XXVIII. *On the geometrical representation of the powers of quantities, whose indices involve the square roots of negative quantities.* By the *Rev.* JOHN WARREN, *M.A. late Fellow and Tutor of Jesus College, Cambridge. Communicated by the President.*

Read June 4, 1829.

ABOUT three months ago I wrote a paper intitled "Consideration of the objections raised against the geometrical representation of the square roots of negative quantities," which paper was communicated to the Royal Society by Dr. YOUNG, and read on the 19th of February last. At that time I had only discovered the manner of representing geometrically quantities of the form $a + b\sqrt{-1}$, and of geometrically adding and multiplying such quantities, and also of raising them to powers, either whole or fractional, positive or negative; but I was not then able to represent geometrically quantities of the form $\overline{a + b\sqrt{-1}}^{\,m + n\sqrt{-1}}$, that is, quantities raised to powers, whose indices involve the square roots of negative quantities. My attention, however, has since been drawn to these latter quantities in consequence of an observation which I met with in M. MOUREY's work on this subject (the work which I mentioned in my former paper); the observation is as follows:

"Les limites dans lesquelles je me suis restreint m'ont forcé à passer sous silence plusieurs espèces de formules, telles sont celles-ci

$$a^{\sqrt{-1}},\ a_{\sqrt{-1}},\ \sin(\sqrt{-1})\ \&c., \&c., \&c.$$

Je les discute amplement dans mon grand ouvrage, et je démontre que toutes expriment des lignes directives situées sur le même plan que 1 et $\underset{1}{1}$."

where $a_{\sqrt{-1}}$ and $\underset{1}{1}$ in M. MOUREY's notation signify respectively $a\binom{1}{1}^{\frac{\sqrt{-1}}{4}}$ and $\binom{1}{1}^{\frac{1}{4}}$ according to my notation.

From this observation it was evident that M. Mourey had arrived at the geometrical representation of all algebraic quantities whatever, and that in a larger work he entered fully into the subject; but from his Preface it appeared also, that this larger work existed only in manuscript, and that circumstances would not permit the author to publish it at present. I was induced therefore to pursue my own investigations further; and the result was, that I found (as M. Mourey had stated) that all algebraic quantities whatever are capable of geometrical representation, and are represented by lines all situated in the same plane: and my view in what I am now writing is to communicate this result to algebraists.

This paper, therefore, is intended as a continuation of my "Treatise on the geometrical representation of the square roots of negative quantities;" and the object of it is to extend the geometrical representation to the powers of quantities, whose indices involve the square roots of negative quantities.

Art. 1. Def.) Mathematicians apply the words 'possible' and 'impossible' to algebraic quantities, the former signifying either positive or negative quantities; the latter, quantities involving the square roots of negative quantities. In this sense, as a matter of convenience, these words will be used in this paper; it being understood at the same time that by the word 'impossible' no impossibility is necessarily implied, but on the contrary, that the quantities called impossible have a real existence, and are capable of geometric representation.

2. Def.) Logarithms, according to the common definition given by mathematicians, must be possible quantities; therefore as a general definition of logarithms will be given in this paper, it will be desirable for the sake of distinction to give a more limited name to the common definition of logarithms, and accordingly they will be called possible logarithms; also for the like reason, the powers of quantities according to the common definition of powers, will be called possible powers.

3. Def.) Let ϱ be any quantity whatever, and let ϱ be inclined to unity at an angle $= \theta$, and let r be a positive quantity equal in length to ϱ, and let the possible hyperbolic logarithm of r be v; then $v + \theta\sqrt{-1}$ is called the general hyperbolic logarithm of ϱ, and expressed thus ϱ'.

4. Cor. 1.) $\int \frac{d\varrho}{\varrho} = \varrho'$

For (by Treatise, Art. 168.) $\int \frac{d\varrho}{\varrho} = v + \theta\sqrt{-1}$.

$\therefore \int \frac{d\varrho}{\varrho} = \varrho'$

5. Cor. 2.) If $v + \theta\sqrt{-1} = \varrho'$; then $v + \overline{\theta + pc}\sqrt{-1}$ is also a value of ϱ', where p is any whole number, either positive or negative, and c is the circumference of a circle whose radius $= 1$.

For, since ϱ is inclined to unity at an angle $= \theta$, it is also inclined to unity at an angle $= \theta + p\,c$,

$$\therefore \text{(by Art. 3.) } v + \overline{\theta + p\,c}\,.\sqrt{-1} = \varrho'$$

6. Cor. 3.) Hence, if θ be positive and less than c, $\underset{p}{\varrho'} = v + \overline{\theta + p\,c}\,.\sqrt{-1}$.

7. Cor. 4.) Hence $\underset{p}{\varrho'} = \underset{q}{\varrho'} + \overline{p - q}\,.\,c\sqrt{-1}$.

$$\begin{aligned}
\text{For}\quad & \underset{p}{\varrho'} = v + \overline{\theta + p\,c}\,.\sqrt{-1},\\
\text{and}\quad & \underset{q}{\varrho'} = v + \overline{\theta + q\,c}\,.\sqrt{-1},\\
\therefore\quad & \underset{p}{\varrho'} = \underset{q}{\varrho'} + \overline{p - q}\,.\,c\sqrt{-1}.
\end{aligned}$$

8. Cor. 5.) Hence $\underset{p}{\varrho'} = \underset{o}{\varrho'} + p\,c\sqrt{-1}$.

9. Cor. 6.) If ϱ be a positive quantity, $\underset{o}{\varrho'} =$ possible hyperbolic logarithm of ϱ.

10. Cor. 7.) Hence, if ϱ be any quantity whatever, and r be a positive quantity equal in length to ϱ, and ϱ be inclined to unity at an angle $= \theta$, θ being positive and less than c; $\underset{p}{\varrho'} = \underset{o}{r'} + \overline{\theta + p\,c}\,.\sqrt{-1}$.

11. Def.) Let a and ϱ be any two quantities whatever, then $\frac{\varrho'}{a'}$ is called the general logarithm of ϱ in a system whose base is a.

12. Cor. 1.) If $\underset{p}{a}$ be the base of a system of general logarithms, then the general logarithm of $\underset{p}{a}$ in that system is 1.

13. Cor. 2.) Let E be a quantity such that $\underset{o}{\mathrm{E}'} = 1$, and let ϱ be any quantity whatever; then $\varrho' =$ general logarithm of ϱ in a system whose base is $\underset{o}{\mathrm{E}}$.

For, let $v =$ general logarithm of ϱ in a system whose base is $\underset{o}{E}$,

then (by Art. 11.) $v = \dfrac{\varrho'}{\underset{o}{E'}} = \dfrac{\varrho'}{1} = \varrho'$.

14. Cor. 3.) Hence the quantity $\underset{o}{E}$ in the preceding article is the base of the system of general hyperbolic logarithms.

15. Def.) Let a and m be any quantities whatever, and let ϱ be a quantity such that one of its general logarithms in a system, whose base is a, is m; then ϱ is called the m^{th} general power of a, and expressed thus a^m; and the m^{th} general power of $\underset{p}{a}$ is expressed thus $\left(\underset{p}{a}\right)^m$.

16. Cor.) Hence $\varrho' = m\, a'$.

17. Let a be any quantity whatever, and let b be a positive quantity in length equal to a, and let a be inclined to unity at an angle $= \alpha$, where α is positive and less than c the circumference of the circle, and let $\varrho = \left(\underset{p}{a}\right)^m$; then, if m be a possible quantity, ϱ will be in length $= \left(\underset{o}{b}\right)^m$, and will be inclined to unity at an angle $= m \,.\, \overline{\alpha + p\,c} \,.\, \sqrt{-1}$.

For, since $\varrho = \left(\underset{p}{a}\right)^m$, one of the values of ϱ' is $m\,\underset{p}{a'}$,

Let $\underset{q}{\varrho'}$ be that value of ϱ',

then $\underset{q}{\varrho'} = m\,\underset{p}{a'}$

$= m\,\underset{o}{b'} + m \,.\, \overline{\alpha + p\,c} \,.\, \sqrt{-1}$;

Let r be a positive quantity, in length $= \varrho$, and let ϱ be inclined to unity at an angle $= \theta$, θ being positive and less than c,

$$\text{then } \underset{q}{\varrho'} = \underset{o}{r'} + \overline{\theta + q\,c} \,.\, \sqrt{-1},$$

$$\therefore \underset{o}{r'} + \overline{\theta + q\,c} \,.\, \sqrt{-1} = m\,\underset{o}{b'} + m \,.\, \overline{\alpha + p\,c} \,.\, \sqrt{-1},$$

$$\therefore \underset{o}{r'} = m\,\underset{o}{b'}, \text{ and } \theta + q\,c = m \,.\, \overline{\alpha + p\,c}$$

$$\therefore r = \left(\underset{o}{b}\right)^m,$$

$$\therefore \varrho \text{ is in length } = \left(\underset{o}{b}\right)^m;$$

But ϱ is inclined to unity at an angle $= \theta + q\,c$,

$\therefore$ ϱ is inclined to unity at an angle $= m\,.\,\overline{\alpha + p\,c}$;

$\therefore$ ϱ is in length $= \left(\underset{o}{b}\right)^m$, and is inclined to unity at an angle $= m\,.\,\overline{\alpha + p\,c}$.

18. Let a be a positive quantity, and let ϱ be the m^{th} possible power of $\underset{o}{a}$; then ϱ will also be the m^{th} general power of $\underset{o}{a}$.

For since a is a positive quantity, and ϱ the m^{th} possible power of $\underset{o}{a}$, (by Treatise, Art. 65.) ϱ is a positive quantity;

Also, from the nature of possible logarithms, since ϱ is the m^{th} possible power of $\underset{o}{a}$; possible hyperbolic logarithm of $\varrho = m \times$ possible hyperbolic logarithm of a,

that is (by Art. 9.) $\underset{o}{\varrho'} = m\,\underset{o}{a'}$,

$\therefore$ (by Art. 15.) ϱ is the m^{th} general power of $\underset{o}{a}$.

19. Let a be any quantity whatever, and let ϱ be the m^{th} possible power of $\underset{p}{a}$, then ϱ will also be the m^{th} general power of $\underset{p}{a}$.

For let b be a positive quantity, in length $= a$,
and let a be inclined to unity at an angle $= \alpha$, α being positive and less than c the circumference of the circle,
then, since ϱ is the m^{th} possible power of $\underset{p}{a}$,

(by Treatise, Art. 59, 60.) length of $\varrho = m^{\text{th}}$ possible power of b

$$= \text{(by Art. 18.)} \left(\underset{o}{b}\right)^m;$$

And (by Treatise, Art. 63, 64.) ϱ is inclined to unity at an angle $= m\,.\,\overline{\alpha + p\,c}$;
$\therefore$ (by Art. 17.) ϱ is the m^{th} general power of $\underset{p}{a}$.

20. Let $\left(\underset{p}{a}\right)^m = \left(\underset{q}{a}\right)^m$, where a is any quantity whatever, and m either irrational or impossible; then $p = q$.

For let $\left(\underset{p}{a}\right)^m$ or $\left(\underset{q}{a}\right)^m = b$,

then, since $b = \left(\underset{p}{a}\right)^m$, one value of $b' = m\,\underset{p}{a'}$,

let $\underset{x}{b'}$ be that value,

$$\text{then } \underset{x}{b'} = m \underset{p}{a'};$$

$$\text{In like manner let } \underset{y}{b'} = m \underset{q}{a'};$$

$$\text{then } \underset{x}{b'} - \underset{y}{b'} = m \underset{p}{a'} - m \underset{q}{a'},$$

$$\therefore \text{(by Art. 7.) } \overline{x - y} \cdot c \sqrt{-1} = m \cdot \overline{p - q} \cdot c \sqrt{-1},$$

$$\therefore x - y = m \cdot \overline{p - q},$$

where m is either irrational or impossible; therefore, since x, y, p, q are either $= 0$ or are whole numbers either positive or negative, the conditions of the equation cannot be satisfied unless $p = q$,

$$\therefore p = q.$$

21. Let $\left(\underset{p}{a}\right)^m = b$, and let $\left(\underset{q}{b}\right)^n = f$, where a, m, n, are any quantities whatever;

then, if $\underset{q}{b'}$ be that value of b', which is equal to $m \underset{p}{a'}$, $\left(\underset{p}{a}\right)^{mn} = f$.

$$\text{For, since } f = \left(\underset{q}{b}\right)^n$$

$$f' = n \underset{q}{b'}$$

$$= m\, n \underset{p}{a'}$$

$$\therefore f = \left(\underset{p}{a}\right)^{m\,n}.$$

22. $\left(\underset{o}{\text{E}}\right)^{m + n\sqrt{-1}} = \left(\underset{o}{\text{E}}\right)^m \cdot \left(\underset{1}{1}\right)^{\frac{n}{c}}$, where m and n are any possible quantities whatever, and c is the circumference of the circle, and $\underset{o}{\text{E}}$ the base of the hyperbolic logarithms.

For let $\left(\underset{o}{\text{E}}\right)^{m + n\sqrt{-1}} = \varrho$,

then one of the values of ϱ' is $m + n\sqrt{-1}$,

let $\underset{q}{\varrho'}$ be that value of ϱ',

then $\underset{q}{\varrho'} = m + n\sqrt{-1}$;

let r be a positive quantity, in length $= \varrho$, and let ϱ be inclined to unity at an angle $= \theta$, θ being positive and less than c the circumference of the circle,

then $\underset{q}{\varrho'} = \underset{o}{r'} + \overline{\theta + q\, c} \cdot \sqrt{-1}$;

$$\therefore \underset{o}{r'} + \overline{\theta + q\,c} \,.\, \sqrt{-1} = m + n\sqrt{-1},$$

$$\therefore \underset{o}{r'} = m, \text{ and } \theta + q\,c = n,$$

$$\therefore r = \binom{E}{o}^{m};$$

$$\text{But } \varrho = r\binom{1}{1}^{\frac{\theta}{c}} = r\binom{1}{1}^{\frac{\theta + qc}{c}},$$

$$\therefore \varrho = r \,.\, \binom{1}{1}^{\frac{n}{c}}$$

$$= \binom{E}{o}^{m} \,.\, \binom{1}{1}^{\frac{n}{c}}$$

$$\therefore \binom{E}{o}^{m + n\sqrt{-1}} = \binom{E}{o}^{m} \,.\, \binom{1}{1}^{\frac{n}{c}}.$$

23. Cor.) Hence $\binom{E}{o}^{n\sqrt{-1}} = \binom{1}{1}^{\frac{n}{c}}$.

24. $\binom{E}{o}^{m} \,.\, \binom{E}{o}^{n} = \binom{E}{o}^{m+n}$, where m and n are any quantities whatever, and $\underset{o}{E}$ the base of the hyperbolic logarithms.

For let $\left.\begin{array}{r} m = p + q\sqrt{-1} \\ n = s + t\sqrt{-1} \end{array}\right\}$ where p, q, s, t are possible quantities,

then $$\binom{E}{o}^{m} \,.\, \binom{E}{o}^{n} = \binom{E}{o}^{p + q\sqrt{-1}} \,.\, \binom{E}{o}^{s + t\sqrt{-1}}$$

$$= \binom{E}{o}^{p} \,.\, \binom{1}{1}^{\frac{q}{c}} \,.\, \binom{E}{o}^{s} \,.\, \binom{1}{1}^{\frac{t}{c}}$$

$$= \text{(by Treatise, Art. 88.) } \binom{E}{o}^{p+s} \,.\, \binom{1}{1}^{\frac{q+t}{c}}$$

$$= \binom{E}{o}^{p + s + \overline{q+t} \,.\, \sqrt{-1}}$$

$$= \binom{E}{o}^{m+n}.$$

25. $\binom{a}{p}^{m} \,.\, \binom{a}{p}^{n} = \binom{a}{p}^{m+n}$, where a, m, n are any quantities whatever.

For let $\underset{p}{a'} = s$,

then $\left(\binom{a}{p}^m\right)' = m\,s$,

$\therefore \binom{a}{p}^m = \binom{\mathrm{E}}{o}^{m\,s}$,

In like manner $\binom{a}{p}^n = \binom{\mathrm{E}}{o}^{n\,s}$, and $\binom{a}{p}^{m+n} = \binom{\mathrm{E}}{o}^{\overline{m+n}\,.\,s}$,

$$\therefore \binom{a}{p}^m \cdot \binom{a}{p}^n = \binom{\mathrm{E}}{o}^{m\,s} \cdot \binom{\mathrm{E}}{o}^{n\,s}$$

$$= \binom{\mathrm{E}}{o}^{\overline{m+n}\,.\,s}$$

$$= \binom{a}{p}^{m+n}.$$

26. $\dfrac{1}{\binom{a}{p}^m} = \binom{a}{p}^{-m}$, where a and m are any quantities whatever.

For let $\underset{p}{a'} = n$,

then $\left(\binom{a}{p}^m\right)' = m\,n$, and $\left(\binom{a}{p}^{-m}\right)' = -\,m\,n$,

$\therefore \binom{a}{p}^m = \binom{\mathrm{E}}{o}^{m\,n}$, and $\binom{a}{p}^{-m} = \binom{\mathrm{E}}{o}^{-m\,n}$;

Let $m\,n = s + t\sqrt{-1}$, where s and t are possible quantities,

then $\binom{a}{p}^m = \binom{\mathrm{E}}{o}^{s+t\sqrt{-1}} = \binom{\mathrm{E}}{o}^s \cdot \binom{1}{1}^{\frac{t}{c}}$,

$$\therefore \frac{1}{\binom{a}{p}^m} = \frac{1}{\binom{\mathrm{E}}{o}^s \cdot \binom{1}{1}^{\frac{t}{c}}} = \binom{\mathrm{E}}{o}^{-s} \cdot \binom{1}{1}^{-\frac{t}{c}} = \binom{\mathrm{E}}{o}^{-s-t\sqrt{-1}}$$

$$= \binom{\mathrm{E}}{o}^{-m\,n}$$

$$= \binom{a}{p}^{-m}.$$

27. Let a and b be any quantities whatever, and f a quantity such that $\underset{p}{a'} + \underset{q}{b'} = \underset{s}{f'}$; then $a\,b = f$.

For let $\underset{p}{a'} = x$, and $\underset{q}{b'} = y$,

then $\underset{s}{f'} = x + y$,

$$\therefore\ a = \binom{\mathrm{E}}{o}^{x},\ b = \binom{\mathrm{E}}{o}^{y},\ f = \binom{\mathrm{E}}{o}^{x+y},$$

$$\therefore\ a\,b = \binom{\mathrm{E}}{o}^{x} \cdot \binom{\mathrm{E}}{o}^{y} = \binom{\mathrm{E}}{o}^{x+y} = f.$$

28. Let a and b be any quantities whatever, and f a quantity such that $\underset{p}{a'} - \underset{q}{b'} = \underset{s}{f'}$; then $\frac{a}{b} = f$.

For, since $\underset{p}{a'} - \underset{q}{b'} = \underset{s}{f'}$,

$$\underset{p}{a'} = \underset{q}{b'} + \underset{s}{f'},$$

$\therefore$ (by preceding Art.) $a = b\,f$,

$$\therefore\ \frac{a}{b} = f.$$

29. Let a and b be any quantities whatever, and let a be inclined to unity at an angle $= \alpha$, and b at an angle $= \beta$, α and β being each positive and less than c the circumference of the circle, and let $a\,b = f$;

then $\underset{p}{a'} + \underset{q}{b'} = \underset{p+q}{f'}$, if $\alpha + \beta$ be less than c,

$= \underset{p+q+1}{f'}$, if $\alpha + \beta$ be not less than c.

For let g be a positive quantity, in length $= a$,

h - - - - - - - - - - - - - - - - - - $= b$,

then $a = g \binom{1}{1}^{\frac{\alpha}{c}}$,

$b = h \binom{1}{1}^{\frac{\beta}{c}}$,

$$\therefore f = g\,h \binom{1}{1}^{\frac{\alpha+\beta}{c}};$$

also $\underset{p}{a'} = \underset{o}{g'} + \overline{\alpha + p\,c} \cdot \sqrt{-1}$,

$\underset{q}{b'} = \underset{o}{h'} + \overline{\beta + q\,c} \cdot \sqrt{-1}$,

$$\therefore\ \underset{p}{a'} + \underset{q}{b'} = \underset{o}{g'} + \underset{o}{h'} + \overline{\alpha + \beta + \overline{p + q} \cdot c} \cdot \sqrt{-1};$$

but g'_o = possible hyperbolic logarithm of g,

h'_o = - h,

$\therefore\ g'_o + h'_o$ = - $g\,h$

$$= \left(\underset{o}{g\,h}\right)';$$

$$\therefore\ a'_p + b'_q = \underset{o}{g\,h'} + \overline{\alpha + \beta + \overline{p + q}\,.\,c}\,.\,\sqrt{-1};$$

but, since $g\,h\,.\left(\begin{smallmatrix}1\\1\end{smallmatrix}\right)^{\frac{\alpha+\beta}{c}} = f$,

$$\left(\underset{o}{g\,h}\right)' + \overline{\alpha + \beta}\,\sqrt{-1} = f'_o, \text{ if } \alpha + \beta \text{ be less than } c,$$

$$= f'_1, \text{ - - - - - - not less - -},$$

$$\therefore\ \left(\underset{o}{g\,h}\right)' + \overline{\alpha + \beta + \overline{p + q}\,.\,c}\,.\,\sqrt{-1} = f'_{p+q}, \text{ if } \alpha + \beta \text{ be less than } c,$$

$$= f'_{p+q+1}, \text{ - - - - - not less - -};$$

$$\therefore\ a'_p + b'_q = f'_{p+q}, \text{ if } \alpha + \beta \text{ be less than } c,$$

$$= f'_{p+q+1}, \text{ - - - - - not less - - .}$$

30. Cor.) Hence if a and b be any quantities whatever, and a be inclined to unity at an angle $= \alpha$, and b at an angle $= \beta$, α and β being each positive and less than c the circumference of the circle, and $\frac{a}{b} = f$;

then $a'_p - b'_q = f'_{p-q}$, if α be not less than β,

$= f'_{p-q-1}$, if α be less than β.

31. Let a and b be any quantities whatever, and let a be inclined to unity at an angle $= \alpha$ and b, at an angle $= \beta$, α and β being each positive and less than c the circumference of the circle, and let $a\,b = f$, and let m be any quantity whatever;

$$\text{then } \left(\underset{p}{a}\right)^m . \left(\underset{q}{b}\right)^m = \left(\underset{p+q}{f}\right)^m, \text{ if } \alpha + \beta \text{ be less than } c$$

$$= \left(\underset{p+q+1}{f}\right)^m, \text{ - - - - not less - - .}$$

For, first, let $\alpha + \beta$ be less than c,

$$\text{then (by Art. 29.) } \underset{p}{a'} + \underset{q}{b'} = \underset{p+q}{f'},$$

$$\therefore\ m\underset{p}{a'} + m\underset{q}{b'} = m\underset{p+q}{f'},$$

$$\therefore\ \left(\binom{a}{p}^m\right)' + \left(\binom{b}{q}^m\right)' = \left(\binom{f}{p+q}^m\right)',$$

$$\therefore\ \text{(by Art. 27.) } \binom{a}{p}^m \cdot \binom{b}{q}^m = \binom{f}{p+q}^m;$$

Next, let $\alpha + \beta$ be not less than c

$$\text{then (by Art. 29.) } \underset{p}{a'} + \underset{q}{b'} = \underset{p+q+1}{f'}$$

$$\therefore\ \binom{a}{p}^m \cdot \binom{b}{q}^m = \binom{f}{p+q+1}^m.$$

32. Let a and b be any quantities whatever, and let a be inclined to unity at an angle $= \alpha$, and b at an angle $= \beta$, α and β being each positive and less than c the circumference of the circle, and let $\frac{a}{b} = f$, and let m be any quantity whatever;

$$\text{then } \frac{\binom{a}{p}^m}{\binom{b}{q}^m} = \binom{f}{p-q}^m, \text{ if } \alpha \text{ be not less than } \beta,$$

$$= \binom{f}{p-q-1}^m, \text{ if } \alpha \text{ be less than } \beta.$$

For this may be proved nearly in the same manner as the preceding article.

33. Let m be any quantity whatever, and $\underset{o}{\mathrm{E}}$ the base of the hyperbolic logarithms;

$$\text{then } \binom{\mathrm{E}}{o}^m = 1 + m + \frac{m^2}{1.2} + \frac{m^3}{1.2.3} + \&c.$$

For let $m = p + q\sqrt{-1}$, where p and q are possible quantities,

$$\text{then } \binom{\mathrm{E}}{o}^m = \binom{\mathrm{E}}{o}^{p+q\sqrt{-1}} = \binom{\mathrm{E}}{o}^p \cdot \binom{1}{1}^{\frac{q}{c}}$$

$$= \left(1 + p + \frac{p^2}{1.2} + \&c.\right) \cdot \left(1 + q\sqrt{-1} - \frac{q^2}{1.2} - \&c.\right)$$

$$= 1 + (p + q\sqrt{-1}) + \frac{(p+q\sqrt{-1})^2}{1.2} + \&c.$$

$$= 1 + m + \frac{m^2}{1.2} + \&c.$$

34. Let a be a quantity inclined to unity at an angle less than $\frac{c}{4}$, where c is the circumference of the circle;

then $\underset{o}{a'} = 2\left\{\frac{a-1}{a+1} + \frac{1}{3}\left(\frac{a-1}{a+1}\right)^3 + \frac{1}{5}\left(\frac{a-1}{a+1}\right)^5 + \&c.\right\}.$

For let $a = b\left(\begin{smallmatrix}1\\1\end{smallmatrix}\right)^{\frac{\alpha}{c}}$, where b is a positive quantity, and $\frac{\alpha}{c}$ positive and less than $\frac{1}{4}$,

then $\underset{o}{a'} = \underset{o}{b'} + \alpha\sqrt{-1}$;

Now, since $a = b\left(\begin{smallmatrix}1\\1\end{smallmatrix}\right)^{\frac{\alpha}{c}}$, we have (by Treatise, Art. 135.)

$$\left(\begin{smallmatrix}a\\o\end{smallmatrix}\right)^x = 1 + (\mathrm{B} + \alpha\sqrt{-1})\,x + \frac{(\mathrm{B} + \alpha\sqrt{-1})^2 x^2}{1.2} + \&c.$$

where x is any possible quantity,

and $\mathrm{B} = 2\left\{\frac{b-1}{b+1} + \frac{1}{3}\left(\frac{b-1}{b+1}\right)^3 + \frac{1}{5}\left(\frac{b-1}{b+1}\right)^5 + \&c.\right\}$

$= $ possible hyperbolic logarithm of b

$= \underset{o}{b'}$;

Also (by Treatise, Art. 132.)

$$\left(\begin{smallmatrix}a\\o\end{smallmatrix}\right)^x = 1 + \mathrm{A}\,x + \frac{\mathrm{A}^2 x^2}{1.2} + \&c.,$$

where $\mathrm{A} = 2\left\{\frac{a-1}{a+1} + \frac{1}{3}\left(\frac{a-1}{a+1}\right)^3 + \frac{1}{5}\left(\frac{a-1}{a+1}\right)^5 + \&c.\right\}$,

$\therefore$ equating the coefficients,

$$\begin{aligned}\mathrm{A} &= \mathrm{B} + \alpha\sqrt{-1}\\ &= \underset{o}{b'} + \alpha\sqrt{-1}\\ &= \underset{o}{a'},\end{aligned}$$

$$\therefore\ \underset{o}{a'} = 2\left\{\frac{a-1}{a+1} + \frac{1}{3}\left(\frac{a-1}{a+1}\right)^3 + \frac{1}{5}\left(\frac{a-1}{a+1}\right)^5 + \&c.\right\}.$$

35. Let a be a quantity inclined to unity at an angle less than $\frac{c}{4}$, and let $a - 1$ be in length less than unity;

Then $\underset{o}{a'} = a - 1 - \frac{1}{2}(a-1)^2 + \frac{1}{3}(a-1)^3 - \&c.$

For this may be proved (by Treatise, Art. 128.) nearly in the same manner as the preceding article.

36. Let a be a quantity inclined to unity at an angle greater than $\frac{3c}{4}$ and less than c;

Then $\underset{-1}{a'} = 2\left\{\frac{a-1}{a+1} + \frac{1}{3}\left(\frac{a-1}{a+1}\right)^3 + \frac{1}{5}\left(\frac{a-1}{a+1}\right)^5 + \&c.\right\}$.

For this may be proved nearly in the same manner as Art. 34.

37. Let a be a quantity inclined to unity at an angle greater than $\frac{3c}{4}$ and less than c, and let $a-1$ be in length less than unity;

Then $\underset{-1}{a'} = a - 1 - \frac{1}{2}(a-1)^2 + \frac{1}{3}(a-1)^3 - \&c.$

For this may be proved nearly in the same manner as Art. 34.

38. Let a be a quantity inclined to unity at an angle less than $\frac{c}{4}$, and let m be any quantity whatever;

Then $\left(\underset{p}{a}\right)^m = 1 + (A + pc\sqrt{-1})\,m + \left(\frac{A + pc\sqrt{-1}}{1.2}\right)^2 m^2 + \&c.$

where $A = 2\left\{\frac{a-1}{a+1} + \frac{1}{3}\left(\frac{a-1}{a+1}\right)^3 + \&c.\right\}$,

or $a - 1 - \frac{1}{2}(a-1)^2 + \&c.$, if $a-1$ be in length less than unity.

For $\underset{o}{a'} = A$

$\therefore \underset{p}{a'} = A + pc\sqrt{-1}$

$\therefore \left(\left(\underset{p}{a}\right)^m\right)' = (A + pc\sqrt{-1})\,m,$

$\therefore \left(\underset{p}{a}\right)^m = \left(\underset{o}{E}\right)^{(A+pc\sqrt{-1})m}$

$= \text{(by Art. 33.) } 1 + (A + pc\sqrt{-1})\,m + \frac{(A+pc\sqrt{-1})^2}{1.2} m^2 + \&c.$

39. Let a be a quantity inclined to unity at an angle greater than $\frac{3c}{4}$ and less than c, and let m be any quantity whatever;

Then $\left(\underset{p}{a}\right)^m = 1 + (A + \overline{p+1}\,.\,c\sqrt{-1})\,m + \frac{(A + \overline{p+1}\,c\sqrt{-1})^2}{1.2} m^2 + \&c.$,

where $A = 2\left\{\frac{a-1}{a+1} + \frac{1}{3}\left(\frac{a-1}{a+1}\right)^3 + \&c.\right\}$,

or $a - 1 - \frac{1}{2}(a-1)^2 + \&c.$, if $a-1$ be in length less than unity.

For this may be proved nearly in the same manner as the preceding article.

40. Let $u = \binom{x}{p}^m$, where x and m are any quantities whatever, and let m remain constant whilst x and u vary,

Then $\frac{du}{dx} = m \binom{x}{p}^{m-1}$.

For, since $u = \binom{x}{p}^m$,

$$u' = m\, x'$$

$$\therefore \text{(by Art. 4.)}\ \frac{1}{u} \cdot \frac{du}{dx} = m \cdot \frac{1}{x},$$

$$\therefore \frac{du}{dx} = m \cdot \frac{u}{x}$$

$$= m \frac{\binom{x}{p}^m}{x}$$

$$= m \binom{x}{p}^{m-1}.$$

41. Let $z = 1 + x$, and let z be inclined to unity at an angle $= \theta$, θ being positive and less than c, and let x be in length less than unity, and let m be any quantity whatever;

then $\binom{z}{p}^m = \binom{1}{p}^m \cdot \left\{1 + m\,x + \frac{m \cdot \overline{m-1}}{1.2} x^2 + \&c.\right\}$, if θ be less than $\frac{c}{4}$,

$= \binom{1}{p+1}^m \cdot \left\{1 + m\,x + \frac{m \cdot \overline{m-1}}{1.2} x^2 + \&c.\right\}$, if θ be greater than $\frac{3c}{4}$.

For, first, let θ be less than $\frac{c}{4}$,

and let $\binom{z}{p}^m = A + B\,x + C\,x^2 + \&c.$,

then differentiating

$$m \binom{z}{p}^{m-1} = B + 2\,C\,x + \&c.,$$

$$m \cdot \overline{m-1} \binom{z}{p}^{m-2} = 2\,C + \&c.,$$

$$\&c. \qquad = \&c.;$$

now let $x = 0$,

then $\binom{z}{p}^m$ becomes $\binom{1}{p}^m$

$\binom{z}{p}^{m-1}$ becomes $\binom{1}{p}^{m-1} = \binom{1}{p}^m$

&c. becomes &c.

$$\therefore A = \binom{1}{p}^m$$

$$B = m \,.\, \binom{1}{p}^m$$

$$C = \frac{m\,.\,\overline{m-1}}{1\,.\,2} \,.\, \binom{1}{p}^m$$

&c. = &c.

$$\therefore \binom{z}{p}^m = \binom{1}{p}^m \,.\, \left\{1 + m\,x + \frac{m\,.\,m-1}{1\,.\,2}\,x^2 + \text{\&c.}\right\};$$

Next, let θ be greater than $\frac{3\,c}{4}$,

In this case, when $x = 0$,

$\binom{z}{p}^m$ becomes $\binom{1}{p+1}$;

$$\therefore \binom{z}{p}^m = \binom{1}{p+1}^m \,.\, \left\{1 + m\,x + \frac{m\,.\,\overline{m-1}}{1\,.\,2}\,x^2 + \text{\&c.}\right\}.$$

42. Let $\binom{E}{o}^{x \binom{E}{o}^{m\sqrt{-1}}} = \varrho$, where x is a positive quantity, and m a possible quantity, and let x and ϱ vary whilst m remains constant; then ϱ will trace out a logarithmic spiral, which cuts its radii vectors at an angle $= m$.

For let r be a positive quantity in length $= \varrho$,

then, since $\binom{E}{o}^{m\sqrt{-1}} = \binom{1}{1}^{\frac{m}{c}} = \cos m + \sin m \,.\, \sqrt{-1}$,

$$\varrho = \binom{E}{o}^{x \cos m + x \sin m \,.\, \sqrt{-1}}$$

$$= \binom{E}{o}^{x \cos m} \,.\, \binom{1}{1}^{\frac{x \sin m}{c}},$$

$\therefore r = \binom{E}{o}^{x \cos m}$, and ϱ is inclined to unity at an angle $= x \,.\, \sin m$,

$$\therefore \underset{o}{r'} = x \,.\, \cos m,$$

$$\therefore\ x = \frac{1}{\cos m} \cdot \underset{o}{r'},$$

$\therefore$ ρ is inclined to unity at an angle $= \frac{\sin m}{\cos m} \cdot \underset{o}{r'} = \tan m \cdot \underset{o}{r'}$,

which is the property of a logarithmic spiral which cuts its radii vectors at an angle $= m$,

$\therefore$ the curve traced out by ρ is a logarithmic spiral which cuts its radii vectors at an angle $= m$.

43. Cor. 1.) The logarithmic spiral in the last article will cut the positive direction at a distance $= 1$.

For let $x = 0$, then $\rho = \binom{\mathrm{E}}{o}^{o} = 1$,

$\therefore$ one of the values of ρ is 1,

$\therefore$ the spiral cuts the positive direction at a distance $= 1$.

44. Cor. 2.) When m is such that $\tan m = 0$, the spiral becomes a straight line; and when m is such that $\tan m$ is infinite, the spiral becomes a circle.

45. Let a be any quantity whatever, and x any possible quantity, and let $\binom{a}{p}^{x} = \rho$, and let ρ and x vary while a and p remain constant; then ρ will trace out a logarithmic spiral.

For let $\underset{p}{a'} = n \binom{\mathrm{E}}{o}^{m\sqrt{-1}}$, where n is positive and m possible,

then $\left(\binom{a}{p}^{x}\right)' = n\, x \binom{\mathrm{E}}{o}^{m\sqrt{-1}}$,

$$\therefore\ \binom{\mathrm{E}}{o}^{n x \binom{\mathrm{E}}{o}^{m\sqrt{-1}}} = \binom{a}{p}^{x} = \rho,$$

$\therefore$ (By Art. 42.) since m is constant, and $n\,x$ and ρ variable, ρ will trace out a logarithmic spiral.

46. Cor. 1.) The spiral will cut the positive direction at a distance $= 1$, and will cut its radii at an angle $= m$.

47. Cor. 2.) ρ becomes equal to a in its $p + 1^{\text{th}}$ revolution in the spiral, reckoning from the time at which it was equal to 1.

For let b be a positive quantity in length $= a$, and let a be inclined to unity at an angle $= \alpha$, where α is positive and less than c,

then $\underset{o}{b'} + \overline{\alpha + p\, c} \,.\, \sqrt{-1} = \underset{p}{a'}$

$$= n \left(\underset{o}{\mathrm{E}}\right)^{m\sqrt{-1}},$$

$$\therefore x \underset{o}{b'} + x \,.\, \overline{\alpha + p c} \,.\, \sqrt{-1} = n x \left(\underset{o}{\mathrm{E}}\right)^{m\sqrt{-1}} = n x \cos m + n x \sin m \,.\, \sqrt{-1},$$

$$\therefore x \,.\, \overline{\alpha + p c} = n x \sin m,$$

but $n x$ sin m is the angle at which ϱ (considered as the radius vector of the spiral) is inclined to unity.

$\therefore x \,.\, \overline{\alpha + p c}$ is the angle at which ϱ, as radius vector of the spiral, is inclined to unity;

but when $x = 1$, $\varrho = a$ and angle $x \,.\, \overline{\alpha + p c}$ becomes $\alpha + p c$,

and α is less than c,

$\therefore \varrho$ becomes equal to a in its $p + 1^{\text{th}}$ revolution.

48. Cor. 3.) If a be positive but not $= 1$, and $p = 0$, the spiral becomes a straight line; if $a = 1$, and p be not $= 0$, the spiral becomes a circle; and if $a = 1$, and $p = 0$, the spiral becomes a point.

49. Cor. 4.) If a be any quantity whatever, and $\underset{p}{a'} = n \left(\underset{o}{\mathrm{E}}\right)^{m\sqrt{-1}}$, n being positive and m possible; and if a logarithmic spiral be described having its pole in the origin of a, and cutting the positive direction at a distance $= 1$ and passing through the extremity of a in its $p + 1^{\text{th}}$ revolution; then the spiral will cut its radii at an angle $= m$.

50. Let $\left(\underset{p}{a}\right)^{n}\left(\underset{o}{\mathrm{E}}\right)^{m\sqrt{-1}} = \varrho$, where a is any quantity whatever, and m any possible quantity, and n any positive quantity; and let a logarithmic spiral be described having its pole in the origin of a and ϱ, and cutting the positive direction at a distance $= 1$, and passing through the extremity of a in its $p + 1^{\text{th}}$ revolution; and let a second logarithmic spiral be described having the same pole with the first spiral, and also cutting the positive direction at a distance $= 1$, and cutting the first spiral at an angle $= m$; then ϱ will be a radius vector of the second spiral.

For let $\underset{p}{a'} = l \left(\underset{o}{\mathrm{E}}\right)^{k\sqrt{-1}}$, where l is positive and k possible,

then (by Art. 49.) the first spiral will cut its radii at an angle $= k$,

and since the second spiral cuts the first at an angle $= m$,

the second spiral will cut its radii at an angle $= k + m$;

$$\text{But } \underset{p}{a'} = l \left(\underset{o}{\mathrm{E}}\right)^{k\sqrt{-1}},$$

$$\therefore \left(\left(\underset{p}{a}\right)^{n\left(\underset{o}{\mathrm{E}}\right)^{m\sqrt{-1}}}\right)' = n \left(\underset{o}{\mathrm{E}}\right)^{m\sqrt{-1}} . \, l \left(\underset{o}{\mathrm{E}}\right)^{k\sqrt{-1}} = l\, n \left(\underset{o}{\mathrm{E}}\right)^{\overline{k+m}\,.\sqrt{-1}},$$

$$\therefore \varrho = \left(\underset{o}{\mathrm{E}}\right)^{l n \left(\underset{o}{\mathrm{E}}\right)^{\overline{k+m}\,.\sqrt{-1}}},$$

$\therefore$ (by Art. 42 and 43.) ϱ is a radius vector of a logarithmic spiral which cuts its radii at an angle $= k + m$, and cuts the positive direction at a distance $= 1$, that is, ϱ is a radius vector of the second spiral.

51. Cor. 1.) If $\left(\underset{p}{a}\right)^{m\sqrt{-1}} = \varrho$, and m be a possible quantity; ϱ will be a radius vector in a spiral (described as in the preceding article) which cuts the spiral, in which $\underset{p}{a}$ is, at a right angle.

52. Cor. 2.) Hence if a be a positive quantity, and $p = 0$, the spiral, in which a is, will become a straight line, and the spiral, in which ϱ is, will be perpendicular to it, that is, will be a circle; but if $a = 1$, and p be not $= 0$, the spiral, in which a is, will become a circle, and the spiral, in which ϱ is, will be perpendicular to it, that is, will be a straight line, and ϱ will be a positive quantity.

53. Let $a^m = \varrho$, and let a and m be any quantities whatever; then the values of ϱ are in geometric progression.

For $\left(\underset{p}{a}\right)^m$ represents any one value of ϱ,

$\therefore$ if we substitute for p successively 0, 1, 2, 3, &c., also -1, -2, -3, &c., we shall obtain all the values of ϱ;

$$\text{Let } \underset{o}{a'} = n,$$

$$\text{then (by Art. 8.) } \underset{p}{a'} = n + p\, c \sqrt{-1},$$

$$\therefore \left(\left(\underset{p}{a}\right)^m\right)' = n\, m + p\, c\, m \sqrt{-1},$$

$$\therefore \left(\underset{p}{a}\right)^m = \left(\underset{o}{\mathrm{E}}\right)^{n m + p c m \sqrt{-1}},$$

now, if we substitute for p successively 0, 1, 2, &c., also -1, -2, &c., the values of $\binom{\mathrm{E}}{o}^{mn+pcm\sqrt{-1}}$ will be in geometric progression,

$\therefore$ the values of ϱ are in geometric progression.

54. Cor. 1.) Hence all the values of ϱ are radii vectors of the same logarithmic spiral.

55. Cor. 2.) If m be impossible or irrational, ϱ will have an infinite number of different values; but if m be rational the values will recur, and the number of different values will be equal to the denominator of m, when m is expressed as a fraction in its lowest terms.

56. Any geometric series being given, it is required to find quantities a and m, such that, a^m may have values equal to each of the terms of the series.

Let b be any term of the series, and r the common ratio,

$$\text{and let } b = \binom{a}{p}^m,$$

$$\text{then } b\,r = \binom{a}{p+1}^m = \binom{a}{p}^m \cdot \binom{1}{1}^m,$$

$$\therefore r = \binom{1}{1}^m;$$

Now $\underset{1}{1}' = c\sqrt{-1}$, where c is the circumference of the circle,

$$\therefore \left(\binom{1}{1}^m\right)' = m\,c\sqrt{-1},$$

$$\therefore r' = m\,c\sqrt{-1},$$

$$\therefore m = \frac{r'}{c\sqrt{-1}}, \ \therefore m \text{ is known};$$

$$\text{Now } b = \binom{a}{p}^m,$$

$$\therefore b' = m\,\underset{p}{a'} = \frac{r'}{c\sqrt{-1}} \cdot \underset{p}{a'},$$

$$\therefore \underset{p}{a'} = c\sqrt{-1} \cdot \frac{b'}{r'},$$

$$\therefore a = \binom{\mathrm{E}}{o}^{c\sqrt{-1}\cdot\frac{b'}{r'}}, \ \therefore a \text{ is known}.$$

57. Cor.) If the series be of the form $1, r, r^2$, &c., we may take $b = 1$ and $b' = 0$; then we shall have $a = 1$, and $a^m = 1^{\frac{r'}{c\sqrt{-1}}}$.

58. Ex.) Let the series be 1, 2, 4, &c.,

$$\text{then } r = 2,$$

$$\text{and } 1^{\frac{r'}{c\sqrt{-1}}} = 1^{\frac{2'}{c\sqrt{-1}}},$$

$\therefore$ the values of $1^{\frac{2'}{c\sqrt{-1}}}$ are 1, 2, 4, &c., also $\frac{1}{2}$, $\frac{1}{4}$ &c.

59. Let it be required to find the values of $\sqrt{-1}^{\sqrt{-1}}$.

$$\underset{1}{1}' = c\sqrt{-1},$$

$$\therefore \left(\left(\underset{1}{1}\right)^{\frac{1}{4}}\right)' = \frac{c}{4}\sqrt{-1},$$

$$\text{but } \sqrt{-1} = \left(\underset{1}{1}\right)^{\frac{1}{4}},$$

$$\therefore \left(\sqrt{-1}\right)' = \frac{c}{4}\sqrt{-1},$$

$\therefore$ since $\frac{c}{4}$ is less than c, $\left(\underset{o}{\sqrt{-1}}\right)' = \frac{c}{4}\sqrt{-1}$,

$$\therefore \left(\underset{p}{\sqrt{-1}}\right)' = \frac{c}{4}\sqrt{-1} + pc\sqrt{-1} = \left(\tfrac{1}{4} + p\right)c\sqrt{-1},$$

$$\therefore \left(\left(\underset{p}{\sqrt{-1}}\right)^{\sqrt{-1}}\right)' = \left(\tfrac{1}{4} + p\right)c\sqrt{-1}\,.\,\sqrt{-1} = -\left(\tfrac{1}{4} + p\right)c,$$

$$\therefore \left(\underset{p}{\sqrt{-1}}\right)^{\sqrt{-1}} = \left(\underset{o}{\mathrm{E}}\right)^{-\left(\frac{1}{4}+p\right)c} = \frac{1}{\left(\underset{o}{\mathrm{E}}\right)^{\left(\frac{1}{4}+p\right)c}},$$

let 0, 1, 2, &c., also -1, -2, &c., be successively substituted for p, then we have the values of $\sqrt{-1}^{\sqrt{-1}}$ as follows, viz.

$$\frac{1}{\left(\underset{o}{\mathrm{E}}\right)^{\frac{c}{4}}},\ \frac{1}{\left(\underset{o}{\mathrm{E}}\right)^{\frac{5c}{4}}},\ \frac{1}{\left(\underset{o}{\mathrm{E}}\right)^{\frac{9c}{4}}},\ \&c.,$$

$$\left(\underset{o}{\mathrm{E}}\right)^{\frac{3c}{4}},\ \left(\underset{o}{\mathrm{E}}\right)^{\frac{7c}{4}},\ \&c.,$$

all in geometric progression.

60. From what has been demonstrated it will be manifest that all algebraic quantities may be geometrically represented, both in length and direction, by lines drawn in a given plane from a given point.

61. With respect to quantities such as sin $(a + b\sqrt{-1})$, cos $(a + b\sqrt{-1})$, &c., quantities strictly speaking not algebraic, it may be observed, that if it be found useful to introduce these quantities into algebra, they may without impropriety be introduced, by giving to sines, cosines, &c., algebraic definitions; thus sin A may be defined to signify

$$\frac{\left(\begin{smallmatrix}1\\1\end{smallmatrix}\right)^{\frac{\mathrm{A}}{c}} - \left(\begin{smallmatrix}1\\1\end{smallmatrix}\right)^{-\frac{\mathrm{A}}{c}}}{2\sqrt{-1}} \text{ or } \frac{\left(\begin{smallmatrix}\mathrm{E}\\o\end{smallmatrix}\right)^{\mathrm{A}\sqrt{-1}} - \left(\begin{smallmatrix}\mathrm{E}\\o\end{smallmatrix}\right)^{-\mathrm{A}\sqrt{-1}}}{2\sqrt{-1}},$$ and we shall have

$$\sin(a+b\sqrt{-1}) = \frac{\left(\begin{smallmatrix}\mathrm{E}\\o\end{smallmatrix}\right)^{a\sqrt{-1}-b} - \left(\begin{smallmatrix}\mathrm{E}\\o\end{smallmatrix}\right)^{-a\sqrt{-1}+b}}{2\sqrt{-1}} = \frac{\left(\begin{smallmatrix}\mathrm{E}\\o\end{smallmatrix}\right)^{-b}\cdot\left(\begin{smallmatrix}1\\1\end{smallmatrix}\right)^{\frac{a}{c}} - \left(\begin{smallmatrix}\mathrm{E}\\o\end{smallmatrix}\right)^{b}\cdot\left(\begin{smallmatrix}1\\1\end{smallmatrix}\right)^{-\frac{a}{c}}}{2\sqrt{-1}},$$

therefore, considering sines, cosines, &c., merely as algebraic quantities, we may make use of the expressions sin $(a + b\sqrt{-1})$, cos $(a + b\sqrt{-1})$, &c., if such expressions be found convenient in algebraic operations.

Jesus College, Cambridge, J. W.
April 22, 1829.