V. *On the figure requisite to maintain the equilibrium of a homogeneous fluid mass that revolves upon an axis.* By James Ivory, *A. M. F. R. S.*

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The theory of the figure of the earth, as delivered in the *Philosophiæ Naturalis Principia Mathematica*, is liable to some objections. In determining the ratio of the axes, the illustrious author assumes that the terrestrial meridian is an ellipse, having the greatest diameter in the plane of the equator. M'Laurin afterwards proved, by a most elegant synthetic process of reasoning, that a homogeneous fluid body, possessed of such a figure as Newton supposed, will fulfil all the conditions of equilibrium arising from the attraction of the particles, and a centrifugal force of rotation. In this manner the assumption of Newton was verified; but the theory was still left imperfect, since it is necessary to determine, by a direct investigation, all the figures of a fluid mass that are consistent with the laws of equilibrium, rather than to show that the same laws will be fulfilled in particular instances. We are indebted to Legendre for the first demonstration that a homogeneous fluid body, revolving about an axis, cannot be in equilibrio by the attraction of its particles, unless it have the figure of an oblate elliptical spheroid. The researches of Legendre were rendered more general by Laplace, who gave a complete theory of the figure of the planets, distinguished by that depth and elegance which is so much admired in all his writings. It is assumed, however

by the eminent geometers we have mentioned, that the figure of the fluid mass is but little different from a sphere which is a restriction not essential to the problem, but introduced for the sake of overcoming some of the difficulties of the investigation. In the following Paper, the figure of a homogeneous fluid body, that revolves about an axis, and is in equilibrio by the attraction of its particles, is deduced by a direct analysis in which no arbitrary supposition is admitted.

1. It is necessary to begin this research, with laying down some general properties of the attractions of bodies; and we cannot better accomplish this end, than by considering the function, which is the sum of all the molecules of a body divided by their respective distances from the attracted point. Conceive any material body to be divided into an indefinitely great number of molecules, one of which is represented by dm; and having drawn three planes intersecting at right angles within the body, let x, y, z, denote the co-ordinates that determine the position of dm, and a, b, c, those that determine the attracted point: then, if we put

$$r = \sqrt{a^2 + b^2 + c^2}$$

$$f = \sqrt{(a-x)^2 + (b-y)^2 + (c-z)^2};$$

r will be the distance of the attracted point from the origin of the co-ordinates, and f that of dm from the attracted point.

$$\text{Now let } V(r) = \int \frac{dm}{f},$$

the sign of integration, extending to all the molecules of the body; and $V(r)$ will be the function alluded to, and which we have to consider.

It need not be mentioned that $V(r)$ is not a function of r, but of the three co-ordinates a, b, c; or it is an abridged

symbol denoting a function of r, and the angles which that line makes with the axes of the co-ordinates.

The distinguishing property of the expression $V(r)$ is this: if we take its fluxions with respect to the variable quantities a, b, c, the differential coefficients $-\left(\frac{d\,.V(r)}{d\,a}\right)$, $-\left(\frac{d\,.V(r)}{d\,b}\right)$, $-\left(\frac{d\,.V(r)}{d\,c}\right)$, will be respectively equal to the accumulated attractions of all the molecules of the body on the attracted point in the directions of a, b, c, and tending to shorten these lines.

Suppose now another body similar to the first in its lineal dimensions, and likewise having the parts similarly situated of the same density. If, therefore, this second body be divided into the same number of similar molecules as the first body; every two molecules, $d\,m$ and $d\,m'$, situated alike, will be of equal density, and their volumes will be proportional to the volumes of the two whole bodies. Suppose, also, that z', y', z', are three rectangular co-ordinates of the molecule $d\,m'$, drawn to planes situated in the second body, similarly to the like planes in the first; and farther, let a', b', c', be the co-ordinates of an attracted point, placed in the same relative situation in the second body, as the former attracted point in the first; then,

$$r' = \sqrt{a'^2 + b'^2 + c'^2}$$

$$f' = \sqrt{(a' - x')^2 + (b' - y')^2 + (c' - z')^2}$$

$$V'(r') = \int \frac{d\,m'}{f'}.$$

It is manifest from what has been said, that r and r', f and f', are homologous lines of the two bodies; and hence,

$$\frac{r}{f} = \frac{r'}{f'}; \quad \frac{d\,m}{r^3} = \frac{d\,m'}{r'^3}:$$

consequently,

$$\frac{1}{r^2} \cdot \frac{dm}{f} = \frac{1}{r'^2} \cdot \frac{dm'}{f'} :$$

and as this is universally true of all the molecules, we have,

$$\frac{V(r)}{r^2} = \frac{V'(r')}{r'^2} .$$

In this form the expression is inconvenient, because both the quantities become infinite when we suppose that the attracted point is placed in the origin of the co-ordinates. But the inconvenience is easily removed by substituting, for r and r', any other two homologous lines. Let r and r', produced if necessary, meet the surfaces; and let R and R′ denote the parts within the two bodies, intercepted between the surface and the origin of the co-ordinates: then it is manifest that $\frac{R}{r} = \frac{R'}{r'}$; and we shall therefore have

$$\frac{V(r)}{R^2} = \frac{V'(r')}{R'^2} .$$

If, therefore, we suppose a series of such bodies as we have been describing, which increase in magnitude from zero to infinity, while they constantly preserve the same proportions in their lineal dimensions, and the same densities of the parts similarly situated; the quantity $\frac{V(r)}{R^2}$, will have constantly the same value in all the bodies, supposing that the attracted points are alike placed in them all. It is manifest, therefore, that $\frac{V(r)}{R^2}$ will depend only on what is common to all the bodies in the series; or it will be a function of the quantities that remain unchanged in passing from one of the bodies to another. But as these quantities are not the same in all positions of the attracted point, it will be proper to distinguish several cases.

First, let the attracted point be in the surface, in which case $r = \mathrm{R}$: then, $\frac{a}{\mathrm{R}}, \frac{b}{\mathrm{R}}, \frac{c}{\mathrm{R}}$ are the only quantities that constantly retain the same values in all the bodies. These quantities remain unchanged, because the line r, or R, always makes the same angles with the axes of the co-ordinates. We therefore have,

$$\frac{\mathrm{V}(\mathrm{R})}{\mathrm{R}^2} = \mathrm{F}.\left\{\frac{a}{\mathrm{R}}, \frac{b}{\mathrm{R}}, \frac{c}{\mathrm{R}}\right\},$$

the letter F being the mark of a function. Hence,

$$\mathrm{V}(\mathrm{R}) = \mathrm{R}^2 \times \mathrm{F}.\left\{\frac{a}{\mathrm{R}}, \frac{b}{\mathrm{R}}, \frac{c}{\mathrm{R}}\right\}.$$

Again, let us put,

$$a = r.\mu;\ b = r\sqrt{1-\mu^2}.\,\mathrm{Cos.}\,\varpi;\ c = \gamma\sqrt{1-\mu^2}.\,\mathrm{Sin.}\,\varpi;$$

and μ will be the cosine of the angle which the line r, or R, makes with the axis of a; and ϖ the angle which the projection of the same line upon the plane of b and c makes with the axis of b: then, when $r = \mathrm{R}$, we have,

$$\mathrm{V}(\mathrm{R}) = \mathrm{R}^2 \times \mathrm{F}.\left\{\mu, \sqrt{1-\mu^2}.\,\mathrm{Cos.}\,\varpi, \sqrt{1-\mu^2}.\,\mathrm{Sin.}\,\mu\right\}.$$

Secondly, suppose that the attracted point is placed within each of the bodies; then the quantities common to them all are these, viz. $\frac{r}{\mathrm{R}}, \frac{a}{r}, \frac{b}{r}, \frac{c}{r}$. Hence,

$$\frac{\mathrm{V}(r)}{\mathrm{R}^2} = \mathrm{F}.\left\{\frac{r}{\mathrm{R}}, \frac{a}{r}, \frac{b}{r}, \frac{c}{r}\right\}.$$

Consequently,

$$\mathrm{V}(r) = \mathrm{R}^2 \times \mathrm{F}.\left\{\frac{r}{\mathrm{R}}, \frac{a}{r}, \frac{b}{r}, \frac{c}{r}\right\},$$

$$\mathrm{V}(r) = \mathrm{R}^2 \times \mathrm{F}.\left\{\frac{r}{\mathrm{R}}, \mu, \sqrt{1-\mu^2}.\,\mathrm{Cos.}\,\varpi, \sqrt{1-\mu^2}.\,\mathrm{Sin.}\,\varpi\right\}.$$

In order to have a more exact notion of this function, we may suppose it to be expanded in a series of the powers of the fraction $\frac{r}{\mathrm{R}}$: then,

$$V(r) = R^2 \times \left\{ P^{\circ} + P^{(1)} \cdot \frac{r}{R} + P^{(2)} \cdot \frac{r^2}{R^2} + \&c. \right\},$$

the coefficients P°, $P^{(1)}$, $P^{(2)}$, &c. being all functions of μ, $\sqrt{1-\mu^2} . \text{Cos.}\, \varpi$, $\sqrt{1-\mu^2} . \text{Sin.}\, \varpi$. When the attracted point coincides with the origin of the co-ordinates, the value of $\frac{V(0)}{R^2}$ is equal to $P^{(0)}$; and when the same point is in the surface, then $\frac{r}{R} = 1$, and the value of $\frac{V(R)}{R^2}$ is equal to

$$P^{(0)} + P^{(1)} + P^{(2)} + \&c.$$

Finally, let the attracted point be without the surface; then the quantities common to all the bodies are these, viz. $\frac{R}{r}$, $\frac{a}{r}$, $\frac{b}{r}$, $\frac{c}{r}$: hence

$$\frac{V(r)}{R^2} = F . \left\{ \frac{R}{r}, \frac{a}{r}, \frac{b}{r}, \frac{c}{r} \right\}.$$

Consequently,

$$V(r) = R^2 \times F . \left\{ \frac{R}{r}, \frac{a}{r}, \frac{b}{r}, \frac{c}{r} \right\};$$

$$V(r) = R^2 \times F . \left\{ \frac{R}{r}, \mu, \sqrt{1-\mu^2}\, \text{Cos.}\, \varpi, \sqrt{1-\mu^2} . \text{Sin.}\, \varpi \right\}.$$

In this case, $\frac{V(r)}{R^2}$ decreases as r increases, and finally vanishes when r is infinite. The expansion must therefore have this form, viz.

$$V(r) = R^2 \times \left\{ Q^{(1)} . \frac{R}{r} + Q^{(2)} . \frac{R^2}{r^2} + Q^{(3)} : \frac{R^3}{r^3} + \&c. \right\},$$

$Q^{(1)}$, $Q^{(2)}$, $Q^{(3)}$, &c. being functions of μ, $\sqrt{1-\mu^2} . \text{Cos.}\, \varpi$, $\sqrt{1-\mu^2} . \text{Sin.}\, \varpi$. When the attracted point is in the surface, $\frac{R}{r^2} = 1$, and the value of $\frac{V(R)}{R^2}$ is equal to

$$Q^{(1)} + Q^{(2)} + Q^{(3)} + \&c.$$

The preceding reasoning is quite general, and will apply

to any material body, whatever be its form. The body may consist of parts not connected by any mathematical law; or, which is the same thing, it is not necessary that the equation of its surface be subject to the law of continuity.

2. The co-ordinates of the molecule dm being x, y, z, let R' denote its distance from the origin of the co-ordinates; and put,

$x = R'.\mu'$; $y = R\sqrt{1-\mu'^2}.\text{Cos.}\varpi'$; $z = R'\sqrt{1-\mu'^2}.\text{Sin.}\varpi'$;

then, since we likewise have,

$a = r.\mu$; $b = r.\sqrt{1-\mu^2}.\text{Cos.}\varpi$; $c = r\sqrt{1-\mu^2}.\text{Sin.}\varpi$;

and,

$$f = \sqrt{(a-x)^2 + (b-y)^2 + (c-z)^2};$$

we shall get,

$$\gamma = \mu\mu' + \sqrt{1-\mu^2}.\sqrt{1-\mu'^2}.\text{Cos.}(\varpi - \varpi'),$$
$$f = \sqrt{r^2 - 2rR'.\gamma + R'^2}.$$

It now becomes necessary to expand $\frac{1}{f}$ in a series of the powers of $\frac{r}{R}$, or of $\frac{R}{r}$. Much has already been written on this expansion. The coefficients have been exhibited in various forms, and many remarkable properties which they possess have been very diligently explored. It would not, therefore, be necessary to add any thing upon this subject, unless it be possible to give to the same quantities a new and more simple form of expression, useful in the present investigation.

If we suppose,

$$\frac{1}{f} = \frac{1}{r}\cdot\left\{1 + C^{(1)}.\frac{R'}{r} + C^{(2)}.\frac{R'^2}{r} \ldots\ldots + C^{(i)}\frac{R'^i}{r^i} + \&c.\right\};$$

the following differential equation has already been proved in the Philosophical Transactions for 1812, viz.

$$i(i+1).C^{(i)}+\frac{d.\left\{(1-\gamma^2)\frac{dC^{(i)}}{d\gamma}\right\}}{d\gamma}=o:$$

and from this equation the value of $C^{(i)}$ is deduced, viz.

$$C^{(i)}=\frac{1.3.5\ldots 2i-1}{1.2.3\ldots\ldots i}\times\left\{\gamma^i-\frac{i(i-1)}{2.2i-1}.\gamma^{i-2}+\&c.\right\}.$$

In the same place another more general differential equation is found, of which the former is only a particular case, viz.

$$(i-n).(i+n+1).(1-\gamma^2)^n\frac{d^n C^{(i)}}{d\gamma^n}+\frac{d.\left\{(1-\gamma^2)^{n+1}\frac{d^{n+1}C^{(i)}}{d\gamma^{n+1}}\right\}}{d\gamma}=o.$$

For the sake of abridging I shall now put

$$\varphi^{(n)}=(1-\gamma^2)^n.\frac{d^n C^{(i)}}{d\gamma^n};$$

and, consequently,

$$\varphi^{(o)}=(1-\gamma^2)^o\frac{d^o C^{(i)}}{d\gamma^o}=C^{(i)}:$$

then, if in the foregoing formula, we make n successively equal to 0, 1, 2, 3 . . . i, we shall get this series of equations, viz.

$$\varphi^{(0)}+\frac{d.\varphi^{(1)}}{i(i+1.d\gamma}=o$$

$$\varphi^{(1)}+\frac{d.\varphi^{(2)}}{i-1.i+2.d\gamma}=o$$

$$\varphi^{(2)}+\frac{d.\varphi^{(3)}}{i-2.i+3.d\gamma}=o$$

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$$\varphi^{(i-1)}+\frac{d.\varphi^{(i)}}{1.2i.d\gamma}=o.$$

Now, by combining these equations, after having taken the fluxions of each a proper number of times, all the intermediate quantities between $\varphi^{(o)}$ and $\varphi^{(i)}$ may be made to disappear; and we shall finally obtain,

$$\varphi^{(o)} - \frac{(-1)^i}{1 . 2 . 3 \ldots 2i} \times \frac{d^i \varphi^{(i)}}{d\gamma^i} = o:$$

and by restoring the expressions that $\varphi^{(o)}$ and $\varphi^{(i)}$ stand for,

$$C^{(i)} = \frac{(-1)^i}{1 . 2 . 3 \ldots 2i} \times \frac{d.^i \left\{ (1-\gamma^2)^i \frac{d^i C^{(i)}}{d\gamma^i} \right\}}{d\gamma^i} .$$

But, from the series equal to $C^{(i)}$, we get

$$\frac{d^i C^{(i)}}{d\gamma^i} = 1 . 3 . 5 \ldots (2i - 1) .$$

Wherefore,

$$C^{(i)} = \frac{(-1)^i}{2 . 4 . 6 \ldots 2i} \times \frac{d^i (1-\gamma^2)^i}{d\gamma^i} .$$

From this very simple expression, the most remarkable properties of the coefficients of the expansion of $\frac{1}{f}$, are very readily deduced.

3. We may suppose that the indefinitely small molecule dm is a parallelopiped, of which the height is equal to dR'; and the length to $\frac{R' d\mu}{\sqrt{1-\mu'^2}}$, the small line described by the motion of R' perpendicular to the plane of y, z; and the breadth to $R' d\varpi' \sqrt{1-\mu'^2}$, the small line described by the motion of R' parallel to the same plane. The volume of the molecule is therefore equal to $dR' \times \frac{R' d\mu'}{\sqrt{1-\mu'^2}} \times R' d\varpi' \sqrt{1-\mu'^2}$; and, if ϱ be put for its density, we shall have

$$dm = \varrho R'^2 dR' d\mu' d\varpi' .$$

Again, we have,

$$f = \sqrt{r^2 - 2rR'\gamma + R'^2},$$

$$V(r) = \int \frac{dm}{f};$$

and, by taking the fluxions with regard to r,

$$-\left(\frac{d.V(r)}{dr}\right) r = \int \frac{dm\,(r^2 - rR'\gamma)}{f^3}.$$

But,

$$r^2 - rR'\gamma = f^2 + rR'\gamma - R'^2;$$

wherefore,

$$-\left(\frac{d.V(r)}{dr}\right) r = \int \frac{dm}{f} + \int \frac{dm\,(rR'\gamma - R'^2)}{f^3};$$

and, by adding the equivalent quantities, $2\,V(r)$ and $2\int \frac{dm}{f}$, we get

$$2\,V(r) - \left(\frac{d.V(r)}{dr}\right) r = 3\int \frac{dm}{f} + \int \frac{dm\,(rR'\gamma - R'^2)}{f^3};$$

and by substituting the value of dm,

$$V(r) - \frac{d.V(r)}{dr} r = \iint d\mu'\, d\varpi' . \int \varrho . \left\{ \frac{3R'^2\,dR'}{f} + \frac{R'^3\,(r\gamma - R')\,dR'}{f^3} \right\}$$

or, which is the same thing,

$$-\left(\frac{d.\frac{V(r)}{r^2}}{dr}\right) r^3 = \iint d\mu'\, d\varpi' . \int \left(\frac{d.\frac{R'^3}{f}}{dR'}\right) \varrho\, dR'.$$

In the present paper, we confine our attention to a homogeneous body, or fluid mass; and, ϱ being constant, we may suppose it equal to unit; then, having integrated the last equation with regard to the variable R', we shall get,

$$-\left(\frac{d.\frac{V(r)}{r^2}}{dr}\right) r^3 = \iint \frac{R'^3\,d\mu'\,d\varpi'}{f}.$$

In this formula, R' is the line drawn from the origin of the co-ordinates to the surface; and the integration with respect to μ', is to extend from $\mu' = 1$ to $\mu' = -1$; and that with

respect to ϖ', from $\varpi' = o$ to $\varpi' = 2\,\pi$, or the whole circumference.

The preceding formula is true, whether the attracted point be without or within the body. There is however a distinction between the two cases. If we multiply by $-\frac{dr}{r^3}$, and then integrate, we shall get

$$\frac{V(r)}{r^2} = \int -\frac{dr}{r^3} \iint \frac{R'^3\, d\mu'\, d\varpi'}{\sqrt{r^2 - 2\,r\,R'\gamma + R'^3}},$$

no constant quantity being necessary when the attracted point is without the body, because both the quantities vanish when r is infinitely great. But when the attracted point is within the body, it is necessary to add a constant quantity, because $\frac{dd\,.\,V(r)}{d\,r^2}$, is not evanescent when $r = o$: in this case, therefore, we have

$$\frac{V(r)}{r^2} = \int -\frac{dr}{r^3} \iint \frac{R'^3\, d\mu'\, d\varpi'}{\sqrt{r^2 - 2\,r\,R'\gamma + R'^2}} + K;$$

and

$$V(r) = r^2 \int -\frac{dr}{r^3} \iint \frac{R'^3\, d\mu'\, d\varpi'}{\sqrt{r^2 - 2\,r\,R'\gamma + R'^2}} + K\,r^2,$$

K being a quantity independent of r.

It is necessary to find an expression of the value of K. For this purpose we have,

$$V(r) = \int \frac{dm}{f} = \iiint \frac{R'^2\, dR'\, d\mu'\, d\varpi'}{\sqrt{r^2 - 2\,r\,R'\gamma + R'^2}}.$$

Expand the denominator in a series of the ascending powers of r; then,

$$\begin{aligned} V(r) = & \iiint R'\, dR'\, d\mu'\, d\varpi' \\ & + r \iiint dR'.\, C^{(1)}\, d\mu'\, d\varpi' \\ & + r^2 \iiint \frac{dR'}{R'} \cdot C^{(2)}\, d\mu'\, d\varpi' \\ & + r^3 \iiint \frac{dR'}{R'^2} \cdot C^{(3)}\, d\mu'\, d\varpi' \\ & + \&c. \end{aligned}$$

The integrations with respect to dR' should be executed from $R' = o$, to the value of R' at the surface of the body; which cannot be done, because all the terms after the two first would be infinite. Conceive a sphere to be described about the origin of the co-ordinates with the radius r; then the whole value of the function $V(r)$ will be equal to its value with respect to the sphere added to its value with respect to the matter between the sphere and the surface of the body. The attracted point being in the surface of the sphere, the first part of the value of $V(r)$ will be equal to $\frac{4\pi}{3} \times r^2$; and the second part will be found by integrating the foregoing expression, so that all the integrals shall vanish when $R' = r$. Thus we get,

$$\begin{aligned} V(r) = \frac{4\pi}{3} r^2 &+ \iint \tfrac{1}{2}(R'^2 - r^2)\, d\mu'\, d\varpi' \\ &+ r\iint (R' - r) \, . \, C^{(1)}\, d\mu'\, d\varpi' \\ &+ r^2 \iint (\text{Log.}\, R' - \text{Log.}\, r) \, . \, C^{(2)}\, d\mu'\, d\varpi' \\ &+ \frac{r^3}{1} \iint \left(\frac{1}{r} - \frac{1}{R'}\right) . \, C^{(3)}\, d\mu'\, d\varpi' \\ &+ \frac{r^4}{2} \iint \left(\frac{1}{r^2} - \frac{1}{R'^2}\right) . \, C^{(4)}\, d\mu'\, d\varpi' \\ &+ \&c. \end{aligned}$$

Now the integral $-\frac{1}{2}\iint r^2\, d\mu'\, d\varpi'$, taken between the limits $\mu' = 1$, $\mu = -1$; and $\varpi' = o$, $\varpi' = 2\pi$; is equal to $-2\pi r^2$. All the other parts of the above expression that contain r, are evanescent; because we have generally,

$$\iint C^{(i)}\, d\mu'\, d\varpi' = o.$$

In order to prove this, it is to be observed that μ, μ' and γ are the cosines of the three sides of a spherical triangle, and $\varpi - \varpi'$ is the angle opposite to the side whose cosine is γ:

now if we put ψ for the angle opposite to the side whose cosine is μ', we may write $d\gamma\, d\psi$ in place of $d\mu'\, d\varpi'$, making γ and ψ vary between the same limits as μ' and ϖ'. This is allowable; not that we must conceive the two fluxions as continually equal to one another, but because the total sum, between the prescribed limits, is, in either case, equal to the whole surface of the sphere. If now we substitute the value of $C^{(i)}$ given in § 2, the foregoing expression will become,

$$\frac{(-1)^i}{2.4.6\ldots 2i} \times \iint \frac{d^i.(1-\gamma^2)^i}{d\gamma^i} \times d\gamma \times d\psi:$$

and the integral is,

$$\frac{(-1)^i}{2.4.6\ldots 2i} \times 2\pi \times \frac{d^{i-1}.(1-\gamma^2)^i}{d\gamma^{i-1}};$$

a quantity which, being divisible by $1-\gamma^2$, is evanescent at both the limits of γ.

Omitting what has been proved to be evanescent in $V(r)$, and collecting into one sum all the parts multiplied by r^2, and separating them from the rest, we get

$$\begin{aligned} V(r) = & \iint \frac{R'^2}{2}.\, d\mu' d\varpi' + r^2 \times \left\{ -\frac{2\pi}{3} + \iint \log. R' \times C^{(2)} d\mu' d\varpi' \right\} \\ & + r \iint R' C^{(1)} d\mu'\, d\varpi' \\ & - \frac{r^3}{1} \iint \frac{C^{(3)} d\mu'\, d\varpi'}{R'} \\ & - \frac{r^4}{2} \iint \frac{C^{(4)} d\mu'\, d\varpi'}{R'^2} \\ & - \&c. \end{aligned}$$

which expression may be thus written in finite terms, viz.

$$V(r) = r^2 \int -\frac{dr}{r^3} \iint \frac{R'^3\, d\mu'\, d\varpi'}{f} + r^2.\left\{ -\frac{2\pi}{3} + \iint \log. R' \times C^{(2)}\, d\mu'\, d\varpi' \right\}$$

as will be evident by expanding $\frac{1}{f}$, and performing the in-

tegrations with respect to dr. By comparing this expression of $V(r)$ with the foregoing one, we get,

$$K = -\frac{2\pi}{3} + \iint \log. R' \times C^{(2)}\, d\mu'\, d\varpi'.$$

From this expression it follows that K has the same value in all similar homogeneous bodies. Suppose another body, similar and homogeneous to the first, and having the axes of the co-ordinates similarly placed: let R' and $R'_{\prime}$ be lines drawn to the surface from the origin of the co-ordinates, and making the same angles with the axes; then, K' denoting the like quantity in the second body as K in the first, we have

$$K = -\frac{2\pi}{3} + \iint \log. R' \times C^{(2)}\, d\mu'\, d\varpi',$$

$$K' = -\frac{2\pi}{3} + \iint \log. R'_{\prime} \times C^{(2)}\, d\mu'\, d\varpi';$$

wherefore,

$$K - K' = \iint \log. \frac{R'}{R'_{\prime}} \times C^{(2)}\, d\mu'\, d\varpi'.$$

But the two bodies being similar, and R' and $R'_{\prime}$ lines similarly drawn in them, it follows that $\frac{R'}{R'_{\prime}}$ will remain unchanged, when μ' and ϖ' vary. Consequently,

$$K - K' = \log. \frac{R'}{R'_{\prime}} \times \iint C^{(2)}\, d\mu'\, d\varpi' = 0.$$

4. Having now laid down the properties of attraction to which we shall have occasion to refer, we are next to consider the conditions necessary to the equilibrium of a fluid mass. These were first reduced to a uniform mode of calculation by CLAIRAUT, in his Theory of the Figure of the earth. They are investigated in all the great treatises of rational mechanics; in the *Mecanique Analytique* of LAGRANGE, the *Mecanique Celeste* of LAPLACE, and the *Mecanique* of POISSON.

The English reader will likewise find the same investigations in a work published in 1821, under the title of *Elementary Illustrations of the Celestial Mechanics,* which is a translation of the first book of the *Mecanique Celeste,* and, in addition to the text, contains much valuable matter. Referring to these works, we shall first merely enumerate the chief properties of the equilibrium of a fluid mass, for the sake of recalling them to the recollection of the reader; and then make such an application of the general principles as our present purpose requires.

A heterogeneous fluid body cannot be in equilibrio, unless the outer surface be every where of the same density; and farther, unless particles of the same density be arranged in distinct strata in the interior of the mass. The pressure upon all equal spaces of every stratum of uniform density, that is, the force acting perpendicularly to them, and pushing them inward, must be equal. Hence, these are called level strata, or *couches de niveau,* because the direction of the accelerating force, or of gravity, is every where perpendicular to them. It is easy to perceive that the densities must decrease in approaching the outer surface. For, in two contiguous strata of different densities, if we take two molecules equal in volume, and placed at the same point of the separating surface; the common gravity acting upon both will produce a greater pressure in the denser molecule. Wherefore, if the denser matter were nearer the outer surface, it would penetrate into the rarer matter below it; which is contrary to the perfect separation of the strata of different densities.

Supposing all these conditions to be fulfilled, it readily follows that the fluid body will be in equilibrio. For the

equal pressure upon every part of the upper surface of a level stratum, being propagated through the interior fluid, will act with equal force in an opposite direction upon every part of the lower surface; and hence, every molecule of the stratum will be equally pressed in all opposite directions.

When the fluid mass is homogeneous, the distinction of the level strata arising from the difference of their densities is lost; but the possibility of dividing it into any number of strata separated by level surfaces, is still a necessary condition of the equilibrium.

The condition, that every level surface must be a continuous curve stretching through the whole fluid mass, imposes a limitation on the forces with which the equilibrium is possible. All these curve surfaces are defined by a common equation between three independent co-ordinates; and as this equation is to be found by integrating an expression containing the co-ordinates and their fluxions, the operation must be practicable, without supposing any relation between the three variable quantities. Hence, the forces acting on the particles of the fluid must be such, that the three differential coefficients shall fulfil what is called the criterion of integrability; otherwise the equilibrium will be impossible. In determining the equilibrium of the fluid placed on the surfaces of the planets, the nature of the forces brought into action is such, that the problem is always free from contradictory conditions.

To come now to the main object we have in view,* conceive that H K I is a body of homogeneous fluid in equilibrio

* Theorie de la Figure de la Terre par Clairaut. Premiere Partie, cap. V. § XXI.

by the action of all the forces that urge its particles. Let x, y, z, be three rectangular co-ordinates of a point K in the surface; and put X, Y, Z, for the accelerating forces that act upon a particle at K respectively in the directions of the co-ordinates, and tending to diminish these lines. Suppose that K varies its position a little in the fluid's surface; then the condition that the resultant of the forces parallel to the co-ordinates, shall be perpendicular to that surface, is expressed by this equation, viz.

$$X\,dx + Y\,dy + Z\,dz = 0.$$

In order that the equilibrium be possible, the expression just set down must be a complete differential; which subjects the forces X, Y, Z, to the criterion of integrability. This condition being fulfilled, the equation of the fluid's surface will be,

$$\int (X\,dx + Y\,dy + Z\,dz) = C,$$

C being an arbitrary constant quantity. If, for the sake of brevity, we represent the preceding integral by φ, we shall have,

$$\varphi = C$$

$$X = \frac{d\varphi}{dx}, \quad Y = \frac{d\varphi}{dy}, \quad Z = \frac{d\varphi}{dz}.$$

Again, let

$$p = \sqrt{X^2 + Y^2 + Z^2};$$

then p is the resultant of the forces X, Y, Z; and it acts on a particle placed at K, in a direction perpendicular to the fluid's surface, and tending inward. It is the gravity at that point.

Suppose now that a stratum of fluid is laid upon the surface H K I, the thickness at K being equal to the indefinitely small line K S. The new pressure at K will be proportional

to the superincumbent matter multiplied by p, or by the gravity which urges a particle inward. But, as the density is constant, the quantity of matter pressing at K, will be proportional to the thickness K S. Wherefore, if δC represent the additional pressure at K, we shall have

$$\delta C = p \times KS.$$

Hence,

$$KS = \frac{\delta C}{p} = \frac{\delta C}{\sqrt{X^2 + Y^2 + Z^2}}.$$

If now we suppose that δC remains constant, and, by means of the formula just set down, determine the thickness at every point; it is evident that the stratum will press equably upon the surface of the fluid H K I, and consequently will not disturb the equilibrium by its pressure. It remains to determine the equation of the upper surface of the stratum. For this purpose we have,

$$\delta C = p \times KS = \frac{p^2}{p} \times KS = \left(X\frac{X}{p} + Y\frac{Y}{p} + Z\frac{Z}{p}\right) \times KS.$$

The co-ordinates of the point K being x, y, z, let those of the point s be $x + \delta x$, $y + \delta y$, $z + \delta z$: then K S being perpendicular to the surface H K I, it is easy to prove that,

$$\delta x = \frac{X}{\sqrt{X^2 + Y^2 + Z^2}} \times KS = \frac{X}{p} \times KS,$$

$$\delta y = \frac{Y}{p} \times KS$$

$$\delta z = \frac{Z}{p} \times KS;$$

wherefore, by substitution, we get

$$\delta C = X\delta x + Y\delta y + Z\delta z;$$

that is,

$$\delta C = \frac{d\varphi}{dx}\delta x + \frac{d\varphi}{\delta y}\delta y + \frac{d\varphi}{dz}\delta z.$$

Consequently,

$$\varphi + \frac{d\varphi}{dx}\delta x + \frac{d\varphi}{dy}\delta y + \frac{d\varphi}{dz}\delta z = C + \delta C.$$

Now this expression is derived from the equation,

$$\varphi = C,$$

on the one hand, by changing C into $C + \delta C$; and, on the other, by substituting, in the function φ, the co-ordinates $x + \delta x$, $y + \delta y$, $z + \delta z$ of the upper surface of the stratum, in place of x, y, z, the co-ordinates of the surface H K I. Thus it appears, that the equation of the new fluid body is derived from that of the first one, merely by varying the constant introduced in the integration.

Before proceeding farther, it is requisite to distinguish carefully two separate cases. The first is, when the particles of the fluid do not attract one another; and the second, when they are endowed with attractive powers. These are plainly two cases essentially different from one another: for, in the first, a stratum added induces no other change than an increase of pressure; but, in the second, besides the pressure a new force is introduced, arising from the attraction which the matter of the stratum exerts upon the fluid body to which it is added.

In the first case, when there are no new forces introduced by attraction, it is manifest from what has been said, that the fluid body of which the equation is,

$$\varphi = C + \delta C$$

is in equilibrio; because the stratum presses equally upon all parts of the surface H K I. If we suppose a second stratum to be laid upon the first, and compute its thickness by the gravity at the surface N O L, in the same manner that the thickness of the first was determined by means of the gravity at the surface H K I, we shall have another fluid body in equilibrio, of which the equation will be,

$$\varphi = C + \delta C + \delta' C$$

$\delta' C$ being equal to the pressure caused by the new stratum. And, in like manner, any number of strata may be added composing a fluid body in equilibrio.

But as strata have been added without disturbing the equilibrium, in like manner any number of strata below the surface HKI, may be successively taken away, so as to leave the remaining fluid in equilibrio. The original body HKI may be thus exhausted, or reduced to an infinitesimal quantity that may be neglected; and then the whole mass, both above and below the surface HKI, will consist of level strata separated by surfaces having a common equation, in which the constant quantity introduced in the integration alone varies in passing from one surface to another. We may therefore conclude that, when the particles of the fluid do not attract one another, the only conditions necessary for the equilibrium are, first, that the force resulting from X, Y, Z, be directed into the interior of the mass; and, secondly, that $X\,dx + Y\,dy + Z\,dz$ be an exact differential.

But this first case can have no application in the theory of the figure of the planets, the leading principle of which is, that every particle of matter attracts every other particle. We must therefore proceed to consider what new conditions are required in the second case, when the particles are possessed of attractive powers.

All the forces, whether attractive or not, that urge the particles of the fluid body HKI, are supposed to be included in the expressions X, Y, Z; and it has been shown that the gravity arising from these forces produces, by its action upon the stratum of which the thickness is KS, equal pressures

upon every point of the surface HKI. The whole mass of fluid, NOL, will therefore be in equilibrio, if it be urged by no other forces. But the attraction of the stratum upon all the matter within it, is a new force brought into play, the efforts of which must be balanced, otherwise the equilibrium could not subsist. Now this new force is distinct from the pressure caused by the gravity, and can never be included in it. Two separate principles must therefore be employed to ensure the equilibrium of the fluid body HKI, when acted upon by the two independent forces. But a fluid body cannot be in equilibrio by the action of external forces upon it, except in one of these two ways: either there must be an equable pressure upon the outer surface; or, all the forces that act upon every separate particle must destroy one another. We are therefore necessarily led to suppose, that the added stratum must possess such a figure as to attract every particle in the inside with equal force in all opposite directions. By the help of this principle, and by no other means, the fluid body HKI, will still continue to be in equilibrio when subjected to the additional pressure, and to the new attractive force.

When more strata are added, they must separately possess the property of attracting every particle in the inside with equal force in opposite directions; by which supposition, we are brought at every step to the same circumstances, as in the case when there is no attraction between the particles. The whole fluid mass being ultimately divided into level strata, the property common to each must belong to the aggregate of any number of them.

On the whole, it is not sufficient for the equilibrium of a

homogeneous fluid body, the particles of which attract one another, that the resultant of the forces X, Y, Z, be directed inward, and that $X\,dx + Y\,dy + Z\,dz$ be an exact differential: to these conditions it is necessary to add that, every particle placed within a stratum bounded by two level surfaces, be in equilibrio by the attraction of the stratum.

The conclusion we have arrived at does not coincide with the usual determination of the equilibrium of a fluid mass, in which no distinction is made between the two cases when the particles attract one another, and when they possess no such powers. The difference arises from this, that no notice is commonly taken of the attraction which the thin level stratum exerts upon the fluid body to which it is added. Every difficulty respecting this point will be removed, if it be impressed on the mind that the gravity at any level surface, and the pressure caused by it, are forces distinct from, and independent of, the attraction of the exterior matter. In estimating the pressure, the exterior fluid is unavoidably regarded merely as inert matter subjected to external force; and when there are active powers inherent in it, the effect of these must be separately investigated. It is said that nothing more is requisite to the equilibrium of a homogeneous fluid, than that the pressure be equable over all the outer surface. For, it is argued, since there is no distinction of density in the interior, it is always possible to trace curves that shall cut at right angles the resultants of all the forces urging the particles; which curves will therefore be level surfaces. But the defectiveness of this reasoning will appear if it be observed that, as every particle of the fluid is attracted by the whole mass, the curve surfaces traced in the manner

described, will be entirely dependent upon the outer surface. If the uppermost stratum, or any number of the uppermost strata be taken away, a part of the attractive force acting upon every particle will be destroyed, and the curve surfaces will no longer be perpendicular to the resultants of the remaining forces urging the particles. Suppose that the strata are taken away successively; then, the figure necessary to the equilibrium of the remaining fluid will change as each stratum is abstracted; which is contrary to the just principles of the equilibrium of a fluid mass. The level surfaces of a homogeneous fluid mass in equilibrio, are determined without ambiguity by varying the arbitrary constant of the general equation. And as there is no doubt that the figure of the outer surface has no relation to any matter placed without it; so any level surface, which is defined by a perfectly similar equation, must be independent of all the exterior matter. Farther, the gravitation acting at any point of the outer surface is a function of the co-ordinates of that point, and has no dependence upon any exterior matter; and, the like force at any level surface being the same function of the co-ordinates of that surface, it must be equally independent of the exterior matter. And although it be admitted that every level surface must be perpendicular to the resultant of all the forces urging the particles, yet it does not follow that no modification of the forces is necessary to the equilibrium. In reality, the foregoing observations prove that, if we reason consistently from what is allowed in the usual determination of the equilibrium of a fluid mass, we shall be led to the same conclusion at which we have already arrived; namely,

that the forces acting at any point in the interior, must be so modified by the figure of the fluid, as to render every level surface, and the gravity at every point of it, independent of the exterior matter.

We may cite as examples of the two different cases of the equilibrium of a homogeneous fluid, the hypothesis of Huyghens respecting the figure of the earth, which falls under the first case; and the Newtonian theory on the same subject, which belongs to the second. Huyghen's supposed an attractive force residing in the earth's centre, and acting with the same intensity at all distances. Therefore, in the case of a revolving mass, every particle is urged by a constant force directed to the centre, and by a centrifugal force proportional to the distance from the axis of rotation. As there is no attraction between particle and particle, a level stratum will act by pressure only upon the fluid below it; and the only condition requisite to the equilibrium, is an equable pressure over all the outer surface. But, according to Newton, every particle attracts every other particle; and a level stratum will act upon the fluid below it, both by the pressure of gravitation and by its own attractive force. In this theory, therefore, the adjustment of the equilibrium requires the joint application of both the principles of the second case.

The method of investigation followed in what goes before, is similar to a process of reasoning in Clairaut's theory of the figure of the earth; and it is certainly surprising that the difference of the two cases was not remarked by that acute geometer. Other authors have very generally adopted a more simple procedure introduced by Euler. It will be

worth while to set before the reader very briefly the steps of EULER's investigation, for the purpose of pointing out the omission with which it is chargeable.

Suppose that the fluid mass in equilibrio is divided into indefinitely small rectangular parallelopipeds by planes parallel to those of the co-ordinates; and let x, y, z, be the co-ordinates of one angle of a parallelopiped which has dx, dy, dz, for its sides, and which we may conceive to be so placed, that $x + dx$, $y + dy$, $z + dz$, are the co-ordinates of the opposite angle. The forces that act upon the parallelopiped are; the pressure of the adjacent fluid upon its six faces; and the accelerating forces X, Y, Z, urging every particle in directions parallel to x, y, z, and tending to increase these lines. The pressure at any point of the fluid must depend upon the situation of that point, or it must be some function φ of the co-ordinates x, y, z: and, according to the principles of the differential calculus, φ will retain the same value over all the three faces of the parallelopiped that comprehend any one of the solid angles. Now, y and z remaining constant, if we substitute $x + dx$ in place of x, φ will be changed into $\varphi + \frac{d\varphi}{dx} . dx$; and the two quantities φ and $\varphi + \frac{d\varphi}{dx} dx$, will represent the intensities of pressure upon the opposite rectangles comprehended by dy and dz: the forces compressing the parallelopiped are therefore $\varphi \times dy\, dz$, and $(\varphi + \frac{d\varphi}{dx} dx)$ $dy\, dz$; and the difference, or $\frac{d\varphi}{dx} dx\, dy\, dz$ is the force causing the parallelopiped to move in a direction tending to diminish x. In like manner, the pressures on the other sides produce the forces $\frac{\varphi}{dy}\, dx\, dy\, dz$ and $\frac{d\varphi}{dz}\, dx\, dy\, dz$, causing the parallelo-

piped to move in directions tending to diminish y and z. Again, the accelerating forces X, Y, Z, acting on every parallelopiped produce the motive forces $X\,dx\,dy\,dz$, $Y dx dy dz$, $Z\,dx\,dy\,dz$, tending to increase the lines x, y, z. But the equilibrium of the parallelopiped requires the equality of the opposite forces: wherefore,

$$\frac{d\phi}{dx} = X, \frac{d\phi}{dy} = Y, \frac{d\phi}{dz} = Z.$$

Hence, we get,

$$d\phi = X\,dx + Y\,dy + Z\,dz.$$

Wherefore if we trace a stratum of the fluid so that ϕ shall every where have the same value, the figure of the stratum will be defined by the equation

$$X\,dx + Y\,dp + Z\,dz = o;$$

which likewise shows that the resultant of the accelerating forces is perpendicular to the stratum.

In what has been said, the equilibrium of every parallelopiped is established with respect to all the outward forces extrinsic to its own matter. If the question relate to no other forces, the whole fluid mass, and all the level strata of which it consists, will be in equilibrio, and the problem is solved. But when the particles of the fluid attract one another, there are forces not yet taken into account, inherent in every parallelopiped, by means of which it will act upon all the exterior matter, and the efforts of which must be balanced, otherwise the equilibrium could not subsist. Now, if we suppose, as before, that all the level strata are possessed of such a figure as to act upon particles in the inside with equal forces in opposite directions, it is evident, that every parallelopiped will be in equilibrio by its action upon all the matter on the

outside of the stratum. With regard to the matter in the inside, a parallelopiped will act upon it effectively; but, the united attraction of all the parallelopipeds in the same stratum upon every interior particle being equal in opposite directions, it will not disturb the equilibrium of the fluid below the stratum. Therefore, when we take into account all the forces that act upon the parallelopipeds; both those urging them externally, and those inherent in their own matter; it is evident, that all the molecules in the same level stratum will be in equilibrio with respect to the matter above them, and that they will press equably upon the fluid body below them, by the action of the gravity alone. The fluid mass will therefore be in equilibrio with respect to all the forces in action. Thus, in every view of the problem, it appears that, when nothing essential is omitted, the particular conformation of the level strata which annihilates their action upon particles in the inside, is just as necessary to the equilibrium of the fluid mass, as the equality between the pressure and the effect of the accelerating forces.

There is another way of arriving at the same conclusion, which, in reality, first led to the suspicion of some defect lurking in the usual determination of the equilibrium of a fluid mass. This new view of the subject, which applies only to the law of attraction that takes place in nature, is contained in the two following propositions.

Proposition I.

If a homogeneous fluid body revolving about an axis, be in equilibrio by the attraction of its particles in the inverse proportion of the square of the distance; any other mass of

the same fluid having a similar figure, and revolving with the same rotatory velocity about an axis similarly placed, will likewise be in equilibrio, supposing that its particles attract one another by the same law.

Suppose that a homogeneous fluid body revolves about the axis AB, and is in equilibrio by the attraction of its particles and the centrifugal force; and let another mass of the same fluid, similar in its figure to the first body, revolve, in the same time, about the axis ab, similarly situated to AB: this latter body will also be in equilibrio.

Conceive that the body in equilibrio is divided into an indefinitely great number of thin level strata; and let the other body be divided into the same number of strata by surfaces similar, and similarly situated to the level surfaces of the first body. Take any point H in a level surface of the body in equilibrio; and in the corresponding surface of the other body, let the point h be similarly situated to H. Farther, suppose the two bodies are similarly divided into the same indefinitely great number of molecules, of which dm and dm' are any two situated alike, and therefore having their volumes and quantities of matter proportional to the volumes and quantities of matter of the two whole bodies: and let f and f' denote the respective distances of the points H and h from dm and dm', and r and r', their respective distances from the axes AB and ab.

The forces with which the molecules dm and dm' attract the points H and h (which must be considered as two equal particles of matter) are proportional to $\frac{dm}{f^2}$ and $\frac{dm'}{f'^2}$: and, in these fractions, the numerators being proportional to the

cubes, and the denominators to the squares, of any two homologous lines of the respective bodies, the attractive forces will be simply proportional to any two such lines. The lines f and f', in the directions of which the forces act, are likewise similarly inclined to the surfaces passing through the two points H and h. It follows, therefore, that the forces with which the similar molecules into which the two bodies are divided, attract the points H and h, are constantly in the same proportion to one another, and act in directions that make like angles with the surfaces passing through the same points. Farther, since the velocity of rotation is the same in the two bodies, the centrifugal forces urging the points H and h will be proportional to the respective distances from the axes A B and $a b$; that is, to r and r', or to any homologous lines of the respective bodies; and the same forces, having their directions in the prolongations of r and r', make like angles with the surfaces passing through H and h. Wherefore all the accelerating forces urging the points H and h, are respectively in the same proportion to one another, and have like inclinations to the surfaces passing through the same points. Consequently, the resultants of the same forces will follow the like proportion, namely, that of any homologous lines; and they will likewise be similarly inclined to the two surfaces. But the resultant of the accelerating forces acting at H, is perpendicular to the level surface passing through that point; wherefore, the resultant of the accelerating forces acting at h, is likewise perpendicular to the surface in which that point is placed, and has to the other resultant the same proportion of any two homologous lines of the respective bodies. And thus, as in the body in equilibrio, the gravity, or the resultant of the accelerating forces

is every where perpendicular to the level surfaces; so in the other body, the like force is every where perpendicular to the surfaces similarly situated.

Take K and k any other two points similarly situated in the same surfaces that contain H and h: and suppose that HM, KN, are the thicknesses of the level stratum, in the upper surface of which H and K are placed; and, in like manner, let hm, kn, be the thicknesses of the like stratum in the other body. Farther, put G, G′ for the resultants of the accelerating forces, or the gravitations, at H and K; and g, g' for the like forces at h, k. Because H and h are points similarly situated, the forces G, g are proportional to any homologous lines of the respective bodies. The same thing is true of the forces G′, g'. Wherefore,

$$G : G' :: g : g'.$$

But the line HM is homologous to hm, and KN, to kn: wherefore,

$$HM : KN :: hm : kn.$$

Consequently,

$$G \times HM : G' \times KN :: g \times hm : g' \times km.$$

But the proportion of G × HM to G′ × KN is equal to that of the pressures of the stratum upon the fluid below it at the points H and K: for the quantities of matter in the stratum are proportional to the thicknesses HM and KN; and the pressures are proportional to the gravitations multiplied by the quantities of matter. In like manner $g \times hm$ and $g' \times kn$, are proportional to the pressures of the stratum upon the fluid below it at the points h and k. Wherefore the pressures at H and K are proportional to the pressures at h and k. And, in general, taking any points similarly placed in the two corresponding surfaces, the pressures of the stratum upon the

fluid below it in one body are in the same proportion to one another, as the pressures of the stratum upon the fluid below it in the other body. But in the body in equilibrio, the pressures at all points are equal; wherefore, in the other body, a stratum likewise presses equably upon the fluid below it. And what is true of each individual stratum, must be true of the accumulated pressure of any number of superincumbent strata.

Thus, in the two bodies, every thing is similar. The forces which urge the particles of one, are, in the case of the other, all increased, or all diminished, in the same proportion, while they act in like directions. If, in the one, the gravity be every where perpendicular to the level surfaces; the like force is perpendicular to the surfaces similarly traced in the other: and if, in the first, all the level strata press equably upon the fluid below them; the same thing is true of the strata into which the second is divided. Wherefore, the equilibrium of one body is a necessary consequence of the equilibrium of the other.

Proposition II.

If a homogeneous fluid mass revolve about an axis, and be in equilibrio by the attraction of its particles in the inverse proportion of the square of the distance; all the level surfaces will be similar to the outer one: and any stratum of the fluid contained between two level surfaces will attract particles in the inside with equal force in opposite directions.

Suppose that the homogeneous fluid body R S T, revolving about the axis A B, is in equilibrio by the centrifugal force, and the attraction of its particles in the inverse proportion of

the square of the distance. The axis of rotation A B, will pass through G, the centre of gravity of the fluid mass. In the interior of the revolving body, trace, round the point G, any surface H I K, similar and similarly situated to the outer surface. Then the whole fluid body R S T, and the part of it bounded by the surface H I K, are similar to one another in their figure; and they revolve about the common axis A B, which cuts them both similarly: wherefore, because the first body is in equilibrio, the latter body will also be in equilibrio, supposing that it revolves by itself, the exterior matter being taken away, or annihilated.*

And, because the body H I K is in equilibrio, when it revolves by itself, the resultant of the forces acting at its surface (namely, the attraction of its own particles and the centrifugal force) will, at every point, be perpendicular to that surface.

Suppose now that all the fluid exterior to the surface H I K is divided into very thin strata by the surfaces O P Q, L M N, similar and similarly situated to the outer surface R S T. Then, understanding by the gravitation at any of the surfaces O P Q, L M N, &c. the resultant of the centrifugal force and the attraction of the fluid matter within that surface, it has been proved that these gravitations are perpendicular to the respective surfaces. Wherefore the uppermost stratum will be pressed perpendicularly upon the surface O P Q by the gravitation at that surface. For the same reason the next stratum will be pressed perpendicularly upon the surface L M N. And, in like manner, the successive strata will be pressed perpendicularly, each upon the surface on which

* Proposition I.

it lies, by the respective gravitations. If we conceive a curve line GHLOR, extending from the common centre G so as to cut all the similar surfaces at right angles; that curve line will mark the directions both of the gravitation and the pressure of the fluid in the interior of the body RST. Wherefore the several surfaces OPQ, LMN, &c. are no other than the level surfaces of the body RST in equilibrio; and each of these surfaces will be pressed by the superincumbent fluid with the same intensity over its whole extent.

But at the same time that the uppermost stratum presses upon the fluid below it, by the gravitation at the surface OPQ, it likewise attracts every particle of matter within the same surface. And, in like manner, every successive stratum both presses on the surface on which it lies by the gravitation at that surface, and attracts all the particles within it. Wherefore the body HIK is not only pressed by the superincumbent fluid, but every particle of it, is likewise attracted by all the exterior matter. These forces are independent on one another. Although the body HIK be in equilibrio with respect to the pressure it sustains, it does not follow that it will likewise be in equilibrio with respect to the attraction which the exterior matter exerts upon it. In order that this latter equilibrium take place, it is necessary that every stratum of the exterior matter be possessed of such a figure as to attract all particles in the inside with equal force in opposite directions.

We have now proved that, if the fluid mass RST be in equilibrio, the interior body HIK will likewise be in equilibrio when it revolves by itself, the exterior matter being taken away, or annihilated; which cannot be the case, unless

the same body H I K be in equilibrio with respect to the pressure and attraction which the exterior matter exerts upon it. It has likewise been proved that all the surfaces O P Q, L M N, &c. similar to the outer surface R S T, are level surfaces; and this ensures the equilibrium of the interior body H I K with respect to the pressure it sustains. Its equilibrium with respect to the attraction of the exterior matter, requires farther, that all the strata between the surfaces R S T, O P Q, L M N, &c. attract every particle within them equally in opposite directions. We are therefore to conclude that the homogeneous fluid body R S T, which revolves about the axis A B, and the particles of which attract one another in the inverse proportion of the square of the distance, cannot be in equilibrio, unless both these conditions be fulfilled; 1st. The level surfaces must be all similar to one another; 2dly. Every stratum contained between two level surfaces must attract particles in the inside with equal force in opposite directions.

In the Proposition that has just been proved, the similarity of the level surfaces is an accidental property connected with the supposed law of attraction. In the general hypothesis of an attractive power between the particles, the conditions of equilibrium are no more than these: 1st. The resultant of the accelerating forces acting at every point of the outer surface must be directed at right angles towards that surface: 2dly. All the level strata must possess such a figure as to attract particles in the inside with equal force in opposite directions. It may not be altogether superfluous to prove, by a synthetic demonstration, that these conditions are sufficient for the equilibrium. This is done in the following Proposition.

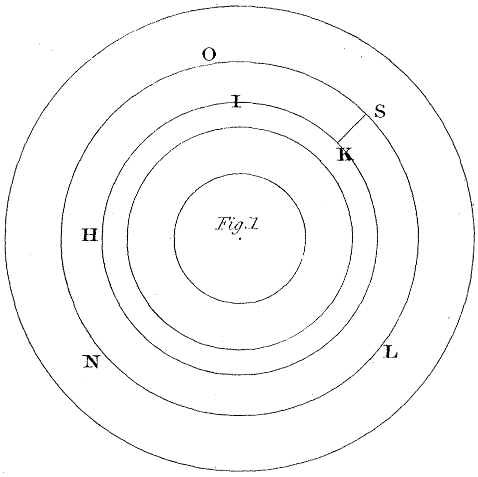

Fig. 1.

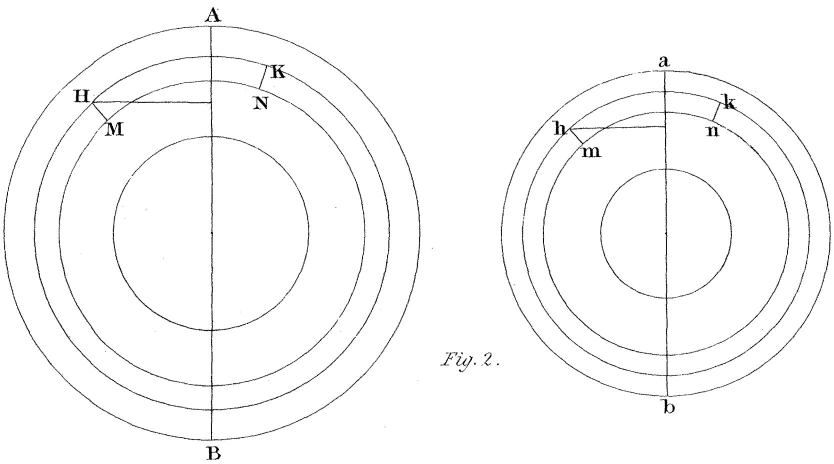

Fig. 2.

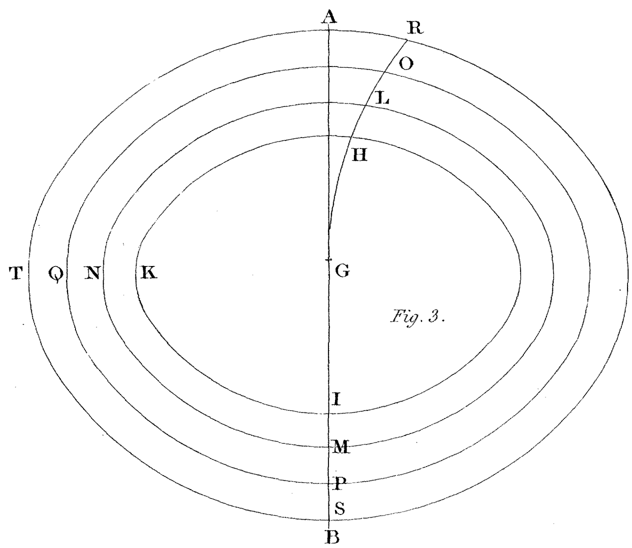

Fig. 3.

Proposition III.

If a homogeneous fluid mass fulfil the two above-mentioned conditions, it is in equilibrio.

Let the equation of the outer surface of the fluid mass, be

$$\varphi = C;$$

φ representing a function of three rectangular co-ordinates, x, y, z. Then the accelerating forces parallel to the axes of x, y, z, will be respectively equal to $\frac{d\varphi}{dx}, \frac{d\varphi}{dy}, \frac{d\varphi}{dz}$; and the condition that the resultant of these forces is perpendicular to the surface of the fluid will be expressed by the differential equation,

$$\frac{d\varphi}{dx} dx + \frac{d\varphi}{dy} dy + \frac{d\varphi}{dz} dz = 0.$$

We shall suppose that the whole fluid mass is divided into thin strata by level surfaces, which are determined by making the constant quantity C vary by insensible degrees in the equation of the outer surface. Farther, let the thickness of the uppermost stratum be denoted by the line k, drawn perpendicular to the outer surface from a point of which the co-ordinates are x, y, z; and let $x - \delta x$, $y - \delta y$, $z - \delta z$, be the co-ordinates of the other extremity of k in the under surface of the stratum. The equation of this surface will be found by substituting $x - \delta x$, $y - \delta y$, $z - \delta z$, in place of x, y, z, in the function φ, and by changing C into $C - \delta C$; it will therefore be,

$$\varphi - \frac{d\varphi}{dx}\delta x - \frac{d\varphi}{dy}\delta y - \frac{d\varphi}{dz}\delta z = C - \delta C:$$

and by subtracting this from the equation of the outer surface, we get

$$\frac{d\phi}{dx}\delta x + \frac{d\phi}{dy}\delta y + \frac{d\phi}{dz}\delta z = \delta C.$$

Again; if we put,

$$p = \sqrt{\left(\frac{d\phi}{dx}\right)^2 + \left(\frac{d\phi}{dy}\right)^2 + \left(\frac{d\phi}{dz}\right)^2};$$

then, because the line k is perpendicular to the outer surface, it follows, from the known properties of curve surfaces, that the cosines of the angles which k makes with the co-ordinates, are respectively equal to,

$$\frac{1}{p}\times\frac{d\phi}{dx};\ \frac{1}{p}\times\frac{d\phi}{dy};\ \frac{1}{p}\times\frac{d\phi}{dz}:$$

and hence,

$$\delta x = \frac{k}{p}\times\frac{d\phi}{dx};\ \delta y = \frac{k}{p}\times\frac{d\phi}{dy};\ \delta z = \frac{k}{p}\times\frac{d\phi}{dz}:$$

and by substituting these values in the preceding formula, we obtain,

$$\frac{k}{p}\times\left\{\left(\frac{d\phi}{dx}\right)^2 + \left(\frac{d\phi}{dy}\right)^2 + \left(\frac{d\phi}{dz}\right)^2\right\} = \frac{k}{p}\times p^2 = k\times p = \delta C.$$

Now p is the resultant of the accelerating forces at the surface: and the line k, or the thickness, is proportional to the quantity of matter in the stratum at the same point; wherefore $k\times p$ is the pressure; and the formula,

$$k\times p = \delta C,$$

shows that the uppermost stratum presses upon the fluid below it equally at all points.

As the attraction of the whole fluid mass is one of the component forces of the gravitation p, the attraction of the stratum must enter as a part of the same force. But it is evidently only an infinitely small part of it; and consequently produces only an infinitely small part of the pressure $p\times k$. We may therefore consider the gravitation at the

outer surface, and the pressure of the uppermost stratum upon the fluid below it, as both independent of the attractive force of the matter of the stratum. But the attraction of the same matter upon all the particles within the stratum is a force of the same order with the pressure $p \times k$, and comparable with it, and which must not be neglected. Thus it appears, that the uppermost stratum acts upon the fluid below it both by pressure and by attraction; and, as all the level strata are derived from one another by the same law, it follows, that every stratum in the interior likewise acts upon the fluid below it both by pressure and by attraction.

Now it has already been shown, that the pressure of the uppermost stratum is the same over all the surface of the fluid below it; and the same thing, it is manifest, is equally true of any level stratum. Wherefore, since the strata press equably upon one another, any fluid body in the interior, bounded by a level surface, will be in equilibrio with respect to the pressure it sustains from all the superincumbent strata. But, according to the second condition in the hypothesis of the proposition, the same body will also be in equilibrio with respect to the attraction of all the exterior strata. Thus, every interior fluid body bounded by a level surface, is in equilibrio with respect to all the forces which the exterior matter exerts upon it. And, as this is true independently of the dimensions of the interior body, we may suppose that it is ultimately reduced to a quantity infinitely small, which exerts no force, and is in equilibrio by the external forces acting upon it. Then the whole fluid mass will be resolved into level strata, that are in equilibrio with respect both to the pressures and to the attractive forces, which they exert

upon one another. We are therefore to conclude, that the two conditions of the proposition are sufficient to ensure the equilibrium of the fluid mass, and that both are necessary to produce the effect.

5. Having established the physical properties of a homogeneous fluid mass in equilibrio, the investigation of its figure, which is now brought within the power of analysis, is not attended with much difficulty.

If a homogeneous body of fluid revolve about an axis, and be in equilibrio by the centrifugal force and the attraction of its particles, the axis of rotation will pass through the centre of gravity. This point is to be supposed at rest; since it is not the effect of any external forces that we have to consider, but merely the mutual action of the particles upon one another. If one of the planes of the co-ordinates pass through the centre of gravity at right angles to the axis of rotation, the other two will intersect one another in the same axis. Let a, b, c, denote the co-ordinates of an attracted point (which must be considered as some small particle of the fluid containing a given quantity of matter) placed any where in the mass, a being parallel to the axis of rotation; and put $r = \sqrt{a^2 + b^2 + c^2}$. Suppose also that $V(r)$ denotes the sum of all the molecules of the body divided by their respective distances from the attracted point: then $\frac{d.V(r)}{da}$, $\frac{d.V(r)}{db}$, $\frac{d.V(r)}{dc}$, will be the accumulated attractive forces exerted upon the attracted point by the whole mass, in the directions of a, b, c, and tending to increase these lines. Again; let ω be the centrifugal force at the distance from the axis of rotation equal to unit; then, the distance of the attracted point from the same axis being equal to $\sqrt{b^2 + c^2}$, the centrifugal force

urging it from the axis, will be $\omega \times \sqrt{b^2 + c^2}$; and the effect of the same force to lengthen b and c, will be equal to $\omega \times \sqrt{b^2 + c^2} \times \frac{b}{\sqrt{b^2 + c^2}}$ and $\omega \times \sqrt{b^2 + c^2} \times \frac{c}{\sqrt{b^2 + c^2}}$, or to $\omega\, b$ and $\omega\, c$. Hence, the forces acting upon the attracted point, and tending to increase a, b, c, are respectively,

$$\frac{d\,.\,\mathrm{V}(r)}{d\,a},$$

$$\frac{d\,.\,\mathrm{V}(r)}{d\,b} + \omega\,.\,b,$$

$$\frac{d\,.\,\mathrm{V}(r)}{d\,b} + \omega\,.\,c.$$

Now, the resultant of these forces must be perpendicular to the level surface of which a, b, c are the co-ordinates; which condition is expressed by this differential equation, viz.

$$\frac{d\,.\,\mathrm{V}(r)}{d\,a}\,d\,a + \frac{d\,.\,\mathrm{V}(r)}{d\,b}\,d\,b + \frac{d\,.\,\mathrm{V}(r)}{d\,c}\,d\,c + \omega\,(b\,d\,b + c\,d\,c) = o:$$

and the integral of this, viz.

$$\mathrm{V}(r) + \frac{\omega}{2}(b^2 + c^2) = \mathrm{C},$$

is the general equation of all the level surfaces. Let μ denote the cosine of the angle which the line r makes with the axis of rotation; and the foregoing equation will become,

$$\mathrm{V}(r) + \frac{\omega}{2}\,.\,r^2\,(1 - \mu^2) = \mathrm{C}.$$

And if we put R for the radius of the outer surface of the fluid body, we shall have, for the equation of that surface,

$$\mathrm{V}(\mathrm{R}) + \mathrm{R}^2 \times \frac{\omega}{2}(1 - \mu^2) = \mathrm{C}, \ldots \text{(A)}$$

which is one condition of the equilibrium of the fluid mass.

The equation just found is an essential condition of the equilibrium, although it is not the only one. As it merely expresses that the resultant of all the accelerating forces is perpendicular to the fluid's surface, it is not confined to a homogeneous body, nor to one entirely fluid, but is true in

every case when a fluid in equilibrio covers, either entirely or partially, the surface of any body, however variable in structure or density. Now, from the equation, we get,

$$\frac{1}{R^2} = \frac{\frac{V(R)}{R^2} + \frac{\omega}{2}(1 - \mu^2)}{C};$$

and, as it has been proved in §. 1, that $\frac{V(R)}{R^2}$ is always a function of three rectangular co-ordinates of a point in the surface of a sphere, it follows that $\frac{1}{R^2}$, and consequently R, must be like functions. This inference, being founded on considerations of the most general nature, cannot but include every case of a fluid in equilibrio, placed upon the surface of a revolving body.

Again, suppose that $R_{,}$ is the radius of any level surface which contains the attracted point within it; and let $V_{,}(r)$ denote the sum of all the molecules of the fluid within the level surface, divided by their respective distances from the attracted point: then,

$$V(r) - V_{,}(r)$$

will be the sum of the molecules in the stratum of fluid contained between the outer surface and the level surface, divided by their respective distances from the attracted point; and the attractive forces of the stratum upon the attracted point in the directions of a, b, c, will be respectively,

$$\frac{d.\left\{V(r) - V_{,}(r)\right\}}{da},$$

$$\frac{d.\left\{V(r) - V_{,}(r)\right\}}{db},$$

$$\frac{d.\left\{V(r) - V_{,}(r)\right\}}{dc}.$$

Now, in the case of a homogeneous mass of fluid in equilibrio, these forces must be evanescent; and that too, for every point within the stratum, which requires that

$$V(r) - V_{,}(r),$$

shall be a constant quantity independent of a, b, c. And this is the remaining condition necessary to the equilibrium.

Now, if we put,

$$Q = r^2 f - \frac{dr}{r^3} \iint \frac{R'^3 \, d\mu' \, d\varpi'}{\sqrt{r^2 - 2rR'\gamma + R'^2}}$$

$$Q_{,} = r^2 f - \frac{dr}{r^8} \iint \frac{R'^3_{,} \, d\mu' \, d\varpi'}{\sqrt{r^2 - 2rR'_{,}\gamma + R'^2_{,}}};$$

we shall get,

$$V(r) = Q + Kr^2$$

$$V_{,}(r) = Q_{,} + K_{,}r^2 :^*$$

consequently,

$$V(r) - V_{,}(r) = Q - Q_{,} + r^2 (K - K_{,}).$$

A very little attention will show that Q and $Q_{,}$, contain no terms multiplied by r^2. For if we expand Q into a series of the ascending powers of r, the term containing r^2, will be

$$r^2 \times \iint C^{(2)} \, d\mu' \, d\varpi',$$

which, by the nature of the function $C^{(2)}$, is equal to zero. Wherefore the foregoing expression cannot be independent of a, b, c, unless

$$K - K_{,} = o; \text{ and } K = K_{,}.$$

But it has already been shown that the equality of K to $K_{,}$ requires that the radii R' and $R'_{,}$, which make the same angles with the axes of the co-ordinates, and consequently are in the same straight line, be constantly in the same proportion to one another :* and hence we obtain this property of a homo-

* Section 3.

geneous fluid mass in equilibrio, namely, that all the level surfaces are similar to the outer one.

Again, since R and $R'_{,}$ are always in the same proportion to one another, $Q - Q_{,}$ will be independent of a, b, c, if we make r disappear in Q. Now, by expanding the expression of Q, and equating the co-efficients of the several powers of r to zero, we get,

$$Q = \int\int R'^2 \, d\mu' \, d\varpi$$

$$o = \int\int R' . C^{(1)} d\mu' \, d\varpi'$$

$$o = \int\int \frac{C^{(3)} d\mu' \, d\varpi'}{R'}$$

$$o = \int\int \frac{C^{(4)} d\mu' \, d\varpi'}{R'}$$

and generally,

$$o = \int\int \frac{C^{(i)} d\mu' \, d\varpi'}{R'^{i-2}}$$

In the first place, all these equations are satisfied if we suppose R' constant; that is, if the figure of the fluid be a sphere. But the supposition of a sphere is inconsistent with the equation (A), unless ω be evanescent. Wherefore, a homogeneous fluid body of a spherical figure cannot be in equilibrio by the attraction of its particles, unless it have no rotatory motion.

Again, it follows from what has been shown, that R' is a function of μ', $\sqrt{1-\mu'^2}$. Sin ϖ, $\sqrt{1-\mu'^2}$. Cos ϖ'; μ' being the cosine of an arc θ' reckoned from the pole of a great circle on the sphere, and ϖ' the angle between θ' and a given great circle passing through the same pole. Now if we suppose that R' is an even function, or that it contains only the squares and the combinations of the squares of $\mu'_{,}$, $\sqrt{1-\mu'^2}$. Sin ϖ', $\sqrt{1-\mu'^2}$. Cos ϖ'; the values of R', which are always

positive, will be the same in quantity, at points diametrically opposite on the sphere, at which points μ', Sin. ϖ' and Cos. ϖ', are different only in their signs. And because,

$$\gamma = \mu\,\mu' + \sqrt{1-\mu^2}\,.\,\sqrt{1-\mu'^2}\,\text{Cos.}\,(\varpi - \varpi'),$$

it is obvious that γ will likewise have the same value and different signs at any two points diametrically opposite on the sphere; and the same property will belong to every function of γ that contains only the odd powers. Now we have

$$R'\,.\,C^{(1)}\,d\,\mu'\,d\,\varpi' = -R'\,.\,C^{(1)}\,d\,\theta'\,\text{Sin}\,\theta'\,d\,\varpi'\,;$$

and, as θ' increases from o to π, and ϖ' from o to 2π, it is obvious that the fluxions will be the same in quantity, but will have different signs, at any two points diametrically opposite on the sphere; because the sign of $C^{(1)}$, which is an odd function of γ, alone changes. Wherefore the integral will decrease just as much in one hemisphere as it increases in the other; and being extended to the whole sphere, it will be equal to zero. In the same manner it is proved that

$$\iint \frac{C^{(i)}\,d\,\mu'\,d\,\varpi'}{R'^{\,i-2}} = o,$$

whenever i is an odd number. Thus all the equations we are considering, in which i is an odd number, are satisfied by the supposition that R' is an even function of μ', $\sqrt{1-\mu'^2}\,.$ Sin ϖ', $\sqrt{1-\mu'^2}\,.$ Cos ϖ'.

It remains to consider the cases when i is an even number, viz.

$$\iint \frac{C^{(4)}\,d\,\mu'\,d\,\varpi'}{R'^2} = o$$

$$\iint \frac{C^{(6)}\,d\,\mu'\,d\,\varpi'}{R'^4} = o$$

$+$ &c.

For this purpose the following theorem is premised, viz.

Theorem. If m, m', m'' denote any positive integer numbers, such that $m + m' + m''$ is less than i; then

$\int\int \mu'^{m} . (\sqrt{1-\mu'^2} . \text{Sin.}\,\varpi')^{m'} . (\sqrt{1-\mu'^2} . \text{Cos.}\,\varpi')^{m''} . C^{(i)} d\mu' d\varpi' = 0,$

the integral being extended from $\mu' = 1$ to $\mu' = -1$, and from $\varpi' = 0$ to $\varpi' = 2\pi$.

As expressions of this kind have been very amply discussed, and as the theorem follows very readily from the properties generally known, I shall not stop to give the demonstration.

It follows from the theorem that the equation,

$$\int\int \frac{C(4) d\mu' d\varpi'}{R'^2} = 0,$$

cannot be true, if $\frac{1}{R'^2}$ contain any even power of the quantities μ', $\sqrt{1-\mu'^2} . \text{Sin.}\,\varpi'$, $\sqrt{1-\mu'^2} . \text{Cos.}\,\varpi'$, above the square; or if it contain any product of two or more of the squares of the same quantities. Wherefore the most general value of $\frac{1}{R'^2}$, consistent with the above equation, is

$$\frac{1}{R'^2} = A\mu'^2 + B(1-\mu'^2)\,\text{Sin.}^2\,\varpi' + C(1-\mu'^2)\,\text{Cos.}^2\,\varpi'.$$

It may be observed, that this expression would not be more general by adding an absolute quantity, as D: for, since

$$\mu'^2 + (1-\mu'^2)\,\text{Sin.}^2\,\varpi' + (1-\mu'^2)\,\text{Cos.}^2\,\varpi' = 1,$$

such a quantity would blend itself with the other terms. But the same value of $\frac{1}{R'^2}$ will likewise satisfy all the equations,

$$\int\int \frac{C^{(i)} d\mu' d\varpi'}{R'^{i-2}} = 0,$$

in which i is an even number. For, because

$$\frac{1}{R'^{i-2}} = (A\mu'^2 + B(1-\mu'^2)\,\text{Sin.}^2\,\varpi' + C(1-\mu'^2)\,\text{Cos.}^2\,\varpi')^{\frac{i-2}{1}},$$

it follows that the expansion of $\frac{1}{R^{i-2}}$ will produce no quantities in the integral except such as are evanescent by the theorem.

The most general value of $\frac{1}{R'^2}$, consistent with one of the conditions of the equilibrium, has now been found. If we write $\frac{1}{k^2}, \frac{1}{k'^2}, \frac{1}{k''^2}$, for A, B, C, we shall get,

$$\frac{1}{R'^2} = \frac{\mu'^2}{k^2} + \frac{(1-\mu'^2)\,\mathrm{Sin.}^2\,\varpi'}{k'^2} + \frac{(1-\mu'^2)\,\mathrm{Cos.}^2\,\varpi'}{k''^2};$$

an equation which belongs to an ellipsoid of which k, k', k'', are the three semi-axes. It is therefore proved that a homogeneous fluid mass cannot be in equilibrio by the attraction of its particles, and a centrifugal force of rotation, unless its figure be included in the ellipsoids. But it is still to be shown that the same figure is consistent with the other condition of the equilibrium. For the sake of abridging, put

$$S = \mu'^2 + \frac{k^2}{k'^2}(1-\mu'^2)\,\mathrm{Sin.}^2\,\varpi' + \frac{k^2}{k''^2}(1-\mu'^2)\,\mathrm{Cos.}^2\,\varpi';$$

then, $R'^2 = \frac{k^2}{S}$; and $R' = \frac{k}{\sqrt{S}}$.

The value of Q being reduced to the first term of its expansion, we have,

$$Q = \int\!\!\int R'^2\,d\mu'\,d\varpi' = k^2.\int\!\!\int \frac{d\mu'\,d\varpi'}{S}:$$

and hence,

$$V(R) = Q + K.R^2 = k^2.\int\!\!\int \frac{d\mu'\,d\varpi'}{S} + KR^2.$$

Let this value be substituted in the equation (A), and we shall obtain,

$$R^2 = \frac{k^2\int\!\!\int \frac{d\mu'\,d\pi'}{S} - C}{K + \frac{\omega}{2}(1-\mu^2)}$$

And the solution of the problem is now reduced to show that this formula is similar to the equation

$$R'^2 = \frac{k^2}{S}.$$

When this is done, the relation between the figure of the fluid mass and the given rotatory velocity will be found by making the two expressions of R and R′ coincide, so that both shall belong to the same surface.

In the first place, the integral in the numerator of the value of R^2 is a function of k, k', k''; and as any value may be assigned to C, the whole numerator may be regarded as an arbitrary quantity. The denominator is therefore all that remains to be considered. Now,

$$K = -\frac{2\pi}{3} + \iint \text{Log. } R' \times C^{(2)} d\mu' d\varpi';$$

and, since Log. $R' =$ Log. $k - \frac{1}{2}$ Log. S, we shall get,

$$K = -\frac{2\pi}{3} + \tfrac{1}{2}\iint - \text{Log. } S \times C^{(2)} d\mu' d\varpi',$$

because, $\quad \text{Log. } k \times \iint C^{(2)} d\mu' d\varpi' = 0.$

Again, $\quad C^{(2)} = \frac{1}{2.4} \cdot \frac{d^2.(1-\gamma^2)^2}{d\gamma^2} = \frac{3}{2}\gamma^2 - \frac{1}{2};$

and, $\gamma = \mu\mu' + \sqrt{1-\mu^2} \cdot \sqrt{1-\mu'^2} \cdot \text{Cos. } (\varpi - \varpi');$

or, if we write $m, n, p,$ for μ, $\sqrt{1-\mu^2}$. Sin. ϖ, $\sqrt{1-\mu^2}$. Cos. ϖ, we shall obtain,

$$\gamma = m\mu + n\sqrt{1-\mu'^2} \text{ . Sin. } \varpi + p\sqrt{1-\mu'^2} \text{ . Cos. } \varpi':$$

and hence,

$$\begin{aligned} K = -\frac{2\pi}{3} &+ \frac{m^2}{2}\iint - \left(\frac{3}{2}\mu'^2 - \frac{1}{2}\right) \text{Log. S . } d\mu' d\varpi' \\ &+ \frac{n^2}{2}\iint - \left\{\frac{3}{2}(1-\mu'^2)\text{ Sin.}^2\varpi' - \frac{1}{2}\right\} \text{Log.S.} d\mu' d\varpi' \\ &+ \frac{p^2}{2}\iint - \left\{\frac{3}{2}(1-\mu'^2)\text{ Cos.}^2\varpi' - \frac{1}{2}\right\} \text{Log.S.} d\mu' d\varpi' \\ &+ 3mn\iint - \mu'\sqrt{1-\mu'^2}\text{. Sin. } \varpi' \text{ Log. S. } d\mu' d\varpi' \\ &+ 3mp\iint - \mu'\sqrt{1-\mu'^2}\text{. Cos. } \varpi' \text{ Log. S . } d\mu' d\varpi' \\ &+ 3np\iint - (1-\mu'^2)\text{ Sin.}\varpi' \text{ Cos.}\varpi' \text{ Log. S.} d\mu' d\varpi'. \end{aligned}$$

But, because $\mu' = \text{Cos.}\,\theta'$, the three last integrals will become,

$$\int\!\!\int \text{Sin.}^2\,\theta'\ \text{Cos.}\,\theta'\ \text{Sin.}\ \varpi'\ \text{Log. S}\,.\,d\,\theta'\,d\,\varpi'$$

$$\int\!\!\int \text{Sin.}^2\,\theta\ \text{Cos.}\,\theta'\ \text{Cos.}\ \varpi'\ \text{Log. S}\,.\,d\,\theta'\,d\,\varpi'$$

$$\int\!\!\int \text{Sin.}^3\,\theta'\ \text{Sin.}\ \varpi'\ \text{Cos.}\,\varpi'\ \text{Log. S}\,.\,d\,\theta'\,d\,\varpi':$$

and, attending to the expression of S, it will follow that, in the two first integrals, if we suppose ϖ' to remain constant and θ' to vary from o to 180°; the fluxions will be equal, but will have different signs, at equal distances from o and 180°: wherefore the integrals, taken between the prescribed limits, are evanescent. In the third integral, if we suppose θ' to remain constant and ϖ' to vary from o to 360°, the fluxions will be equal, but will have different signs, at equal distances from o and 180° in the first semicircle: and at equal distances from 180° and 360° in the second semicircle: wherefore the whole integral is evanescent. Rejecting therefore the three last terms of the value of K, and representing the three remaining integrals by L, M, N, we shall get,

$$\begin{aligned} K + \tfrac{\omega}{2}(1 - \mu^2) = {} & \left(\tfrac{L}{2} - \tfrac{3}{2}\pi\right) . \mu^2 \\ & + \left(\tfrac{M}{2} - \tfrac{3}{2}\pi + \tfrac{\omega}{2}\right) . (1 - \mu^2)\ \text{Sin.}^2\,\varpi \\ & + \left(\tfrac{N}{2} - \tfrac{3}{2}\pi + \tfrac{\omega}{2}\right) . (1 - \mu^2)\ \text{Cos.}^2\,\varpi. \end{aligned}$$

This is the denominator in the formula for R^2, and it is entirely similar to the expression of S. Wherefore the two values of R^2 and R'^2 are alike in point of form; and the figure of the fluid mass that corresponds to the given rotatory velocity will be determined by making them coincide.

We are now to conclude that a homogeneous fluid mass cannot be in equilibrio by the attraction of its particles and a centrifugal force of rotation, unless it have the figure of an

ellipsoid; and farther, that an ellipsoid may be found that will fulfil all the conditions of the equilibrium, unless there be some cases in which the necessary relations between the figure and the given velocity of rotation, lead to equations that cannot be solved.

6. In order to apply the foregoing solution, it becomes necessary to compute the integrals L, M, N; or, in other words, to find the attractive forces of an ellipsoid upon a point in the surface. If we extend the problem generally to a point within or without the figure, it is attended with some difficulty; and it is usual to deduce the latter case from the former, which is more easily solved. There is however a great analogy between the two cases; or rather the distinction between them may be dispensed with; since the supposition of a point within the figure is equivalent to that of a point in the surface, which is the extreme case of a point without the figure. In this view the problem admits of a general solution deducible, by a short analysis, from the transformations used in this Paper.

Suppose that k, k', k'', represent the semi-axes of an ellipsoid; and let x, y, z, respectively parallel to the axes, denote the three co-ordinates of a point in the surface of the figure. Farther let a, b, c, be the co-ordinates of an attracted point without the figure; and conceive another ellipsoid, the surface of which passes through the attracted point, and which has its principal sections in the same planes with the principal sections of the given ellipsoid, and also the differences of the squares of its semi-axes h, h', h'', equal to the differences of the squares of k, k', k''; that is, $h'^2 - h^2 = k'^2 - k^2$, and $h''^2 - h^2 = k''^2 - k^2$. The equations of the two curve surfaces will thus be,

$$\frac{x^2}{k^2}+\frac{y^2}{k'^2}+\frac{z^2}{k''^2}=1$$

$$\frac{a^2}{h^2}+\frac{b^2}{h'^2}+\frac{c^2}{h''^2}=1,$$

$$h'^2-h^2=k'^2-k^2;\ h''^2-h^2=k''^2-k^2.$$

Again, as in the former part of this Paper, let

$$x=\text{R}\,\mu' \qquad a=r\,\mu$$

$$y=\text{R}\sqrt{1-\mu'^2}.\,\text{Sin.}\,\varpi', \qquad b=r\sqrt{1-\mu^2}\,\text{Sin.}\,\varpi$$

$$z=\text{R}\sqrt{1-\mu'^2}.\,\text{Cos.}\,\varpi', \qquad c=r\sqrt{1-\mu^2}\,\text{Cos.}\,\varpi\,;$$

then R and r are respectively radii of the two ellipsoids.

Farther, assume

$$\frac{h}{k}\,x=\frac{h}{k}\,\text{R}\,\mu'=\text{R}'p'$$

$$\frac{h'}{k'}\,y=\frac{h'}{k'}\,\text{R}\sqrt{1-\mu'^2}\,\text{Sin.}\,\varpi'=\text{R}'\sqrt{1-p'^2}\,\text{Sin.}\,q'$$

$$\frac{h''}{k''}\,z=\frac{h''}{k''}\,\text{R}\sqrt{1-\mu'^2}\,\text{Cos.}\,\varpi'=\text{R}'\sqrt{1-p'^2}\,\text{Cos.}\,q'$$

$$\frac{k}{h}\,a=\frac{k}{h}\,r\,\mu=r'\,p$$

$$\frac{k'}{h'}\,b=\frac{k'}{h'}\,r\sqrt{1-\mu^2}\,\text{Sin.}\,\varpi=r'\sqrt{1-p^2}\,\text{Sin.}\,q$$

$$\frac{k''}{h''}\,c=\frac{k''}{h''}\,r\sqrt{1-\mu^2}\,\text{Cos.}\,\varpi=r'\sqrt{1-p^2}\,\text{Cos.}\,q.$$

From these formulæ we get

$$\text{R}'^2=\frac{h^2}{k^2}\,x^2+\frac{h'^2}{k'^2}\,y^2+\frac{h''^2}{k''^2}\,z^2,$$

$$\text{R}^2=x^2+y^2+z^2;$$

and hence,

$$\text{R}'^2-\text{R}^2=\left\{\frac{x^2}{k^2}+\frac{y^2}{k'^2}+\frac{z^2}{k''^2}\right\}(h^2-k^2)=h^2-k^2.$$

In like manner, it is shown that

$$r^2-r'^2=h^2-k\,.$$

Wherefore,

$$\text{R}^2+r^2=\text{R}'^2+r'^2.$$

And again,

$$a x + b y + c z = \mathrm{R} r \gamma = \mathrm{R}' r' \gamma';$$

$$\gamma = \mu \mu' + \sqrt{1-\mu^2} . \sqrt{1-\mu'^2} \text{ Cos. } (\varpi - \varpi')$$

$$\gamma' = p p' + \sqrt{1-p^2} . \sqrt{1-p'^2} \text{ Cos. } (q - q').$$

Wherefore,

$$\mathrm{R}^2 - 2 \mathrm{R} r \gamma + r^2 = \mathrm{R}'^2 - 2 \mathrm{R}' r' \gamma' + r'^2.$$

Farther, from the assumed equations we readily derive these values, viz.

$$\text{Tan. } \varpi' = \frac{h''}{h'} . \frac{k'}{k''} . \text{ Tan. } q',$$

$$\frac{\mu'}{\sqrt{1-\mu'^2}} = \frac{p'}{\sqrt{1-p'^2}} . \frac{k}{h} . \frac{h' h''}{\mathrm{M}},$$

$$\mathrm{M} = \sqrt{k'^2 h''^2 \text{ Sin.}^2 q' + h'^2 k''^2 \text{ Cos.}^2 q'}$$

$$\mathrm{R} \sqrt{1-\mu'^2} = \mathrm{R}' \sqrt{1-p'^2} . \frac{\mathrm{M}}{h' h''}.$$

And, if we now take the fluxions of the first and second of these formulæ; observing, in the second operation, to make q', and consequently M, constant; we shall get,

$$d \varpi' = d q' . \frac{k' k'' . h' h''}{\mathrm{M}^2},$$

$$\frac{d \mu'}{(1-\mu'^2)^{\frac{3}{2}}} = \frac{d p'}{(1-p'^2)^{\frac{3}{2}}} . \frac{k h' h''}{h \mathrm{M}}$$

$$\mathrm{R}^3 (1-\mu'^2)^{\frac{3}{2}} = \mathrm{R}'^3 (1-p'^2)^{\frac{3}{2}} . \frac{\mathrm{M}^3}{h'^3 h''^3}.$$

And, by combining these formulæ, we obtain,

$$\mathrm{R}^3 d \mu' d \varpi' = \frac{k k' k''}{h h' h''} . \mathrm{R}'^3 d p \; d q'.$$

But, in § 3 it has been shown, that,

$$-\left(\frac{d . \frac{\mathrm{V}(r)}{r^2}}{d r}\right) r^3 = \iint \frac{\mathrm{R}^3 d \mu' d \varpi'}{\sqrt{\mathrm{R}^2 - 2 \mathrm{R} r \gamma + r^2}}:$$

and hence, by substitution,

$$-\left(\frac{d . \frac{\mathrm{V}(r)}{r^2}}{d r}\right) r^3 = \frac{k k' k''}{h h' h''} . \iint \frac{\mathrm{R}'^3 d p' d q'}{\sqrt{\mathrm{R}'^2 - 2 \mathrm{R}' r' \gamma' + r'^2}}.$$

Now, in the equation of the ellipsoid of which k, k', k'' are the axes, if we substitute $\frac{R' p'}{h}$, $\frac{R' \sqrt{1-p'^2}\,\text{Sin.}\,q'}{h'}$, $\frac{R' \sqrt{1-p'}\,\text{Cos.}\,q'}{h''}$ for the equivalent quantities, $\frac{x}{k}$, $\frac{y}{k'}$, $\frac{z}{k''}$; we shall obtain,

$$\frac{1}{R'^2} = \frac{p'^2}{h^2} + \frac{(1-p'^2)\,\text{Sin.}^2\,q'}{h'^2} + \frac{(1-p'^2)\,\text{Cos.}^2\,q'}{h''}.$$

And, by a like procedure in the other ellipsoid, we obtain,

$$\frac{1}{r'^2} = \frac{p^2}{k^2} + \frac{(1-p^2)\,\text{Sin.}^2\,q}{k'^2} + \frac{(1-p^2)\,\text{Cos.}^2\,q}{h''^2}.$$

Thus R′ is a radius of the ellipsoid that passes through the attracted point, and r' a radius of the given ellipsoid which is entirely within the first figure. The last integral may therefore be developed in a series of the ascending powers of r': and then, applying the same reasoning as in the former case of the developement of Q,* it will be found that all the terms are evanescent, except the first. Thus the general term of the expansion is,

$$r'^i \times \iint \frac{C^{(i)}\,dp'\,dq'}{R'^{i-2}}:$$

and, when i is an odd number, this integral is equal to zero; because the increment at any point on the surface of the sphere, is just equal to the decrement at the point diametrically opposite: and, when i is an even number, it is evanescent; because $\frac{1}{R'^{i-2}}$ contains no terms except such as are evanescent in the integral, according to the theorem in § 5. Wherefore, the integral being reduced to the first term of its expansion, we get,

$$-\left(\frac{d\,.\,\frac{V(r)}{r^2}}{dr}\right) r^3 = \iint R'^2\,dp'\,dq';$$

p' and q' being taken between the same limits, as μ' and ϖ'.

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Now, we have

$$R'^2 = \frac{h^2}{p'^2 + \frac{h^2}{h'^2}(1-p'^2)\,\text{Sin.}^2 q' + \frac{h^2}{h''^2}(1-p'^2)\,\text{Cos.}^2 q'};$$

or, which is the same thing,

$$R'^2 = \frac{h^2\,h'^2\,h''^2}{h''^2\,(h^2+e^2p'^2)\,\text{Sin.}^2 q' + h'^2\,(h^2+e'^2p'^2)\,\text{Cos.}^2 q'},$$

$$h'^2 - h^2 = e^2,\ h''^2 - h^2 = e'^2;$$

wherefore

$$\iint R'^2\,dp'\,dq' = \iint \frac{h^2\,h'^2\,h''^2\,.\,dp'\,dq'}{h''^2\,(h^2+e^2p'^2)\,\text{Sin.}^2 q' + h'^2\,(h^2+e'^2p'^2)\,\text{Cos.}^2 q'}.$$

This expression is now integrable with regard to q': and we get, between the limits $q' = o$ and $q' = 2\,\pi$,

$$\iint R'^2\,dp'\,dq' = 2\,\pi\,.\,h\,h'\,h''.\int \frac{h\,dp'}{\sqrt{(h^2+e^2p'^2)\,.\,(h^2+e'^2p'^2)}}.$$

The integral now found increases as much, while p' decreases from 1 to o, as it does, while p' decreases from o to -1: wherefore the whole value will be the same, if we make p' vary between the limits 1 and o; and then double the result: thus,

$$\iint R'^2\,dp'\,dq' = 4\,\pi\,.\,h\,h'\,h''.\int \frac{h\,dp'}{\sqrt{(h^2+e^2p'^2)\,.\,(h^2+e'^2p'^2)}}.$$

Finally put $\frac{p'}{h} = \frac{1}{x}$; then we get,

$$\iint R'^2\,dp'\,dq' = 4\,\pi\,.\,h\,h'\,h''.\int \frac{-dx}{\sqrt{(x^2+e^2)\,(x^2+e'^2)}};$$

the limits of x being, $x = h$ and $x = \infty$.

Wherefore, by substitution, we get,

$$-\left(\frac{d\,.\,\frac{V(r)}{r^2}}{dr}\right)r^3 = 4\,\pi\,.\,k\,k'\,k''.\int \frac{-dx}{\sqrt{(x^2+e^2)\,(x^2+e'^2)}}.$$

Now, multiply by $-\frac{dr}{r^3}$, and then integrate; and, having multiplied by r^2 after the integration, we shall obtain,

$$V(r) = 2\,\pi\,.\,k\,k'\,k''.\int \frac{-dx}{\sqrt{(x^2+e^2)\,(x^2+e'^2)}}$$

$$-\,2\,\pi\,.\,k\,k'\,k''.\,r^2.\int \frac{1}{r^2}\cdot\frac{-dx}{\sqrt{(x^2+e^2)\,(x^2+e'^2)}}.$$

In the expression under the sign of integration, r increases from its value at the given attracted point till it becomes infinitely great; the angles which it makes with the axes of the co-ordinates remaining constantly the same. But if we substitute the values of a, b, c in the equation,

$$\frac{a^2}{h^2} + \frac{b^2}{h'^2} + \frac{c^2}{h''^2} = 1;$$

we shall get,

$$\frac{1}{r^2} = \frac{\mu^2}{h^2} + \frac{(1-\mu^2)\,\text{Sin.}^2\,\varpi}{h^2+e^2} + \frac{(1-\mu^2)\,\text{Cos.}^2\,\varpi}{h^2+e'^2};$$

and, by writing x^2 for h^2,

$$\frac{1}{r^2} = \frac{\mu^2}{x^2} + \frac{(1-\mu^2)\,\text{Sin.}^2\,\varpi}{x^2+e^2} + \frac{(1-\mu^2)\,\text{Cos.}^2\,\varpi}{x^2+e'^2};$$

in which expression, μ and ϖ remaining the same, x will vary from h to be infinite, while r increases from its value at the given attracted point to be infinite. Wherefore, by substitution, we get,

$$\begin{aligned} V(r) &= 2\pi\,.\,k\,k'\,k''.\int\frac{-dx}{(x^2+e^2)^{\frac{1}{2}}(x^2+e'^2)^{\frac{1}{2}}} \\ &- 2\pi\,.\,k\,k'\,k''.\,r^2\mu^2.\int\frac{-dx}{x^2(x^2+e^2)^{\frac{1}{2}}(x^2+e'^2)^{\frac{1}{2}}} \\ &- 2\pi\,.\,k\,k'\,k''.\,r^2(1-\mu^2)\,\text{Sin.}^2\,\varpi.\int\frac{-dx}{(x^2+e^2)^{\frac{3}{2}}(x^2+e'^2)} \\ &- 2\pi\,.\,k\,k'\,k''.\,r^2(1-\mu^2)\,\text{Cos.}^2\,\varpi.\int\frac{-dx}{(x^2+e^2)^{\frac{1}{2}}(x+e'^2)^{\frac{3}{2}}}. \end{aligned}$$

Let M denote the mass of the ellipsoid, then $M = \frac{4}{3}\pi\,.\,k\,k'\,k''$: wherefore, by substituting a, b, c, for the equivalent quantities, we finally get,

$$\begin{aligned} V(r) &= \frac{3M}{2}\,.\int\frac{-dx}{(x^2+e^2)^{\frac{1}{2}}(x^2+e'^2)^{\frac{1}{2}}} \\ &- a^2\,.\,\frac{3M}{2}\,.\int\frac{-dx}{x^2(x^2+e^2)^{\frac{1}{2}}(x^2+e'^2)^{\frac{1}{2}}} \\ &- b^2\,.\,\frac{3M}{2}\,.\int\frac{-dx}{(x^2+e^2)^{\frac{3}{2}}(x+e'^2)^{\frac{1}{2}}} \\ &- c^2\,.\,\frac{3M}{2}\,.\int\frac{-dx}{(x^2+e^2)^{\frac{1}{2}}(x^2+e'^2)^{\frac{3}{2}}} \end{aligned}$$

To this expression we must join the equation of the surface of the ellipsoid that passes through the attracted point, viz.

$$\frac{a^2}{h^2} + \frac{b^2}{h^2 + e^2} + \frac{c^2}{h^2 + e'^2} = 1,$$

by means of which h, the limit of x in the several integrals, is to be determined. When the attracted point is in the surface of the given ellipsoid, it is plain that $h = k$; and the limit of x is, therefore, one of the semi-axes.

Thus an expression of $V(r)$ has been found, that is general for all positions of the attracted point; nothing more being requisite than to determine the limit of x in every particular case. The several integrals are closely connected with one another; they are in forms well known to geometers, and susceptible of many transformations; but, in a general solution, it seems most simple to leave the expression as it is above exhibited.

But although a general expression of $V(r)$ has been found, yet it does not immediately make known the attractive forces acting upon a point. These forces, estimated in directions parallel to the co-ordinates, are represented by the partial fluxions of $V(r)$ relatively to the co-ordinates; but, in performing the operations, it must be observed that x is a function of r, and consequently of the co-ordinates. Thus the attractions of the ellipsoid, respectively parallel to a, b, c, are equal to,

$$-\frac{d \,.\, V(r)}{da} - \frac{d \,.\, V(r)}{dx} \cdot \frac{dx}{da}$$

$$-\frac{d \,.\, V(r)}{db} - \frac{d \,.\, V(r)}{dx} \cdot \frac{dx}{db}$$

$$-\frac{d \,.\, V(r)}{dc} - \frac{d \,.\, V(r)}{dx} \cdot \frac{dx}{dc}.$$

But, according to the foregoing value of $V(r)$,

$$\frac{d \,.\, V(r)}{dx} = \frac{\frac{3}{2} M}{\sqrt{(x^2 + e^2)(x^2 + e'^2)}} \cdot \left\{ -1 + \frac{a^2}{x^2} + \frac{b^2}{x^2 + e^2} + \frac{c^2}{x^2 + e'^2} \right\}:$$

and as this quantity is to be valued at the limit, or when $x = h$, we have $\frac{d . V(r)}{dx} = o$. Wherefore the expressions of the attractive forces are reduced to, $-\frac{d . V(r)}{da}$, $-\frac{d . V(r)}{db}$, $-\frac{d . V(r)}{dc}$; that is, to the partial fluxions of $V(r)$, supposing that x is independent of the co-ordinates, a, b, c.

The oblate ellipsoid of revolution, corresponds to the supposition, $e^2 = e'^2$; and, in this case, we get,

$$V(r) = \tfrac{3}{2} M . \int \frac{-dx}{x^2 + e^2} - a^2 . \tfrac{3}{2} M . \int \frac{-dx}{x^2 (x^2 + e^2)}$$
$$- (b^2 + c^2) . \tfrac{3}{2} M . \int \frac{-dx}{(x^2 + e^2)^2}:$$

the equation for finding h, the limit of x, being,

$$\frac{a^2}{h^2} + \frac{b^2 + c^2}{h^2 + e^2} = 1.$$

And, when the attracted point is in the surface of the oblate spheroid of revolution, h is equal to k; and, if we put $\lambda = \frac{e}{k}$, we get,

$$V(r) = \frac{3M}{2k} . \frac{\text{Arc. Tan. } \lambda}{\lambda} - a^2 . \frac{3M}{2k^3} . \frac{\lambda - \text{Arc. Tan. } \lambda}{\lambda^3} - (b^2 + c^2) . \frac{3M}{4k^3} . \frac{\text{Arc. Tan. } \lambda - \frac{\lambda}{1 + \lambda^2}}{\lambda^3}.$$

7. It may not be improper to apply the foregoing solution to find the relation between the figure and the velocity of rotation in the case of an oblate ellipsoid of revolution. As it has been proved that the supposed figure will satisfy one of the conditions of equilibrium,* nothing more is requisite than to employ the other condition, namely, that contained in the equation (A), to determine the relation sought.

Let k be the semi-axis of revolution, and $\sqrt{k^2 + e^2}$, the radius of the equator; if a, b, c be three rectangular co-ordinates of a point in the surface, a being parallel to k, the equation of the figure will be,

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$$\frac{a^2}{k^2} + \frac{b^2 + c^2}{k^2 + e^2} = 1.$$

Put $a^2 = R^2 \mu^2$; $b^2 + c^2 = R^2 (1 - \mu^2)$; $e^2 = k^2 \times \lambda^2$; then, by substitution, we shall get,

$$R^2 (1 + \lambda^2 \mu^2) = k^2 (1 + \lambda^2).$$

Again, from the formula (A), we get

$$V(R) + R^2 . \frac{\omega}{2} (1 - \mu^2) = C,$$

ω being the centrifugal force at the distance 1 from the axis of rotation. This equation must be made identical with the former one, and for this purpose we must substitute in it the value of V (R) reduced to a proper form. Now, M being the mass of the spheroid, we have

$$M = \frac{4\pi}{3} k (k^2 + e^2) = \frac{4\pi}{3} k^3 (1 + \lambda^2);$$

and for the sake of abridging, if we put,

$$A = \frac{\lambda - \text{Arc. Tan.}\, \lambda}{\lambda^3}$$

$$B = \frac{\text{Arc. Tan.}\, \lambda - \frac{\lambda}{1 + \lambda^2}}{2\lambda^3};$$

and likewise attend to the values of a^2 and $b^2 + c^2$, we shall get by the formula in § 6,

$$V(R) = 2\pi k^2 . \frac{(1 + \lambda^2)\, \text{Arc. Tan.}\, \lambda}{\lambda^2} - R^2 . 2\pi \left\{ (1 + \lambda^2) B + (1 + \lambda^2)(A - B) . \mu^2 \right\}:$$

and hence, by substitution, the equation we are considering, when brought to the same form of expression as the first one, will become,

$$R^2 . \left\{ 1 + \frac{(1 + \lambda^2)(A - B) + \frac{\omega}{4\pi}}{(1 + \lambda^2) B - \frac{\omega}{4\pi}} . \mu^2 \right\} = \frac{k^2 (1 + \lambda)^2 \frac{\text{Arc. Tan.}\, \lambda}{\lambda} - \frac{C}{2\pi}}{(1 + \lambda^2) B - \frac{\omega}{4\pi}}.$$

By comparing the two equations, it will appear that the only condition necessary to make them identical, is this, viz.

$$\frac{(1+\lambda^2)(A-B)+\frac{\omega}{4\pi}}{(1+\lambda^2)B-\frac{\omega}{4\pi}}=\lambda^2;$$

for the terms on the right-hand sides will be made to coincide by giving a proper value to the arbitrary quantity C. Hence,

$$\frac{\omega}{4\pi}=(1+\lambda^2)B-A.$$

Now put,

$$\frac{\omega}{\frac{4}{3}\pi}=q;$$

then, restoring the values of A and B, we shall obtain,

$$\frac{2}{9}q=\frac{1}{3}-\left(\frac{1}{3}+\frac{1}{\lambda^2}\right).\left(1-\frac{\text{Arc. Tan. }\lambda}{\lambda}\right).$$

From this formula it appears that $q=o$, both when λ is equal to zero and when it is infinitely great. There is therefore no rotatory motion in either of the extreme cases, when the oblateness is nothing, and when it is infinite; or when the fluid mass is a sphere, and when it is a circular sheet spread out in the plane of the equator. In order to discover whether q is evanescent in any other circumstances, put Tan. $\phi=\lambda$;

then $$\frac{2}{9}q=\frac{1}{3}-\left(\frac{1}{\text{Sin.}^2\phi}-\frac{2}{3}\right)\left(1-\frac{\phi\,\text{Cos.}\,\phi}{\text{Sin.}\,\phi}\right);$$

or, in a series,

$$q=\frac{2}{5}\,\text{Sin.}^2\phi+\frac{1.2}{5.7}\,\text{Sin.}^4\phi-\frac{1.2.4.6}{5.7.9.11}\,\text{Sin.}^8\phi$$
$$-\frac{1.2.4.6.8}{5.7.9.11.13}\,\text{Sin.}^{10}\phi$$
$$-\&c.$$

Now this series being evanescent both when Sin. $\phi=o$, and when Sin. $\phi=1$, it follows that, for every other value of Sin. ϕ, q will be positive; and hence it will first increase from zero to a maximum, and then decrease to the first limit. If

we seek the value of λ that will make q a maximum, we shall find this equation, viz.

$$\text{Arc. Tan. } \lambda = \frac{\lambda}{1+\lambda^2} \cdot \frac{9+7\lambda^2}{9+\lambda^2};$$

from which λ comes out equal to 2·5292. And hence $\sqrt{1+\lambda^2}$, which is the proportion of the equatorial diameter to the polar axis, is equal to 2.7197.

From all this it follows, that if a homogeneous mass of fluid in equilibrio, at rest, and consequently of a spherical figure, begin to revolve about a diameter, it will become more and more oblate as the velocity of rotation increases, till the equatorial diameter have to the polar axis the proportion of 2·7197 to 1: arrived at this point the rotatory velocity must decrease, in order that the fluid in equilibrio continue to have the figure of an ellipsoid of revolution with increasing oblateness; in so much that while the oblateness tends to be infinite, and the fluid to become a circular sheet in the plane of the equator, the velocity of rotation continually approaches to zero.

As the oblateness increases without ceasing, there is but one rotatory velocity with which a spheroid of a given figure will be in equilibrio.

But when a fluid mass is to revolve in a given time, and the figure that will maintain the equilibrium is sought, there are two solutions, if the proposed rotation be within the maximum, and one only, when it reaches that limit.

When the rotatory velocity is greater than the maximum, the equilibrium cannot take place: for, on the one hand, the proposed rotation is inconsistent with the figure of an ellipsoid; and, on the other, it has been proved, that a homoge-

neous fluid cannot be in equilibrio unless it have that figure. In this case, therefore, the fluid would first extend itself, and flatten to a certain degree with a decreasing velocity of rotation, and then oscillate back with an increasing rotatory motion. But the tenacity of the particles would gradually diminish, and finally destroy, the oscillations of the fluid; which would therefore ultimately settle in one of the figures of equilibrium; that is, in an elliptical spheroid of revolution having the equatorial diameter more than 2·7179 times the axis of revolution.

When the oblate figures are little different from spheres, as in the case of the planets, λ, which is equal to the excentricity of the meridian divided by half the polar axis, is so small that we may consider λ^2 as equal to $\text{Sin.}^2\,\phi$, and may reject all the powers of these two quantities. The series for q will thus be reduced to its first term, viz.

$$q = \frac{2}{5}\,\text{Sin.}^2\,\phi = \frac{2}{5}\,\lambda^2.$$

But the polar axis is to the equatorial diameter as 1 to $\sqrt{1+\lambda^2}$, or as 1 to $1 + \frac{1}{2}\lambda^2$: wherefore the same proportion is equal to that of 1 to $1 + \frac{5}{4}q$.

Again, we have $$q = \frac{\omega}{\frac{4}{3}\pi};$$

now, $\frac{4}{3}\pi$ being the mass of a sphere of which the density and the radius are each equal to unit, it will represent the gravitation at the surface; and, if we suppose the same sphere to revolve with the given rotatory velocity, ω will be the centrifugal force at the equator. Wherefore q is the proportion of the centrifugal force at the equator to the gravity; a proportion which remains the same in all spheres that have the

same density and the same velocity of rotation, because both the quantities increase as the radius of the sphere. Hence, in a planet of small oblateness, the value of q to the degree of approximation mentioned, is equal to the proportion of the centrifugal force to the gravitation at the equator; and the proportion of 1 to $1 + \frac{5}{4} q$ is equal to that of the polar axis to the equatorial diameter.

8. In the determination of the equilibrium of a homogeneous fluid mass investigated in this Paper, two conditions are found necessary when the particles are endowed with attractive powers; whereas, in the usual solution of the problem one only is deemed sufficient, namely, that contained in equation (A), which expresses that the resultant of the accelerating forces acting upon the particles in the outer surface shall be every where perpendicular to that surface. It is extremely remarkable that, of the two conditions, the one which is usually omitted, alone and without reference to the other, ascertains the kind of the figures of equilibrium.

M'LAURIN first proved synthetically that the ellipsoid, whatever be the degree of oblateness, fulfils all the conditions requisite for maintaining the equilibrium of a homogeneous fluid mass that revolves about an axis. If therefore the equation (A) were alone sufficient for the equilibrium, the ellipsoid must be deducible from it, not in particular suppositions and approximately, but generally, and by an accurate process of reasoning. But this has not been accomplished, nor even attempted, by any geometer. No application has hitherto been made of the hydrostatical theory, except in the case of spheroids little different from spheres.

If a homogeneous fluid of a spherical form at rest, and

consequently in equilibrio, begin to revolve about a diameter with a rotatory velocity causing a centrifugal force at the equator, very small in proportion to the gravitation; the sphere will acquire a small degree of oblateness at the poles, and the new surface of equilibrium must come under the equation (A). Now, from these considerations alone, without any reference to the other condition of equilibrium, it has been proved by LEGENDRE and LAPLACE, that the particles of the fluid will arrange themselves very nearly in an ellipsoid of revolution, the deviation being proportional to the square and higher powers of the oblateness. But, as the coincidence of the true figure of equilibrium with the ellipsoid is not exact, the result seems to be inconsistent with what M'LAURIN has so ably and elegantly demonstrated to be true. This argument will acquire greater force, and will even become conclusive, against the theory which makes the equation (A) the only condition of the equilibrium, if we consider the oblateness as a finite quantity, and push the approximation so as to take in the square and higher powers:* for, by this procedure, we obtain a series of figures in which the ellipsoid is not included.

There is a great analogy between the modern theory of spheroids little different from spheres, and the assumption of NEWTON, who tacitly supposed that the fluid sphere, in the nascent change of its form, will become, either exactly or very nearly, an elliptical spheroid, oblate at the poles. Both views of the subject leave us in ignorance of the exact form of the surface of equilibrium, although, in the supposed circumstances, it is proved in the one, and assumed in the other, that it is nearly an ellipsoid.

* Mec. Celeste, Vol. ii, p. 105, No. 37.

The figures of the earth and of the planets being entirely deduced from the properties of spheroids little different from spheres, it may not be improper to conclude this Paper with a short exposition of a theory that occupies so conspicuous a place in the celestial mechanics, and which is so intimately connected with the subject we have been discussing.

For this purpose resume the expansion of V (r) already given in § 3, viz.

$$\begin{aligned} V(r) = & \iint R'^2 \, d\mu' \, d\varpi' + r \iint R' . C^{(1)} \, d\mu' \, d\varpi' \\ & + r^2 . \left\{ -\frac{2\pi}{3} + \iint \text{Log. } R' . C^{(2)} \, d\mu' \, d\mu' \right\} \\ & - \frac{r^3}{1} \iint \frac{C^{(3)} \, d\mu' \, d\varpi'}{R'} \\ & \quad \vdots \\ & - \frac{r^i}{i-2} \iint \frac{C^{(i)} \, d\mu' \, d\varpi'}{R'^{\,i-2}} \\ & \&c. \end{aligned}$$

The spheroid being nearly a sphere, we may suppose $R' = a . (1 + \alpha . y')$; α being a small coefficient of which the square and other powers are to be neglected; and y' a function of the angles that determine the position of R'. The expansion supposes that the attracted point is within the spheroid; but it will apply when the same point is in the surface, in which case, $r = R = a(1 + \alpha . y)$. Now, let the values of R and R′ be substituted, and we shall obtain,

$$\begin{aligned} V(R) = & \iint \frac{a^2}{2} (1 + 2\alpha y') \, d\mu' \, d\varpi' - \frac{2\pi}{3} a^2 (1 + 2\alpha y) \\ & + a^2 \alpha . \iint y' . C^{(1)} \, d\mu' \, d\varpi' \\ & + a^2 \alpha . \iint y' . C^{(2)} \, d\mu' \, d\varpi' \\ & + a^2 \alpha . \iint y' . C^{(3)} \, d\mu' \, d\varpi' \\ & + \&c. \end{aligned}$$

This expression is to be substituted in the equation (A): and it is to be observed that $\iint \frac{a^2}{2}\, d\mu'\, d\varpi' = 2\,\pi\, a^2$; and that ω is of the same order with α. Hence we get,

$$\begin{aligned}C = \tfrac{4\pi}{3}\, a^2 - \tfrac{4\pi}{3}\, a^2 y &+ a^2\alpha\,.\iint y'\, d\mu'\, d\varpi' \\ &+ a^2\alpha\,.\iint y'\,.\,C^{(1)}\, d\mu'\, d\varpi' \\ &+ a^2.\left\{\alpha\,.\iint y'\,.\,C^{(2)}\, d\mu'\, d\varpi' + \tfrac{\omega}{2}\,(1-\mu^2)\right\} \\ &+ a^2\alpha\,.\iint y'\,.\,C^{(3)}\, d\mu'\, d\varpi' \\ &+ \text{\&c.}\end{aligned}$$

This is the approximate equation of the surface, when the equation (A) is alone taken into account; and it is to be proved that this equation cannot subsist unless it belong to an elliptical spheroid of revolution.

In the first place, the nature of the function y must depend upon the integrals by which its value is expressed. But all the integrals are independent of the angles that determine the position of the attracted point in the surface, unless so far as those angles enter into the expressions $C^{(1)}$, $C^{(2)}$, $C^{(3)}$, &c. which are all functions of γ. Now γ is a function of μ, $\sqrt{1-\mu^2}\,\text{Sin.}\,\varpi$, $\sqrt{1-\mu^2}\,.\,\text{Cos.}\,\varpi$: and hence it follows, that y and y' are functions of three rectangular co-ordinates of a point in the surface of a sphere.

In the second place, every function of three rectangular co-ordinates is susceptible of an arrangement, by which it will be converted into a series of the same integrals contained in the foregoing equation. Let

$$f = \sqrt{1 - 2\,\varepsilon\,\gamma + \varepsilon^2}:$$

then, as is well known, we shall have,

$$4\pi y = \iint \left(\frac{1}{f} + 2\varepsilon \frac{d \cdot \frac{1}{f}}{d\varepsilon} \right) y' \, d\mu' \, d\varpi',$$

provided we make $\varepsilon = 1$, after the integration. Now for $\frac{1}{f}$ substitute its developement,* and make $\varepsilon = 1$: then,

$$\begin{aligned} 4\pi y = & \iint y' \, d\mu' \, d\varpi' \\ & + 3\iint y' . \mathrm{C}^{(1)} \, d\mu' \, d\varpi' \\ & + 5\iint y' . \mathrm{C}^{(2)} \, d\mu' \, d\varpi' \\ & \quad \cdot \\ & \quad \cdot \\ & \quad \cdot \\ & + (2i+1)\iint y' . \mathrm{C}^{(1)} \, d\mu' \, d\varpi' \\ & + \&c. \end{aligned}$$

This expression is identical when y and y' are functions of three rectangular co-ordinates. It is analytically true of every function that can be algebraically transformed into an expression of three rectangular co-ordinates; and thus it may be said to comprehend every function of two variable angles.

We have now obtained two expressions of y in the same quantities. But it is easy to prove that the same function can be so expressed only one way. The two values of y must therefore be identical; and all the terms that cannot be made to coincide, must be evanescent. Hence we obtain

$$\begin{aligned} o &= \iint y' . \mathrm{C}^{(1)} \, d\mu' \, d\varpi' \\ o &= \iint y' . \mathrm{C}^{(3)} \, d\mu' \, d\varpi' \\ & \cdot \\ & \cdot \\ & \cdot \\ o &= \iint y' . \mathrm{C}^{(i)} \, d\mu' \, d\varpi' \\ & \&c. \end{aligned}$$

* Section 2.

from which it is easy to infer that the most general value of y' is thus expressed, viz.

$$p' = A \,.\, \mu'^2 + B\,(1 - \mu'^2)\,\text{Sin.}^2\varpi' + C\,(1 - \mu'^2)\,\text{Cos.}^2\varpi'.$$

It deserves to be remarked, that the equations just found are the very same that result from the second condition of equilibrium, when, for R$'$, we substitute $a\,(1 + \alpha \,.\, y')$, and neglect the powers of α.

Now, leaving out the evanescent terms, the two foregoing expressions will become,

$$C = \frac{4\pi}{3}a^2 - \frac{4\pi}{3}a^2 y + a^2\alpha \,.\, \int\!\!\int y'\,d\mu'\,d\varpi'$$
$$+ a^2 \,.\, \left\{\alpha \int\!\!\int y' \,.\, C^{(2)}\,d\mu'\,d\varpi' + \frac{\omega}{2}(1 - \mu^2)\right\}$$
$$4\pi y = \int\!\!\int y'\,d\mu'\,d\varpi' + 5\int\!\!\int y' \,.\, C^{(2)}\,d\mu'\,d\varpi' :$$

and farther, if we exterminate the integral containing $C^{(2)}$ from the first, we shall obtain,

$$C = \frac{4\pi}{3}a^2 + \frac{4}{5}a^2\alpha \,.\, \int\!\!\int y'\,d\mu'\,d\varpi'$$
$$- a^2 \,.\, \left\{\frac{8\pi}{15} \,.\, \alpha y - \frac{\omega}{2}(1 - \mu^2)\right\}$$
$$4\pi y = \int\!\!\int y'\,d\mu'\,d\varpi' + 5 \cdot \int\!\!\int y' \,.\, C^{(2)}\,d\mu'\,d\varpi'.$$

The first of these equations proves that y is a function of μ only, and that the spheroid sought is one of revolution. The second is satisfied by putting,

$$y = f\,(1 - \mu^2)$$
$$y' = f\,(1 - \mu'^2) :$$

wherefore, by substituting these values, the first will become,

$$C = \frac{4\pi}{3}a^2\left(1 + \frac{8}{5}\alpha f\right) - a^2\left(\frac{8\pi}{15} \,.\, \alpha f - \frac{\omega}{2}\right)(1 - \mu^2).$$

Hence we finally get,

$$q = \frac{\omega}{\frac{4}{3}\pi}$$

$$\alpha f = \frac{5}{4} q$$

$$C = \frac{4\pi}{3} a^2 . (1 + 2q)$$

$$\alpha y = \alpha f (1 - \mu^2) = \frac{5}{4} q (1 - \mu^2)$$

$$a (1 + \alpha y) = a \left\{ 1 + \frac{5}{4} q (1 - \mu^2) \right\}.$$

Such is the method of investigation for which we are indebted to LEGENDRE and LAPLACE in its fundamental principles: for, when all the operations necessary for applying it extensively and readily are fully explained, it becomes a great branch of analysis. The result is no more than an approximation, both on account of the quantities omitted, and because no attention is paid to one of the conditions of equilibrium. Considering the near approach of all the planets to the spherical form, the method of calculation may be deemed sufficiently accurate for determining the figure of the fluids that cover their surfaces; but it is not the less necessary to place the physical theory on a clear and sure foundation. As the subject is usually treated, there is an obscurity, and a want of evidence, arising from the inconsistency between the hydrostatical theory and what is proved by M'LAURIN, which is extremely embarrassing, but which entirely disappears, when we take into account all the physical conditions requisite to maintain the equilibrium of a homogeneous fluid mass that revolves upon an axis.

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