XXVIII. *On the Astronomical Refractions.* *By* J. Ivory, *A. M. F. R. S.*

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1. It was known to the ancient astronomers that there is a difference between the real and apparent places of the stars, arising from the refraction of light in its passage through the atmosphere. Tycho Brahe' was the first who attempted to free his observations from the effect of this irregularity. Since his time, the astronomical refraction has become more and more an object of attention, as it is found to have the greatest influence on the delicate exactness of modern observations. In the course of the last twenty years, many researches on this subject have been published by philosophers of the first note, who have applied all the resources, both of theory and practice, to overcome the difficulties which it presents. By these means our knowledge has been greatly extended; but the problem of the refractions must still be considered as the most imperfect part of modern astronomy.

The first hypothesis for bringing the astronomical refraction under a regular mode of calculation was proposed by Cassini. He supposed that the atmosphere is a spherical shell consisting of a transparent fluid uniform in its density, which reaches to a certain height above the earth's surface. In this manner the change in the direction of the light coming from a star, is effected at the outer surface of the pellucid

medium, and it is computed by the most elementary principles of optics. This hypothesis, although extremely simple, leads to a rule for the refractions which, to a certain extent, is as accurate as any other. Perhaps it is owing to its great simplicity, that the method of Cassini seems not to have met from astronomers with the attention it deserves. Another hypothesis attributes a variable density to the atmosphere, but assumes that the rate of decrease is exactly proportional to the height ascended. This supposition is in some degree less inaccurate than that of Cassini. Most of the formulæ for the refractions that have obtained any extensive use in astronomy may be deduced from it. Kramp took a more extended view of the problem, and one less exceptionable, as approaching nearer to nature. He conducted his calculations by the real laws that regulate the density of the air, namely, the pressure and temperature. Laplace coincides with Kramp in the general view he takes of this theory; but, in treating it, he has given new proofs of that sagacity and mathematical skill, which has enabled him to accomplish so much in physical science. The table, computed by the theory of Laplace, first published in 1806, perhaps at this day gives the law of the mean refractions with greater accuracy than any other, whether founded on theory or observation.

In the hypothesis of Cassini, the atmosphere extends no higher than five miles above the earth's surface. On the supposition of a density decreasing at an equable rate, its height is limited to ten miles; but in the view of the problem taken by Kramp and Laplace, the atmosphere extends indefinitely into space. In the two first hypotheses, the horizontal refraction is considerably less than the observed

quantity; in the last, it is much greater than the truth. Now we may suppose an infinite number of atmospheres, gradually increasing in height, to be interposed between the two extreme cases; and as the horizontal refraction appears to in crease with the height, there must be some intermediate case which will quadrate with observation in this respect. If we reflect that all these atmospheres will agree in giving the refractions actually observed by astronomers as far as 70° or 80° from the zenith, it is natural to think that the one which likewise coincides with nature at the horizon, will deviate but little from the truth in the intermediate 10°. At any rate we may conjecture, that the height of the atmosphere is an element in the problem that ought not to be neglected. It may be argued indeed that the infinite atmosphere considered by KRAMP and LAPLACE, will hardly be different, mathematically speaking, from one of such considerable altitude as we must suppose in the case of the earth; and that in reality, all very high atmospheres may be reckoned as forming only one case, or at least as leading to results differing from one another only by insensible shades, that may safely be neglected in practice. This observation is probably well founded; and, beyond a certain limit, it must undoubtedly be true; but in a problem of such capital importance in astronomy, the point deserves at least to be examined; more especially as it may lead to some more certain knowledge than we have yet acquired, with respect to the extent and constitution of our atmosphere.

We have no direct knowledge of the height of the atmosphere, except what is derived from the duration of the twilight, and from the great elevation at which meteors are

occasionally observed in it. From these sources we learn that the air extends forty or fifty miles above the earth's surface, and even at that altitude still continues to possess a density sufficient for refracting and reflecting the rays of light.

The authors* who have written on the height and figure of the atmosphere have likewise assigned a boundary, beyond which it cannot reach. But in this they have rather fixed a limit to the domain peculiarly belonging to the earth, than reasoned upon any distinguishing properties of the atmosphere. If we conceive a body that circulates round the earth by the force of gravitation in the time of a diurnal revolution, the path which it describes will mark the limit where the centrifugal force arising from the rotatory motion of the earth, will just balance the opposite centripetal force. Therefore any body that participates of the rotatory motion common to all, if placed beyond the boundary we have mentioned, would continually recede from the earth, and would be lost in the immensity of space; if placed within the same boundary, it would fall to the common centre. The radius of the orbit described by the revolving body is about 25,000 miles, or something more than three diameters of the terrestrial globe. Now the air surrounding the earth cannot reach so far; for if it did, it would be continually dissipated; a supposition which is extremely improbable, since we are acquainted with no source from which a constant waste of so necessary a fluid might be supplied.

But if we would acquire more correct notions as to the height of the atmosphere, we must consider more closely the

* D'Alembert, Opus. Tom. 6. Laplace, Mec. Celeste, Liv. 3. Cap. 7.

principles on which it must depend. Conceive a cylinder of air extending indefinitely in a vertical direction, and let it be divided into equal parts of a moderate length, so that the density of every division may be considered as uniform: then, if we abstract from the diminution of gravity and the increase of the centrifugal force, which are inconsiderable within 200 or 300 miles of the earth's surface, the weight of the air in every portion of the cylinder will be proportional to its density. Now, if we admit that the elastic force is likewise proportional to the density, as it would be in an atmosphere of uniform temperature, it will follow, that the weights of the several divisions of the cylinder will vary in the same proportion as their elasticities. But in the lowest part of the cylinder, the weight of the small quantity of air contained in one division, is incomparably less than its elastic force, which is an equipoise to the whole atmosphere: and the same thing will therefore be true of every portion of the cylinder, however high it is placed. Hence an atmosphere constituted as we have supposed, must necessarily be infinite in its extent. For if it were finite, since there is no pressure at the surface, the weight of a volume of air situated there would be in equilibrium with its elastic force, whereas it has been proved that the former is always an inconsiderable part of the latter.

But in the foregoing reasoning, a cause is neglected which diminishes the elasticity of the air as we ascend above the earth's surface, without affecting the force of gravity in any degree. In the higher parts of the atmosphere a continually increasing degree of cold is found to prevail. Now, the effect of cold is to contract all bodies in their dimensions;

and therefore, by the operation of this cause, as we ascend in the atmosphere, the expansive force of a given volume of air is constantly diminished and brought nearer to an equality with its weight. To estimate this effect with greater precision, let p', z', t', denote the barometric pressure, the density, and the temperature by the centigrade thermometer, at the earth's surface; and let the same letters, without the accent, denote the same things at any height x; then, if $\beta = \frac{3}{800}$, the expansion for one centigrade degree, the known laws that obtain in the expansion of elastic fluids, will lead to this formula, viz.

$$\frac{p}{p'} = \frac{1 + \beta t}{1 + \beta t'} \times \frac{z}{z'}$$

Now here $\frac{p}{p'}$ is the measure of the elastic force at the height x in parts of the same force at the surface; and we see that it depends on the temperature as well as on the relative density $\frac{z}{z'}$. At the earth's surface the quantity $\frac{1 + \beta t}{1 + \beta t'}$ is equal to unit, but it continually decreases as the temperature becomes less in ascending. We cannot conceive that it will become negative, nor can we set any bounds to its approach to zero. But when $\frac{1 + \beta t}{1 + \beta t'}$ is evanescent, or when $t = - 266°$, the elastic force of the air will cease, and gravity will stop the farther dilatation of the atmosphere. This reasoning is independent of the law of the densities; and it proves both that the atmosphere may be finite in its extent, and that it may have a finite density at its upper surface. But it may be objected, that the effect of temperature on the air's elasticity has been verified only to a certain extent; and that in the case of air of great rarity, and subjected to extreme

degrees of cold, the law of dilatation and contraction may be very different from what it has been observed to be in the limited range of our experiments. This observation is probably well founded, but it will not destroy the force of what has been advanced. We know that air always gives out heat when it is compressed into a less volume, and absorbs heat when it expands. As long therefore as that fluid retains its elasticity, so long, we must conclude, will temperature continue to modify the changes of bulk which that force produces. The law of dilatation and contraction may no doubt undergo some change in different circumstances, but every expansion must be productive of cold, and every new degree of cold must diminish the elastic force of a given volume of air. Gravity continuing to act with nearly the same energy, while the elastic force of the air is continually diminished, these two forces will at length become equivalent, and will counterbalance one another, which is all that is necessary for imposing a limit to the extent of the atmosphere. We have proved that air, if it were confined by the action of gravity alone, would extend indefinitely into space; and it is not unreasonable to consider the effect of temperature as a contrivance for securely attaching to the terrestrial globe a fluid so necessary in every point of view to the economy of nature.

Since it is found that all elastic fluids follow the same laws in regard to heat and pressure, the foregoing reasoning is equally true, whether we conceive the atmosphere as composed of one homogeneous fluid, or as a collection of many elastic gases and vapours, however much they may differ from one another in specific gravity.

It may even be possible to form some reasonable conjecture

as to the actual height of the finite atmosphere. Gay Lussac ascended in a balloon to the altitude of 3816 English fathoms, or nearly $4\frac{1}{4}$ miles above the level of the Seine at Paris; the proportion of the heights of the barometer in the balloon and at the surface of the earth being 0.467 nearly, which is therefore the relative elasticity of the air. The temperatures, as observed at the extremities of the elevation, were $30^\circ.8$, and $-9^\circ.5$ on the centigrade scale; and if we increase 0.467 to what it would have been, had the temperature remained unchanged during the ascent, we shall find 0.500, which is the density of the air at the height ascended in parts of the density at the surface of the earth. Thus, in the decreasing scale of elasticities, the diminution is from 1 to 0.467; but, in the decreasing scale of densities, it is only from 1 to 0.500. The quantities of the one scale continually fall behind those of the other at a rate that must bring them to zero, whatever be the gradation of the latter. If we divide 3816 fathoms, the whole height ascended, by $40^\circ.3$, the difference of temperature, the elevation for depressing the thermometer one degree will come out equal to 95 fathoms: and if we suppose that the same rate prevails in all parts of the atmosphere, the whole height will be 266×95 fathoms, or nearly 29 miles. The observations of the twilight show that this is less than the true altitude; and hence we must infer, that the thermometer falls at a slower rate in the higher, than in the lower, parts of the atmosphere. But, taking the observed rate of 95 fathoms for the first 40 degrees, and allowing, on an average, a double, or even a triple, elevation for the remaining 226°, we shall still find that the atmosphere will extend only to a moderate height above the earth's surface.

2. The first investigation that presents itself in the problem of the refractions, is to find the velocity of light at any given height in the atmosphere. The only physical principle wanted for this purpose, is the refractive power of air according to its density. HAUKSBEE first determined by experiment, that air refracts light in proportion to its density; and this result has been confirmed by succeeding philosophers. There is even good reason to think that the conclusion of HAUKSBEE is not materially affected by the variable quantities of aqueous vapour contained in the atmosphere at different times. Admitting, then, the principle we have mentioned, we must conceive that the light coming from the sun, or from a star, moves in vacuo with a uniform velocity till it reaches the atmosphere. It is there deflected from its course by the spherical and concentric shells of air it meets with, each of which acts upon it with a force perpendicular to its surface, and directed to the centre of the earth. Now, as all the light enters the atmosphere with the same velocity; and as the deflecting forces are of the same intensity at the same distance from the common centre to which they tend; it follows, that the new velocities acquired by the action of the forces, will be independent of the direction of the light's motion, and will be the same at the same distance above the earth's surface. Let a denote the radius of the earth; x, any height in the atmosphere; and ρ, the density of the air at that height: conceive also a shell of air having the thickness δx, and the increased density $\varrho + \delta\rho$; then we shall have to consider the relation between the velocities of light when it passes out of a medium having the density ϱ, into another medium having the density $\varrho + \delta\rho$; or, since the density common to both

media has no effect in altering the velocity, we may consider the more simple case, when light passes out of a vacuum into a medium possessed of the density $\delta\varrho$. It is to be observed that the forces, with which matter acts on the rays of light, extend to distances that are imperceptible to our senses, and incapable of being measured; and, on this account, what has been said is modified in no respect by the thinness of the shell of air. However small δx, the thickness of the shell is supposed to be, it may still be considered as infinitely great in comparison of the range of the corpuscular force with which the light is refracted by the air. If we now put v for the velocity with which the light enters the shell of air, and express by an equation the physical principle already mentioned, namely, that the refractive power of air is proportional to its density, we shall get,

$$(v + \delta v)^2 - v^2 = \mathrm{K}\,\delta\rho,^*$$

K expressing a constant coefficient to be determined by experiment. And, because v and ϱ are functions of the same variable quantity x, the foregoing equation may be translated into the language of the differential calculus, in which case it will become,

$$d \cdot v^2 = \mathrm{K} \times d\varrho:$$

and, by integrating,

$$v^2 = 1 + \mathrm{K}\varrho,$$

$$v = \sqrt{1 + \mathrm{K}\varrho};$$

unit representing the primitive velocity of the light in vacuo.

Let us next consider the trajectory described by the light in its passage through the atmosphere. Conceive two perpendiculars to be let fall upon the tangents drawn to the

* NEWTON's Optics, Book 2, Part 3, Prop. X.

trajectory from the points where the light enters into, and passes out of, the spherical shell of air; then, if y represent the latter of these two lines, $y+dy$ will be equal to the former. The distance of the intersection of the two tangents from the centre of the earth being $a+x$, $\sqrt{(a+x)^2-y^2}$ will be the distance of the perpendiculars from the same intersection; and, on a circle described with the radius $\sqrt{(a+x)^2-y^2}$, dy is the arc that subtends the small angle contained by the two tangents; wherefore, if dr denote the measure of the small angle, we shall have

$$dy = dr \times \sqrt{(a+x)^2-y^2};$$

and,

$$dr = \frac{dy}{\sqrt{(a+x)^2-y^2}}.$$

Again; because the light is continually deflected in a direction tending to the centre of the earth, equal areas will be described round that centre in equal times by the motion in the trajectory; but the areas described in equal times are proportional to the velocities multiplied by the perpendiculars falling upon the tangents from the centre of forces: wherefore, the product $v \times y$ will have always the same magnitude at every point of the curve. Let v' be the velocity of the light at the surface of the earth, or at the point where the trajectory enters the eye of the observer; and put y' for the perpendicular upon the tangent drawn from the same point of the curve: then,

$$v \times y = v' \times y'$$

$$y = \frac{v'}{v} \times y'.$$

Suppose also that θ is the apparent zenith distance of the star, or the angle which the last-mentioned tangent makes with

the vertical of the observer, then $y = a \operatorname{Sin.} \theta$: again, if ϱ' denote the density of the air at the surface of the earth, we get, by the formula before investigated, $v' = \sqrt{1 + K \rho'}$, $v = \sqrt{1 + K \rho}$: wherefore

$$y = \frac{v'}{v} \times y' = a \operatorname{Sin.} \theta \times \sqrt{\frac{1 + K \rho'}{1 + K \rho}};$$

and, if we put $\omega = 1 - \frac{\varrho}{\rho'}$,

$$y = a \operatorname{Sin.} \theta \times \sqrt{\frac{1 + K \varrho'}{1 + K \rho' - K \rho' \omega}};$$

and finally,

$$\alpha = \frac{\frac{1}{2} K \varrho'}{1 + K \rho'},$$

$$y = \frac{a \operatorname{Sin.} \theta}{\sqrt{1 - 2 \alpha \omega}}.$$

Let this value of y be substituted in the expression of $d r$ already obtained, then

$$d r = \frac{\alpha \, d \omega}{1 - 2 \alpha \omega} \times \frac{\operatorname{Sin.} \theta}{\sqrt{\left(1 + \frac{x}{a}\right)^2 (1 - 2 \alpha \omega) - \operatorname{Sin.}^2 \theta}}.$$

Whatever opinion we adopt concerning the height of the atmosphere, $\frac{x}{a}$ may be considered as a very small quantity. For, in every hypothesis, the density of the air is attenuated so fast in ascending, that it may be taken as evanescent at an altitude extremely small in proportion to the earth's radius. The quantity α is also a very small fraction; and hence it will be sufficiently accurate if, in the foregoing expression, we put

$$\left(1 + \frac{x}{a}\right)^2 (1 - 2 \alpha \omega) = 1 + 2 \frac{x}{a} - 2 \alpha \omega;$$

by which means we obtain,

$$d r = \frac{\alpha \, d \omega}{1 - 2 \alpha \omega} \times \frac{\operatorname{Sin.} \theta}{\sqrt{\operatorname{Cos.}^2 \theta + 2 \frac{x}{a} - 2 \alpha \omega}}$$

In this formula ω and r increase in ascending above the origin

of the curve placed at the eye of the observer; and, at any determinate height, r is the angle contained by two tangents drawn from the extremities of the intercepted arc; or it is the sum of the angles which the two tangents make with the chord of the arc. When the curve is continued to the boundary of the atmosphere, or at least so high that the air has no longer power to deflect the light from its rectilineal course, the chord may be considered as parallel to the tangent at the remote extremity; and then r is the astronomical refraction. The formula is perfectly general, and will apply in all hypotheses of density, since no particular relation is established between the variable quantities ω and x.

3. But there are relations between the pressure of the air and its density and temperature, which must be attended to in the solution of this problem. Let p' and τ' denote the barometric pressure and the temperature on the centigrade scale at the surface of the earth, and put the same letters, without the accent, for the same things at the height x: then, if $\beta = \frac{3}{800}$, the expansion for one degree of the thermometer, we shall have

$$\frac{p}{p'} = \frac{1 + \beta\tau}{1 + \beta\tau'} \times \frac{\rho}{\rho'}.$$

In order to prove the truth of this formula, we may suppose a volume of air to be inclosed in a manometer, the pressure being p', the density ρ', and the temperature τ': then, if the pressure be changed to p, the temperature remaining the same, the density will become,

$$\frac{p}{p'} \times \rho':$$

and, if the temperature be now likewise changed to τ, the new density will be equal to

$$\frac{p}{p'} \times \frac{1 + \beta\tau'}{1 + \beta\tau} \times \rho':$$

but we have put ρ for the density when the pressure is p and the temperature τ; wherefore,

$$\frac{p}{p'} \times \frac{1 + \beta\tau'}{1 + \beta\tau} \times \rho' = \varrho,$$

which is equivalent to the foregoing formula.

From what has just been proved we get,

$$\frac{p'}{(1 + \beta\tau')\rho'} = \frac{p}{(1 + \beta\tau)\rho};$$

which shows that the quotient of the pressure divided by the density reduced to the fixed temperature zero, has always the same value. Therefore if l denote this constant value, we shall have,

$$p = l \times (1 + \beta\tau) \times \rho;$$

and l is a quantity to be determined by experiment.

Suppose now that a tube or cylinder of air extends from the surface of the earth to the top of the atmosphere; then the barometric column p, will be equal to the pressure of all the air in the cylinder above the height x. Let the barometer be lowered down through the small space dx; the mercury will rise a small height dp; and we shall have

$$dp = -dx \times \rho,$$

an equation which merely expresses that the small column of mercury dp is equivalent to the weight of the column of air having its length equal to dx, and its density to ρ. Divide the left side of the equation by p', and the right side by the equivalent quantity $l \times (1 + \beta\tau') \times \rho'$; then,

$$\frac{dp}{p'} = -\frac{dx}{l \times (1 + \beta\tau')} \times \frac{\rho}{\rho'};$$

and by integrating,

$$\frac{p}{p'} = \int \frac{-dx}{l(1 + \beta\tau')} \times \frac{\rho}{\rho'}.$$

In order to simplify, I shall now write P for the relative pressure $\frac{p}{p'}$, and likewise put $s = \frac{x}{l \times (1 + \beta\tau')}$; then, observing that $\frac{\rho}{\rho'} = 1 - \omega$, what has now been investigated will be expressed by these equations, viz.

$$\left.\begin{aligned} P &= \int - ds\,(1 - \omega) \\ P &= \frac{1 + \beta\tau}{1 + \beta\tau'} \times (1 - \omega) \\ s &= \frac{x}{(1 + \beta\tau')}. \end{aligned}\right\} (A)$$

The quantity $\frac{1 + \beta\tau}{1 + \beta\tau'}$, is equal to the proportion of the relative elasticity of the air to its relative density; and it may depend upon the moisture diffused in the atmosphere, as well as upon the temperature. Whatever be the true form of this function, it must be evanescent at the boundary of the atmosphere. The reason of this will readily appear, if we consider first, that, at the surface of the earth, the elasticity of a given volume of air is incomparably greater than its weight; and, secondly, that in a finite atmosphere, there must be an equality between the same two forces at the upper surface. With regard to the density, we may form two suppositions; it may either be evanescent at the top of the atmosphere, or it may have some very small finite value. But in reality we know that, in ascending, the density of the air decreases with considerable rapidity; so that if it do not decrease so as to be absolutely evanescent, it must finally become so small, that we may safely consider it as equal to zero.

To the equations already investigated we must add another, which is requisite to the solution of this Problem, although it has been universally neglected. By equating the

two foregoing values of p, we shall obtain

$$\frac{1+\beta\tau}{1+\beta\tau'} = \frac{\int -ds\,(1-\omega)}{1-\omega};$$

and by taking the fluxions,

$$\frac{d\,.\left(\frac{1+\beta\tau}{1+\beta\tau'}\right)}{ds} = -1 + \frac{d\omega}{ds}\,.\,\frac{\int -ds\,(1-\omega)}{(1-\omega)^2}:$$

but $ds = \frac{dx}{l(1+\beta\tau')}$; therefore, observing that τ decreases when x increases,

$$-\beta l \times \frac{d\tau}{dx} = -1 + \frac{d\omega}{ds}\,.\,\frac{\int -ds\,(1-\omega)}{(1-\omega)^2};$$

now, at the surface of the earth, $\mathrm{P} = \int -ds\,(1-\omega) = 1$; wherefore,

$$\frac{d\omega}{ds}\,(\text{when } s=0) = 1 - \beta l \times \frac{d\tau}{dx}.$$

Suppose that μ represents the height through which the thermometer must be carried at the surface of the earth, in order to depress the mercury one degree; it is obvious, that $\frac{1}{\mu}$ is the numerical value of $\frac{d\tau}{dx}$: wherefore,

$$\frac{d\omega}{ds}\,(\text{making } s=0) = 1 - \frac{\beta l}{\mu}. \quad \text{(B)}$$

The quantity $\frac{d\omega}{ds}$ is derived from the function of the height that represents the decrease of density. It appears that the value of it at the surface of the earth depends upon μ; and terrestrial observations show that this quantity is subject to great irregularities, which are not well understood. It is found that the refractions near the horizon are liable to variations equally irregular and unknown. There can be little doubt that both these effects are produced by the same causes, which disturb the gradation of heat, and the arrangement of the strata of air near the earth's surface.

It will now be necessary to resume the former value of dr,

and in it to substitute s for x. Now, $s = \frac{x}{l(1+\beta\tau')} = \frac{x}{a} \times \frac{a}{l(1+\beta\tau')}$: if therefore we put $i = \frac{l(1+\beta\tau')}{a}$, we shall get $is = \frac{x}{a}$, and

$$dr = \frac{\alpha\, d\omega}{1 - 2\alpha\omega} \times \frac{\text{Sin.}\,\theta}{\sqrt{\text{Cos.}^2\theta + 2is - 2\alpha\omega}};$$

and, by expanding $\frac{1}{1-2\alpha\omega}$,

$$dr = \alpha\,\text{Sin.}\,\theta \times \frac{d\omega}{\sqrt{\text{Cos.}^2\theta + 2is - 2\alpha\omega}}$$
$$+ \alpha^2\,\text{Sin.}\,\theta \times \frac{2\omega\, d\omega}{\sqrt{\text{Cos.}^2\theta + 2is - 2\alpha\omega}}$$
$$+ \&c.$$

The second term of this expansion has to the first a less proportion than that of α to 1, while ω increases from 0 to $\frac{1}{2}$; and a greater proportion, while ω increases from $\frac{1}{2}$ to 1: and hence, on account of the smallness of α, we may combine both terms in one, viz.

$$dr = \alpha(1+\alpha)\,\text{Sin.}\,\theta \times \frac{d\omega}{\sqrt{\text{Cos.}^2\theta + 2is - 2\alpha\omega}}. \quad (C)$$

4. In order to appreciate justly the several formulæ on which this theory depends, it is necessary to know the values of the quantities that must be found from observation. Of these, the coefficient α has been determined both astronomically, and by direct experiments on the refractive power of the air. From the comparison of a great number of astronomical observations, DE LAMBRE found $K\rho' = \cdot 000588094$, at the temperature of melting ice, and the mercury in the barometer standing at 29.921 English inches. In the same circumstances, M. M. BIOT and ARAGO, by very accurate experiments on the refraction of air inclosed in a prism, found .000588768 for the value of the same quantity. Adopting

the number of DE LAMBRE, which is that employed in the calculation of the French tables of refraction, we get

$$\alpha = \frac{\frac{1}{2} K \rho'}{1 + K \rho'} = .000293876;$$

and, by reducing to the mean temperature of 10° on the centigrade scale, or 50° of FAHRENHEIT, and to the standard barometer 30 English inches, we finally obtain

$$\alpha = .0002835,$$

$$\text{Log.} - 4.4525531.$$

From a numerous set of observations Dr. BRINKLEY has deduced a value somewhat less than the preceding; and hence it appears, that there is still some small degree of uncertainty in the determination of this coefficient. It is to be expected that the unequal mixture of moisture, by altering the density of the air, will produce variations in the value of α. But it has been determined that, when a quantity of aqueous vapour is added to a volume of air, the density is diminished nearly in the same proportion that the refractive power of the vapour is greater than the refractive power of the air. A compensation is thus effected; and the mixed medium is hardly different from dry air of the like density in its action on light.

The value of l must be found by means of the formula

$$p = l \times (1 + \beta\tau)\varrho.$$

Here we must conceive that ϱ is measured in parts of the density of mercury; and, as $(1 + \beta\tau)\varrho$ is the density of the air reduced to the fixed temperature zero, the equation merely expresses that the density of air is proportional to the pressure when the temperature remains unchanged. Now, M. M. BIOT and ARAGO have found that the specific gravity of air under the pressure of 0.76 metres, and at the temperature of

melting ice, is to the specific gravity of mercury at the same temperature, as 1 to 10467: hence we have $p = 0.76, \rho = \frac{1}{10467}$, $\tau = o$; and, by the substitution of these numbers, we get,

$$l = 10467 \times 0.76 \text{ metres},$$

or, in English fathoms,

$$l = 4349.8.$$

This is the length of l at the temperature of melting ice; but, if the temperature be changed, it will vary directly as the volume of the air, and inversely as that of the mercury. If now we take for the radius of the earth $(= a)$, a mean between half the polar axis and the radius of the equator, and reduce the foregoing value of l to the mean temperature of 50° of FAHRENHEIT, we shall get,

$$\left.\begin{array}{l} l = 4504.8 \\ a = 3481280 \end{array}\right\} \text{fathoms}$$

$$\frac{l}{a} = .001294; \text{ Log.} - 3.1119343.$$

The value of μ, or the height through which the thermometer must be carried at the earth's surface, in order to depress the mercury one degree, has not been determined with much certainty or exactness. The greatest irregularity is found to prevail, in regard to this element, in observations made on different heights and at different times. This is, no doubt, to be attributed in part to local peculiarities affecting the thermometer. The most accurate way of determining this element would be by means of observations made in balloons elevated to moderate heights. RAMOND, from 38 barometrical measurements, makes the mean depression for one centesimal degree equal to 164.7 metres, or 90 fathoms; HUMBOLDT found 161 metres, or 88 fathoms; and the ascent

of GAY LUSSAC gives 174 metres, or 95 fathoms. With these several values we shall find $\frac{\beta l}{\mu}$ equal to 0.188, 0.192, and 0.177 respectively; and we may adopt $\frac{1}{5}$ as an approximation. Thus,

$$\frac{\beta l}{\mu} = \frac{1}{5}$$

$$\frac{d\omega}{ds}\,(\text{making } s = o) = 1 - \frac{\beta l}{\mu} = \frac{4}{5}.$$

5. In one particular case of this problem we are possessed of many skilful observations made in the course of the trigonometrical surveys of England and France. We allude to the terrestrial refraction, which regards that part of the trajectory described by the light in its passage from a terrestrial object to the eye of the observer. As this case is immediately deduced from the equations that have been investigated, the comparison of the result with observations may, in some degree, instruct us how far the theory will agree with nature.

We have found this equation, viz.

$$\frac{d\omega}{ds}\,(\text{making } s = o) = \frac{4}{5};$$

which being accurately true at the surface of the earth, it may, without sensible error, be extended to a small height above the surface. In the case of the terrestrial refraction we thus have,

$$\omega = \frac{4}{5}\, s;$$

and, if this value be substituted in the expression of dr, we get,

$$dr = \alpha\,\text{Sin.}\,\theta \times \frac{\frac{4}{5} ds}{\sqrt{\text{Cos.}^2\,\theta + 2\,(i - \frac{4}{5}\alpha)\, s}}.$$

By integrating

$$r = \frac{4}{5} \times \frac{\alpha\,\text{Sin.}\,\theta}{i - \frac{4}{5}\alpha} \times \left\{ \sqrt{\text{Cos.}^2\,\theta + 2\,(i - \frac{4}{5}\,\alpha)\, s} - \text{Cos.}\,\theta \right\}:$$

and, if the observed object be just 90° from the zenith, then

$$r = \frac{4\alpha}{5} \times \frac{\sqrt{2s}}{\sqrt{i - \frac{4}{5}\alpha}}.$$

Now $s = \frac{x}{l(1+\beta\tau')}$; and neglecting the effect of temperature, $s = \frac{x}{l}$. Let ν be the angle at the earth's centre contained by lines drawn to the observer and the object; then, x being the height between the surface of the earth and the tangent to that surface drawn from the place of the observer, we have $a.\nu^2 = 2x$; and hence $2s = \frac{2x}{l} = \frac{a}{l}\nu^2 = \frac{\nu^2}{i}$: consequently,

$$\frac{r}{2} = \frac{2}{5} \times \frac{\alpha}{i} \cdot \frac{\nu}{\sqrt{1 - \frac{4}{5} \cdot \frac{\alpha}{i}}};$$

and, in numbers,

$$\frac{r}{2} = \frac{\nu}{10.36}.$$

Now r is the sum of the angles which the tangents, drawn from the extremities of the arc intercepted between the observer and the object, make with the chord of the arc; and, as the curvature will vary but little in a small extent, the two angles may be considered as equal, and $\frac{r}{2}$ will be the refraction at the eye of the observer. When the terrestrial refraction, as found by actual observation, is compared with the angle at the earth's centre, it is very irregular, varying from $\frac{1}{2}$ to $\frac{1}{24}$. In a case, where such excessive irregularities occur, no great confidence can be placed in a mean, even of a great number of observations; more especially as local peculiarities have so much effect, that the mean at one place does not agree with the mean at another. In the English Survey, $\frac{1}{10}$ is allowed for the terrestrial

refraction; the French mathematicians make it equal to $\frac{1}{12}$; and the result found above falls between these limits.

6. In the expression of the refraction (C), the quantities i and α are very small fractions; and $\cos.^2 \theta$ varies from 1 to 0 as the zenith distance increases from 0 to 90°. For a considerable extent from the zenith $\cos.^2 \theta$ will greatly exceed i and α; and so long as this is the case, we may find the value of r by expanding the radical quantity in a series. Proceeding in this manner, and retaining only the two first terms of the expansion, we shall get

$$dr = \alpha(1+\alpha)\text{Tan.}\,\theta \times \left\{d\omega - \frac{isd\omega - \alpha\omega d\omega}{\text{Cor.}^2\theta}\right\}:$$

and, by integrating from $\omega = o$ to $\omega = 1$,

$$r = \alpha\,\text{Tan.}\,\theta \times \left\{1 + \alpha - \frac{i\int s\,d\omega - \frac{1}{2}\alpha}{\text{Cor.}^2\theta}\right\},$$

the terms multiplied by α, $i\alpha$, α^2 being alone retained. Now

$$s(1-\omega) = \int ds(1-\omega) - \int s\,d\omega;$$

and because $s(1-\omega) = o$, both when $\omega = o$ and $\omega = 1$, if we take the whole integrals between these limits, we get

$$\int s\,d\omega = \int ds(1-\omega).$$

But $\int ds(1-\omega)$ between the limits $\omega = o$ and $\omega = 1$ has the same value that $\int - ds(1-\omega)$ has, between the limits $\omega = 1$ and $\omega = o$; and this last integral is equal to the whole pressure, or to unit: wherefore

$$\int s\,d\omega = 1;$$

and, by substitution,

$$r = \alpha\,\text{Tan.}\,\theta \times \left\{1 + \alpha - \frac{i - \frac{1}{2}\alpha}{\text{Cos.}^2\theta}\right\}.$$

By means of this formula, which was first found by LAPLACE, the French tables of refraction are computed as far as 74° from the zenith. The quantities i and α, depending only upon

the temperature and pressure of the lowest stratum of air and upon the radius of the earth, the formula involves no hypothesis concerning the gradation of heat or density. But if the expansion of the expression of the refraction be extended to more terms, we meet with quantities that cannot be integrated without supposing a relation between s and ω, that is, without introducing a supposition respecting the constitution of the atmosphere.

The ultimate deviation of the light of a star from its primitive direction depends upon the augmentation of the velocity which the light acquires in its passage through the atmosphere, and likewise upon the different obliquities with which it crosses the several strata of air. Now, the first of these two things is the same for all stars and for all constitutions of the atmosphere; for it is the same when the density of the lowest stratum of air continues the same. But the second is different for stars that are differently placed with regard to the zenith: and it varies also with the densities of the strata that compose the atmosphere. It is therefore certain that the formula of Laplace is rigorously exact in no case whatever. But when a star is near the zenith, the variations in the obliquity of the light in passing through the several strata of air, are inconsiderable; and the formula will be nearly true. However, there is always some error, which accumulates as the zenith distance increases, and will at length become sensible. Delambre tells us that in comparing the observations of different days, he found errors arising from refraction that amounted to 6″ or 7″ at 75° from the zenith;* and the observations of a very accurate astronomer

* Astron. Vol. 1. p. 320.

show that similar inequalities are perceptible much nearer the zenith.* Now these inequalities do not arise from any thing imperfect in the manner of observing; they are undoubtedly produced by alterations in the remote parts of the atmosphere, which do not affect the barometer or the thermometer placed at the Observatory. It appears, therefore, that the peculiar constitution of the atmosphere has a perceptible influence on the refraction at 75° from the zenith; and when LAPLACE's formula is made to extend to 74°, it is carried to its utmost limit.

However mutable we may suppose the condition of the atmosphere to be, there must be a mean state equally removed from the opposite extremes. Now, a table of refractions that should have this mean state of the atmosphere for its basis, would be the most advantageous of any. For although, with respect to single observations, the errors of such a table might be as great as in some other hypotheses, yet, in a numerous set of observations made at different times, so as to embrace all the usual changes, the inequalities of an opposite kind would counterbalance one another. But, to a certain distance from the zenith, LAPLACE's formula is sufficiently exact for practical purposes; and it has the advantage of taking away the necessity of having recourse to precarious suppositions respecting the constitution of the atmosphere.

As the formula we are considering contains nothing except what is common to every atmosphere, it must be deducible from the hypothesis of CASSINI; and it may be worth while to establish this point by a strict investigation. CASSINI supposed that the earth is surrounded by a pellucid spherical

* Dr. BRINKLEY's Paper, Philosophical Transactions, 1821, p. 342.

shell of uniform density, reaching to a certain altitude, and possessed of the same weight with the real atmosphere. The height of this homogeneous stratum of air will therefore be equal to the quantity l, before investigated. Suppose that the light of a star is refracted at the upper surface of this atmosphere in a straight line directed to the eye of the observer, and making an angle θ with the vertical line; then the perpendicular let fall upon the refracted ray from the earth's centre will be equal to a Sin. θ: and, in a right-angled triangle, of which $a + l$ is the hypothenuse, a Sin. θ one side, and ϕ the angle at the top of the atmosphere opposite to that side, we have,

$$\text{Sin.}\,\phi = \frac{a\,\text{Sin.}\,\theta}{a+l} = \frac{\text{Sin.}\,\theta}{1+\frac{l}{a}} = \frac{\text{Sin.}\,\theta}{1+i},$$

$$\text{Cos.}\,\phi = \frac{\sqrt{\text{Cos.}^2\theta + 2i + i^2}}{1+i},$$

$$\text{Tan.}\,\phi = \frac{\text{Sin.}\,\theta}{\sqrt{\text{Cos.}^2\theta + 2i + i^2}}.$$

It is manifest that ϕ is the angle of refraction; and if r be the refraction, or the angle between the incident and refracted light, $\phi + r$ will be the angle of incidence: and Sin. $(\phi + r)$ will be to Sin. ϕ, as the velocity of the light in air to the velocity in vacuo, that is, as $\sqrt{1 + K\rho'}$ to 1, or as $\frac{1}{\sqrt{1-2\alpha}}$ to 1: wherefore,

$$\text{Sin.}\,(\phi + r) = \frac{\text{Sin.}\,\phi}{\sqrt{1-2\alpha}}$$

$$\text{Cos.}\,(\phi + r) = \frac{\sqrt{\text{Cos.}^2\phi - 2\alpha}}{\sqrt{1-2\alpha}}.$$

But,

$$\text{Sin.}\,r = \text{Sin.}\,(\phi + r)\,\text{Cos.}\,\phi - \text{Cos.}\,(\phi + r)\,\text{Sin.}\,\phi;$$

therefore,

$$\text{Sin.}\, r = \frac{\text{Sin.}\,\phi\ \text{Cos.}\,\phi - \text{Sin.}\,\phi\sqrt{\text{Cos.}^2\phi - 2\alpha}}{\sqrt{1-2\alpha}}.$$

Now $r = \text{Sin.}\, r + \frac{1}{6}\text{Sin.}^3 r + \text{\&c.}$; and, at the horizon where r is greatest, all the terms of the series after the first will not amount to $\frac{1}{20}$ of a second: thus $r = \text{Sin.}\, r$, and

$$r = \frac{\text{Sin.}\,\phi\ \text{Cos.}\,\phi - \text{Sin.}\,\phi\sqrt{\text{Cos.}^2\phi - 2\alpha}}{\sqrt{1-2\alpha}};$$

by expanding the radical quantity in the numerator,

$$r = \frac{\text{Tan.}\,\phi}{\sqrt{1-2\alpha}} \times \left\{\alpha + \tfrac{1}{2}\cdot\frac{\alpha^2}{\text{Cos.}^2\phi} + \tfrac{1}{2}\cdot\frac{\alpha^3}{\text{Cos.}^4\phi} + \tfrac{5}{8}\cdot\frac{\alpha^4}{\text{Cos.}^6\phi} + \text{\&c.}\right\}.$$

When $\text{Cos.}\,\theta = 0$, $\text{Tan.}\,\phi = \frac{1}{\sqrt{2i}}$, and $\frac{1}{\text{Cos.}^2\phi} = \frac{1}{2i}$; and hence, even in this extreme case, the term last set down of the foregoing series, and all the following terms, may be rejected; therefore, because $\frac{1}{\text{Cos.}^2\phi} = 1 + \text{Tan.}^2\phi$, we have

$$r = \frac{\alpha + \frac{1}{2}\alpha^2}{\sqrt{1-2\alpha}}\text{Tan.}\,\phi + \frac{\frac{1}{2}\alpha^2 + \alpha^3}{\sqrt{1-2\alpha}}\text{Tan.}^3\phi + \frac{\frac{1}{2}\alpha^3}{\sqrt{1-2\alpha}}\text{Tan.}^5\phi;$$

and farther, by rejecting the very small quantities $\alpha^3\,\text{Tan.}\,\phi$, $\alpha^3\,\text{Tan.}^3\phi$, $\alpha^4\,\text{Tan.}^5\phi$, &c. we obtain, with sufficient accuracy,

$$r = (\alpha + \tfrac{3}{2}\alpha^2)\,\text{Tan.}\,\phi + \frac{\alpha^2}{2}\text{Tan.}^3\phi + \frac{\alpha^3}{2}\text{Tan.}^5\phi;$$

and finally, by substituting the value of $\text{Tan.}\,\phi$,

$$r = \frac{(\alpha + \frac{3}{2}\alpha^2)\,\text{Sin.}\,\theta}{\left\{\text{Cos.}^2\theta + 2i + i^2\right\}^{\frac{1}{2}}} + \frac{1}{2}\cdot\frac{\alpha^2\,\text{Sin.}^3\theta}{\left\{\text{Cos.}^2\theta + 2i + i^2\right\}^{\frac{3}{2}}} + \frac{1}{2}\cdot\frac{\alpha^3\,\text{Sin.}^5\theta}{\left\{\text{Cos.}^2\theta + 2i + i^2\right\}^{\frac{5}{2}}}.$$

If we put $\text{Sin.}\,\theta = 1$, $\text{Cos.}\,\theta = 0$, we shall obtain the horizontal refraction in the hypothesis of CASSINI, viz.

$$r = \frac{\alpha}{\sqrt{2i}} \times \left\{1 + \tfrac{3}{2}\alpha + \tfrac{1}{4}\cdot\frac{\alpha}{i} + \tfrac{1}{8}\cdot\frac{\alpha^2}{i^2}\right\}.$$

To find the refractions near the zenith, we must develope the radical quantities and retain, as formerly, such terms only as are multiplied by α, αi, α^2: in this manner we get,

$$r = \alpha \operatorname{Tan.} \theta \cdot \left\{ 1 + \frac{3}{2}\alpha - \frac{i}{\operatorname{Cos.}^2 \theta} + \frac{\alpha}{2} \operatorname{Tan.}^2 \theta \right\};$$

or,

$$r = \alpha \operatorname{Tan.} \theta \times \left\{ 1 + \alpha - \frac{i - \frac{1}{2}\alpha}{\operatorname{Cos.}^2 \theta} \right\},$$

and this is no other than LAPLACE's formula, which is thus deducible from the most simple, as well as the most complicated, hypothesis.

The same formula may be thus written, viz.

$$r = \alpha(1+\alpha) \cdot \left\{ \operatorname{Tan.} \theta - \frac{i - \frac{1}{2}\alpha}{\alpha(1+\alpha)^2} \times \alpha(1+\alpha) \operatorname{Tan.} \theta \times \frac{1}{\operatorname{Cos.}^2 \theta} \right\};$$

and, the second term of this expression being inconsiderable in comparison of the first, we get, for an approximate value, $r = \alpha(1+\alpha) \operatorname{Tan.} \theta$; and again, if we substitute r for $\alpha(1+\alpha) \operatorname{Tan.} \theta$ we shall obtain,

$$n = \frac{i - \frac{1}{2}\alpha}{\alpha(1+\alpha)^2},$$

$$r = \alpha(1+\alpha) \cdot \left\{ \operatorname{Tan.} \theta - n r \times \frac{d \cdot \operatorname{Tan.} \theta}{d\theta} \right\}.$$

But this value of r is no other than the two first terms of the developement of $\alpha(1+\alpha) \operatorname{Tan.} (\theta - nr)$; and hence,

$$r = \alpha(1+\alpha) \operatorname{Tan.} (\theta - nr),$$

an expression which must be considered as an approximation of the same order with the formula of LAPLACE, and it must be restricted within the same limits. It is to be observed, however, that the two forms of expression will not be entirely equivalent unless the same values of α and i be, in every case, substituted in both; which implies that n will vary a little according to the pressure and temperature of the air.

The formula for the refractions near the zenith is common

to every constitution of the atmosphere. In proceeding farther, our reasoning must comprehend all the varieties of temperature and density that actually take place in nature from the surface of the earth to the utmost height at which the air possesses power to refract the rays of light. Even if we should succeed in this, it would be chimerical to expect that a formula can be found that would apply to single observations without great occasional inequalities. This is not to be ascribed to any fault of the theory; it arises from the nature of the observations themselves. If we examine a set of observed refractions, it will be easy to discover instances in which the true refraction has diminished when, according to the instruments employed, it ought to have increased; and, the contrary. The refractions are therefore affected by circumstances of which the observer has no intimation, and which cannot enter into any theory. The real causes of such anomalies is undoubtedly the irregular changes that take place in the remote parts of the atmosphere, which are not indicated by the barometer or the thermometer. We must conceive that the atmosphere is perpetually oscillating about a mean state, which it ought to be the aim of theory to discover. The test of success in the research must be looked for, not in the perfect agreement of the theory with every single instance, but in the disappearance of the unavoidable errors in a sufficient number of observations made at different times.

7. There is no ground in experience for attributing to the gradation of heat in the atmosphere any other law than that of an equable decrease as the altitude increases. This law prevails very nearly at least to the greatest heights to which

we have been able to ascend. The mean elevation for one degree of depression of the centigrade thermometer is very nearly 90 English fathoms; and in the great height ascended by GAY LUSSAC, rather more than $4\frac{1}{4}$ miles, the same quantity comes out equal to 95 fathoms. To this great extent the law of a uniform decrease of temperature holds good, without much deviation from the truth. It therefore seems to be the assumption most likely to guide us aright in approximating to the true constitution of the atmosphere.

The law we have mentioned is expressed by this equation, viz.

$$\frac{1+\beta\tau}{1+\beta\tau'} = 1 - \frac{s}{m+1};$$

$m+1$ being a constant quantity which, in the case of nature, will be determined by equation (B). Now if we substitute this value of $\frac{1+\beta\tau}{1+\beta\tau'}$ in the formula (A), and then equate the two values of P, we shall get,

$$\left(1 - \frac{s}{m+1}\right) \cdot (1-\omega) = \int - ds\,(1-\omega);$$

and, hence,

$$\frac{d\omega}{1-\omega} = \frac{ds}{m+1} \times \frac{m}{1-\frac{s}{m+1}};$$

consequently,

$$1 - \omega = \left(1 - \frac{s}{m+1}\right)^m.$$

Thus we obtain,

$$P = \left(1 - \frac{s}{m+1}\right)^{m+1}$$

$$1 - \omega = \left(1 - \frac{s}{m+1}\right)^m$$

$$\frac{1+\beta\tau}{1+\beta\tau'} = 1 - \frac{s}{m+1}.$$

In these equations, the hypothesis of CASSINI corresponds to $m=o$; that of a density decreasing uniformly as the altitude

increases, to $m = 1$; and, when m is infinitely great, the same equations become,

$$P = c,^{-s}$$

$$1 - \omega = c,^{-s}$$

$$\frac{1 + \beta\tau}{1 + \beta\tau'} = 1,$$

c being the base of the hyperbolic logarithms; and they now belong to an atmosphere in which the density is proportional to the pressure, and the heat is the same in every part. These three suppositions, with some modifications of them, are the foundations of all the theories that have been advanced with regard to the variations of density in the atmosphere. They are the simplest cases that come under the foregoing formulæ, and likewise those that are suggested by the most obvious physical hypotheses. But in reality these considerations afford no good ground of preference; since, whatever value we give to m, the general laws relating to the heat and pressure of the air, are equally well represented. The refractions near the zenith will likewise be the same, whatever number m stands for. We may therefore adopt that value of m which will give the true refractions near the horizon; or that one, which will satisfy equation (B), in which case the gradation of heat will coincide with that actually observed at the surface of the earth. More especially if, by the same value of m, we can conciliate both the above-mentioned conditions, we may conclude that the solution of the problem must agree well with observation. But, in order to continue this research, it is necessary to find a method that will enable us to compute the refractions for any proposed value of m.

If we make $z = \frac{s}{m+1}$, then

$$s = m + 1 \cdot z,$$
$$1 - \omega = (1 - z),^m$$
$$\omega = 1 - (1 - z),^m$$
$$2is - 2\alpha\omega = 2i(m+1)z - 2\alpha\left\{1 - (1-z)^m\right\}:$$

and, again,

$$\lambda = \frac{\alpha}{i},$$
$$\mathrm{a} = m + 1 - \lambda$$
$$\psi = 1 - (1 - z)^{m-1}$$
$$2is - 2\alpha\omega = 2i\mathrm{a}z - 2i\lambda(1 - z)\psi$$
$$d\omega = m\,dz\,(1 - z)^{m-1}.$$

The expression of the refraction (equation C) will therefore become,

$$r = \alpha(1 + \alpha)\,\text{Sin.}\,\theta \times \int \frac{m\,dz\,(1 - z)^{m-1}}{\sqrt{\text{Cos.}^2\theta + 2i\mathrm{a}z - 2i\lambda(1-z)\psi}}:$$

and by expanding the radical quantity,

$$\Delta = \sqrt{\text{Cos.}^2\,\theta + 2i\mathrm{a}z}$$

$$r = \alpha(1 + \alpha)\,\text{Sin.}\,\theta \times \left\{ \int \frac{m\,dz\,(1-z)^{m-1}}{\Delta} + i\lambda \int \frac{m\,dz\,(1-z)^m\psi}{\Delta^3} + \frac{1 \cdot 3}{1 \cdot 2} \cdot i^2\lambda^2 \cdot \int \frac{m\,dz\,(1-z)^{m+1}\psi^2}{\Delta^5} + \&c. \right.$$

And, in this expression, it is not necessary to integrate generally, but merely to find the definite integrals between the limits $z = o$ and $z = 1$.

Now by taking the fluxions of the quantity $\frac{(1-z)^{m+n-1}\psi^n}{\Delta^{2n-1}}$, we have

$$\frac{(1-z)^{m+n-1}\psi^n}{\Delta^{2n-1}} = -(2n-1) \cdot i\mathrm{a} \cdot \int \frac{dz\,(1-z)^{m+n-1}\psi^n}{\Delta^{2n+1}}$$

$$+\int \frac{dz}{\Delta^{2n-1}} \cdot \frac{d\cdot(1-z)^{m+n-1}\psi^n}{dz};$$

and because the function without the sign of integration is evanescent at both the limits $z=0$, $z=1$, we shall get, with regard to the definite integrals,

$$\int \frac{dz\,(1-z)^{m+n-1}\psi^n}{\Delta^{2n+1}} = -\frac{1}{2n-1}\cdot\frac{1}{ia}\cdot\int\frac{dz}{\Delta^{2n-1}}\cdot\frac{d\cdot(1-z)^{m+n-1}\psi^n}{dz}.$$

By operating in like manner with the quantities $\frac{1}{\Delta^{2n-3}}\cdot\frac{d\cdot(1-z)^{m+n-1}\psi^n}{dz}$ and $\frac{1}{\Delta^{2n-5}}\cdot\frac{dd\cdot(1-z)^{m+n-1}\psi^n}{dz^2}$, we shall obtain,

$$\int\frac{dz}{\Delta^{2n-1}}\cdot\frac{d\cdot(1-z)^{m+n-1}\psi^n}{dz} = \frac{1}{2n-3}\cdot\frac{1}{ia}\cdot\int\frac{dz}{\Delta^{2n-3}}\cdot\frac{dd\cdot(1-z)^{m+n-1}\psi^n}{dz^2}$$

$$\int\frac{dz}{\Delta^{2n-3}}\cdot\frac{dd\cdot(1-z)^{m+n-1}\psi^n}{dz^2} = \frac{1}{2n-5}\cdot\frac{1}{ia}\cdot\int\frac{dz}{\Delta^{2n-5}}\cdot\frac{d^3(1-z)^{m+n-1}\psi^n}{dz^3}.$$

And if we continue the like operations till we come to the quantity, $\frac{1}{\Delta}\cdot\frac{d^n\cdot(1-z)^{m+n-1}\psi^n}{dz^n}$, which is no longer divisible by ψ; and then combine all the results, we shall get,

$$\int\frac{dz\,(1-z)^{m+n-1}\psi^n}{\Delta^{2n+1}} = \frac{1}{1\cdot3\cdot5\ldots2n-1}\cdot\frac{1}{i^n a^n}\cdot\int\frac{dz}{\Delta}\cdot\frac{d^n\cdot(1-z)^{m+n-1}\psi^n}{dz^n}.$$

By the application of this formula all the integrals in the value of r will be reduced to others in which the exponent of Δ is unit; viz.

$$r = \alpha(1+\alpha)\,\text{Sin.}\,\theta \times \Big\{ \int\frac{m\,dz\,(1-z)^{m-1}}{\Delta}$$

$$+\lambda\,.\,m\,.\int\frac{dz}{\Delta}\cdot\frac{d\,.\,(1-z)^m\psi}{a\,.\,dz}$$

$$+\frac{\lambda^2}{1\,.\,2}\cdot m\,.\int\frac{dz}{\Delta}\cdot\frac{dd\,.\,(1-z)^{m+1}\psi^2}{a^2.\,dz^2}$$

$$+\frac{\lambda^3}{1\,.\,2\,.\,3}\cdot m\,.\int\frac{dz}{\Delta}\cdot\frac{d^3\,(1-z)^{m+2}\psi^3}{a^3.\,dz^3}$$

$$+\ \&c.$$

In the extreme case when m is infinitely great, we have and,

$$1 - \omega = c^{-s};$$

$$r = \alpha(1+\alpha)\ \text{Sin.}\,\theta \times \int \frac{ds\,c^{-s}}{\sqrt{\text{Cos.}^2\theta + 2is - 2i\lambda(1-c^{-s})}}:$$

and if we expand this expression and apply the like reasoning as before, we shall obtain

$$\Delta = \sqrt{\text{Cos.}^2\theta + 2is}$$

$$\begin{aligned} r = \alpha(1+\alpha)\ \text{Sin.}\,\theta \times \Big\{ & \int \frac{ds\,c^{-s}}{\Delta} \\ & + \lambda . \int \frac{ds}{\Delta} . \frac{d\,.\,c^{-s}(1-c^{-s})}{ds} \\ & + \frac{\lambda^2}{1\,.\,2} . \int \frac{ds}{\Delta} . \frac{dd\,.\,c^{-s}(1-c^{-s})^2}{ds^2} \\ & + \frac{\lambda^3}{1\,.\,2\,.\,3} . \int \frac{ds}{\Delta} . \frac{d^3\,.\,c^{-s}(1-c^{-s})^3}{ds^3} \\ & + \&c. \end{aligned}$$

an expression which has already been given by KRAMP and LAPLACE, and is no other than the limit of the foregoing formula when m is infinitely great.

The calculation of the refractions is now reduced to such integrals as $\int \frac{dz\,(1-z)^{p-1}}{\Delta}$, p being any number; and the valuing of these must next engage our attention.

8. In the first place, when $\theta = 90°$, as in the case of the refractions at the horizon, then $\text{Cos.}^2\,\theta = o$, and $\Delta = \sqrt{2iaz}$: now, put $z = t^2$, and

$$\int \frac{dz\,(1-z)^{p-1}}{\Delta} = \frac{2}{\sqrt{2ia}} . \int dt\,(1-t^2)^{p-1},$$

the integral being taken between the limits $t = o$ and $t = 1$.

When p is a whole number,

$$\int dt\,(1-t^2)^{p-1} = \frac{2\,.\,4\,.\,6\,...\,2\,(p-1)}{3\,.\,5\,.\,7\,....\,2p-1};$$

which will apply conveniently in all cases unless when p is a great number.

When p is great, assume

$$1-t^2 = c^{-\frac{x^2}{p-1}}:$$

then

$$t^2 = \frac{x^2}{p-1} - \frac{1}{2}\cdot\frac{x^4}{(p-1)^2} + \frac{1}{6}\cdot\frac{x^6}{(p-)^3} - \frac{1}{24}\cdot\frac{x^8}{(p-1)^4} + \&c.:$$

and by extracting the square root,

$$t = \frac{1}{\sqrt{p-1}} \times \left\{ x - \frac{1}{4}\cdot\frac{x^3}{p-1} + \frac{5}{96}\cdot\frac{x^5}{(p-1)^2} \right.$$

$$\left. - \frac{1}{128}\cdot\frac{x^7}{(p-1)^3} + \frac{79}{92160}\cdot\frac{x^9}{(p-1)^4} - \&c. \right\}.$$

Hence,

$$\int dt(1-t^2)^{p-1} = \int\frac{dx\,c^{-x^2}}{\sqrt{p-1}}\cdot\left\{1 - \frac{3}{4}\cdot\frac{x^2}{p-1} + \frac{25}{96}\cdot\frac{x^4}{(p-1)^2} - \&c.\right\}:$$

now, the limits of the integrals being $t=0$, $t=1$, and $x=0$, $x=\infty$; we get

$$\int dt(1-t^2)^{p-1} = \frac{1}{2}\cdot\frac{\sqrt{\pi}}{\sqrt{p-1}} \times \left\{1 - \frac{3}{8}\cdot\frac{1}{p-1}\right.$$

$$\left. + \frac{25}{128}\cdot\frac{1}{(p-1)^2} - \frac{105}{1024}\cdot\frac{1}{(p-1)^3} + \frac{1659}{32768}\cdot\frac{1}{(p-1)^4} - \&c.\right\}.$$

By employing proper reductions, any proposed case may be brought to another in which this series will converge swiftly.

In the next place, when $\text{Cos.}^2\,\theta$ is not evanescent, put

$$z = u - e^2(u - u^2);$$

then,

$$(1-z)^{p-1} = (1-u)^{p-1}.\,(1+e^2u)^{p-1}$$

$$\Delta = \sqrt{\text{Cos.}^2\,\theta + 2\,i\,\text{a}\,(1-e^2)\,u + 2\,i\,\text{a}\,e^2u^2}:$$

in order to determine e, assume,

$$\Delta = \text{Cos.}\,\theta + e\,u\sqrt{2\,i\,\text{a}};$$

then,

$$\frac{\sqrt{2\,i\,\text{a}}}{\text{Cos.}\,\theta} = \frac{2e}{1-e^2}$$

$$\frac{dz}{\Delta} = \frac{2e}{\sqrt{2\,i\,\text{a}}} \times du.$$

Hence,

$$p\int\frac{dz(1-z)^{p-1}}{\Delta}=\frac{2e}{\sqrt{2ia}}\cdot p\,du\,.\,(1-u)^{p-1}\,.\,(1+e^2u)^{p-1};$$

consequently,

$$p\,.\int\frac{dz(1-z)^{p-1}}{\Delta}=\frac{2e}{\sqrt{2ia}}\times\Big\{p\int du\,(1-u)^{p-1}$$
$$+e^2\,.\,p\,.\,p-1\,.\int du\,.\,u(1-u)^{p-1}$$
$$+e^4\,.\,p\,.\,p-1\,.\,\frac{p-2}{2}\,.\int du\,.\,u^2(1-u)^{p-1}$$
$$+\ \&c.$$

and, by integrating between the limits $u=0$, $u=1$,

$$p\int\frac{dz(1-z)^{p-1}}{\Delta}=\frac{2}{\sqrt{2ia}}\,.\,\Big\{e+\frac{p-1}{p+1}e^3+\frac{p-1\,.\,p-2}{p+1\,.\,p+2}\,.\,e^5+\&c.\Big\}.$$

This series will stop when p is a whole number; and e being always less than 1, it will converge fast unless when p is a very great number.

9. The horizontal refraction has not been determined by astronomers with much exactness. The quantity most generally adopted is 33′ 46″.3, which is that of the French tables, and is very little different from the determination of BRADLEY: it supposes the mean temperature of 50° of FAHRENHEIT and the barometrical pressure equal to 29.92 English inches. At the same temperature, and with the mean pressure 30 inches, it is equal to

2031″.5.

If we would compare with this the horizontal refraction in the hypothesis of CASSINI, we have only to substitute in the formula found in No. 6, the values of α and i given in No. 4: the result will come out equal to

1218″.6.

The case, when the density decreases in the same proportion that the altitude increases, corresponds to $m=1$ in the

formula of No. 7; and ψ being $= o$, we get,

$$r = \alpha\,(1 + \alpha)\ \mathrm{Sin.}\ \theta \times \int \frac{dz}{\Delta};$$

and, at the horizon,

$$r = \alpha(1 + \alpha) \cdot \int \frac{dz}{\sqrt{2iaz}} = \frac{2\alpha(1+\alpha)}{\sqrt{2ia}}.$$

Now, a being equal to $m + 1 - \lambda$, we have in this case $a = 2 - \lambda$; wherefore,

$$r = \frac{2\alpha(1+\alpha)}{\sqrt{2i(2-\lambda)}} = \frac{\alpha(1+\alpha)}{\sqrt{i\left(1 - \frac{\lambda}{4}\right)}} = 1671''.$$

In both these hypotheses, although the refractions near the zenith agree with nature, yet, at the horizon, they fall greatly short of observation.

At the other extreme, when m is infinitely great, the term which is multiplied by λ^n in the expression of the refraction given in No. 7, is thus expressed, viz.

$$\frac{\lambda^n}{1.2.3\ldots n} \times \int \frac{ds}{\Delta} \cdot \frac{d^n.\, c^{-s}(1 - c^{-s})^n}{ds^n}:$$

but, at the horizon, $\Delta = \sqrt{2is}$; therefore,

$$\frac{\lambda^n}{1.2.3\ldots n} \times \frac{1}{\sqrt{2i}} \times \int \frac{ds}{\sqrt{s}} \cdot \frac{d^n.\, c^{-s}(1 - c^{-s})^n}{ds^n}:$$

and, by expanding and performing the operations indicated, the same term will become

$$\frac{\lambda^n}{1.2.3\ldots n} \times \frac{1}{\sqrt{2i}} \times \int \frac{ds}{\sqrt{s}} \cdot \left\{ \pm c^{-s} \mp n\,.\, 2^n.\, c^{-2s} \pm n.\frac{n-1}{2} \cdot 3^n\, c^{-3s} \mp \&c. \right\},$$

the upper or lower sign taking place according as n is even or odd. If now we put $s = t^2$, and then integrate between the limits $t = o$, $t = \infty$, we shall get,

$$\frac{\lambda^n}{1.2.3\ldots n} \times \frac{\sqrt{\pi}}{\sqrt{2i}} \times \left\{ \pm 1 \mp n \,.\, \frac{2^n}{\sqrt{2}} \pm n.\frac{n-1}{2} \cdot \frac{3^n}{\sqrt{3}} \mp \&c. \right\}.$$

Hence if, in the case of the horizontal refraction, we assume

$$r = \frac{\alpha(1+\alpha)\sqrt{\pi}}{\sqrt{2i}} \times \left\{ 1 + A^{(1)}\lambda + A^{(2)}\lambda^2 \ldots . A^{(n)}\lambda^n . \text{ \&c.} \right\},$$

we shall have

$$A^{(n)} = \frac{1}{1.2.3..n} \times \left\{ \pm 1 \mp n . \frac{2^n}{\sqrt{2}} \pm n . \frac{n-1}{2} . \frac{3^n}{\sqrt{3}} \mp \text{ \&c.} \right\}.$$

By means of this formula KRAMP has found,

$$A^{(1)} = 0.414214,$$
$$A^{(2)} = 0.269649,$$
$$A^{(3)} = 0.200865,$$
$$A^{(4)} = 0.160253,$$
$$A^{(5)} = 0.132935,$$
&c.

And with these values the horizontal refraction, in an atmosphere of uniform temperature, will come out equal to

$$2254''.5.$$

In this case, therefore, the refractions, at the horizon greatly exceed the truth, although at the zenith they agree with observation.

It is therefore certain that if we augment m, by successively putting $m = 2$, $m = 3$, &c., we shall at length find an atmosphere that will agree with nature both at the zenith and the horizon. But if we reflect that there must be an intimate connection between the quantity of the refractions and the gradation of heat in the atmosphere, we shall probably be spared some repetitions of the same operations, by determining m so as to satisfy Equation (B). Now we have

$$1 - \omega = \left(1 - \frac{s}{m+1}\right)^m;$$

consequently,

$$\frac{d\omega}{ds}\left(\text{when } s = o\right) = \frac{m}{m+1} = \frac{4}{5};$$

and hence $m = 4$.

But in the formula of No. 7, when $m = 4$, $a = 5 - \lambda$, $\psi = 1 - (1 - z)^3$; and if we perform the operations indicated, and for the sake of brevity put

$$Q^{(1)} = 4 \int \frac{dz\,(1-z)^3}{\Delta}$$

$$Q^{(2)} = 7 \int \frac{dz\,(1-z)^6}{\Delta}$$

$$Q^{(3)} = 10 \int \frac{dz\,(1-z)^9}{\Delta}$$

$$Q^{(4)} = 13 \int \frac{dz\,(1-z)^{12}}{\Delta}$$

we shall get,

$$r = \alpha\,(1+\alpha)\,\text{Sin.}\,\theta \times \Big\{ Q^{(1)} + \lambda \cdot \frac{4}{5-\lambda} \cdot \left(-Q^{(1)} + Q^{(2)}\right)$$
$$+ \frac{\lambda^2}{1.2} \cdot \frac{4}{5-\lambda} \cdot \frac{5\,Q^{(1)} - 16\,Q^{(2)} + 11\,Q^{(3)}}{5-\lambda}$$
$$+ \frac{\lambda^3}{1.2.3} \cdot \frac{4}{5-\lambda} \cdot \frac{-30\,Q^{(1)} + 216\,Q^{(2)} - 396\,Q^{(3)} + 210\,Q^{(4)}}{(5-\lambda)^2}$$
$$+ \&c.$$

This is the general value of the refraction when $m=4$: but, at the horizon, we get

$$Q^{(1)} = \frac{2}{\sqrt{2i(5-\lambda)}} \times 4\int dt\,(1-t^2)^3 = \frac{2}{\sqrt{2i(5-\lambda)}} \times \frac{64}{35}$$

$$Q^{(2)} = \frac{2}{\sqrt{2i(5-\lambda)}} \times 7\int dt\,(1-t^2)^6 = \frac{2}{\sqrt{2i(5-\lambda)}} \times \frac{1024}{429}$$

$$Q^{(3)} = \frac{2}{\sqrt{2i(5-\lambda)}} \times 10\int dt\,(1-t^2)^9 = \frac{2}{\sqrt{2i(5-\lambda)}} \times \frac{131072}{46189}$$

$$Q^{(4)} = \frac{2}{\sqrt{2i(5-\lambda)}} \times 13\int dt\,(1-t^2)^{12} = \frac{2}{\sqrt{2i(5-\lambda)}} \times \frac{4194304}{1300075};$$

and, with these values, the series for the horizontal refraction will become

$$r = \frac{2\alpha(1+\alpha)}{\sqrt{2i(5-\lambda)}} \times \Big\{ 1.82857 + \lambda \times 0.46717$$
$$+ \lambda^2 \times 0.18959$$
$$+ \lambda^3 \times 0.08836$$
$$+ \&c.$$

and by completing the calculation, we shall get,

$$r = 2041''.3.$$

This result is very near 2031″ 5, the horizontal refraction usually adopted; the difference 9″.8 being much less than the uncertainty in the determination of this quantity. But it will be more satisfactory to compare the refractions in this hypothesis at all altitudes with those admitted by astronomers. In order to find a formula for this purpose we have only to substitute for $Q^{(1)}$, $Q^{(2)}$, $Q^{(3)}$, the series investigated in No. 8; and we may leave out the term multiplied by λ^3, since the amount of it is less than 1″ even at the horizon. Thus we get,

$$\frac{\sqrt{2i(5-\lambda)}}{\text{Cos. } \theta} = \frac{2e}{1-e^2};$$

$$r = \frac{2\alpha(1+\alpha)\text{Sin. } \theta}{\sqrt{2i(5-\lambda)}} \times \left\{ e + \frac{3}{5}e^3 + \frac{1}{5}e^5 + \frac{e^7}{35} \right.$$
$$+ \frac{4\lambda}{5-\lambda} \cdot \left(\frac{3}{20}e^3 + \frac{13}{60}e^5 + \frac{19}{210}e^7 + \frac{e^9}{22} + \frac{e^{11}}{132} \right)$$
$$\left. + \frac{2 \cdot \lambda^2}{(5-\lambda)^2} \cdot \left(\frac{1}{3}e^5 + \frac{193}{273}e^7 + \frac{94}{243}e^9 + \frac{146}{429}e^{11} \right) \right\}$$

And, by substituting the numerical values, we shall find,

$$\text{Tan. } \phi = 19.0462371 + \text{Sec. } \theta - 20.$$

		Log. of Coeff.
$r =$	$1048''.95 \times \text{Tan. } \frac{1}{2}\phi \text{ Sin. } \theta$ . . .	3.0207558
$+$	$658.21 \times \text{Tan.}^3 \frac{1}{2}\phi \text{ Sin. } \theta$ . . .	2.8183661
$+$	$252.92 \times \text{Tan.}^5 \frac{1}{2}\phi \text{ Sin. } \theta$ . . .	2.4029800
$+$	$59.64 \times \text{Tan.}^7 \frac{1}{2}\phi \text{ Sin. } \theta$ . . .	1.7755092
$+$	$11.61 \times \text{Tan.}^9 \frac{1}{2}\phi \text{ Sin. } \theta$ . . .	1.0648048
$+$	$2.95 \times \text{Tan.}^{11} \frac{1}{2}\phi \text{ Sin. } \theta$ . . .	0.4706968

But it is to be observed, that the logarithm of $\frac{30}{29.921}$ has been subtracted from the logarithm of every coefficient, in order to bring the formula to the same barometrical pressure with

the Table of Refractions published in the *Connaissance des Tems*, with which it is proposed to compare it. The comparison is contained in the following Table:

Zen. Dist.	Formula.	Con. Des. T.	Diff.
°	′ ″	′ ″	″
45	0 58.2	58.2	0
60	1 40.6	1 40.6	0
70	2 38.8	2 38.8	0
80	5 19.3	5 19.8	0.5
85	9 51.7	9 54.3	2.6
86	11 44.2	11 48.3	4.1
87	14 21.5	14 28.1	6.6
88	18 11.9	18 22.2	10.3
89	24 8.6	24 21.2	12.6
90	33 54.3	33 46.3	— 8.0

The formula agrees exactly with the table till 80° of zenith distance, when the difference is 0″.5. But if we turn to the *Tables Astronomiques*, published in 1806, by the French Board of Longitude, we shall find that there is a small correction to be subtracted from the mean refractions; and when this is taken into account, the perfect agreement between the formula and the table will be restored. In like manner there are subtractive corrections to be applied at all other zenith distances; and these increase very swiftly in approaching the horizon. To explain the reason of this, it must be observed that the French table was originally constructed for 32° of Fahrenheit, and was reduced to the mean temperature of 50°, on the supposition that the refractions vary in the same proportion with the density of the air; by which procedure the change in their quantity that arises from the variations of the

elementary quantities in the algebraic formula is neglected. Had the computation been rigorously made, as BESSEL has since done in the table he published in 1818, the mean refractions of the French table would have been less, instead of being greater, than the results of the foregoing formula. But it was the opinion of the eminent astronomers under whose direction the table was published, that the refractions near the horizon are too uncertain to require attention to minute accuracy.

It appears therefore, as far as we can form an exact judgment, that the formula approaches very near the true mean refractions. It will afterwards be shown that the hypothesis from which it has been deduced, likewise represents, with considerable accuracy, the pressures and densities actually observed in the atmosphere at different heights. But in one respect there is a deviation from nature. According to the supposition $m = 4$, the total height of the atmosphere is equal to $5 \times l$, or about 25 miles, which, in all probability, is hardly equal to half the real height. It therefore becomes necessary to inquire what influence this circumstance will have on the quantity of the refractions.

10. Continuing to represent the density, or $1 - \omega$, by $(1 - z)^m$, we may assume

$$P = (1 - f) \cdot (1 - z)^{m+1} + f(1 - z)^{2m},$$

f being an arbitrary quantity. Then, from the formulæ (A), we get $s = \int \frac{-dP}{1-\omega}$, and $\frac{1+\beta\tau}{1+\beta\tau'} = \frac{P}{1-\omega}$; and hence

$$s = (m + 1) \cdot (1 - f) \cdot z + 2f \cdot \left\{1 - (1 - z)^m\right\},$$

$$\frac{1+\beta\tau}{1+\beta\tau'} = (1 - z) \cdot \left\{1 - f + f(1 - z)^{m-1}\right\};$$

and it is to be observed, that this last quantity is always

evanescent at the top of the atmosphere, however great m is supposed to be. We likewise get

$$\frac{d\omega}{ds}\,(\text{making } s = o) = \frac{m}{m+1+(m-1)f} = \frac{4}{5};$$

wherefore

$$f = \frac{1}{4}\cdot\frac{m-4}{m-1}.$$

In this formula, $f = o$, when $m = 4$, which is the case already considered; and $f = \frac{1}{4}$, when m is infinitely great. Between these two extreme cases, there are contained an infinite number of atmospheres gradually extending higher above the earth's surface, till the total height from being about 25 miles becomes unlimited. In all these different atmospheres $\frac{d\omega}{ds}$ has the same value when $s = o$; and therefore they all agree with one another, and with nature, in having the same gradation of heat at the earth's surface. But the rate at which the heat decreases is different in every one; being equable only when $m = 4$, and in all the rest becoming slower as the height increases. As all this is easily made out from the foregoing equations, it will not be necessary to enter into any detail on the subject.

When m is less than 4, f becomes negative: but these cases are excluded, since they belong to atmospheres still less elevated than when $m = 4$. They are excluded too for another reason: for, although the rate of the decrease of heat at the earth's surface agrees with nature, yet it increases in ascending, which is contrary to experience.

It remains to determine the refractions in the different atmospheres included in the formula. As we already know the horizontal refraction in one extreme case, it will be sufficient to seek its amount in the other extreme case. Now, if we put

$u = mz$, then, supposing m infinitely great, we get

$$1 - \omega = (1 - z)^m = c^{-u},$$

$$s = (1 - f)\, u + 2f(1 - c^{-u}):$$

consequently,

$$2is - 2\alpha\omega = 2i(1-f).u + 2i(2f - \lambda).(1 - c^{-u}):$$

and, if we make $a = (1-f)\, k = \frac{2f - \lambda}{1 - f}$; then $2is - 2\alpha\omega = 2ia.u + 2ia.k(1 - c^{-u})$.

Hence,

$$r = \alpha(1 + \alpha) \text{ Sin. } \theta \times \int \frac{du\, c^{-u}}{\sqrt{\text{Cos.}^2 \theta + 2ia.u + 2ia.k(1 - c^{-u})}}.$$

Applying to this expression the method already employed in No. 9, we shall get, in the case of the horizontal refraction,

$$r = \frac{\alpha(1+\alpha)\sqrt{\pi}}{\sqrt{2i(1-f)}} \times \left\{1 - A^{(1)}.k + A^{(2)}.k^2 - A^{(3)}k^3 + A^{(4)}.k^4 - A^{(5)}k^5\right\}.$$

Now, $f = \frac{1}{4}$; $\lambda = 0.21909$; $k = 0.37455$: with tnese numbers r comes out equal to $2059''.7$ when all the terms of the series that are set down, are taken in except the last; and to $2057''.4$, when all the terms are taken in: the more correct value of r is therefore $2058''.5$, which is just $17''.2$ more than in the other extreme case of $m = 4$. Thus the refractions undergo hardly any change in all the atmospheres comprehended in the formula; although their height increases from about 25 miles to be infinitely great; and although the rate of the decrease of heat, which has always the same initial value, varies differently in each.

Reflecting on what has just been proved, it is extremely probable that, for every value of m between the two extreme cases, the densities and pressures will be found, at least to a great height, very nearly the same as in the real atmosphere. We can hardly account, on any other supposition, for the

near coincidence of the refractions in so many different cases with the observed quantities. In order to examine this point, we may take the case of GAY LUSSAC's ascent; the data obtained by observation, as they are given by RAMOND,* being as follows, viz.

$$\text{Log. } P = -\ 1.6361109$$
$$\tau' = \ 30^\circ.8$$
$$\tau = -\ 9.5.$$

With these numbers, by means of the formula $P = \frac{1+\beta\tau}{1+\beta\tau'} \times (1-\omega)$, we get,

$$1-\omega = 0.5004,$$

which may be reckoned the density by observation; and we must now compare it with the result of the theory.

Now when $m=4, f=o$; and we have these equations, viz.

$$P = \left(1-\frac{s}{5}\right)^5$$
$$1-\omega = \left(1-\frac{s}{5}\right)^4;$$

consequently,

$$1-\omega = P^{\frac{4}{5}}:$$

and, by substituting the foregoing value of P, we find,

$$1-\omega = 0.5115,$$

which is greater than the value deduced from observation by about $\frac{1}{50}$ of the whole.

Again, we have generally,

$$1-\omega = (1-z)^m$$
$$P = (1-f)(1-z)^{m+1} + 2f(1-z)^{2m}:$$

wherefore,

$$P = (1-f)(1-\omega)^{1+\frac{1}{m}} + 2f(1-\omega)^2;$$

* Memoires sur la Formule Barometrique, 1811. Examples at the end.

and, when m is infinite, $f = \frac{1}{4}$,

$$P = \tfrac{3}{4}(1-\omega) + \tfrac{1}{4}(1-\omega)^2.$$

By solving this equation with the given value of P, we get,

$$1 - \omega = 0.4951,$$

which is less than the true quantity by about $\frac{1}{100}$ of the whole.

It is remarkable that the two results lie on opposite sides of the true quantity: from which it follows, that a value of m greater than 4 may be found, that will accord exactly with the observation of GAY LUSSAC. No confidence, however, could be placed in a calculation founded on a single instance, where an enormous difference in the results would be produced by a small error in the quantities determined by experiment. But at any rate, what has just been remarked, agrees very well with all the arguments that have been advanced to prove the finite extent of the atmosphere surrounding the earth.

For farther illustration, some other observed heights have been selected from the same work of RAMOND, the calculations being made in the same manner. The results are contained in the following table.

Places.	By Observation.				By Theory.		Heights.
	Logarithms P.	τ'	τ	Density.	Density.		Fathoms.
					$m = 4$	$m = \infty$	
Puy de Dome, in Auvergne,	—1.94529	18°.6	11°.7	0.9035	0.9041	0.9035	583
Mont Perdu, in the Pyrenees,	—1.88944	20	7.5	0.8106	0.8157	0.8132	1185
Pic du Midi, High Pyrenees.	—1.86738	25.4	10.4	0.7768	0.7833	0.7798	1429
Etna - - - -	—1.82811	23.1	4.4	0.7196	0.7286	0.7232	1825
Chimborazo, in the Andes,	—1.69582	25.3	— 1.6	0.5468	0.5710	0.5580	3215
GAY LUSSAC's ascent,	—1.63611	30.8	— 9.5	0.5004	0.5115	0.4951	3816

It appears, therefore, that in all the atmospheres comprehended in the assumed formula, the density corresponding to a given pressure and temperature coincides very nearly with what is actually found by experiment. But although this be admitted, it may still be questioned, whether the height at which any proposed pressure takes place will agree equally well with observation. Now, in the real atmosphere, the height belonging to any pressure is usually deduced from the formula for barometrical measurements; and it will be sufficient to show, that the same formula is true in all the atmospheres we are considering.

In the first place, when $m = 4$, we have

$$P = \left(1 - \frac{s}{5}\right)^5$$

$$\frac{1 + \beta\tau}{1 + \beta\tau'} = 1 - \frac{s}{5}.$$

From the first of these equations we get

$$\text{Log.}\, \frac{1}{P} = 5 \log.\, \frac{1}{1 - \frac{s}{5}} = s\left(1 + \frac{1}{2} \cdot \frac{s}{5}\right),$$

neglecting the cube and the higher powers of s: and hence,

$$s = \left(1 - \frac{1}{2} \cdot \frac{s}{5}\right) \log.\, \frac{1}{P}.$$

But, from the other equation, we get

$$1 + \frac{1 + \beta\tau}{1 + \beta\tau'} = 2 - \frac{s}{5};$$

and hence,

$$\frac{1 + \beta \cdot \frac{\tau + \tau'}{2}}{1 + \beta\tau'} = 1 - \frac{1}{2} \cdot \frac{s}{5}.$$

Now substitute this value in the foregoing equation, and likewise, for s, write the equivalent quantity $\frac{x}{l(1 + \beta\tau')}$; and we shall obtain,

$$x = l \times \left\{1 + \beta \cdot \frac{\tau + \tau'}{2}\right\} \cdot \text{Log.} \frac{1}{P},$$

which is no other than the usual formula for barometrical measurements, x being the height corresponding to the relative pressure P.

Again, when m is infinite, we have already found,

$$1 - \omega = c^{-u},$$

$$P = (1-f)c^{-u} + fc^{-2u} = c^{-u} \cdot \left\{1 - f(1 - c^{-u})\right\},$$

$$s = (1-f)u + 2f(1 - c^{-u})$$

$$\frac{1+\beta\tau}{1+\beta\tau'} = \frac{P}{1-\omega} = 1 - f(1 - c^{-u}).$$

Hence,

$$\text{Log.} \frac{1}{P} = u + \log. \frac{1}{1 - f(1 - c^{-u})};$$

and, neglecting the cube and higher powers of $1 - c^{-u}$,

$$\text{Log.} \frac{1}{P} = u + f(1 - c^{-u}) + \frac{f^2}{2} \cdot (1 - c^{-u})^2.$$

But, we easily get,

$$\frac{1 + \beta \cdot \frac{\tau + \tau'}{2}}{1 + \beta\tau'} = 1 - \tfrac{1}{2} f(1 - c^{-u});$$

wherefore, by multiplying,

$$\text{Log.} \frac{1}{P} \times \frac{1 + \beta \cdot \frac{\tau + \tau'}{2}}{1 + \beta\tau'} = u + f(1 - c^{-u}) - \frac{f}{2} u (1 - c^{-u}).$$

Now, because

$$fu = f(1 - c^{-u}) + \frac{f}{2}(1 - c^{-u})^2 + \&c.$$

we get, by substituting and neglecting the same powers of $(1 - c^{-u})$ as before,

$$\text{Log.} \frac{1}{P} \times \frac{1 + \beta \cdot \frac{\tau + \tau'}{2}}{1 + \beta\tau'} = u + f(1 - c^{-u}) - \frac{f}{2}(1 - c^{-u})^2.$$

Farther,

$$s = u - fu + 2f(1 - c^{-u});$$

but,

$$fu = f\left(1 - c^{-u}\right) + \frac{f}{2}\left(1 - c^{-u}\right)^2;$$

consequently,

$$s = u + f\left(1 - c^{-u}\right) - \frac{f}{2}\left(1 - c^{-u}\right)^2.$$

We have therefore

$$s = \frac{1}{P} \times \frac{1 + \beta \cdot \frac{\tau + \tau'}{2}}{1 + \beta\tau'};$$

and because $s = \frac{x}{l(1 + \beta\tau')}$, we get as before

$$x = l \times \log. \frac{1}{P} \times \left\{1 + \beta \cdot \frac{\tau + \tau'}{2}\right\}.$$

The very exact coincidence in the properties of all the atmospheres comprehended in the assumed formula, with the phenomena actually observed at the surface of the earth, accounts in a satisfactory manner for the near approach of the refractions in every case to the quantities determined by astronomers. It appears that, although the refractions near the zenith are affected in a degree hardly perceptible by the peculiar constitution of the atmosphere, yet, near the horizon, they depend entirely on the same arrangement of the strata of air indicated by terrestrial experiments. The causes of the irregularities observable in these last, likewise disturb the celestial phenomenon in a more remarkable manner. In measuring the height of a column of air, the accidental disturbances to which the atmosphere is continually subject, are in some measure corrected by means of the temperatures observed at both extremities of the column; but, in computing the refractions, the astronomer has no guide but the thermometer placed at his Observatory. In the remote parts of the atmosphere, there may occur innumerable changes deflecting the light of a star from its proper course, of which

he has no intimation, and for which he can make no allowance. In comparison of this great source of error we may reckon as of small account, the inaccuracies that are owing to the neglect of the moisture diffused in the atmosphere, or to our want of an exact knowledge of the law of density in regard to temperature. There can hardly be any other remedy than that of which astronomers so often avail themselves, whenever an ignorance of the real causes obliges them to assimilate the phenomena to the effect of chance; namely, to multiply observations in different circumstances, with the view of making the inequalities of an opposite description compensate one another.

From the foregoing discussion we may draw this conclusion: that an atmosphere constituted like that of the earth, must have an altitude of at least 25 miles, in order that the refractions from the zenith to the horizon be such as they are actually observed to be. But an atmosphere agreeing with nature in the quantity of the refractions may be found, that shall have any proposed altitude greater than the minimum quantity.

We may infer from the duration of the twilight, that the atmosphere of the earth must have an altitude equal to 50 miles, or even more; which corresponds to the supposition of m equal to, or greater than 10. But all these cases are so little different, as to the refractions, from the extreme case when m is infinitely great, that we may suppose them to coincide with it. The most probable supposition with respect to the mean law of density, seems therefore to be contained in these equations, viz.

$$1 - \omega = c^{-u}$$

$$s = (1 - f)u + 2f(1 - c^{-u});$$

f being nearly equal to $\frac{1}{4}$. In this hypothesis it has been shown that the same pressures, densities, and temperatures, would take place at the same altitudes as in the real atmosphere, as far at least as observation enables us to determine. We may therefore presume, that it is not far from that mean state which would prevail, if the regular disposition of the strata of air were not continually deranged by disturbing causes. It remains now to find the refractions in this hypothesis, and to compare them with the quantities observed by astronomers.

12. We have hitherto supposed that $\alpha, i, \lambda = \frac{\alpha}{i}$, are quantities varying with the pressure and temperature of the air; but it will now be necessary to restrict those symbols to the particular values that take place at the mean temperature of 50° of FAHRENHEIT, and the mean pressure of 30 English inches. We shall use the expressions $\alpha\left(1 + \frac{\delta\alpha}{\alpha}\right), i\left(1 + \frac{\delta i}{i}\right)$, $\lambda + \delta\lambda$, to denote the like quantities as altered by the changes in the atmosphere. What was before signified by

$$2is - 2\alpha\omega, \text{ or } 2i(s - \lambda\omega),$$

will now be thus written, viz.

$$2i\left(1 + \frac{\delta i}{i}\right) \cdot \left\{s - (\lambda + \delta\lambda)\,\omega\right\};$$

and, by substituting the assumed value of s, and rejecting quantities of the second order with regard to the variations, the same expression will become,

$$\begin{aligned} &2i \cdot \left\{(1-f)\,u + 2f \cdot c^{-u}\right\} \\ &+ 2i \cdot \frac{\delta i}{i} \cdot \left\{(1-f)\,u + 2fc^{-u}\right\} \\ &- 2i \cdot \delta\lambda \cdot \left(1 - c^{-u}\right); \end{aligned}$$

or, more simply,

$$2iy + 2i\,.\,\frac{\delta i}{i}y - 2i\delta\lambda(1 - c^{-u}),$$

by putting $y = (1-f)u + 2fc^{-u}$. We shall therefore have this expression of the refraction, viz.

$$r = \left(1 + \frac{\delta\alpha}{\alpha}\right).\,\alpha(1+\alpha)\,\text{Sin.}\,\theta \times \int \frac{duc^{-u}}{\sqrt{\text{Cos.}^2\theta + 2iy + 2i\,.\,\frac{\delta i}{i}y - 2i\delta\lambda(1-c)^{-u}}}$$

and by expanding

$$r = \left(1 + \frac{\delta\alpha}{\alpha}\right).\,\alpha(1+\alpha)\,\text{Sin.}\,\theta \times \int \frac{duc^{-u}}{\sqrt{\text{Cos.}^2\theta + 2iy}}$$

$$-i\,.\,\frac{\delta i}{i}\,.\,\alpha(1+\alpha)\,\text{Sin.}\,\theta \times \int \frac{duc^{-u}y}{\left\{\text{Cos.}^2\theta + 2iy\right\}^{\frac{3}{2}}}$$

$$+ i\delta\lambda\,.\,\alpha(1+\alpha)\,\text{Sin.}\,\theta \times \int \frac{du\,.\,c^{-u}(1 - c^{-u})}{\left\{\text{Cos.}^2\theta + 2iy\right\}^{\frac{3}{2}}}.$$

Let p denote the observed height of the mercury in the barometer reduced to the fixed temperature of 50° of Fahrenheit; τ the temperature of the air on the same scale; and $\beta = \frac{1}{480}$ the expansion for one degree: then,

$$\alpha\left(1 + \frac{\delta\alpha}{a}\right) = \frac{\alpha}{1 + \beta(\tau - 50)} \times \frac{p}{30}$$

$$i\left(1 + \frac{\delta i}{i}\right) = i\,(1 + \beta(\tau - 50))$$

$$\lambda + \delta\lambda = \frac{\alpha}{i} \times \frac{1 + \frac{\delta\alpha}{a}}{1 + \frac{\delta i}{i}} = \frac{\lambda}{1 + 2\beta(\tau - 50)} \times \frac{p}{30}.$$

consequently,

$$\frac{\delta i}{i} = \frac{\tau - 50}{480}$$

$$\delta\lambda = -\frac{2\lambda}{480}(\tau - 50) - \frac{\lambda}{30}(30 - p).$$

By substituting these values, we get,

$$r = \left(1 + \frac{\delta\alpha}{\alpha}\right).\,\alpha(1+\alpha)\,\text{Sin.}\,\theta \times \int \frac{du\,.\,c^{-u}}{\sqrt{\text{Cos.}^2\theta + 2iy}}$$

$$-(\tau-50)\times\frac{\alpha(1+\alpha)\,\text{Sin.}\,\theta}{480}\times\left\{i\int\frac{du\,c^{-u}.y}{\left\{\text{Cos.}^2\theta+2iy\right\}^{\frac{3}{2}}}+2i\lambda.\int\frac{du\,c^{-u}(1-c^{-u})}{\left\{\text{Cos.}^2\theta+2iy\right\}^{\frac{3}{2}}}\right\}$$

$$-(30-p).\frac{\lambda.\alpha(1+\alpha)\,\text{Sin.}\,\theta}{30}.i\int\frac{du\,c^{-u}(1-c^{-u})}{\left\{\text{Cos.}^2\theta+2iy\right\}^{\frac{3}{2}}}.$$

Let us now assume

$$r=\left(1+\frac{\delta\alpha}{\alpha}\right).\delta\theta+(\tau-50)\frac{d\delta\theta}{d\tau}-(30-p).\frac{d\delta\theta}{dp},$$

$\delta\theta$ being the mean refraction at the apparent zenith distance θ; then, by equating the like parts of the equivalent expressions, we get,

$$\delta\theta=\alpha(1+\alpha)\,\text{Sin.}\,\theta\times\int\frac{du\,c^{-u}}{\sqrt{\text{Cos.}^2\theta+2iy}},$$

$$\frac{d\delta\theta}{d\tau}=-\frac{\alpha(1+\alpha)\,\text{Sin.}\,\theta}{480}\times\left\{i\int\frac{du\,c^{-u}y}{\left\{\text{Cos.}^2\ +2iy\right\}^{\frac{3}{2}}}+2i\lambda.\int\frac{du\,c^{-u}(1-c^{-u})}{\left\{\text{Cos.}^2\theta+2iy\right\}^{\frac{3}{2}}}\right\}$$

$$\frac{d\delta\theta}{dp}=\frac{\lambda\alpha(1+\alpha)}{30}\times\int\frac{du.c^{-u}(1-c^{-u})}{\left\{\text{Cos.}^2\theta+2iy\right\}^{\frac{3}{2}}};$$

each of which expressions must be separately considered.

Now we have

$$y=u+(f-\lambda).(1-c^{-u})-f(c^{-u}-1+u);$$

and if we substitute this value of y in the expression of $\delta\theta$, and then expand the radical quantity, retaining only the terms of the first order, we shall get,

$$\Delta=\sqrt{\text{Cos.}^2\theta+2iu}$$

$$\delta\theta=\alpha(1+\alpha)\,\text{Sin.}\,\theta\times\left\{\int\frac{du\,c^{-u}}{\Delta}\right.$$

$$-i(f-\lambda).\int\frac{du.c^{-u}(1-c^{-u})}{\Delta^3}$$

$$\left.+if.\int\frac{du.c^{-u}(c^{-u}-1+u)}{\Delta^3}\right\}.$$

It will presently be shown that the other terms of the expansion may be neglected; because $f-\lambda$ is very small; and because the function multiplied by f becomes inconsiderable after integration. Since we want only the definite integrals between the limits $u=o$ and $u=\infty$, by applying the method already used, we shall obtain,

$$\delta\theta = \alpha(1+\alpha)\,\text{Sin.}\,\theta \times \left\{ \int \frac{du\,c^{-u}}{\Delta} - (f-\lambda).\int \frac{du}{\Delta}.\frac{d.c^{-u}(1-c^{-u})}{du} + f.\int \frac{du}{\Delta}.\frac{d.c^{-u}(c^{-u}-1+u)}{du} \right\}$$

or, which is the same thing,

$$\delta\theta = \alpha(1+\alpha)\,\text{Sin.}\,\theta \times \left\{ \int \frac{du\,c^{-u}}{\Delta} - (f-\lambda).\int \frac{du(2c^{-2u}-c^{-u})}{\Delta} + f.\int \frac{du(2c^{-u}-2c^{-2u}-uc^{-u})}{\Delta} \right\}.$$

In order to estimate the error produced by the terms of the expansion left out, we may compare the amount of the foregoing formula at the horizon, with the exact value of the horizontal refraction already computed. Now, when $\text{Sin.}\,\theta = 1$, $\text{Cos.}\,\theta = o$, $\Delta = \sqrt{2iu}$; and the expression will become,

$$\delta\theta = \frac{\alpha(1+\alpha)}{\sqrt{2i}} \times \left\{ \int \frac{du\,c^{-u}}{\sqrt{u}} - (f-\lambda).\int \frac{du(2c^{-2u}-c^{-u})}{\sqrt{u}} + f\int \frac{du(2c^{-u}-2c^{-2u}-uc^{-u})}{\sqrt{u}} \right\}.$$

And if we now make $u=t^2$, and then integrate between the limits $t=o$, $t=\infty$, we shall get

$$\delta\theta = \frac{\alpha(1+\alpha)\sqrt{\pi}}{\sqrt{2i}} \times \left\{ 1-(f-\lambda).(\sqrt{2}-1) + f\left(\tfrac{3}{2}-\sqrt{2}\right) \right\}.$$

Hence $$\delta\theta = 2055''.6.$$

But we before found the horizontal refraction equal to $2058''.5$; and the difference is therefore no more than $2''.9$, which is of no moment, since the refractions just at the horizon are alone affected by the error.

In order to reduce the expression of $\delta\theta$ to the most simple form for calculation, we have,

$$\triangle c^{-u} - \text{Cos.}\,\theta = i\int\frac{duc^{-u}}{\Delta} - \text{Cos.}^2\theta\int\frac{duc^{-u}}{\Delta} - 2i\int\frac{du.uc^{-u}}{\Delta};$$

the integrals commencing when $u = o$: and when they are extended to $u = \infty$, we get

$$\int\frac{du.uc^{-u}}{\Delta} = \frac{1}{2}\int\frac{du.c^{-u}}{\Delta} - \frac{\text{Cos.}^2\theta}{2i}\int\frac{duc^{-u}}{\Delta} + \frac{\text{Cos.}\,\theta}{2i}.$$

Now put

$$N = \frac{\text{Cos.}^2\theta}{2i}\int\frac{du.c^{-u}}{\Delta} - \frac{\text{Cos.}\,\theta}{2i}$$

$$M = 2\int\frac{duc^{-2u}}{\Delta} - \int\frac{duc^{-u}}{\Delta};$$

then by substitution we shall finally get

$$\delta\theta = \alpha(1+\alpha)\,\text{Sin.}\,\theta \times \left\{\int\frac{duc^{-u}}{\Delta} + \lambda.M - f\left(2M - \frac{1}{2}\int\frac{duc^{-u}}{\Delta} - N\right)\right\}.$$

The expression of $\frac{d\delta\theta}{d\tau}$ may be put in this form, viz.

$$\frac{d\delta\theta}{d\tau} = -\frac{\alpha(1+\alpha)\,\text{Sin.}\,\theta}{480}.\left\{\frac{1}{2}.\int\frac{duc^{-u}}{\sqrt{\text{Cos.}^2\theta + 2iy}} - \frac{\text{Cos.}^2\theta}{2}.\int\frac{duc^{-u}}{\left\{\text{Cos.}^2\theta + 2iy\right\}^{\frac{3}{2}}} + 2i\lambda.\int\frac{duc^{-u}(1-c^{-u})}{\left\{\text{Cos.}^2\theta + 2iy\right\}^{\frac{3}{2}}}\right\}:$$

and it will be sufficiently accurate to write u for y in the denominators of the two last terms: then

$$\frac{d\,\delta\theta}{d\tau} = -\frac{1}{480}\cdot\frac{\delta\theta}{2} + \frac{\alpha(1+\alpha)\,\mathrm{Sin.}\,\theta}{480}\cdot\left\{i\times 2\lambda\int\frac{du\,c^{-u}(1-c^{-u})}{\Delta^3}\right.$$

$$\left.-\frac{\mathrm{Cos.}^2\,\theta}{2}\cdot\int\frac{du\,c^{-u}}{\Delta^3}\right\}:$$

and, by the same procedure as before,

$$\frac{d\,\delta\theta}{d\tau} = -\frac{1}{480}\cdot\frac{\delta\theta}{2} + \frac{\alpha(1+\alpha)\,\mathrm{Sin.}\,\theta}{480}\cdot\left\{2\lambda\mathrm{M} - \frac{\mathrm{Cos.}^2\,\theta}{2}\int\frac{du\,c^{-u}}{\Delta^3}\right\}.$$

But we have

$$\frac{1}{\mathrm{Cos.}\,\theta} - \frac{c^{-u}}{\Delta} = \int\frac{du\,c^{-u}}{\Delta} + i\int\frac{du\,c^{-u}}{\Delta^3},$$

the integrals commencing when $u = o$; and, when $u = \infty$, we get,

$$\frac{\mathrm{Cos.}^2\,\theta}{2}\cdot\int\frac{du\,c^{-u}}{\Delta^3} = -\frac{\mathrm{Cos.}^2\,\theta}{2i}\int\frac{du\,c^{-u}}{\Delta} + \frac{\mathrm{Cos.}\,\theta}{2i} = -\mathrm{N}.$$

Wherefore,

$$\frac{d\,\delta\theta}{d\tau} = -\frac{1}{480}\cdot\frac{\delta\theta}{2} + \frac{\alpha(1+\alpha)\,\mathrm{Sin.}\,\theta}{480}\cdot\left\{2\lambda\mathrm{M} + \mathrm{N}\right\}:$$

and, by substituting the value of $\delta\theta$,

$$\frac{d\,\delta\theta}{d\tau} = -\frac{\alpha(1+\alpha)\,\mathrm{Sin.}\,\theta}{480}\times\left\{\frac{1}{2}\int\frac{du\,c^{-u}}{\Delta} + \frac{5}{2}\lambda\mathrm{M} + \mathrm{N}\right.$$

$$\left.-\frac{f}{2}\left(2\mathrm{M} - \frac{1}{2}\int\frac{du\,c^{-u}}{\Delta} - \mathrm{N}\right)\right\}.$$

Lastly by writing u for y in the expression of $\frac{d\,\delta\theta}{dp}$, we readily obtain

$$\frac{d\,\delta\theta}{dp} = +\frac{\alpha(1+\alpha)\,.\,\lambda\,\mathrm{Sin.}\,\theta}{30}\times\mathrm{M}.$$

Thus the quantities $\delta\theta$, $\frac{d\,\delta\theta}{d\tau}$, $\frac{d\,\delta\theta}{dp}$, ultimately involve only two different integrals, viz. $\int\frac{du\,c^{-u}}{\Delta}$ and $\int\frac{du\,c^{-2u}}{\Delta}$; the values of which we must next endeavour to investigate.

13. The whole integral $\int \frac{du\,c^{-u}}{\sqrt{\text{Cos.}^2\,\theta + 2iu}}$, extending from $u = o$ to $u = \infty$, is composed of these two parts, viz.

$$\int \frac{du\,c^{-u}}{\sqrt{\text{Cos.}^2\,\theta + 2iu}} + c^{-m} . \int \frac{du\,c^{-u}}{\sqrt{\text{Cos.}^2\,\theta + 2im + 2iu}};$$

the first part being contained between the limits $u = o$, and $u = m$; and the second part, which arises from substituting $m + u$ for u in the first part, being extended from $u = o$ to $u = \infty$.

To begin with the first part: put

$$u = m\,(1 - e^2)\,z + m\,e^2\,z^2;$$

and the limits of u being o and m, the limits of z will be o and 1. In order to determine e, assume

$$\Delta = \sqrt{\text{Cos.}^2\,\theta + 2im(1 - e^2)z + 2ime^2z^2} = \text{Cos.}\,\theta + ez\sqrt{2im}:$$

then

$$\frac{\sqrt{2im}}{\text{Cos.}\,\theta} = \frac{2e}{1 - e^2},$$

$$\frac{du}{\Delta} = \frac{2e}{\sqrt{2im}} \times m\,dz,$$

$$\int \frac{du\,c^{-u}}{\Delta} = \frac{2e}{\sqrt{2im}} \times \int m\,dz\,.\,c^{-mz + e^2 m\,(z - z^2)}$$

Let the integral sought, viz. $\int \frac{du\,c^{-u}}{\Delta}$ between the limits $u = o$, $u = m$, be expressed in a series of this form, viz.

$$\int \frac{du\,c^{-u}}{\Delta} = \frac{2}{\sqrt{2im}} \times \left\{ A^{(o)}\,.\,e + A^{(1)}\,e^3 \ldots\ldots + A^{(n)}\,e^{2n+1} \ldots\ \&c. \right\}$$

then if we develope the foregoing exponential value in a series of the powers of e, and equate the like terms of the equivalent expressions, we shall get,

$$A^{(n)} = \frac{m^{n+1}}{1\,.\,2\,.\,3 \ldots\ldots n} . \int dz\,(z - z^2)^n\,c^{-mz},$$

the integral being taken between the limits $z = o$, $z = 1$.

But the integral between the limits mentioned, is equal to the difference of the two values of the same integral taken, the one between the limits $z = o$, $z = \infty$, and the other between the limits $z = 1$, $z = \infty$. Now, by writing $1 + z$ for z, the expression

$$\int dz\,(z - z^2)^n c^{-mz},$$

will be changed into

$$(-1)^n . c^{-m} . \int dz\,(z + z^2)^n c^{-mz};$$

and it is obvious that the value of the former between the limits $z = 1$, $z = \infty$, is equal to the value of the latter between the limits $z = o$, $z = \infty$. It follows therefore from what has been said, that we shall have

$$A^{(n)} = \frac{m^{n+1}}{1 . 2 . 3 \ldots n} \times \left\{ \int dz\,(z - z^2)^n c^{-mz} - (-1)^n . c^{-m} . \int dz\,(z + z^2)^n c^{-mz} \right\},$$

each of the integrals being extended from $z = o$ to $z = \infty$.

Again, p being any whole number, we have, between the limits $z = o$, $z = \infty$,

$$\int dz . z^p . c^{-mz} = \frac{1 . 2 . 3 \ldots p}{m^{p+1}}.$$

Wherefore if we expand the binomial quantities in the value of $A^{(n)}$, and integrate the terms separately, we shall obtain

$$A^{(n)} = 1 - n . \frac{n+1}{m} + n . \frac{n-1}{2} . \frac{n+1 . n+2}{m^2} - n . \frac{n-1}{2} . \frac{n-2}{3} . \frac{n+1 . n+2 . n+3}{m^3} + \&c.$$
$$- (-1)^n . c^{-m} . \left(1 + n . \frac{n+1}{m} + n . \frac{n-1}{2} . \frac{n+1 . n+2}{m^2} + \&c.\right).$$

By this means we get the first part of the integral sought in a series that has all its terms positive, and that will always converge because e never exceeds unit.

Let us next consider the second, or supplemental part, viz.

$$c^{-m} \times \int \frac{du\,c^{-u}}{\sqrt{\text{Cos.}^2\theta + 2im + 2iu}}.$$

Now $\text{Cos.}^2\theta = \frac{(1-e^2)^2}{4e^2} \times 2im$: and, if we substitute this value, the integral will become

$$\frac{2}{\sqrt{2im}} \times \frac{c^{-m}}{2} \times \frac{2e}{1+e^2} \cdot \int \frac{du\, c^{-u}}{\sqrt{1+\left(\frac{2e}{1+e^2}\right)^2 \cdot \frac{u}{m}}}:$$

and, by expanding the denominator and integrating between the limits $u = o$, $u = \infty$, we shall get this value of the quantity sought, viz.

$$\frac{2}{\sqrt{2im}} \cdot \frac{c^{-m}}{2} \times \left\{ \begin{array}{l} \frac{2e}{1+e^2} - \frac{1}{m} \cdot \frac{1}{2^2}\left(\frac{2e}{1+e^2}\right)^3 \\ + \frac{1 \cdot 3}{m^2} \cdot \frac{1}{2^4} \cdot \left(\frac{2e}{1+e^2}\right)^5 \\ - \frac{1 \cdot 3 \cdot 5}{m^3} \cdot \frac{1}{2^6}\left(\frac{2e}{1+e^2}\right)^7 \\ + \&c. \end{array} \right\}$$

This series will converge in its first terms: and the results being alternately too great and too small, we can thus estimate the degree of approximation.

By uniting the two parts, we get this expression for the whole integral between the limits $u = o$, $u = \infty$, m being an arbitrary quantity, viz.

$$\int \frac{du\, c^{-u}}{\Delta} = \frac{2}{\sqrt{2im}} \cdot \left\{ A^{(0)}e + A^{(1)}e^3 + A^{(2)}e^5 + \&c. \right\}$$
$$+ \frac{2}{\sqrt{2im}} \times \frac{c^{-m}}{2} \times \left\{ \begin{array}{l} \frac{2e}{1+e^2} - \frac{1}{m} \cdot \frac{1}{2^2}\left(\frac{2e}{1+e^2}\right)^3 \\ + \frac{1 \cdot 3}{m^2} \cdot \frac{1}{2^4}\left(\frac{2e}{1+e^2}\right)^5 \\ - \frac{1 \cdot 3 \cdot 5}{m^3} \cdot \frac{1}{2^6} \cdot \left(\frac{2e}{1+e^2}\right)^7 \\ + \&c. \end{array} \right\}$$

The supplemental part is less than

$$\frac{2}{\sqrt{2im}} \times \frac{ec^{-m}}{1+e^2}:$$

it is therefore very small when m is a considerable number, in which case the value of the integral will be found with sufficient exactness by means of the first series alone. But, with regard to the foregoing expression, we must not omit to remark what is a very curious instance of the artifices that must sometimes be resorted to in order to bring an analytical expression within the boundary of arithmetical computation. If the supplemental part be developed in a series of the powers of e, it will consist of precisely the same terms, but with opposite signs, as that part of the first series which is multiplied by c^{-m}. In reality, therefore, the exact value of the integral is what remains of the first series, when the part multiplied by c^{-m} is thrown out; which is also very manifest from the mode of investigation. But the series so obtained is imperfectly computable. It belongs to that class called semi-convergent; which converge indeed to a certain degree in their first terms, but afterwards become divergent. By adding and subtracting the same quantity in two different shapes, an expression is produced consisting of two parts, that can be calculated separately to any degree of exactness.

For the sake of brevity, let the supplemental part be represented by $\frac{2}{\sqrt{2im}} \cdot c^{-m} \cdot \mathrm{R}$: then, if we separate from the first series the part of $\mathrm{A}^{(0)}$ multiplied by c^{-m}, we shall have

$$\int \frac{du\,c^{-u}}{\Delta} = \frac{2}{\sqrt{2im}} \times \left\{e + \mathrm{A}^{(1)} e^3 + \mathrm{A}^{(2)} e^5 + \&c.\right\}$$

$$+ \frac{2c^{-m}}{\sqrt{2im}} \times (\mathrm{R} - e):$$

And it follows, from what has been said, that the subsidiary part of this expression is no other than the expansion of R deprived of its first term. In like manner if we separate the parts of $A^{(0)}$, $A^{(1)}$, $A^{(2)}$ which involve c^{-m}, we shall get

$$\int \frac{duc^{-u}}{\Delta} = \frac{2}{\sqrt{2im}} \times \left\{ e + \left(1 - \frac{2}{m}\right) e^3 + \left(1 - \frac{6}{m} + \frac{12}{m^2}\right) e^5 + A^{(3)} e^7 + \&c. \right\}$$
$$+ \frac{2c^{-m}}{\sqrt{2im}} \cdot \left\{ R - e + \left(1 + \frac{2}{m}\right) e^3 - \left(1 + \frac{6}{m} + \frac{12}{m^2}\right) e^5 \right\};$$

And here the subsidiary part is the expansion of R wanting the three first terms. On account of the factor c^{-m}, the subsidiary parts decrease without limit as m increases; and thus the value of the integral can always be found to any required degree of exactness, in a series coinciding with the rigorous expression in its first terms, at the same time that it converges in its remaining terms.

Now, let $m = 8$: then

$$A^{(3)} = \frac{13}{64} + \frac{235}{64} c^{-8} = 0.204357$$
$$A^{(4)} = \frac{21}{256} - \frac{2141}{256} c^{-8} = 0.079225$$
$$A^{(5)} = \frac{19}{1024} + \frac{23029}{1024} c^{-8} = 0.026099$$
$$A^{(6)} = \frac{127}{4096} - \frac{287575}{4096} c^{-8} = 0.007453$$
$$A^{(7)} = -\frac{1347}{16384} + \frac{4106939}{16384} c^{-8} = 0.001876$$
$$A^{(8)} = \frac{22237}{65536} - \frac{66205285}{65536} c^{-8} = 0.000422$$

And hence, neglecting the parts of $A^{(0)}$, $A^{(1)}$, $A^{(2)}$, that involve c^{-m}, we get

$$\frac{4\sqrt{i}}{\text{Cos. } \theta} = \frac{2e}{1 - e^2}$$

$$\int \frac{du\,c^{-u}}{\Delta} = \frac{1}{2\sqrt{i}} \times \left\{ e + \tfrac{3}{4}e^3 + \tfrac{7}{16}e^5 + 0.204357.e^7 \right.$$
$$+ 0.079225.e^9$$
$$+ 0.026099.e^{11}$$
$$+ 0.007453.e^{13}$$
$$+ 0.001876.e^{15}$$
$$+ 0.000422.e^{17}$$

Although we are sure that this value is a near approximation to the truth, yet it may not be superfluous to examine whether it be sufficiently exact for the purpose intended. Now, the part of $\delta\theta$ depending on this integral, is

$$\alpha(1+\alpha)\,\text{Sin.}\,\theta . \int \frac{du\,c^{-u}}{\Delta}:$$

and, this being valued by means of the foregoing series in the case of Sin. $\theta = 1$, and $e = 1$, in which circumstances the error of the approximation is greatest, the result will be

$$\frac{\alpha(1+\alpha)}{2\sqrt{i}} \times 2.506932 = 2038''.2\,.$$

But, when Sin. $\theta = 1$, $\Delta = \sqrt{2iu}$; and the quantity we are considering will become

$$\frac{\alpha(1+\alpha)}{\sqrt{2i}} . \int \frac{du\,c^{-u}}{\sqrt{u}}:$$

and if we put $u = t^2$, and integrate between the limits $t = o$, $t = \infty$, the exact value will be equal to

$$\frac{\alpha(1+\alpha)\sqrt{\pi}}{\sqrt{2i}} = 2037''.8\,.$$

It appears, therefore, that the error of the approximation when it is greatest, or when $e = 1$, does not amount to half a second. But as the error is expressed by a series of terms multiplied by e^7, e^9, &c. it diminishes very rapidly as e decreases, and becomes altogether insensible when θ is less than 90°.

It remains to find the value of the integral $\int \frac{du c^{-2u}}{\Delta}$ between the limits $u = o$, $u = \infty$. By substitution we get

$$2\int \frac{du c^{-2u}}{\Delta} = \frac{2e}{\sqrt{2im}} \times \int 2\,m\,dz . c^{-2mz + 2me^2(z - z^2)}$$

from which it is manifest that we shall obtain the value sought by substituting $2m$ in place of m in the coefficients of the former series; thus

$$2\int \frac{du c^{-2u}}{\Delta} = \frac{2}{\sqrt{2im}} \times \left\{ e + \left(1 - \frac{2}{2m}\right) e^3 + \left(1 - \frac{6}{2m} + \frac{12}{4m^2}\right) e^5 + A'^{(3)} e^7 + A'^{(4)} e^9 + \&c. \right\},$$

$A'^{(3)}$, $A'^{(4)}$, &c. denoting what $A^{(3)}$, $A^{(4)}$, &c. become when $2m$ is substituted for m. Hence, making $m = 8$, $2m = 16$, we get

$$A'^{(3)} = \frac{233}{512} + \frac{1031}{512} c^{-16} = 0.455078.$$

$$A'^{(4)} = \frac{1121}{4096} - \frac{13041}{4096} c^{-16} = 0.273681$$

$$A'^{(5)} = \frac{4823}{32768} + \frac{183353}{32768} c^{-16} = 0.147188$$

$$A'^{(6)} = \frac{18691}{262144} - \frac{2851507}{262144} c^{-16} = 0.071299$$

$$A'^{(7)} = \frac{65689}{2097152} + \frac{48804183}{2097152} c^{-16} = 0.031326$$

$$A'^{(8)} = \frac{210889}{16777216} - \frac{914559193}{16777216} c^{-16} = 0.012564$$

Wherefore,

$$2\int \frac{du c^{-2u}}{\Delta} = \frac{1}{2\sqrt{i}} \times \left\{ e + \frac{3}{8} e^3 + \frac{43}{64} e^5 + 0.455078 . e^7 \right.$$
$$+ 0.273681 . e^9$$
$$+ 0.147188 . e^{11}$$
$$+ 0.071299 . e^{13}$$
$$+ 0.031326 . e^{15}$$
$$+ 0.012564 . e^{17}$$

Having now found the values of the two integrals on which the expression of the refraction depends, we get immediately

$$M = \frac{1}{2\sqrt{i}} \times \left\{ \begin{array}{l} \frac{1}{8} \cdot e^2 \\ + 0.234375 . e^5 \\ + 0.250721 . e^7 \\ + 0.194456 . e^9 \\ + 0.121089 . e^{11} \\ + 0.063846 . e^{13} \\ + 0.029450 . e^{15} \\ + 0.012142 . e^{17} \end{array} \right. \qquad \lambda . M = \frac{1}{2\sqrt{i}} \times \left\{ \begin{array}{l} 0.027386 . e^3 \\ + 0.051349 . e^5 \\ + 0.054930 . e^7 \\ + 0.042602 . e^9 \\ + 0.026529 . e^{11} \\ + 0.013987 . e^{13} \\ + 0.006451 . e^{15} \\ + 0.002659 . e^{17} \end{array} \right.$$

And again, we have

$$N = \frac{\text{Cos.}^2 \theta}{2i} \int \frac{du c^{-u}}{\Delta} - \frac{\text{Cos.}\, \theta}{2i}:$$

but

$$\frac{\text{Cos.}^2 \theta}{2i} = \frac{m}{4} \cdot \frac{(1-e^2)^2}{e 2} = 2 . \frac{(1-e^2)^2}{e^2}; \frac{\text{Cos.}\, \theta}{2i} = \frac{1}{2\sqrt{i}} \times 2 . \frac{1-e^2}{e}:$$

and hence, if we put $\int \frac{du c^{-u}}{\Delta} = \frac{1}{2\sqrt{i}} \times \psi$; so that ψ stands for the series in the value of the integral; we shall get

$$N = \frac{1}{2\sqrt{i}} \times 2 \left\{ \frac{(1-e^2)^2}{e^2} . \psi - \frac{1-e^2}{e} \right\}.$$

Wherefore, by substituting the value of ψ, viz.

$$\psi = e + \tfrac{3}{4} e^3 + \tfrac{7}{16} e^5 + A^{(3)} . e^7 + A^{(4)} . e^9 + \&c.$$

we shall find

$$N = \frac{1}{2\sqrt{i}} \times \left\{ \begin{array}{l} -\frac{1}{2} e - \frac{1}{8} e^3 + 2 \left(\frac{3}{4} - \frac{7}{8} + A^{(3)}\right) . e^5 \\ \qquad + 2 \left(\frac{7}{16} - 2 A^{(3)} + A^{(4)}\right) . e^7 \\ \qquad + 2 \left(A^{(3)} - 2 A^{(4)} + A^{(5)}\right) . e^9 \\ \qquad + 2 \left(A^{(4)} - 2 A^{(5)} + A^{(6)}\right) . e^{11} \\ \qquad + 2 \left(A^{(5)} - 2 A^{(6)} + A^{(7)}\right) . e^{13} \\ \qquad + 2 \left(A^{(6)} - 2 A^{(7)} \quad * \quad\right) . e^{15} \\ \qquad + 2 \left(A^{(7)} \quad * \quad * \quad\right) . e^{17} \end{array} \right.$$

And, in numbers,

$$N = \frac{1}{2\sqrt{i}} \times \left\{ -\tfrac{1}{2}e - \tfrac{1}{8}e^3 + 0.158714.e^5 \right.$$
$$+ 0.216022.e^7$$
$$+ 0.144012.e^9$$
$$+ 0.068960.e^{11}$$
$$+ 0.026138.e^{13}$$
$$+ 0.007402.e^{15}$$
$$+ 0.003752.e^{17}$$

To find $\delta\theta$, it only remains to substitute the numerical values of $\int \frac{duc^{-u}}{\Delta}$, λ M, and N, in the expression investigated in No. 12; then,

$$\delta\theta = \frac{\alpha(1+\alpha)}{2\sqrt{i}} \times \text{Sin.}\,\theta \times e \times \left\{ 1 + 0.777386.e^2 \right.$$
$$+ (0.488849 - 0.091286.f).e^4$$
$$+ (0.259287 - 0.183242.f).e^6$$
$$+ (0.121827 - 0.205287.f).e^8$$
$$+ (0.052628 - 0.160168.f).e^{10}$$
$$+ (0.021440 - 0.097828.f).e^{12}$$
$$+ (0.008327 - 0.050560.f).e^{14}$$
$$+ (0.003081 - 0.020321.f).e^{16}$$

The two first terms of this expression do not contain f; and they give that part of the refractions near the zenith, which has no dependence upon the constitution of the atmosphere. As there is some uncertainty in the value of f, it may be determined either so as to make the horizontal refraction coincide with the quantity adopted by astronomers; or so as to make the formula represent some very exact observations made at low altitudes, from 2° to 7° above the horizon. With regard to altitudes less than 2°, it is not clear that the astronomical refractions do not participate of the extreme irregularity that attends the terrestrial refractions,

which would render such observations unfit to be employed in this research. But in the present state of our knowledge it may be doubted, whether a more satisfactory determination of f can be obtained than what we have hitherto assumed, namely, $f = \frac{1}{4}$. With this value, we get

$$\begin{aligned}\delta\theta = \frac{\alpha(1+\alpha)}{2\sqrt{i}} \times \text{Sin.}\,\theta \times e \times \{ 1 &+ 0.777386 \,.\, e^2 \\ &+ 0.466028 \,.\, e^4 \\ &+ 0.213477 \,.\, e^6 \\ &+ 0.070505 \,.\, e^8 \\ &+ 0.012586 \,.\, e^{10} \\ &- 0.003017 \,.\, e^{12} \\ &- 0.004313 \,.\, e^{14} \\ &- 0.001999 \,.\, e^{16}\end{aligned}$$

$$\text{Log.}\,\frac{\alpha(1+\alpha)}{2\sqrt{i}} = 2.9101040.$$

$$\text{Tan.}\,\phi = 19.1580271 + \text{Sec.}\,\theta - 20\,;\ e = \text{Tan.}\,\tfrac{1}{2}\phi.$$

If we make $\theta = 90°$, and $e = 1$, we get,

$$\delta\theta = \frac{\alpha(1+\alpha)}{2\sqrt{i}} \times 2.530653 = 2057''.5.$$

This is the horizontal refraction by the formula: and as the exact value of the same quantity was before found equal to 2058''. 5, the error arising from the method of approximation amounts only to 1'' at the horizon. But all the quantities neglected being of the orders e^7, e^9, &c. the error will be altogether insensible unless when e is extremely near 1, that is, at very low altitudes.

The foregoing expression may be put in another form, which, in some cases, is more convenient for calculation.

$$\text{Since } \frac{2e}{1-e^2} = \frac{4\sqrt{i}}{\text{Cos.}\,\theta}\,;\ \text{we get } \frac{e}{2\sqrt{i}} = \frac{1-e^2}{\text{Cos.}\,\theta}:$$

and hence, by substitution,

$$\delta\theta = \alpha\,(1+\alpha)\,\text{Tan.}\,\theta \times \left\{ \begin{array}{l} 1 - 0.222614 \,.\, e^2 \\ - 0.311358 \,.\, e^4 \\ - 0.252551 \,.\, e^6 \\ - 0.142972 \,.\, e^8 \\ - 0.057919 \,.\, e^{10} \\ - 0.015603 \,.\, e^{12} \end{array} \right.$$

$$\text{Log.}\,\alpha\,(1+\alpha) = 1.7671011.$$

This transformation can be of use only to a certain distance from the zenith; for at the horizon Tan. θ is infinite, and the factor $1 - e^2$ is equal to zero. The expression set down is sufficient for finding the refractions exact to $\frac{1}{100}$ of a second as far as $85°$ from the zenith.

And, if we take the logarithms of both sides of the last expression, we shall get

$$\text{Log.}\,\delta\theta = \text{Log. Tan.}\,\theta + 1.76710 \begin{array}{l} \\ - 0.096680 \,.\, e^2 \\ - 0.145982 \,.\, e^4 \\ - 0.141413 \,.\, e^6 \\ - 0.114530 \,.\, e^8 \\ - 0.089474 \,.\, e^{10} \\ - 0.073278 \,.\, e^{12} \end{array}$$

which formula is very convenient near the zenith, and is sufficient for finding the logarithms of the refractions exact to five figures, as far as $84°$ from the zenith. It is to be observed, that while θ increases from zero, e increases from a limit, from which it varies very little till θ becomes a considerable arc.

In order to have $\frac{d\delta\theta}{d\tau}$ and $\frac{d\delta\theta}{dp}$ it is only requisite to substi-

tute the numerical values already found in the expressions investigated in No. 12: thus,

$$\begin{aligned}\frac{d\,\delta\,\theta}{d\,\tau} = -\frac{\alpha\,(1+\alpha)}{2\sqrt{i}} \times \frac{1}{480} \times \text{Sin.}\,\theta \times e \times \Big\{ & 0\,.\,31846\,.\,e^{2}\\ & + 0\,.\,49442\,.\,e^{4}\\ & + 0\,.\,43262\,.\,e^{6}\\ & + 0\,.\,26447\,.\,e^{8}\\ & + 0\,.\,12831\,.\,e^{10}\\ & + 0\,.\,05260\,.\,e^{12}\\ & + 0\,.\,01815\,.\,e^{14}\\ & + 0\,.\,00807\,.\,e^{16}\end{aligned}$$

$$\text{Log.}\ \frac{\alpha\,(1+\alpha)}{2\sqrt{i}} \times \frac{1}{480} = 0\,.\,2288628.$$

$$\begin{aligned}\frac{d\,\delta\,\theta}{d\,p} = \frac{\alpha\,(1+\alpha)}{2\sqrt{i}} \times \frac{\lambda}{30} \times \text{Sin.}\,\theta \times e \times \Big\{ & 0\,.\,125\,.\,e^{2}\\ & + 0\,.\,23437\,.\,e^{4}\\ & + 0\,.\,25072\,.\,e^{6}\\ & + 0\,.\,19446\,.\,e^{8}\\ & + 0\,.\,12109\,.\,e^{10}\\ & + 0\,.\,06385\,.\,e^{12}\\ & + 0\,.\,02945\,.\,e^{14}\\ & + 0\,.\,01214\,.\,e^{1}\end{aligned}$$

$$\text{Log.}\ \frac{\alpha\,(1+\alpha)}{2\sqrt{i}} \times \frac{\lambda}{30} = 0\,.\,7736018.$$

By means of the foregoing formulæ the table annexed to this paper was computed. In the first column are placed the distances from the zenith: the second contains the values of $\delta\,\theta$, or the mean refractions at the temperature of 50° of FAHRENHEIT and the barometric pressure 30 English inches: the third contains the logarithms of the refractions: and, when the zenith distance is greater than 75°, the values of $\frac{d\,\delta\,\theta}{d\,\tau}$ and $\frac{d\,\delta\,\theta}{d\,p}$ are added in two other columns.

The use of this table will be clear, from the subjoined formula for computing by it the true refraction, supposing that τ is the temperature by Fahrenheit's thermometer, and p the height of the barometer in English inches.

$$r = \frac{1}{1 + \beta(\tau - 50)} \times \frac{p}{30} \times \delta\theta + \frac{d\delta\theta}{d\tau}(\tau - 50) - \frac{d\delta\theta}{dp}(30 - p).$$

The first term is the mean refraction corrected for the observed temperature and pressure in the same manner usually practised by astronomers. When the zenith distance does not exceed 75°, the two remaining terms are to be accounted as evanescent; and, even when the zenith distance is 80° or a little more, the same terms may, on most occasions, be omitted: otherwise the two terms, amounting generally to some seconds, are to be added to the first term with their proper signs.

Three subsidiary Tables are added for facilitating the corrections for the barometer and thermometer. Table II. contains the logarithms of $\frac{1}{1 + \beta(\tau - 50)} = \frac{1}{1 + \frac{\tau - 50}{480}}$, for 40° on either side of the mean temperature 50°; negative indices being avoided by substituting the arithmetical complements. Table III. contains the logarithms, or the arithmetical complements, of $\frac{p}{30}$ for the values of p between 31 and $27\frac{1}{2}$. Table IV. contains the small corrections, positive or negative, to be applied to the numbers in Table III. in order to reduce the observed length of the barometric column to the mean temperature of 50°. The numbers of this Table are the logarithms of $\frac{1}{1 + \frac{\tau - 50}{10000}}$, equal to $-\frac{\tau - 50}{10000} \times \cdot 434$.

14. Instead of applying the new Tables to particular in-

stances, it will be more compendious to compare it with other tables that have been long in the hands of astronomers, and the characters of which are well established. The table has been constructed with the same elementary quantities as the French table, at least as far as regards the refractive power and the weight of the air, which quantities alone influence the magnitude of the refractions near the zenith. But, in comparing the two tables, an allowance must be made for the difference of the standard barometers; and this requires that the refractions in the new table be all diminished by $\frac{8}{3000} = \frac{1}{375}$. Now, taking the refractions at 45° and 80° from the zenith, we get

$$58''.36\left(1 - \frac{1}{375}\right) = 58''.2,$$

$$320.19\left(1 - \frac{1}{375}\right) = 319''.3;$$

and, in the French table, we find the first of these numbers exactly, and 319″.8 in place of the second. But we must not forget that there is a small subtractive correction to be applied to the mean refractions of the French astronomers, which is usually neglected, although it will be found among the tables of refraction (Table V.) inserted in the *Tables Astronomiques*, published by them in 1806. This correction amounts to 0″.5 at 80° from the zenith; and the former number is thus reduced to 319″.3, the same as in the new table. We may therefore conclude, that when we calculate rigorously, the mean refractions of the new table are the same as those of the French astronomers, as far as 81° or 82° from the zenith.

But at lower altitudes there will no longer be the same

perfect agreement between the two tables: first, on account of the difference in the hypotheses respecting the constitution of the atmosphere; secondly, because the tables are differently constructed.

The French refractions at low altitudes are computed by a formula which the sagacity of LAPLACE deduced from a hypothesis respecting density, that must be a near approach to the law that actually obtains in nature. The formula is so constructed as to give the horizontal refraction adopted by astronomers; but we may still judge, in some measure, of the accuracy of the hypothesis, by comparing the rate of the decrease of heat at the earth's surface with the result of actual observation. Now, in the hypothesis of LAPLACE, I have found

$$\frac{d\omega}{ds}\,(\text{making } s = o) = 0.7159\,;$$

which is a near approach to 0.8, the quantity assumed in this Paper. But the difference, although it seem very little, has nevertheless a great influence on the constitution of the atmosphere, as will be obvious if, by means of the equation

$$\frac{d\omega}{ds}\,(s = o) = 1 - \frac{\beta \times l}{\mu},$$

we compute the value of μ resulting from the preceding value of $\frac{d\omega}{ds}$. It will be found that $\mu = 59\frac{1}{2}$; which is the elevation in fathoms that in this hypothesis will depress the centigrade thermometer one degree; and it is no more than about $\frac{2}{3}$ of the true quantity. It follows, therefore, that the theory of LAPLACE does not strictly accord with the actual condition of the atmosphere, which must affect the accuracy of the refractions near the horizon.

The French tables are also liable to some inaccuracy at low altitudes, from the manner in which they are constructed. The calculations are originally made for the temperature zero of the centigrade scale, and the barometric pressure 29.92 English inches; and from the numbers so computed, the refractions are in every case deduced, on the supposition that they vary in the same proportion as the density of the air. But, besides this alteration of their quantity, the refractions undergo other variations, as the elementary quantities of the formula change with the state of the air; namely, those contained in the second and third terms of the foregoing rule for calculating by the new table. Now, the variations here alluded to are neglected in the French table, although they are of considerable amount near the horizon. They are neglected, however, not because the eminent mathematicians who constructed the table were not aware of their existence, but because they deemed them of little moment in a case of so great uncertainty as the refractions at low altitudes. Properly speaking, the table in the *Connaissance des Tems* is not one of *mean refractions;* that is, the numbers in it are not the same that would be found by substituting, in the fundamental formula, the elementary quantities reduced to the proposed standards of temperature and pressure. The true mean refractions computed in this manner would all be less than the quantities actually contained in the table. In practice, therefore, there is a kind of compensation that takes place between the excess of the numbers in the table above the exact values of the mean refractions, and the manner of correcting for the barometer and thermometer; a compensation which is very happy in many

instances, but which cannot fail to leave a final error in a numerous set of observations.

We are in possession of another table of refractions, computed with great care, published in 1818, by M. BESSEL, in his *Astronomiæ Fundamenta.* This table must be considered as having the authority of actual observation as far as 86° from the zenith; since, to that extent, it represents with great exactness the observations of Dr. BRADLEY, which served as the basis of its construction. At lower altitudes, the refractions in it are confessedly too great. To compare the new table with that of M. BESSEL is, therefore, the same as to make a direct appeal to experience.

The astronomical refractions were first discussed with a due attention to all the circumstances of the problem in the Treatise published by KRAMP.* This author gives the name of specific elasticity to the quotient of the relative pressure divided by the relative density of the air; it is therefore equal to $\frac{1+\beta\tau}{1+\beta\tau'}$ in the formulæ of the present paper. He represents it by $c^{-\varepsilon s}$, ε being a small fraction; which function, therefore, contains the law for the gradation of heat according to KRAMP. Now, if we substitute $c^{-\varepsilon s}$ for $\frac{1+\beta\tau}{1+\beta\tau'}$ in the foregoing formulæ, and then equate the two values of P, we shall get

$$(1-\omega)\times c^{-\varepsilon s} = \int -ds\,(1-\omega);$$

and hence

$$1-\omega = c^{-\frac{1}{\varepsilon}(c^{\varepsilon s}-1)+\varepsilon s}.$$

This is the rigorous expression of the density in the hypo-

* Analyse des Refractions Astronomiques et Terrestres, 1798.

thesis of KRAMP; but, as it is too complicated for calculation, he deduces from it this more simple value, viz.

$$1 - \omega = c^{-(1-\varepsilon)s}$$

by retaining only the part of the expansion of the function in the index that contains the first power of s.

In all this KRAMP is followed by BESSEL, whose aim is to determine the value of ε that will best represent all the observations of Dr. BRADLEY, without paying any regard to the terrestrial phenomena, or to any farther theoretical considerations whatever.

Now, there is an essential distinction between the rigorous expression of the density, and the approximate value used instead of it. The latter belongs to a finite atmosphere, and the former to one of unlimited extent. To prove this, we need only substitute $c^{-(1-\varepsilon)s}$ for $1 - \omega$ in the equation,

$$P = \int - ds\,(1 - \omega);$$

and then we shall get

$$P = \frac{c^{-(1-\varepsilon)s}}{1-\varepsilon} - \frac{\varepsilon}{1-\varepsilon};$$

the constant quantity being necessary, because $P = 1$ when $s = o$. But as P cannot be negative, we have $P = o$ at the top of the atmosphere; and the total height will therefore be determined by the equation

$$c^{-(1-\varepsilon)s} - \varepsilon = 0.$$

If we could suppose that ε is a very small fraction, and the height of the atmosphere very great, what has just been observed would be of little consequence. But, at the surface of the earth, we ought to have $\frac{d\omega}{ds} = 1 - \varepsilon = \frac{4}{5}$, and $\varepsilon = \frac{1}{5}$; which would limit the atmosphere to about double the height

in the hypothesis of Cassini. Bessel determines $\varepsilon = \frac{1}{28}$ nearly; which is quite inconsistent with the value of $\frac{d\omega}{ds}$ at the surface of the earth, and with the elevation necessary for depressing the thermometer one degree, as found by experiment. Accordingly, although the refractions in his table represent Dr. Bradley's observations with great exactness as far as 86° from the zenith; yet, at lower altitudes, they diverge greatly from the truth; and the horizontal refraction comes out very nearly the same as in an atmosphere of uniform temperature. In this last hypothesis the refractions agree with nature as far as between 70° and 80° from the zenith; and, by means of the arbitrary quantity ε, they are bent to a conformity with observation a few degrees farther.

The preceding remarks have been made with the view of showing how it happens that the refractions in M. Bessel's table agree with observation to a certain extent, and afterwards differ so widely from the true quantities. In comparing the two tables, we must attend to the points in which they are different from one another. In the table of M. Bessel the constant of refraction is a little less than in the new table: the mean temperature is $48\frac{3}{4}$° of Fahrenheit in the former, and 50° in the latter; and the standard barometers are 29.6 inches and 30 inches. Now, supposing the two tables to represent the true mean refractions equally well, the differences we have mentioned will hardly have any other effect than to introduce a constant factor, by means of which the one table would be reduced to the other. The logarithms of the numbers in the two tables ought, therefore, to have constantly

the same difference; and how far this is actually the case, will appear by the inspection of the following table.

Distance from zenith.	Log. γ.		Difference.
	N. Table.	Bessel.	
45°	1.76612	1.75961	0.00651
55	1.92039	1.91385	0.00654
65	2.09568	2.08910	0.00658
75	2.33184	2.32510	0.00674
80	2.50541	2.49849	0.00692
81	2.54874	2.54175	0.00699
82	2.59624	2.58923	0.00701
83	2.64875	2.64174	0.00701
84	2.70740	2.70042	0.00698
85	2.77367	2.76683	0.00684
86	2.84951	2.84321	0.00630
87	2.93754	2.93246	0.00508
88	3.04122	3.03903	0.00219

As far therefore as 86° from the zenith, it appears that, in a practical point of view at least, the law of the refractions is the same in both tables; and the real difference between them is no more than a small variation in the constant of refraction. But, from 86° or 87° to the horizon, the two tables diverge so much from one another, that no comparison can be instituted between them.

The first instance of a rule for correcting the mean refractions different from the common one, which supposes that the variations are proportional to the changes in the density of the air, occurs in a formula of the eminent astronomer, T. Mayer, of Gottingen. The rule is given in the author's lunar tables without the demonstration; and it has been very generally misunderstood and decried;* so very uncertain is

* See the Article Refraction in the *Tables Astronomiques.*

even an improvement in the abstruser parts of science. Doubts are entertained whether the rule was found by theory, or constructed in conformity to actual observation. The latter supposition cannot but seem very improbable when we attend to the formula; which is, besides, deduced very readily from the method of investigation pursued in this paper. M. Bessel is the first astronomer who has accurately computed all the variations of the refractions produced by the changes of temperature and pressure; and we shall next compare the new table in this respect with the result of his theory. Now, in his table, at the zenith distances 83°, 85°, 88°, the total corrections for the temperature τ are respectively, $-0''.9821(\tau-50)$, $-1''.3678(\tau-50)$, $2''.9944(\tau-50)$: but each of these quantities involves the usual correction proportional to the change of the air's density, equal to $-\frac{\delta\theta}{480}\times(\tau-50)$; and, when this part is separated, they will stand as under;

$$\begin{array}{ll} \text{at } 83^{\circ}, & -0''.9131(\tau-50)-0''.0690(\tau-50), \\ \quad 85, & -1.2178(\tau-50)-0.1500(\tau-50), \\ \quad 88, & -2.2792(\tau-50)-0.7150(\tau-50), \end{array}$$

the latter parts being equal to $\frac{d\delta\theta}{d\tau}\times(\tau-50)$ in the notation of this paper. In the new Table the values of $\frac{d\delta\theta}{d\tau}\times(\tau-50)$ are respectively, $-0.074(\tau-50)$, $-0.159(\tau-50)$, $-0.722(\tau-50)$; differing insensibly from the calculations of M. Bessel.

To complete this examination of the new Table, we shall add some particular instances. We begin with the two examples at p. 159 of the *Connaissance des Tems*.

EXAMPLE I.

$\theta = 86° \, 14' \, 42''$

Therm. 8°.75 Cent. = 47° ¾ FAHR.

Barom. 0^m.741 = 29.17 In.

86° 10′	2.86325
4 42″	662
	2.86987
Therm.	0.00204
Barom. 9.98781 } +9 }	9.98790
Log. r - -	2.85981
$r = 724.1 =$	12′ 4″.1
$-0.28 \times -2\frac{1}{4}$	+ 0.6
$+0.8 \times -0.45$	− 0.4
	12 4.3
By observation	12 4.2
Error	+ 0.1
Error of French T.	2.2

EXAMPLE II.

$\theta = 86° \, 15' \, 20''$

Therm. 8°⅛ = 46°.9 FAHR.

Barom. 0^m.766 = 30.16 In.

86° 10′	2.86325
5 20″	752
	2.87077
Therm. - -	0.00281
Barom. 0.00232 } + 13 }	0.00245
Log. r - -	2.87603
$r = 751.7 =$	12′ 31″.7
$-0.28 \times -3 =$	+ 0.8
$-0.16 \times -0.45 =$	+ 0.1
	12 32.6
By observation	12 32.5
Error	+ 0.1
Error of French T.	2.9

The next instance is more to the purpose, being the mean of 42 sub-polar observations of α Lyræ by Dr. BRINKLEY.*

$\theta = 87° \, 42' \, 10''$

Therm. 35°

Barom. 29.5

87° 40′	3.00466
2 10″	390
	3.00856
Therm. - -	0.01379
Barom. 9.99270 } + 64 }	9.99334
Log. r - -	0.01569
$r = 1036.8 =$	17′ 16″.8
-0.6×-15	+ 9 .0
-1×0.5	− 0 .5
	17 25.3
By observation - - -	17 26.5
Error - - - - - - -	1.2
Error of French Tables	5.5

* See his Paper on the Refractions, Irish Transact. 1815.

The error is here very small, as it ought to be in a mean of so many good observations. Half a degree subtracted from the thermometer would bring out an exact result; and some small difference may be fairly ascribed to the uncertainty arising from the different temperatures of the interior and exterior thermometers.

But it will be satisfactory to exhibit the errors of every particular observation. In the Irish Transactions for 1815, Dr. BRINKLEY has given 44 sub-polar observations of α Lyræ with the observed refractions; and these are contained in the following Table, extracted from the *Connaissance des Tems* for 1819, p. 418, the column of the errors of the new Table being added.

Date.	Barometer.	Thermometer. In.	Thermometer. Out.	Observed zenith distance.	Observed Refractions.	Corrections for F. Table.	Corrections of N. Table.
1809, Jan. 22	29.25	25		87° 42′ 1″.6	17′ 57″.4	+ 23.7	+ 12.0
Feb. 18	30.01	43½		40.7	24.8	+ 3.4	+ 3.8
20	29.78	43½		40.6	24.2	+ 10.7	+ 11.1
March 5	30.09	42½		33.0	34.7	+ 8.7	+ 8.6
12	30.05	44		22.1	46.2	+ 26.0	+ 26.5
1810, Feb. 13	29.94	34	30	57.0	3.1	− 3.5	− 8.1
19	30.02	32	29.5	5.9	55.6	+ 10.2	+ 3.0
March 17	29.62	36	33	31.0	33.4	+ 9.4	+ 5.5
1811, Jan. 18	29.90	33½	32	12.2	38.1	− 0.2	− 6.2
23	30.27	35	32½	41 55.1	56.6	+ 9.1	+ 4.6
28	29.35	27½	21½	58.5	54.6	+ 22.7	+ 13.3
Feb. 3	29 44	31½	30	42 34.3	20.4	− 7.7	− 14.8
7	29.24	39	38	52.5	3.2	− 2.8	− 4.2
8	29.28	39	35	51.2	4.7	− 2.3	− 4.0
12	29.03	38	34	58.4	16 58.4	− 2.6	− 5.8
13	28.91	35	33	43 3.3	53.7	− 10.2	− 14.0
Dec. 28	29.39	30½	25½	42 3.0	17 38.7	+ 12.2	+ 4.6
1812, Jan. 2	29.07	31½	30	22.0	21.2	+ 6.9	+ 0.3
3	28.95	29½	26½	34.0	9.5	− 5.8	− 13.7
4	29.11	27½	23½	41 56.2	47.6	+ 24.6	+ 15.4
7	29.93	32	31	42 2.1	42.6	+ 0.6	− 6.6
21	29.64	34	28½	1.2	47.9	+ 20.7	+ 15.3
30	29.18	39	35	36.4	19.2	+ 17.2	+ 25.2
Feb. 7	29.42	38	33	27.2	26.4	+ 13.9	+ 11.6
Dec. 22	29.66	33	26½	41 48.0	50.7	+ 21.1	+ 15.4
1813, Jan. 1	29 64	36	31	42 9.1	32.7	+ 9.4	+ 5.4
3	29.90	42½	40	23.0	19.5	+ 0.7	+ 0.8
11	29.52	36	31½	11.8	33.2	+ 14.2	+ 10.0
19	30.04	36	32	41 58.2	49.2	+ 12.0	+ 8.2
26	30.16	33	28	46.2	18 3.2	+ 16.1	+ 9.9
Feb. 6	29.40	39	38	42 46.8	17 5.6	− 5.1	− 7.1
15	28.50	40	38	43 24.8	16 29.6	− 10.0	− 11.1
18	29.26	39	37½	0.0	55.0	− 12.0	− 13.6
22	29.24	42	36½	42 52.3	17 3.3	+ 4.0	+ 4.1
Dec. 26	30.19	35½	31½	41 55.8	43.6	+ 0.1	− 4.7
27	30.01	36½	34	42 21.2	18.5	− 17.3	− 21.6
31	29.88	35½	33½	1.0	40.0	+ 7.5	+ 3.1
1814, Jan. 1	29.69	35	32½	21.2	20.1	− 8.4	− 12.8
4	29.11	26½	23	41 59.6	42.7	+ 17.4	+ 6.9
22	29.88	21	17	25.7	18 22.2	+ 18.3	+ 4.0
26	28.95	33	32½	42 56.2	16 52.8	− 16.5	− 21.6
27	28.78	32½	30½	49.8	59.4	− 4.2	− 9.9
29	28.63	31½	29	51.5	58.4	− 2.1	− 8.4
Feb. 13	29.67	41½	39	47.1	17 6.3	− 8.4	− 8.9

The errors of the French Table are $+ 340.8 - 119.1$; amounting to 459.9 when the signs are neglected; and giving a mean error equal to $\frac{340.8 - 119.1}{44}$, or $5''$.

In the new Table we have $+ 218.6 - 196.3$; the total sum being 414.9, and the mean error $\frac{218.6 - 196.3}{44}$, or half a second.

In both views the advantage is in favour of the new Table.

The inspection of the foregoing Table will show how fruitless it would be to expect a near agreement in every single instance between observation and any table of refractions whatever. Thus, the zenith-distance is less, and the barometer and thermometer are both nearer the standard mean quantities on the 12th of March, than on the 5th of the same month, 1809; on all these accounts, the refraction should be less on the former day than on the latter; whereas, according to observation, it is greater by $11''.5$. There is, therefore, no sure test of the accuracy of a Table of Refractions except the smallness of the mean error in a series of observations made at different times.

I shall now subjoin and compare with the new Table, observations of a number of stars at low altitudes, for which I am indebted to the liberality of STEPHEN GROOMBRIDGE, Esq. F. R. S. The reductions necessary for finding the true refractions were made by that astronomer; and the practice of estimating the temperature of the air by the exterior thermometer, which he recommends as answering best with his method of observing, is followed in calculating from the

Tables. I have not thought it necessary to insert all the observations communicated by Mr. GROOMBRIDGE; but such only, where the altitudes are less than five degrees.

The correction for temperature is variously computed by astronomers. Some use the interior, and others the exterior thermometer; and some prefer taking a mean between the two. But it may be affirmed with some degree of certainty, that the practice of computing by the exterior thermometer can be perfectly correct only when the temperature is the same within and without the Observatory. If we suppose that this is the case at first, and that afterwards the air within the Observatory is heated above, or cooled below, the external temperature; the consequence must certainly be, that the apparent place of a star will undergo some alteration. On the other hand, if the heat be equally distributed within the Observatory, and remain constant, while the temperature on the outside varies; it is not clear whether any change at all would be observed in the place of a star, more especially if the change of temperature were small.* But this is a point that can be determined only by careful experiments; and, until some light be thrown upon it, no great improvement can be expected in our knowledge of the astronomical refraction.†

* See Dr. BRINKLEY's Paper, Philosophical Transactions, 1821, p. 335.

† N. B. In calculating the refractions, the temperature of the mercury in the barometer is estimated by the interior, that of the air by the exterior, thermometer.

Stars.	No. of Obs.	Mean Apparent Altitude.	Mean Observed Refractions.	Mean Bar.	Mean Ther. In.	Mean Ther. Out.	Corrections of French Table.	Corrections of New Table.
		° ′ ″	′ ″		°	°	″	″
11 Lacertæ.	16	85 4 10.5	10 10.5	29.91	47.5	38.3	−5.3	− 5.1
κ Androm.	12	85 4 37.8	10 3.7	29.72	51	43.5	−2.2	− 1.1
ξ Cygni.	15	85 10 51.3	10 23.3	29.92	46.6	38.5	−4.2	− 3.8
μ Ursæ Maj.	10	85 53 57.3	11 55.8	29.83	41	32.8	−3.7	− 3.5
ι Androm.	8	86 6 22.2	12 10.4	29.73	49.4	39.8	−3.3	− 2.6
γ Androm.	10	86 53 9.1	13 46.7	29.75	60.3	54.3	−7.0	− 1.2
ο Androm.	12	86 58 29.9	14 41.5	29.93	48.6	38.9	−2.9	− 2.7
β Bootis.	5	87 8 27.5	15 21.8	29.70	38.4	28.7	−9.7	−14.2
η Aurigæ.	9	87 18 57.8	15 14.2	29.85	60.2	56.3	−2.9	+ 5.1
ζ Aurigæ.	13	87 29 7.6	15 44.9	29.78	62.4	56.6	−6.4	+ 2.3
β Persei.	17	87 59 51.9	18 7.1	29.89	59.7	52.5	−6.6	+ 0.4
γ Cygni.	25	88 29 50.5	21 37.6	30.05	46.5	38.1	−0.9	− 7.2
ε Persei.	16	88 41 17.4	22 23.0	30.01	58.3	49.3	+6.9	+ 10.8

If we reject the observations of β Bootis and ε Persei, the errors of the French Table are all negative; and, in the New Table, the negative amount to more than triple the positive errors. Two different reasons may be assigned for the preponderance of the negative errors: it may be alledged that the refractions in the Tables are too great; or it may be said that, by using the exterior thermometer, the calculated refractions are increased more than in proportion to the real temperature of the air. The latter of these reasons is quite sufficient to account for the discordance; and it will receive additional force, if we attend to the great differences between the exterior and interior thermometers. In this case we cannot, therefore, draw any conclusion with the same confidence as in the preceding observations of Dr. Brinkley; but we may safely affirm, that the errors of the Table are not greater than the uncertainty of estimating the temperature of the air by the exterior thermometer.

TABLE I.

Mean Refractions for the Temperature of 50° of Fahrenheit, *and the Barometric Pressure 30 inches.*

Distance from Zenith.	$\delta\theta$	Log. $\delta\theta$	Difference.	Distance from Zenith.	$\delta\theta$	Log. $\delta\theta$	Difference.
°	′ ″			°	″		
0	0 0			30	33.72	1.5279	173
1	1.02	0.0085	3012	31	35.09	1.5452	170
2	2.04	0.3097	1763	32	36.49	1.5622	168
3	3.06	0.4860	1252	33	37.93	1.5790	164
4	4.08	0.6112	974	34	39.39	1.5954	162
5	5.11	0.7086	796	35	40.89	1.6116	160
6	6.14	0.7882	675	36	42.42	1.6276	159
7	7.17	0.8557	587	37	44.00	1.6435	156
8	8.21	0.9144	519	38	45.61	1.6591	155
9	9.25	0.9663	466	39	47.27	1.6746	155
10	10.30	1.0129	424	40	48.99	1.6901	154
11	11.35	1.0553	388	41	50.75	1.7055	152
12	12.42	1.0941	359	42	52.57	1.7207	151
13	13.49	1,1300	334	43	54.43	1.7358	152
14	14.56	1.1634	313	44	56.35	1.7510	151
15	15.66	1.1947	294	45	58.36	1.76611	1512
16	16.75	1.2241	278	46	1 0.43	1.78123	1514
17	17.86	1.2519	265	47	2.57	1.79637	1518
18	18.98	1.2784	252	48	4.80	1.81155	1523
19	20.11	1.3036	241	49	7.11	1.82678	1530
29	21.26	1.3277	230	50	9.52	1.84208	1539
21	22.42	1.3507	222	51	12.02	1.85747	1551
22	23.60	1.3729	215	52	14.64	1.87298	1565
23	24.80	1.3944	207	53	17.38	1.88863	1577
24	26.01	1.4151	201	54	20.24	1.90440	1596
25	27.24	1.4352	195	55	23.25	1.92036	1617
26	28.49	1.4547	189	56	26.41	1.93653	1638
27	29.76	1.4736	185	57	29.73	1.95291	1664
28	31.05	1.4921	181	58	33.23	1.96955	1691
29	32.38	1.5102	177	59	36.93	1.98646	1722
30	33.72	1.5279		60	40.85	2.00368	

TABLE I. *continued.*

Distance from Zenith.	$\delta\theta$	Log. $\delta\theta$	Difference.
° '	' ''		
60	1 40.85	2.00368	1756
61	45.01	2.02124	1794
62	49.44	2.03918	1836
63	54.17	2.05754	1881
64	59.22	2.07635	1932
65	2 4.65	2.09567	1988
66	10.48	2.11555	2048
67	16.78	2.13603	2116
68	23.61	2.15719	2191
69	31.04	2.17910	2275
70 00	39.16	2.20185	388
10	40.59	2.20573	390
20	42.04	2.20963	393
30	43.52	2.21356	396
40	45.02	2.21752	398
50	46.53	2.22150	402
71 00	48.08	2.22552	404
10	49.65	2.22956	407
20	51.25	2.23363	410
30	52.87	2.23773	413
40	54.53	2.24186	417
50	56.21	2.24603	419
72 00	57.92	2.25022	423
10	59.66	2.25445	425
20	1.43	2.25870	429
30	3.23	2.26299	433
40	5.06	2.26732	436
50	6.93	2.27168	440
73 00	8.83	2.27608	443
10	10.77	2.28051	447
20	12.74	2.28498	450
30	14.75	2.28948	454
40	16.80	2.29402	458
50	18.88	2.29860	462
74 00	21.01	2.30322	

Distance from Zenith.	$\delta\theta$	Log. $\delta\theta$	Difference.	$\frac{d\,\delta\theta}{d\tau}$
° '	' ''			
74 00	3 20.01	2.30322	467	
10	23.18	2.30789	470	
20	25.39	2.31259	475	
30	27.66	2.31734	479	
40	29.95	2.32213	483	
50	32.30	2.32696	488	
75 00	34.70	2.33184	493	0.009
10	37.16	2.33677	497	
20	39.65	2.34174	502	
30	42.21	2.34676	507	
40	44.82	2.35183	512	
50	47.48	2.35695	517	
76 00	50.21	2.36212	523	0.012
10	53.00	2.36735	528	
20	55.85	2.37263	533	
30	58.76	2.37796	538	
40	4 1.74	2.38334	545	
50	4.79	2.38879	551	
77 00	7.91	2.39430	557	0.015
10	11.11	2.39987	563	
20	14.39	2.40550	569	
30	17.74	2.41119	576	
40	21.19	2.41695	583	
50	24.72	2.42278	589	
78 00	28.33	2.42867	596	0.018
10	32.04	2.43463	603	
20	35.84	2.44066	611	
30	39.75	2.44677	618	
40	43.76	2.45295	626	
50	47.88	2.45921	635	
79 00	52.12	2.46556	642	0.023
10	56.47	2.47198	650	
20	5 0.94	2.47848	659	0.026
30	5.54	2.48507	669	
40	10.28	2.49176	677	
50	15.16	2.49853	688	
80 00	20.19	2.50541		0.030

TABLE I. *continued.*

Distance from Zenith.	$\delta\theta$	Log. $\delta\theta$	Diff.	$\frac{d\delta\theta}{d\tau}$	$\frac{d\delta\theta}{dp}$	Distance from Zenith.	$\delta\theta$	Log. $\delta\theta$	Diff.	$\frac{d\delta\theta}{d\tau}$	$\frac{d\delta\theta}{dp}$
° ′	′ ″					° ′	′ ″				
80 00	5 20.19	2.50541	696	0.030	0.04	85 00	9 53.84	2.77367	1191	0.159	0.25
10	25.36	2.51237	707	0.031		10	10 10.35	2.78558	1219	0.171	0.26
20	30.70	2.51944	716	0.034		20	27.73	2.79777	1248	0.184	0.28
30	36.20	2.52660	727	0.034		30	46.03	2.81025	1277	0.198	0.31
40	41.88	2.53387	738	0.036		40	11 5.30	2.82302	1309	0.213	0.33
50	47.74	2.54125	749	0.038		50	25.66	2.83611	1340	0.229	0.36
81 00	53.79	2.54874	759	0.040	0.05	86 00	47.15	2.84951	1374	0.248	0.39
10	6 0.04	2.55635	772	0.042		10	12 9.88	2.86325	1410	0.269	0.43
20	6.50	2.56407	785	0.044		20	33.97	2.87735	1447	0.292	0.47
30	13.18	2.57192	797	0.046	0.07	30	59.51	2.89182	1484	0.317	0.51
40	20.09	2.57989	811	0.049		40	13 26.61	2.90666	1523	0.345	0.56
50	27.26	2.58800	824	0.051		50	55.40	2.92189	1565	0.376	0.62
82 00	34.68	2.59624	838	0.053	0.08	87 00	14 26.04	2.93754	1608	0.410	0.68
10	42.37	2.60462	851	0.057		10	58.71	2.95362	1654	0.448	0.75
20	50.33	2.61313	866	0.060		20	15 33.60	2.97016	1701	0.490	0.83
30	58.59	2.62179	883	0.063	0.10	30	16 10.89	2.98717	1749	0.538	0.91
40	7 7.19	2.63062	899	0.067		40	50.8	3.00466	1801	0.593	1.01
50	16.13	2.63961	914	0.071		50	17 33.6	3.02267	1855	0.654	1.13
83 00	25.40	2.64875	931	0.074	0.11	88 00	18 19.6	3.04122	1909	0.722	1.26
10	35.05	2.65806	949	0.079		10	19 9.0	3.06031	1967	0.799	1.41
20	45.10	2.66755	967	0.084		20	20 2.2	3.07998	2026	0.887	1.59
30	55.58	2.67722	986	0.089	0.13	30	59.6	3.10024	2089	0.987	1.79
40	8 6.50	2.68708	1006	0.095		40	22 1.7	3.12113	2155	1.101	2.02
50	17.90	2.69714	1026	0.101		50	23 8.9	3.14268	2221	1.231	2.29
84 00	29.80	2.70740	1047	0.107	0.16	89 00	24 21.8	3.16489	2290	1.380	2.61
10	42.24	2.71787	1069	0.114		10	25 40.9	3.18779	2361	1.551	2.98
20	55.25	2.72856	1092	0.122		20	27 7.1	3.21140	2434	1.749	3.41
30	9 8.88	2.73948	1115	0.130	0.20	30	28 40.8	3.23574	2509	1.977	3.93
40	23.16	2.75063	1139	0.139		40	30 23.2	3.26083	2584	2.241	4.54
50	38.12	2.76202	1165	0.149		50	32 15.0	3.28667	2667	2.549	5.26
85 00	53.84	2.77367		0.159	0.25	90 00	34 17.5	3.31334		2.909	6.12

TABLE II.—*Thermometer.*

°		Diff.	°		Diff.
50	0.00000		50	0.00000	
49	0.00090	91	51	9.99910	
48	0.00181		52	9.99820	
47	0.00272		53	9.99730	
46	0.00363	92	54	9.99640	
45	0.00455		55	9.99550	
44	0.00546		56	9.99460	90
43	0.00638		57	9.99371	
42	0.00730		58	9.99282	
41	0.00822		59	9.99193	
40	0.00914		60	9.99104	
39	0.01006	93	61	9.99016	89
38	0.01099		62	9.98927	
37	0.01192		63	9.98839	
36	0.01285	94	64	9.98751	
35	0.01379		65	9.98663	
34	0.01472		66	9.98575	88
33	0.01566		67	9.98488	
32	0.01660		68	9.98401	
31	0.01754		69	9.98314	
30	0.01848		70	9.98227	
29	0.01942	95	71	9.98140	87
28	0.02037		72	9.98054	
27	0.02132		73	9.97967	
26	0.02227		74	9.97881	
25	0.02323	96	75	9.97795	
24	0.02418		76	9.97709	
23	0.02513		77	9.97623	
22	0.02609		78	9.97537	86
21	0.02706	97	79	9.97452	
20	0.02803		80	9.97367	
19	0.02900		81	9.97282	
18	0.02997		82	9.97197	
17	0.03094		83	9.97112	
16	0.03191		84	9.97027	85
15	0.03288	98	85	9.96943	
14	0.03386		86	9.96859	
13	0.03484		87	9.96775	
12	0.03582		88	9.96691	
11	0.03680	99	89	9.96607	84
10	0.03779		90	9.96524	

TABLE III.—*Barometer.*

°		Diff.	°		Diff.
30	0.00000		30	0.00000	
30.1	0.00145		29.9	9.99855	145
30.2	0.00289	144	29.8	9.99709	
30.3	0.00432		29.7	9.99563	
30.4	0.00575		29.6	9.99417	146
30.5	0.00718	143	29.5	9.99270	
30.6	0.00860		29.4	9.99123	147
30.7	0.01002	142	29.3	9.98975	148
30.8	0.01143		29.2	9.98826	
30.9	0.01284	141	29.1	9.98677	
31.0	0.01424	140	29.0	9.98528	149
			28.9	9.98378	
			28.8	9.98227	
			28.7	9.98076	151
			28.6	9.97924	
			28.5	9.97772	
			28.4	9.97620	152
			28.3	9.97466	
			28.2	9.97313	
			28.1	9.97158	
			28.0	9.97004	154
			27.9	9.96848	
			27.8	9.96692	156
			27.7	9.96536	157
			27.6	9.96379	
			27.5	9.96221	158

TABLE IV.

°	+		°	—
50	0.00000		50	0.00000
40	0.00043		60	0.00043
30	0.00087		70	0.00087
20	0.00130		80	0.00130
10	0.00173		90	0.00173

This Table of refractions has been constructed merely with the view of comparing the theory contained in the Paper with observation. The elements are precisely the same as those of the French Table in all other respects excepting the quantity f, which is assumed equal to $\frac{1}{4}$, from the exact manner in which this value seems to represent terrestrial observations. But it would be more satisfactory to determine the same quantity by the comparison of many observed refractions at low altitudes, between the distances of 85° and 88° from the zenith; and by this means a Table might be constructed that would be deserving of greater confidence.

J. IVORY.