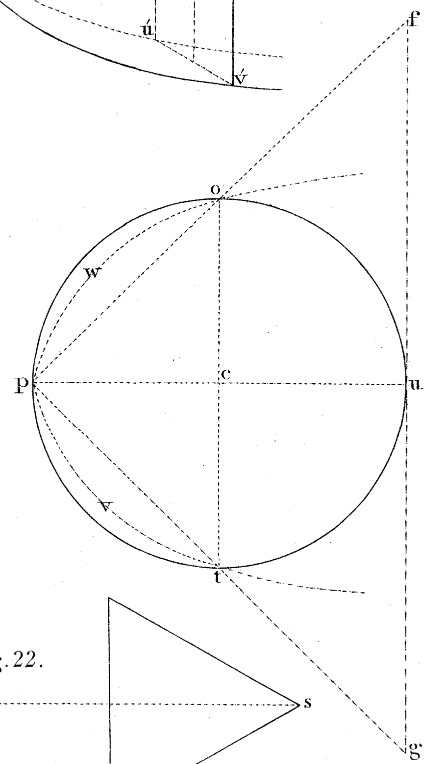

Fig. 18.

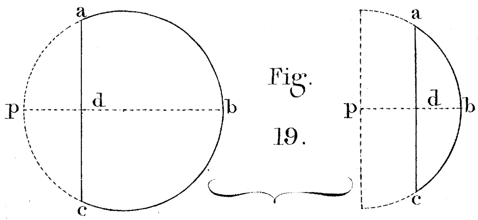

Fig. 19.

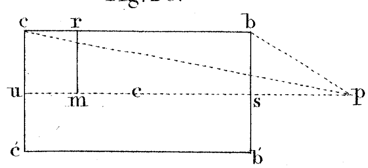

Fig. 20.

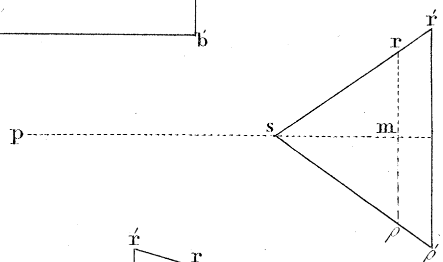

Fig. 21.

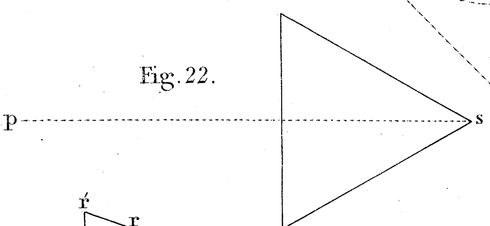

Fig. 22.

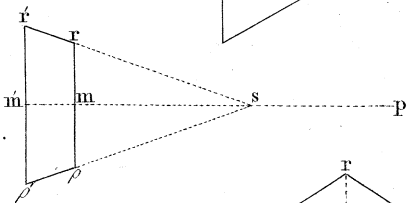

Fig. 23.

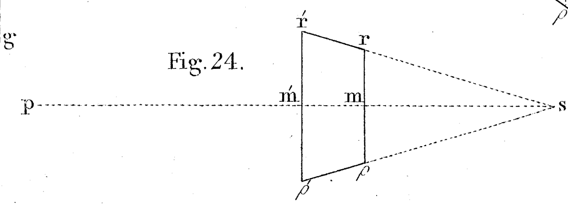

Fig. 24.

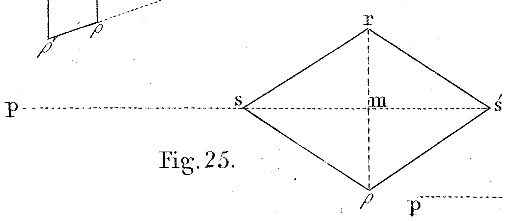

Fig. 25.

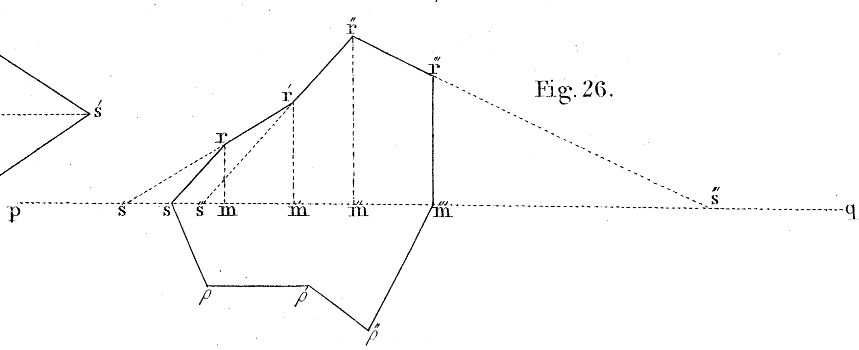

Fig. 26.

where $\mu = \beta\alpha^2(1+r^2)$, $\nu = 1+\gamma\alpha^2(1+r^2)$, $\mu' = \beta\alpha^2(1+r'^2)$, $\nu' = 1+\gamma\alpha^2(1+r'^2)$,

$\varphi = \frac{4r\beta\alpha^2}{\sqrt{\nu}\,(1+\gamma\alpha^2)} \mathrm{L}\left(\frac{\sqrt{\nu x}}{\sqrt{\mu}} + \sqrt{\frac{\nu x}{\mu}+1}\right)$, or $= \frac{4r\beta\alpha^2}{\sqrt{-\nu}\,(1+\gamma\alpha^2)}$ arc $\left(\text{sine} = \frac{\sqrt{-\nu x}}{\sqrt{\mu}}\right)$ as ν is positive or negative,

$\varphi' = \frac{4r'\beta\alpha^2}{\sqrt{\nu'}\,(1+\gamma\alpha^2)} \mathrm{L}\left(\frac{\sqrt{\nu' x}}{\sqrt{\mu'}} + \sqrt{\frac{\nu' x}{\mu'}+1}\right)$, or $= \frac{4r'\beta\alpha^2}{\sqrt{-\nu'}\,(1+\gamma\alpha^2)}$ arc $\left(\text{sine}\ \frac{\sqrt{-\nu' x}}{\sqrt{\mu'}}\right)$ as ν' is positive or negative.