XI. *Geometrical Investigations concerning the Phenomena of Terrestrial Magnetism.*

By THOMAS STEPHENS DAVIES, *Esq. F.R.S. Lond. and Ed. F.R.A.S. Royal Military Academy, Woolwich.*

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THOUGH the experiments of MICHEL and COULOMB have satisfactorily determined the law according to which magnetic forces vary as the distance of the needle acted on is made to vary, yet, so far as I know, no one has attempted to deduce from that law any method of accounting for the phenomena of terrestrial magnetism. Till that is done, however, we cannot assure ourselves whether the poles (I use the term to designate the centres from which the forces emanate,) be two or more; nor even whether it be necessary to consider them infinite in number and distributed over the whole surface or through the whole mass of the magnet. The agreement of the results as to quantity with the actual phenomena would be decisive in favour of any hypothesis. The *necessary consequences* have not, however, yet been deduced from any one hypothesis whatever: and even had it been otherwise, there is so much uncertainty attached to magnetical observations, and so many anomalous and unaccountable discrepancies and disturbances continually mingling in the registered results, that it is not possible, in the present state of the numerical data, to bring any hypothesis fairly to the trial, however complete the mathematical development of its consequences may be.

Notwithstanding the great difficulty of conducting a series of observations in a perfectly unexceptionable manner, and the utter impossibility, with our present knowledge, of properly determining the correction to be made at any given place and period with any given instrument, there are yet several features in the phenomena which are of too decided a character to be overlooked in comparing the results of any theory with observation. We may not, indeed, be able to avoid considerable discrepancies in our comparison, but still there should at least be a general tendency towards agreement, and in no case a direct reversal of the phenomena presenting itself as the result of any hypothesis which prefers its claims to our adoption. In respect to terrestrial magnetism, no direct attempt has, however, been made to embody the results of any hypothesis in a series of appropriate formulæ; and hence the conjectures which have been made respecting such agreements have been made from extremely vague and inconclusive considerations.

The duality of the terrestrial magnetic poles is the oldest hypothesis, and perhaps that whose consequences will be found most easy to examine. The hasty comparisons

that have been made between its *supposed* results and the observations made on the needle at certain places, and especially respecting the Halleyan lines, and HANSTEEN'S poles of greatest intensity, have caused the hypothesis to be rejected by many persons, who, if they had looked more closely into the question, could not have failed to discover that their conclusions were altogether premature, and probably erroneous. I speak now of the broad features of the phenomena compared with a popular rather than a calculated series of deductions from the hypothesis. Whether, however, when the results come to be more closely tested by an appeal to the numerical values of the quantities in question, the same accordance would be found, is a question altogether different: and it is one which we are not at present in a condition, for want of numerical data, upon which to offer a distinct opinion, much less are we entitled to express a positive decision concerning it.

The present series of papers is chiefly intended to deduce the mathematical consequences of the theory of two poles situated arbitrarily within the earth, and especially to investigate the singular points and lines which result from the intersection of the earth's surface with other surfaces related to the magnetic poles. Amongst these are the magnetic equator, the points at which the needle is vertical, the lines of equal dip, the Halleyan lines, the isodynamic lines, and the Hansteen poles. If it shall appear in the course of these developments that the general features of all these are roughly represented by the hypothesis of two poles, then it will be a strong argument in favour of that simple theory; but should a result, in any one of these cases, follow from that hypothesis which is very decidedly opposed to the corresponding observed phenomena, we shall be compelled, if our observations are authentic and to be depended on as unaffected by an extreme degree of foreign influence, to abandon it altogether.

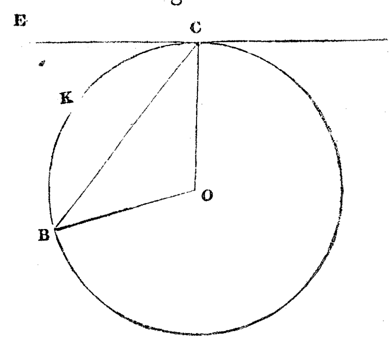
Our hope of being able to separate the disturbing from the primary forces must depend altogether upon their relative quantities. The success of astronomical research has hinged wholly on the relative smallness of the disturbances in comparison with the primary forces that govern the motions of a planet: and if the same order of magnitude should exist in the magnetic forces in question, the same success will, there is every reason to hope, follow in due time. If not, the research should be placed at once amongst the *desperata*. However, till some method is discovered of ascertaining whether such is the case or not, we should leave no effort untried either to accomplish the proposed object, or to render manifest its impracticability. With that view the present investigations, which are conducted in a manner altogether untried before by any one, are offered to the attention of geometers, as being calculated, besides exhibiting the general consequences of the dual hypothesis, in some degree to point out where we should look for the influence of foreign forces, and especially showing that in reference to one great circle, the want of symmetry in the results at positions taken symmetrically with respect to it, gives us great cause to suspect the action of such foreign forces.

The nature of this paper does not allow of much numerical experiment upon the observation-data; but still, in illustration of the method of determining the position of the magnetic axis, I have entered into a little. The results are not very favourable to the hypothesis; but when it is considered that the observations were selected almost at hazard, all made with different instruments, by various persons, and in geological regions extremely dissimilar, we could hardly, in the confessedly imperfect state of the art of observation, expect to obtain satisfactory results. Taking all things into account, the results are, unfavourable though they be, as satisfactory as we could expect. However, whatever conclusion may be drawn from them, they furnish at least a pattern for calculating any better and more consistent observations we may at any future time be able to procure; and if any that are beyond question can be obtained, it will enable us to bring the *linearity* of the magnetic axis to an indisputable test. The *duality*, should the linearity be established, can be at once put to the test by means of the process in Section III.

Now that this method of investigation is proposed, it will doubtless occur to some of my readers that a more *direct* course would have been to assume the undetermined coordinates $a_, b_, c_,$, $a_{,,} b_{,,} c_{,,}$, of the two poles, and express the equation of the sphere in reference to the same axes, and hence the directive effect of the two poles upon a needle placed at a point $x y z$ on the spherical surface. Such a method, they will believe, must also have occurred to me as the most natural; but if they will take the trouble to form the equations of condition that this method will require, they will see the utter impracticability of effecting the reductions under the mere motive of making an experiment upon the results of an hypothesis when no confidence was felt in the numerical data which entered into the formula. The reason for adopting the less direct, but incomparably more simple, preliminary test illustrated by Sections XI. and XII. will then be sufficiently obvious.

I.—Given the dip and variation of the magnetic needle and the geographical coordinates of the place of observation, to find the geographical coordinates of the place where the needle, sufficiently prolonged, will intersect the surface of the earth again.

Fig. 1.



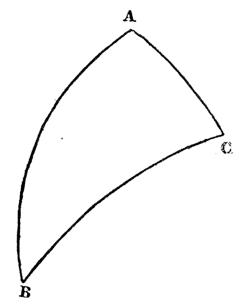
We assume, for reasons too well known to need specification here, that the orthogonal projection of the dipping-needle upon the horizontal plane gives the position of the horizontal needle; or, which comes to the same, that the dipping-needle and the horizontal needle are in the same vertical plane. This plane cuts the sphere in a great circle, which we shall for the present suppose to coincide with the plane of the paper, and to be represented in the annexed figure by C K B. Let E C be the

intersection of the plane B K C with the horizontal plane, and let C B be the line along which the dipping-needle disposes itself.

Join C O, O B, (O being the centre of the circle and, of consequence, of the sphere): then the arc C K B measures the angle C O B, which is twice the angle E C B, or twice the dip of the free needle. This arc, then, is known from observations at several particular places on the earth's surface.

Next, let the spherical triangle A B C denote that whose vertex A is the geographical pole of the earth; C the place of observation; and the angle B that determined as above from observations made at C: and let the angle A C B denote the observed variation of the horizontal needle at C. Then we have the sides A C, C B, and included angle A C B, from which to determine the colatitude A B and polar angle B A C.

Fig. 2.



We have therefore the polar spherical coordinates of the point B, the polar distance A B at once, and the polar angle by adding B A C to the longitude of C with its proper sign.

I shall designate the coordinates of C and B by $\alpha_{,}\ \beta_{,}$ and $\alpha_{,,}\ \beta_{,,}$ respectively, as is done in my paper on Spherical Geometry in the twelfth volume of the Edinburgh Transactions; α denoting the polar distance and β the longitude of the point.

II.—Given the dip, variation, and geographical coordinates of the place of observation, to express the equations of the line in which the dipping-needle disposes itself.

Let $a_{,}\,b_{,}\,c_{,}$ and $a_{,,}\,b_{,,}\,c_{,,}$ denote the coordinates of two points in space: then the equations of the straight line through them are

$$x = \frac{a_{,,} - a_{,}}{c_{,,} - c_{,}}\,z - \frac{a_{,,}c_{,} - a_{,}c_{,,}}{c_{,,} - c_{,}}$$

$$y = \frac{b_{,,} - b_{,}}{c_{,,} - c_{,}}\,z - \frac{b_{,,}c_{,} - b_{,}c_{,,}}{c_{,,} - c_{,}}.$$

But in the present case the points $a_{,}\,b_{,}\,c_{,}$ and $a_{,,}\,b_{,,}\,c_{,,}$ are on the surface of the sphere; and if we consider the axis of the sphere to be the axis of z, the intersection of the meridian with the equator to be the axis of y, and that of the meridian at right angles to it with the equator to be the axis of x, then $\alpha_{,}\ \beta_{,}$ and $\alpha_{,,}\ \beta_{,,}$ being, as before stated, the coordinates of the extremities of the chord in which the dipping-needle disposes itself, we shall have, for determining the equations of the needle, the following values of the constants:

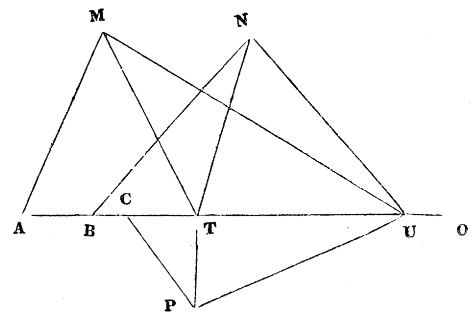
$$\begin{array}{l|l}
c_{,} = r\cos\alpha_{,} & c_{,,} = r\cos\alpha_{,,} \\
b_{,} = r\sin\alpha_{,}\cos\beta_{,} & b_{,,} = r\sin\alpha_{,,}\cos\beta_{,,} \\
a_{,} = r\sin\alpha_{,}\sin\beta_{,} & a_{,,} = r\sin\alpha_{,,}\sin\beta_{,,}.
\end{array}$$

Hence the equations of the needle take the following forms:

$$x = \frac{\sin\alpha_{\prime\prime}\sin\beta_{\prime\prime} - \sin\alpha_{\prime}\sin\beta_{\prime}}{\cos\alpha_{\prime\prime} - \cos\alpha_{\prime}} z - \frac{\sin\alpha_{\prime\prime}\sin\beta_{\prime\prime}\cos\alpha_{\prime} - \sin\alpha_{\prime}\sin\beta_{\prime}\cos\alpha_{\prime\prime}}{\cos\alpha_{\prime\prime} - \cos\alpha_{\prime}} r \quad . \quad . \quad (1.)$$

$$y = \frac{\sin\alpha_{\prime\prime}\cos\beta_{\prime\prime} - \sin\alpha_{\prime}\cos\beta_{\prime}}{\cos\alpha_{\prime\prime} - \cos\alpha_{\prime}} z - \frac{\sin\alpha_{\prime\prime}\cos\beta_{\prime\prime}\cos\alpha_{\prime} - \sin\alpha_{\prime}\cos\beta_{\prime}\cos\alpha_{\prime\prime}}{\cos\alpha_{\prime\prime} - \cos\alpha_{\prime}} r. \quad . \quad (2.)$$

III.—Let M, N, P be the centres of three dipping-needles at known positions on the surface of the earth, and denote the poles by T and U. Then the needles will arrange themselves so as that each shall be in a plane passing through T U; and hence each needle prolonged will cut the magnetic axis T U in some point, as A, B, C, respectively.



Take any point, O, in T U, and refer all the points to this origin; denote the several distances O U, O T, O C, O B, O A, by u, t, c, b, a respectively; the angles M A O, N B O, and P C O, by A, B, C; and the distances M A, N B, P C, by f, g, h.

Then we have

$$\left.\begin{aligned} MT^2 &= (a-t)^2 - 2f(a-t)\cos A + f^2 \\ MU^2 &= (a-u)^2 - 2f(a-u)\cos A + f^2 \\ NT^2 &= (b-t)^2 - 2g(b-t)\cos B + g^2 \\ NU^2 &= (b-u)^2 - 2g(b-u)\cos B + g^2 \\ PT^2 &= (c-t)^2 - 2h(c-t)\cos C + h^2 \\ PU^2 &= (c-u)^2 - 2h(c-u)\cos C + h^2. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3.)$$

Again, by the properties of a needle subjected to the action of the magnetic poles T and U, we have (those needles being small in comparison with its distance from those poles,) the following proportions:

$$\left.\begin{aligned} TA : AU &:: TM^3 : MU^3 \\ TB : BU &:: TN^3 : NU^3 \\ TC : CU &:: TP^3 : PU^3. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4.)$$

Inserting in (4.) the values of the lines T A, &c. given in (3.), we get the three equations

$$\frac{a-t}{a-u} = \left\{\frac{(a-t)^2 - 2f(a-t)\cos A + f^2}{(a-u)^2 - 2f(a-u)\cos A + f^2}\right\}^{\frac{3}{2}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (5.)$$

$$\frac{b-t}{b-u} = \left\{\frac{(b-t)^2 - 2g(b-t)\cos B + g^2}{(b-u)^2 - 2g(b-u)\cos B + g^2}\right\}^{\frac{3}{2}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (6.)$$

$$\frac{c-t}{c-u} = \left\{\frac{(c-t)^2 - 2h(c-t)\cos C + h^2}{(c-u)^2 - 2h(c-u)\cos C + h^2}\right\}^{\frac{3}{2}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (7.)$$

IV.—Given the equations of the dipping-needle at three given places on the surface of the earth to find the magnetic poles.

Let M, N, P be the places on the surface of the earth, and denote the needles as follows, viz.

$$\text{M A by } x = a' z + \alpha' \text{ and } y = b' z + \beta' \quad \ldots \quad (8.)$$

$$\text{N B by } x = a'' z + \alpha'' \text{ and } y = b'' z + \beta'' \quad \ldots \quad (9.)$$

$$\text{P C by } x = a''' z + \alpha''' \text{ and } y = b''' z + \beta''', \quad \ldots \quad (10.)$$

and denote the magnetic axis itself (T U) by

$$x = \bar{a} z + \bar{\alpha} \text{ and } y = \bar{b} z + \bar{\beta}. \quad \ldots \quad (11.)$$

Then, since the line (11.) intersects the lines (8.), (9.), (10.), we have the three equations of condition

$$(\alpha' - \bar{\alpha})(b' - \bar{b}) = (\beta' - \bar{\beta})(a' - \bar{a}) \quad \ldots \quad (12.)$$

$$(\alpha'' - \bar{\alpha})(b'' - \bar{b}) = (\beta'' - \bar{\beta})(a'' - \bar{a}) \quad \ldots \quad (13.)$$

$$(\alpha''' - \bar{\alpha})(b''' - \bar{b}) = (\beta''' - \bar{\beta})(a''' - \bar{a}). \quad \ldots \quad (14.)$$

Taking now as the unknown coordinates of the magnetic poles, T and U, the symbols $a_{,} b_{,} c_{,}$ and $a_{,,} b_{,,} c_{,,}$, we have $\bar{a}\, \bar{b}$, $\bar{\alpha}\, \bar{\beta}$, given functions of $a_{,} b_{,} c_{,}$ and $a_{,,} b_{,,} c_{,,}$; and hence we have in the equations just given three equations for the discovery of these six quantities which determine the poles. The three other requisite equations are thus derived:

By means of (11.) combined separately and successively with (8.), (9.), (10.), we can find the coordinates of the points A, B, C in terms of $a_{,} b_{,} c_{,}$ and $a_{,,} b_{,,} c_{,,}$ and given quantities $a'\, \alpha'\, b'\, \beta'$, &c.; and the coordinates of T and U are themselves $a_{,} b_{,} c_{,}$ and $a_{,,} b_{,,} c_{,,}$. We hence have the several quantities $a - t$, $b - t$, $c - t$, $a - u$, $b - u$, and $c - u$ in terms of $a_{,} b_{,} c_{,}$ and $a_{,,} b_{,,} c_{,,}$. Also the distances M A, N B, P C, that is, f, g, h, are also given in terms of the coordinates of M and A, N and B, P and C respectively, and hence in terms of $a_{,} b_{,} c_{,}$ and $a_{,,} b_{,,} c_{,,}$ and given quantities. And lastly, the equations of the lines T U and M A, N B, P C being given in terms of $a_{,} b_{,} c_{,}$ and $a_{,,} b_{,,} c_{,,}$ and known quantities, the cosine of the inclinations of T U to each of them, that is, of the angles A, B, C, are given so as to involve no quantities but known ones and the coordinates of the magnetic poles. It hence follows, that all the terms which enter into the composition of the equations (5.), (6.), (7.), are functions of the coordinates of the poles and of given quantities. The three remaining requisite equations for the actual determination of the magnetic poles are furnished, then, by those equations marked (5.), (6.), and (7.); the whole six equations which we have laid down being each, obviously, independent of the others under every combination.

V.—The preceding processes show that the determination of the magnetic poles, their duality being admitted, as well as the equality of their intensity, can be effected from three observations of the magnetic needle as to dip and azimuth; and hence

that the problem is now reduced to a purely arithmetical state, requiring only the application of known processes, and perfectly capable of execution, for the actual assignment of the positions of the poles themselves. A very slight attention to the processes themselves must, however, convince us that the operations will be very laborious; but at the same time, the symmetrical forms in which the two triads of equations are presented, might induce us to hope that a greater degree of simplification would result in the final formulæ than our passage through so many operations could at first sight lead us to expect. Such, indeed, proves to be the case; and the results are not altogether destitute of elegance, as well as simplification. Fortunately, however, there is no necessity to even attempt the solution under the present aspect of the problem; and having learnt from it, in its present state, the number of observations necessary for the determination of the poles, we shall exchange it for another, which is in some degree different as to general plan, and considerably more simple in its requisite calculations.

VI.—A necessary consequence of the hypothesis upon which we are proceeding, is,—*that all the needles must intersect the magnetic axis.* If, then, we assume the coordinates of the two poles $a_{\prime}\, b_{\prime}\, c_{\prime}$ and $a_{\prime\prime}\, b_{\prime\prime}\, c_{\prime\prime}$, we have the equation of the magnetic axis as before,

$$\left.\begin{aligned} x &= \frac{a_{\prime\prime} - a_{\prime}}{c_{\prime\prime} - c_{\prime}} z - \frac{a_{\prime\prime} c_{\prime} - a_{\prime} c_{\prime\prime}}{c_{\prime\prime} - c_{\prime}} \\ y &= \frac{b_{\prime\prime} - b_{\prime}}{c_{\prime\prime} - c_{\prime}} z - \frac{b_{\prime\prime} c_{\prime} - b_{\prime} c_{\prime\prime}}{c_{\prime\prime} - c_{\prime}}. \end{aligned}\right\} \quad \ldots \ldots \ldots \ldots \quad (15.)$$

And if we take the equations of four magnetic needles, as

$$\left.\begin{aligned} x &= a' z + \alpha' \text{ and } y = b' z + \beta' \\ x &= a'' z + \alpha'' \text{ and } y = b'' z + \beta'' \\ x &= a''' z + \alpha''' \text{ and } y = b''' z + \beta'' \\ x &= a^{\text{IV}} z + \alpha^{\text{IV}} \text{ and } y = b^{\text{IV}} z + \beta^{\text{IV}}, \end{aligned}\right\} \quad \ldots \ldots \ldots \quad (16—19.)$$

the intersections of (15.) with (16—19.) give four equations, of condition similar to those of (12.), (13.), (14.), from which to determine $a_{\prime}\, b_{\prime}\, c_{\prime}$ and $a_{\prime\prime}\, b_{\prime\prime}\, c_{\prime\prime}$, viz.

$$\left(\alpha' - \frac{a_{\prime\prime} c_{\prime} - a_{\prime} c_{\prime\prime}}{c_{\prime\prime} - c_{\prime}}\right)\left(b' - \frac{b_{\prime\prime} - b_{\prime}}{c_{\prime\prime} - c_{\prime}}\right) = \left(\beta' - \frac{b_{\prime\prime} c_{\prime} - b_{\prime} c_{\prime\prime}}{c_{\prime\prime} - c_{\prime}}\right)\left(a' - \frac{a_{\prime\prime} - a_{\prime}}{c_{\prime\prime} - c_{\prime}}\right). \quad . \quad . \quad (20—24.)$$

These four equations will determine four of the coordinates, as $a_{\prime}\, b_{\prime}$ and $a_{\prime\prime}\, b_{\prime\prime}$, leaving the two others indeterminate. But still the law of force furnishes four other equations from which to determine the two quantities $c_{\prime}$ and $c_{\prime\prime}$; that is, a redundancy of equations, from which redundancy the remaining number may be taken as checks of accurate calculation if the principle be admitted, or as tests of the truth of the principle when we are assured of the accuracy of calculation and of observation.

By this method a greater uniformity of process, and a perfect symmetry in respect to the quantities involved, are obtained; but still the process is very laborious, and it

is probable that the resulting equation will be of a higher degree than really belongs to the problem in its direct form. If so, it will contain foreign factors, which it may be difficult to detect and peculiarize, so as to separate them from the proper solutions of the problem. The method, besides, is not *essentially* different from the last.

Another difficulty also presents itself here; nor is it the only one. The mere intersection of the magnetic axis with the magnetic needle is not a test of the duality in point of number, nor of the equality of intensity in point of force, nor is it confined to any law of variation of force whatever; and hence the mere intersection is not of itself *sufficient* for the determination of the question respecting the duality or the relative intensity of the poles. Still it is one of the *necessary conditions,* though not the *only one,* by which the hypothesis is to be tested; since the poles, being of any number, and of any intensities whatever, *if situated in the same straight line,* will cause the needle to intersect that line, and hence render that phenomenon incapable of determining the number, intensity, or position of the poles; yet wherever this intersection is not fulfilled the duality of the poles cannot be admitted, nor yet the position of the poles, however many they may be, be in one straight line. The determination of $a_{\prime}\, b_{\prime}\, c_{\prime}\, a_{\prime\prime}\, b_{\prime\prime}\, c_{\prime\prime}$ from the equations (20—24.) cannot then be effected completely.

We shall hence proceed in the following manner. A straight line, which constantly touches three given straight lines, but undergoing all the changes compatible with that triple contact, describes the hyperboloid of one sheet. This surface being of the second order, will be cut by a fourth given line in two points; and hence there are two positions which a line resting upon four other lines can take. If, then, we imagine these four lines to be four different positions of the magnetic needle, and the line which rests upon them to be the magnetic axis, we shall perceive at once that in case of any number of poles of any variety of intensity, and acting under any law of variation of force depending upon distance, the magnetic axis can be determined in position from *four* observations of the magnetic needle; and, therefore, of course, in the case which we are examining, where the poles are two, the intensities equal, and the law of force that determined by MICHEL and COULOMB.

Let us take as the equations of the magnetic axis and four of the needles, respectively, the following:

$$\left.\begin{array}{l} x = \overline{a}\ \ z + \overline{\alpha}\ \text{ and } y = \overline{b}\ \ z + \overline{\beta} \\ x = a'\ z + \alpha'\ \text{ and } y = b'\ z + \beta' \\ x = a''\ z + \alpha''\ \text{ and } y = b''\ z + \beta'' \\ x = a'''\ z + \alpha'''\ \text{ and } y = b'''\ z + \beta''' \\ x = a^{\text{IV}}\ z + \alpha^{\text{IV}}\ \text{ and } y = b^{\text{IV}}\ z + \beta^{\text{IV}}. \end{array}\right\} \quad \ldots\ldots\ldots \quad (25\text{—}29.)$$

Then the condition, that the first of these intersects each of the others simultaneously, gives the four equations,

$$(\alpha' - \overline{\alpha})\,(b' - \overline{b}) = (\beta' - \overline{\beta})\,(a' - \overline{a}) \quad \ldots\ldots\ldots \quad (30.)$$

$$(\alpha'' - \overline{\alpha})\,(b'' - \overline{b}) = (\beta'' - \overline{\beta})\,(a'' - \overline{a}) \quad \ldots \quad (31.)$$

$$(\alpha''' - \overline{\alpha})\,(b''' - \overline{b}) = (\beta''' - \overline{\beta})\,(a''' - \overline{a}) \quad \ldots \quad (32.)$$

$$(\alpha^{\text{IV}} - \overline{\alpha})\,(b^{\text{IV}} - \overline{b}) = (\beta^{\text{IV}} - \overline{\beta})\,(a^{\text{IV}} - \overline{a}). \quad \ldots \quad (33.)$$

This reduction is easily effected by subtracting the first from each of the others, in which case we obtain equations of the first degree, giving each of the other three the quantities $\overline{a}\ \overline{b}\ \overline{\alpha}\ \overline{\beta}$ in terms of the fourth, as of $\overline{\alpha}$. These substituted in any one of the four equations give a quadratic equation involving $\overline{\alpha}$; and hence we obtain two values of $\overline{\alpha}$, and hence again of $\overline{b}$, of $\overline{\alpha}$, and of $\overline{\beta}$. We should then obtain, by a simple and direct process, the *equations of the magnetic axis.*

The next inquiry is into the signification of this double result. Are there two magnetic axes which fulfill the condition? If so, are they both occupied by magnets? Or if not, why is one to be selected in preference to the other? Can they both belong to *every* quadruple combination of the magnetic needle?

The last question may be answered at once. If they both belonged to all the combinations of the needles, then they must form two of the directrices of a rule surface, to which the needles themselves were always tangents. The third directrix not being yet fixed, there is no inconsistency in the conclusion thus derived; for the needles are at liberty to rest upon any magnetic surface, whatever be the number or intensity of the poles, or whatever be the parameter which determines the particular stratum of surface which corresponds to the place of observation on the sphere. There is hence nothing to prevent their belonging to every position of the magnetic needle, so far as we at present can discover from the conditions in their arbitrary form. How far this is consistent with the particular data is another question, and will be presently discussed.

There is no *necessity* that they should be both occupied by magnets; and it is at once giving up the duality of the poles, and even their being situated in a right line, to make such an hypothesis. They are both, it is true, solutions of the algebraical problem which we have proposed; but as the algebraical rarely includes *all* the conditions of the physical problem, it is easy to suppose that one of these solutions may be foreign to the inquiry, without violating our knowledge of the nature of the relations subsisting between the algebraical and physical problem. To prove that it actually *is* a foreign result must be subsequent to the determination of the particular values of the coefficients of the quantities involved in the inquiry. All we can say at present is, that there are two axes which fulfill the algebraical condition; but as there is only one which enters into the physical hypotheses, one of these two algebraical axes must be rejected. We cannot, however, ascertain which, except by other conditions than have yet been taken into the formula. For the present, then, we can only compute them both, and take that which best answers to those other physical conditions of which the algebraical problem has taken no account.

VII.—Having obtained the equations of the magnetic axis by the determination of $\bar{a}\,\bar{b}\;\bar{\alpha}\,\bar{\beta}$, we may put the rectilinearity of the positions of the poles to the test at once. For if we take a fifth needle

$$x = a^{\mathrm{v}}\,z + \alpha^{\mathrm{v}} \text{ and } y = b^{\mathrm{v}}\,z + \beta^{\mathrm{v}} \quad \ldots\ldots\ldots \quad (34.)$$

and combine it with (29.), viz.

$$x = \bar{a}\,z + \bar{\alpha} \text{ and } y = \bar{b}\,z + \bar{\beta},$$

so as to ascertain whether they intersect or not, by means of the equation of condition

$$(\alpha^{\mathrm{v}} - \bar{\alpha})\,(b^{\mathrm{v}} - \bar{b}) = (\beta^{\mathrm{v}} - \bar{\beta})\,(a^{\mathrm{v}} - a) \quad \ldots\ldots \quad (35.)$$

The approximation to a fulfilment of this condition for $\alpha^{\mathrm{v}}\,a^{\mathrm{v}}$, $b^{\mathrm{v}}\,\beta^{\mathrm{v}}$, derived from all the other observations upon which reliance can be placed, and to the extent of the probable accuracy of those observations, will establish this hypothesis to the same extent.

In order to estimate the real amount of the error in the application of this equation of condition, we must recollect that the formula itself is derived from that which gives the shortest distance between two given lines, viz.

$$\text{D (dist.)} = \pm\frac{(\alpha^{\mathrm{v}} - \bar{\alpha})\,(b^{\mathrm{v}} - \bar{b}) - (\beta^{\mathrm{v}} - \bar{\beta})\,(a^{\mathrm{v}} - \bar{a})}{\sqrt{(a^{\mathrm{v}} - a)^2 + (b^{\mathrm{v}} - \bar{b})^2 + (\bar{a}\,b^{\mathrm{v}} - a^{\mathrm{v}}\,\bar{b})^2}}. \quad \ldots\ldots \quad (36.)$$

Hence, in order to estimate the number of miles which the needle would be from fulfilling the condition, we must calculate the denominator of the fraction (36.), and divide the result of (35.) by it. By then calculating the angle which D would subtend at the place of observation, we shall be in some degree prepared to judge whether such an error might possibly have arisen from unskilful observation, the imperfect structure of the instrument, from any probable geological or meteorological causes, or from any temporary local disturbance. Or, conversely, were this line satisfactorily determined by a great number of tests, and the instrument and observer well prepared for the task, then we should be able in some degree to estimate the amount of the disturbing forces that climate, geological structure, and local attraction do actually exert at that place, and perhaps in some cases also to form a probable conjecture respecting the separate contribution of each of these causes to the total amount of the disturbance.

VIII.—Let us now suppose the magnetic axis satisfactorily determined and tested; and proceed to inquire whether the poles be two or more, and whether equal or unequal in the intensity of their action: and in the first place we shall suppose the intensities equal.

Recurring to the figure in (III.) and the conditions that are tabulated, as (5.), (6.), (7.), we have the quantities designated as a, b, c, A, B, C, f, g, h, and the corresponding ones for any number of points $\mathrm{M}_{\prime}$ $\mathrm{N}_{\prime}$ $\mathrm{P}_{\prime}$ $\mathrm{Q}_{\prime\prime}$, &c. from actual observation and the calculated equation of the magnetic axis. From any two of these, as (5.), (6.), we then shall be able to compute t and u. The remaining equations, whatever be their

number, will serve as tests of the truth of the dual hypothesis with the poles of equal intensity.

It will be the most convenient method of taking the point O, to assume for it the intersection of the magnetic axis by the perpendicular from the centre of the sphere. Having, then, the distances of the poles from this point, and the equations of the line in which they lie, we can easily determine their coordinates, the great object of our inquiry.

It can be no objection to this process that it necessarily requires the solution of equations of a high order, since it is only the solution of it in the case of given numeral coefficients, and not with literal coefficients. The method of effecting these solutions with rapidity and precision is now well known, and need not here be dwelt upon. We shall have occasion hereafter to employ them in the numerical solution of this special problem*.

IX.—If we suppose the intensities unequal, we can assume their ratios to be that of $F_{,}$ to $F_{,,}$, or R. The relation upon which (5.), (6.), (7.),... are founded no longer holds good in this case. Nevertheless, by reference to (XIV.) we see that the difference is only in the numeral coefficients of the equations, and not in the form or the number of terms, or in any circumstance that alters in the slightest degree the labour or the difficulty of the actual solution. We have however, in consequence of the new quantity R which is thus introduced, to employ one equation more derived from observation†, and one only. Hence still in this case, too, four observations are sufficient, not only for the determination of the actual position of the poles, but also to furnish a test of the accuracy of the hypothesis. As a method, then, this also is complete, and the problem is fully brought within the reach of known and familiar operations.

X.—As a specimen of the method of computing the equations of the magnetic needle, I have given calculations for Chamisso Island, Valparaiso, Paramatta, Port Bowen, Paris, and Boat Island; and that the whole process may be distinctly seen, I have also given the equations of the magnetic axis itself as deduced from the equations of the first four needles, and a comparison of the result with the Paris needle. That result is not very favourable to the theory, provided the observations themselves are considered trustworthy. But since those philosophers who have had most experience in the use of magnetical instruments, and especially of the dipping-needle, are most strongly convinced that there are errors attached to all our present *instruments* and *modes of observation* whose amount vitiates any result obtained by them,

* I refer, of course, to Mr. HORNER's method, published in the Philosophical Transactions for 1819, and in the fifth volume of Professor LEYBOURN's Mathematical Repository. It is unnecessary to add, that all the *effective* methods of solution of algebraical equations that have since appeared have been but imitations of Mr. HORNER's, however much the *notation* and *form* of the reasoning employed in them may differ from his.

† Or if we seek to determine the actual value of $F_{,}$ and $F_{,,}$ we shall want all the four complete observations.

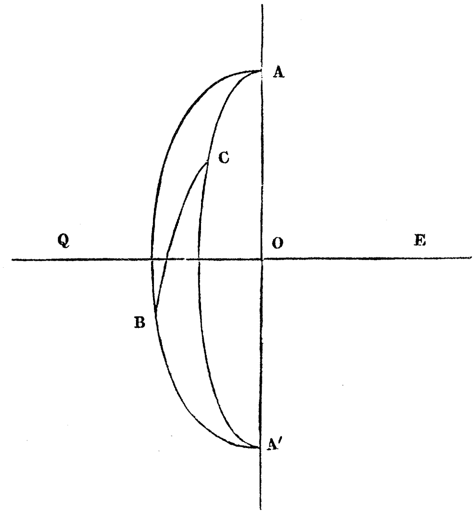
I have not thought it necessary to add any further discussion of the question in the present stage of my investigations, in as much as till the results can be ensured as unaffected by extraneous sources of error, all methods must alike be useless, since they are alike dependent upon data that are at least unsatisfactory if not erroneous. There is some reason, however, to hope, since the attention of the scientific world is now so intensely turned to researches of this nature, that there will at length be discovered some methods of observing which shall be free from this class of errors. However, till this is done, it would be useless to attempt the discussion of the present or any other method of mathematical investigation into its numerical details: and the utmost we can now perform is to lay down *methods of investigation* by which, when satisfactory experimental data are obtained, the question may be brought to a decisive test at once.

I should also state here, that in consequence of the great labour attending the calculations of the axes, I have been led to examine the method of construction by descriptive geometry, (especially on account of the facility and the considerable degree of certainty which may be attached to its solutions,) of the problem of *describing a line which shall rest upon four given lines.* In any case where the data are so uncertain as the present, such a method is sufficiently accurate, since in very few cases will the errors of construction be probably near so great as the errors in the data themselves. The geometrical problem itself is not in practice so simple as could be desired, at least by any method yet made public, but still it offers far greater facilities than the algebraical one. It has, moreover, one important advantage which the algebraical has not, viz. the ready and visual exhibition of those cases which are unfit for this method, or those in which a small error in the data will greatly increase in the result. It should hence be always used before the algebraical. No doubt the algebraical method may be rendered subservient to the same purpose, but then the method is intolerably operose, and hence, practically, almost useless.

By employing such constructions, I find that there is a greater degree of approximation in the few magnetic axes which I have determined by them from existing data than appears compatible with any other theory of the constitution of the terrestrial magnet than that which considers the magnetic force situated in two isolated centres or poles. By this I would not be understood to say that the approximation is close, but simply that, in comparison with all the positions which lines may take, there seems to be *one region* of space, in reference to the coordinate planes or planes of projection, into which they dispose themselves, but dispose themselves very irregularly in it.

As I expect to be favoured, by the kindness of my distinguished colleague Mr. Christie, with the results of the observations of the late lamented Captain Foster, I shall probably resume this branch of the subject at an early period; and hence any further details respecting these constructions which may appear necessary will be with more propriety included in a future than in the present paper.

The equations of the magnetic needle for different places, in reference to rectangular coordinates, when the geographical coordinates, dip, and variation are given.



Let A O A′ be the meridian of Greenwich, E O Q the equator, C the position of a place where a magnetic observation is made, A C B the variation, and B C equal to twice the dip of the needle. Then B is the point on the surface of the earth towards which the dipping-needle is directed, and that in which the straight line which coincides with the needle intersects the earth a second time.

Estimating (as is done in my paper on Spherical Loci before referred to) the positions of places on the surface of the earth by means of the polar angle C A O and radius-vector C A, we have the coordinates of C directly from observation; and by means of the triangle A C B, whose sides B C, C A, and included angle A C B are given, we can compute the coordinates of B. Denote the polar distance and polar angle of C by $\alpha_{,}$ $\beta_{,}$, and those of B by $\alpha_{,,}$ $\beta_{,,}$.

In the next place, by the employment of equations (1.), (2.) of II., we may obtain the equations of the needle, referred to three rectangular axes, the coordinate planes of which are the meridian of Greenwich, the meridian of $\pm 90°$, and the equator. The results for the six different places before mentioned are given in the last column of the following Table. The construction of the Table itself is indicated at the head of each column, in a way that renders further explanation unnecessary.

Place, Date, Observer, and Authority.	Geographical coordinates of the place.	Observed position of the needle.	Spherical coordinates of the place of observation.	Spherical coordinates of the second intersection.	Rectangular equations of the needle.
Port Bowen. 1824. Capt^s Parry & Foster. Phil. Trans. 1826.	Lat. 73° 14′ 0″ N. Long. 88 54 0 W.	Dip. 81° 1′ 24″ N. Var. 124 0 0 W.	$\alpha_{,} =$ 16° 46′ 0″ $\beta_{,} = -$ 88 54 0	$\alpha_{,,} =$ 165° 25′ 32″ $\beta_{,,} = -$ 256 3 55	$x = -$ ·278746 $z -$ ·023702 r $y =$ ·034611 $z -$ ·027756 r
Boat Island. 1821. Captain Parry. Voyage, I.	Lat. 68° 59′ 13″ N. Long. 53 12 56 W.	Dip. 82° 53′ 40″ N. Var. 72 2 0 W.	$\alpha_{,} =$ 21° 0′ 47″ $\beta_{,} = -$ 53 12 56	$\alpha_{,,} =$ 151° 5′ 36″ $\beta_{,,} = =$ 204 45 57	$x = -$ ·270447 $z -$ ·034011 r $y =$ ·122598 $z -$ ·122598 r
Chamisso Island. 1827. Captain Beechey. Voyage, App. p. 732.	Lat. 66° 12′ 0″ N. Long. 161 46 0 W.	Dip. 77° 39′ 0″ N. Var. 32 0 30 W.	$\alpha_{,} =$ 23° 48′ 0″ $\beta_{,} = -$ 161 46 0	$\alpha_{,,} =$ 133° 34′ 36″ $\beta_{,,} =$ 0 27 25	$x = -$ ·081565 $z -$ ·051635 r $y = -$ ·690471 $z +$ ·248470 r
Paris. 1829. M. Arago. Annuaire, 1830.	Lat. 48° 50′ 0″ N. Long. 2 20 0 E.	Dip. 67° 41′ 18″ N. Var. 22 12 5 W.	$\alpha_{,} =$ 41° 10′ 0″ $\beta_{,} =$ 2 20 0	$\alpha_{,,} =$ 96° 10′ 34″ $\beta_{,,} = -$ 162 10 56	$x =$ ·384722 $z -$ ·262819 r $y =$ 1·864500 $z -$ ·745899 r
Valparaiso. 1821. Captain Basil Hall. "Magnetism," Enc. Met.	Lat. 33° 1′ 0″ S. Long. 288 29 0 E.	Dip. 38° 46′ 0″ S. Var. 14 43 0 E.	$\alpha_{,} =$ 123° 1′ 0″ $\beta_{,} =$ 288 29 0	$\alpha_{,,} =$ 156° 58′ 20″ $\beta_{,,} =$ 85 20 23	$x = -$ 3·159630 $z -$ 2·518180 r $y -$ ·621395 $z +$ ·206769 r
Paramatta. 1823. Sir Thomas Brisbane. Phil. Trans. 1829, pt. 3.	Lat. 33° 48′ 50″ S. Long. 151 1 34 E.	Dip. 62° 57′ 0″ S. Var. 8 47 41 E.	$\alpha_{,} =$ 123° 48′ 50″ $\beta_{,} =$ 151 1 34	$\alpha_{,,} =$ 109° 46′ 44″ $\beta_{,,} = -$ 21 23 25	$x = -$ 3·418875 $z -$ 1·500120 r $y =$ 7·349865 $z +$ 3·363313 r

XII.—We may now proceed to compute the equations of that straight line (or lines, for there are two, but determined by the same series of processes,) which rests upon any four of lines determined as those in the last section were found. Thus we shall take as an instance Chamisso, Valparaiso, Paramatta, and Port Bowen, the equations of which are given in the Table.

Assume as the equations of the magnetic axis the two following:

$$x = a z + \alpha,$$
$$y = b z + \beta,$$

and denote the several equations of the needles by equations of the same form, but with the constants accented, viz. $a' \ b'$, $\alpha' \ \beta'$: then the conditions of intersection will, in each case, be expressed by the equation

$$(\alpha' - \alpha)(b' - b) - (\beta' - \beta)(a' - a) = 0.$$

The insertions of the actual values in the four cases mentioned above being made, and the vincula thrown open, we obtain

$$\begin{aligned}
\cdot055920 + \cdot690471\,\alpha - \cdot081565\,\beta + \cdot248470\,a + \cdot051635\,b + \alpha b - a\beta &= 0,\\
-\cdot911471 - \cdot621395\,\alpha - 3\cdot159630\,\beta + \cdot206769\,a + 2\cdot518180\,b + \alpha b - a\beta &= 0,\\
\cdot473095 - 7\cdot349865\,\alpha - 3\cdot418875\,\beta + 3\cdot363313\,a + 1\cdot500120\,b + \alpha b - a\beta &= 0,\\
-\cdot008188 - \cdot034611\,\alpha - \cdot278746\,\beta - \cdot027756\,a + \cdot023702\,b + \alpha b - a\beta &= 0.
\end{aligned}$$

Subtracting each of the last three of these from the first, we obtain

$$\begin{aligned}
b &= \cdot391938 + \cdot531872\,\alpha + 1\cdot247954\,\beta + \cdot017024\,a,\\
b &= -\cdot288008 + 3\cdot550860\,\alpha + 2\cdot304000\,\beta - 2\cdot150420\,a,\\
b &= -2\cdot295058 - 25\cdot957900\,\alpha - 7\cdot059053\,\beta - 9\cdot888800\,a.
\end{aligned}$$

Subtract the two latter from the former of these, then there result

$$\begin{aligned}
a &= -\cdot271254 - 2\cdot674164\,\alpha - \cdot838603\,\beta, \text{ and}\\
a &= -\cdot313706 + 1\cdot392881\,\alpha + \cdot487231\,\beta.
\end{aligned}$$

From equating which values of a we obtain $\beta = \cdot032019 - 3\cdot067660\,\alpha$.

Insert this value of β in that of a, and we find $a = -\cdot298105 - \cdot101755\,\alpha$.

In a similar manner, from the proper substitutions, $b = \cdot426822 - 3\cdot288083\,\alpha$.

And inserting all these values in the first of the four equations, that of the Chamisso needle, we obtain

$$\alpha^2 - \cdot071009\,\alpha = \cdot003821, \text{ and hence } \alpha = \cdot035505 \pm \cdot071274.$$

The two pairs of equations which result from this calculation, then, as those of the magnetic axis, are

$$\left.\begin{aligned}
x &= -\cdot308970\,z + \cdot106779\,r,\\
y &= \cdot075732\,z - \cdot295538\,r;
\end{aligned}\right| \text{ and } \begin{aligned}
x &= -\cdot294465\,z - \cdot035769\,r,\\
y &= \cdot544430\,z + \cdot141744\,r;
\end{aligned}$$

according as the $+$ or $-$ sign is taken in the valuation of α.

In the same way we may proceed to find the magnetic axis which would accord with any other four observations, and by a comparison of these ascertain whether the

discrepancies were such as to admit of account from the errors of observation and the imperfection of instruments. However, it is much simpler to ascertain whether the axis thus determined agrees with the observation made at a fifth or a sixth place. Let us take the Parisian needle as an instance.

By recurring to equation (36.), and putting the values of $\bar{a}\,\bar{b}$ and $\bar{\alpha}\,\bar{\beta}$ just determined and those of $a^{v}\,b^{v}$ and $\alpha^{v}\,\beta^{v}$ found in the table of section XI. for Paris, we have the least distance between the Parisian needle and the magnetic axis either $\cdot173341\,r$ or $\cdot189540\,r$, according as the $+$ or $-$ sign above mentioned is employed.

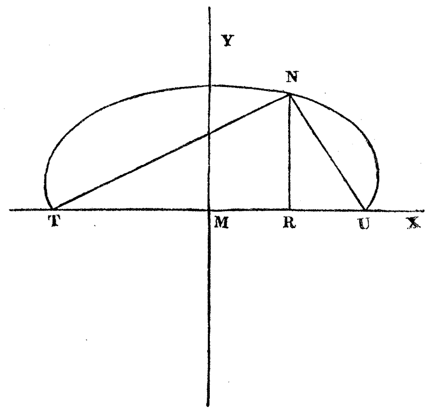
These are between a fifth and a sixth part of the terrestrial radius. We may now, were it necessary, seek the coordinates of the points in which the line of shortest distance intersects the two different magnetic axes and the Paris needle, and thence the equations of the lines drawn from those points to Paris, and thence again the angle formed by the Paris needle, and each of the other lines: that is, we should find the error of observation in the Paris needle if we suppose the magnetic axis correctly determined from the other four needles. But it is unnecessary to go through the computations, as it is easy to see that this angle will not be very different from (but its difference, whatever it be, will be greater than it,) $\tan^{-1}\cdot173341$, and $\tan^{-1}\cdot189540$, that is, about $10°$ or $11°$. The discrepancies in every other case that I have tried are as great as, and in most of them still greater than, in that just examined. The further prosecution of this branch of the inquiry, with our present data, must therefore be abandoned.

XIII.—So far as *method* is concerned, the previous processes are perfectly adapted to decide the question of the duality and the equality of intensity of the magnetic poles. In the absence, however, of data upon which full reliance can be placed, we are not at present able to apply that method to the actual circumstances of the earth. It hence becomes desirable to examine certain other phenomena, to ascertain whether, in their general bearings and character, they also are compatible with the hypothesis of two such centres of force. These are:—the points at which the needle becomes horizontal, constituting the magnetic equator; the points at which the needle becomes vertical; the curves of equal dip; the Halleyan lines, or curves of equal variation; the lines of equal magnetic intensity, or isodynamic lines of HANSTEEN; and those particular cases where the isodynamic line is reduced to a point, which constitutes the "pole" in the language of HANSTEEN and a considerable number of modern writers. The first two of these I shall examine in the present, and the remaining ones in a subsequent paper. I then propose to enter into a minute numerical discussion of such observations as I may be able to obtain in the interim; and of course attempt to ascertain how far any one may be vitiated by instrumental or local circumstances, and how far the geometrical peculiarity of the observation itself may render it unfit for our present method of investigation.

I do not propose this course without being fully sensible of the difficulty and labour it entails upon me; but at the same time I feel perfectly assured that any

assistance which it is in the power of men of science to afford will be freely offered me; and especially in furnishing such observations as they themselves place most reliance on, together with the circumstances under which the observations were made.

XIV.—*On the Magnetic Curve.*



Let $F_{,}$ and $F_{,,}$ denote the intensities of the forces situated in the two poles T, U; and β the angle which the needle, subjected to the action of those forces and situated in a given point N $(x\,y)$, would make with the axis of (x) the magnet itself. Let also

$$r_{,}^2 = y^2 + (x+a)^2$$
$$r_{,,}^2 = y^2 + (x-a)^2.$$

Then the usual considerations give us

$$\frac{\frac{F_{,}y}{r_{,}^3} + \frac{F_{,,}y}{r_{,,}^3}}{\frac{F_{,}(x+a)}{r_{,}^3} + \frac{F_{,,}(x-a)}{r_{,,}^3}} = \tan\beta. \qquad (37.)$$

But if we represent $\tan\beta$ by $\frac{dy}{dx}$, we shall have the differential equation of the curve to which the needle will be a tangent, and which passes through that point, $x\,y$. To find the equation of the curve itself it only remains then to integrate

$$\frac{\frac{F_{,}y}{r_{,}^3} + \frac{F_{,,}y}{r_{,,}^3}}{\frac{F_{,}(x+a)}{r_{,}^3} + \frac{F_{,,}(x-a)}{r_{,,}^3}} = \frac{dy}{dx}. \qquad (38.)$$

Multiply the numerators of all the terms of the first side by y, and also multiply out the denominators; then there will result

$$\frac{F_{,}y^2\,dx - F_{,}y\,(x+a)\,dy}{r_{,}^3} + \frac{F_{,,}y^2\,dx - F_{,,}y\,(x-a)\,dy}{r_{,,}^3} = 0.$$

In the former of these numerators add and subtract $F_{,}(x+a)^2\,dx$, and in the latter $F_{,,}(x-a)^2\,dx$; then we shall obtain another form, which is immediately integrable. It is

$$\frac{F_{,}\{y^2+(x+a)^2\}\,dx - F_{,}(x+a)\,\{y\,dy + (x+a)^2\,dx\}}{r_{,}^3}$$
$$+ \frac{F_{,,}\{y^2+(x-a)^2\}\,dx - F_{,,}(x-a)\,\{y\,dy + (x-a)\,dx\}}{r_{,,}^3} = 0;$$

the integral of which is

$$\frac{F_{,}(x+a)}{r_{,}} + \frac{F_{,,}(x-a)}{r_{,,}} = \frac{c}{a}, \qquad (39.)$$

in which $\frac{c}{a}$ is the arbitrary constant which particularizes the individual curve we

may have occasion to consider. It is determinable from any one condition; as passing through a given point, touching a given line, &c.

But $\frac{x+a}{r_{\prime}}$ and $\frac{x-a}{r_{\prime\prime}}$ are the cosines of the angles which the directions of the component forces make with the axis of x, that is, with the magnetic axis; and hence, denoting these by $\theta_{\prime}$, $\theta_{\prime\prime}$, (being estimated from the same branch of the axis,) we shall have

$$\mathrm{F}_{\prime} \cos \theta_{\prime} + \mathrm{F}_{\prime\prime} \cos \theta_{\prime\prime} = 2 \cos \beta', \qquad (40.)$$

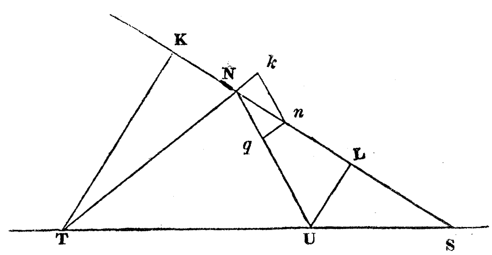
where $\beta' = \cos^{-1} \frac{c}{2a}$.

This equation having the values of $r_{\prime}$, $r_{\prime\prime}$ and β restored, becomes

$$\frac{\mathrm{F}_{\prime}(x+a)}{\sqrt{y^2+(x+a)^2}} + \frac{\mathrm{F}_{\prime\prime}(x-a)}{\sqrt{y^2+(x-a)^2}} = \frac{c}{2a}, \qquad (41.)$$

which, when deprived of its radicals and denominators, is of the *eighth degree*.

We might, however, obtain this property in a different manner; and as it also facilitates the investigation of one or two other theorems which we shall require hereafter, it may with propriety be added here.



Let T and U be the poles, N the centre of the needle. Let, as before, the forces be $\mathrm{F}_{\prime}$ and $\mathrm{F}_{\prime\prime}$; let T N S and U N S, the angles made by each of the component forces and the resultant one, be called $\Delta_{\prime}$ and $\Delta_{\prime\prime}$ respectively. Let $r_{\prime}$ and $r_{\prime\prime}$ be the distances T N, N U, and $\theta_{\prime}$, $\theta_{\prime\prime}$ the angles N T S, N U S; and take N k : N q :: $\frac{\mathrm{F}_{\prime}}{r_{\prime}^2} : \frac{\mathrm{F}_{\prime\prime}}{r_{\prime\prime}^2}$, and complete the parallelogram N $k n q$. Then N n is the position of the needle. Produce it to meet the axis T U in S; and draw the perpendiculars T K and U L upon N S. Then by the composition of forces

$$\frac{\sin \Delta_{\prime}}{\sin \Delta_{\prime\prime}} \left[= \frac{\sin p \,\mathrm{N}\, n}{\sin q \,\mathrm{N}\, n}\right] = \frac{\mathrm{N}p}{\mathrm{N}q} = \frac{\frac{\mathrm{F}_{\prime\prime}}{r_{\prime\prime}^2}}{\frac{\mathrm{F}_{\prime}}{r_{\prime}^2}} = \frac{\mathrm{F}_{\prime\prime}}{\mathrm{F}_{\prime}} \cdot \frac{r_{\prime}^2}{r_{\prime\prime}^2}. \qquad (42.)$$

Denote now the angle N S T by Σ, and T S, U S by t and u respectively. Then

$$\frac{\sin \Delta_{\prime}}{\sin \Delta_{\prime\prime}} = \frac{\frac{t}{r_{\prime}} \sin \Sigma}{\frac{u}{r_{\prime\prime}} \sin \Sigma} = \frac{t\, r_{\prime\prime}}{u\, r_{\prime}}; \qquad (43.)$$

or by comparing (42.) and (43.), we obtain,

$$\frac{t}{u} = \frac{\mathrm{F}_{\prime\prime}}{\mathrm{F}_{\prime}} \cdot \frac{r_{\prime}^3}{r_{\prime\prime}^3}, \qquad (44.)$$

a relation which, when $\mathrm{F}_{\prime} \pm \mathrm{F}_{\prime\prime} = 0$, is already known*.

* Vide LESLIE's Geometrical Analysis, art. "Magnetic Curves"; or Mr. BARLOW's Treatise on Magnetism in the Encyclopædia Metropolitana, p. 794.

Represent now the perpendiculars by $p_{\prime}$ and $p_{\prime\prime}$, and then, by the common formula for the inclination of the tangent to the radius-vector, we have

$$p_{\prime} = \frac{r_{\prime}^2 \, d\,\theta_{\prime}}{d\,s} \text{ and } p_{\prime\prime} = \frac{r_{\prime\prime}^2 \, d\,\theta_{\prime\prime}}{d\,s}.$$

Hence

$$\frac{p_{\prime}}{p_{\prime\prime}} = -\frac{r_{\prime}^2 \, d\,\theta_{\prime}}{r_{\prime\prime}^2 \, d\,\theta_{\prime\prime}} = \frac{t}{u} = \frac{\mathrm{F}_{\prime\prime} r_{\prime}^3}{\mathrm{F}_{\prime} r_{\prime\prime}^3},$$

or

$$\mathrm{F}_{\prime} r_{\prime\prime} \, d\,\theta_{\prime} + \mathrm{F}_{\prime\prime} r_{\prime} \, d\,\theta_{\prime\prime} = 0\,;$$

or again, since $\frac{r_{\prime}}{r_{\prime\prime}} = \frac{\sin\theta_{\prime}}{\sin\theta_{\prime\prime}}$, it becomes finally

$$\mathrm{F}_{\prime} \sin\theta_{\prime} \, d\,\theta_{\prime} + \mathrm{F}_{\prime\prime} \sin\theta_{\prime\prime} \, d\,\theta_{\prime\prime} = 0,$$

the integral of which, as before, is

$$\mathrm{F}_{\prime} \cos\theta_{\prime} + \mathrm{F}_{\prime\prime} \cos\theta_{\prime\prime} = 2\cos\beta'.$$

If now we take $\mathrm{F}_{\prime} + \mathrm{F}_{\prime\prime} = 0$, or $\mathrm{F}_{\prime} = -\mathrm{F}_{\prime\prime}$, we shall have our formula simplified, at the same time that we adopt the hypothesis which seems best to accord with all we yet know of the disposition of the forces in the artificial and also in a natural magnet; and hence it is the most appropriate assumption we can make respecting the constitution of the terrestrial magnet itself. We shall thus have

$$\cos\theta_{\prime} - \cos\theta_{\prime\prime} = 2\cos\beta,$$

where $\cos\beta = \mathrm{F}_{\prime} \cos\beta$; or, since one of these angles is *external* to the triangle N T U formed by the magnet and its polar distances from N, we may substitute instead of $\theta_{\prime\prime}$ its supplement, and then the last equation will take the form

$$\cos\theta_{\prime} + \cos\theta_{\prime\prime} = 2\cos\beta. \qquad (45.)$$

This property was originally given by Professor PLAYFAIR in Professor ROBISON'S article "Magnetism," published in the First Supplement to the Encyclopædia Britannica. See also ROBISON'S Mechanical Philosophy, vol. iv. p. 350. Professor ROBISON, from one or two passages in his writings, seems to have entertained some idea that these curves could be rendered available to an explanation of the phenomena of terrestrial magnetism; but I do not recollect that either he or any one else has suggested *how* this was to be accomplished, nor, much less, attempted to actually accomplish it by such means.

XV.—If, still considering the axis of the magnet as the axis of x, we conceive the magnetic curve to revolve about that axis so as to describe a surface of revolution, its equation obviously is obtained by putting $y^2 + z^2$ instead of y^2 in the equation of the generating curve. That is, the magnetic surface is expressed by the equation

$$\frac{x+a}{\sqrt{y^2+z^2+(x+a)^2}} - \frac{x-a}{\sqrt{y^2+z^2+(x-a)^2}} = \frac{c}{a}. \qquad (46.)$$

In this equation of the magnetic surface, a, or half the length of the magnet, is constant, and c is arbitrary, giving different curves according as the value of that parameter is varied.

Again, since we can refer this surface, or any one of its meridional sections, to any new system of coordinates, we may conceive the last equation (46.) to be so transformed as to represent the same geometrical surface that it now does whilst it is referred to the centre of the earth as the origin, and to the mutual intersections of the equator and two rectangular meridians as axes of coordinates. This transformation, however, by the usual processes, would be extremely difficult—perhaps impracticable —on account of the labour it would require; but this labour may be almost wholly avoided by means of the property (PLAYFAIR'S) of the curve referred to the polar angles expressed in equation (45.).

Let the coordinates of the poles T and U referred to the above-named axes be $a_, b_, c_,$ and $a_{,,} b_{,,} c_{,,}$ respectively; and the individual curve defined by the parameter β. Then viewing $\theta_,$ and $\theta_{,,}$ as the two *internal* angles, we shall have

$$\cos\theta_, = \frac{(x-a_,)^2+(y-b_,)^2+(z-c_,)^2+(a_,-a_{,,})^2+(b_,-b_{,,})^2+(c_,-c_{,,})^2-(x-a_{,,})^2-(y-b_{,,})^2-(z-c_{,,})^2}{2\sqrt{\{(a_,-a_{,,})^2+(b_,-b_{,,})^2+(c_,-c_{,,})^2\}\times\{(x-a_,)^2+(y-b_,)^2+(z-c_,)^2\}}}$$

$$= \frac{(a_,-x)(a_,-a_{,,})+(b_,-y)(b_,-b_{,,})+(c_,-z)(c_,-c_{,,})}{\sqrt{(a_,-a_{,,})^2+(b_,-b_{,,})^2+(c_,-c_{,,})^2}\times\sqrt{(x-a_,)^2+(y-b_,)^2+(z-c_,)^2}} \quad \ldots\ldots \quad (47.)$$

and

$$\cos\theta_{,,} = -\frac{(a_{,,}-x)(a_,-a_{,,})+(b_{,,}-y)(b_,-b_{,,})+(c_{,,}-z)(c_,-c_{,,})}{\sqrt{(a_,-a_{,,})^2+(b_,-b_{,,})^2+(c_,-c_{,,})^2}\times\sqrt{(x-a_{,,})^2+(y-b_{,,})^2+(z-c_{,,})^2}}. \quad \ldots \quad (48.)$$

From (4.), (12.), and (13.), we have at once the equation of the surface, viz.

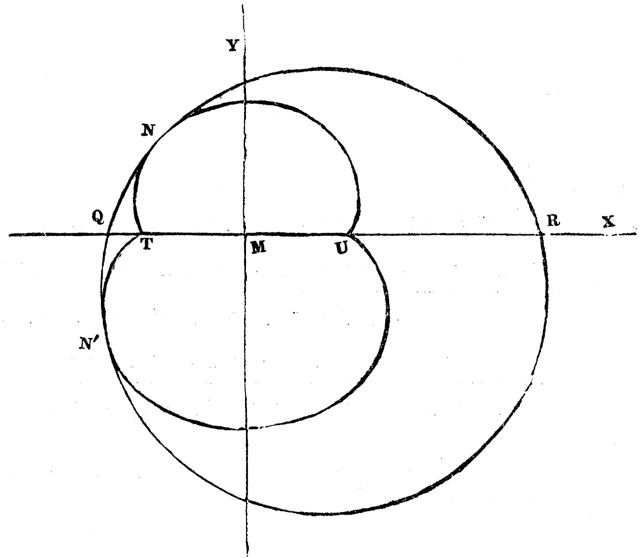
$$\left.\begin{array}{l} \dfrac{(x-a_,)(a_,-a_{,,})+(y-b_,)(b_,-b_{,,})+(z-c_,)(c_,-c_{,,})}{\sqrt{(x-a_,)^2+(y-b_,)^2+(z-c_,)^2}} \\ \quad -\dfrac{(x-a_{,,})(a_,-a_{,,})+(y-b_{,,})(b_,-b_{,,})+(z-c_{,,})(c_,-c_{,,})}{\sqrt{(x-a_{,,})^2+(y-b_{,,})^2+(z-c_{,,})^2}} \\ =\pm 2\cos\beta\sqrt{(a_,-a_{,,})^2+(b_,-b_{,,})^2+(c_,-c_{,,})^2}=\pm 4a\cos\beta \end{array}\right\} \quad \ldots\ldots \quad (49.)$$

This equation, deprived of its radicals and denominators, like the equation of the generating curve, is of the *eighth order*.

Now by varying the parameter β (which defines the particular surface upon which the point we are considering is situated,) by minute increments, we shall have a series of thin strata, each of which is isolated and independent, and which, collectively, extend through all space. If, therefore, we conceive these strata to be infinitesimally thin, we may consider the magnetic influence extended over a series of surfaces, each of which has the property of being *touched* by a small needle placed at any point in that surface. If, moreover, we conceive the sphere and each of these surfaces to be cut by a series of meridian planes passing through the magnetic axis, we shall always find some one magnetic curve on each side of the magnetic axis which will touch the two segments into which the magnetic axis (prolonged if necessary) divides the circle lying in that magnetic meridian plane. At these points the needle will touch the sphere, or, which is the same thing, it will be horizontal to the earth; and

as there will be such points in each of the consecutive meridional sections, there will be a series of consecutive points on the surface of the earth at which the needle will be horizontal. These points lie in a continuous curve, which has been called the *magnetic equator*; and we proceed to inquire into the character of that curve.

XVI.—*On the Magnetic Equator.*



Let T U, as before, be the poles of the terrestrial magnet, and TNU that one of the magnetic curves which touches the corresponding magnetic meridian of the earth QNR in N.

Put for the moment the equation of the circle QNR under the general form

$$(x-c)^2+(y-b)^2=r^2; \quad . \ . \ (50.)$$

and denoting by $g_{\prime}0$ and $g_{\prime\prime}0$ the coordinates of Q and R respectively, we shall find

$$c=\frac{g_{\prime}+g_{\prime\prime}}{2}, \text{ and } r^2=b^2+\left(\frac{g_{\prime}-g_{\prime\prime}}{2}\right)^2. \quad . \ . \ . \ . \ . \ . \ . \ . \ . \quad (51.)$$

Inserting (51.) in (50.), and reducing the equation, the circle is finally expressed by

$$x^2-\overline{g_{\prime}+g_{\prime\prime}}\,x+g_{\prime}g_{\prime\prime}+y^2-2by=0. \quad . \ . \ . \ . \ . \ . \ . \quad (52.)$$

This involves the arbitrary quantity b, which is the parameter upon which the identical circle depends.

Since this circle is to *touch* the magnetic curve at some point N, the values of $\frac{dy}{dx}$ derived from the equations of the circle and magnetic curve at their common point (xy) must be equal; and as the arbitrary constant in the equation of the magnetic curve vanishes by differentiation, we shall have three equations between which to eliminate the indeterminate quantities $\frac{dy}{dx}$ and b. This elimination will leave one equation between x and y, which will designate the locus of the point N.

By differentiating (52.) we obtain

$$\frac{dy}{dx}=-\frac{2x-\overline{g_{\prime}+g_{\prime\prime}}}{2(y-b)}. \quad . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \quad (53.)$$

And the differential equation of the magnetic curve is, from (38.) and $F_{\prime}+F_{\prime\prime}=0$,

$$\frac{dy}{dx}=\frac{\frac{y}{r_{\prime}^3}-\frac{y}{r_{\prime\prime}^3}}{\frac{x+a}{r_{\prime}^3}-\frac{x-a}{r_{\prime\prime}^3}}. \quad . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \quad (54.)$$

Also, from (52.),

$$2(y-b)=\frac{y^2-x^2+\overline{g_{\prime}+g_{\prime\prime}}\,x-g_{\prime}g_{\prime\prime}}{y}. \quad \ldots \ldots \quad (55.)$$

Insert (55.) in (53.); then there results, after combining (53.) and (54.) and slightly reducing

$$\frac{(r_{\prime\prime}^3-r_{\prime}^3)\,y}{(x+a)\,r_{\prime\prime}^3-(x-a)\,r_{\prime}^3}=\frac{(\overline{g_{\prime}+g_{\prime\prime}}-2\,x)\,y}{y^2-x^2+\overline{g_{\prime}+g_{\prime\prime}}\,x-g_{\prime}g_{\prime\prime}}, \quad \ldots \quad (56.)$$

which is a rectangular equation to the curve which is traced out by N. It separates at once into the two components

$$y=0 \quad \ldots \ldots \ldots \quad (57.)$$

$$\frac{r_{\prime\prime}^3-r_{\prime}^3}{(x+a)\,r_{\prime\prime}^3-(x-a)\,r_{\prime}^3}=\frac{\overline{y_{\prime}+g_{\prime\prime}}-2\,x}{y^2-x^2+\overline{g_{\prime}+g_{\prime\prime}}\,x-g_{\prime}g_{\prime\prime}}, \quad \ldots \quad (58.)$$

which last is readily reduced to

$$\{y^2+(x+a)^2-\overline{a+g_{\prime}}\,.\,\overline{a+g_{\prime}}\}\,r_{\prime\prime}^3=\{y^2+(x-a)^2-\overline{a-g_{\prime}}\,.\,\overline{a-g_{\prime\prime}}\}\,r_{\prime}^3. \quad (59.)$$

Squaring both sides, and restoring the values of $r_{\prime}$ and $r_{\prime\prime}$, it becomes an equation of the ninth order; in which, arranging according to powers of x or y, the coefficients become very complex, and altogether unmanageable by any of the usual methods. It is of the form

$$(x-\overline{g_{\prime}+g_{\prime\prime}})\,y^8+\mathrm{A}\,y^6+\mathrm{B}\,y^4+\mathrm{C}\,y^2+\mathrm{D}=0, \quad \ldots \quad (60.)$$

where A, B, C, D are functions of x, which in all cases render the terms not higher than of the ninth degree.

XVII.—Though we cannot completely discuss the course of the curve and the character of its singular points by means of this equation, we may yet learn some particulars of its general features with considerable facility; and as they will be of great use to us in our future inquiries, we shall insert them here.

1. Since y appears only in *even* powers, the curve is composed of pairs of branches, such that the branches in each pair are equal and symmetrically disposed with respect to the axis of x.

2. Since x appears of an *odd* degree, there will be at least one real value of x for every value of y, whether y be positive or negative. There will at least be one pair of equal and symmetrical branches, and these branches will be infinite ones.

3. Since (y^2) appears of the *fourth* degree, there may possibly be four values of y^2 for every specific value of x; but there cannot possibly be an odd number. Of these four possible roots any number may be *minus*, and the corresponding values of y itself be still impossible or imaginary. But by art. 2. there must be at least one pair of real values of y, there must be at least *two* real values of y^2, one of which must be $+$, for every value whatever of x.

4. Also four is the greatest number of pairs of symmetrical branches that can exist.

5. If the system be made to revolve round the axis of symmetry (that of x), it will

generate as many sheets of surface as there are branches in the curve; but as the symmetrical branches of the curve generate superposed sheets of surface, or sheets which are geometrically identical, the number actually described is only half as great as the number of branches. There may hence be one, two, three, or four sheets of surface generated, according as one, two, three, or four of the values of y^2 are real and positive; and upon any point in one of these a minute needle being placed, and a circle described through that point and the points Q R, the needle will find its repose in that plane, and be a tangent to the circle at that point.

6. The intersection of these four sheets of surface with the earth's surface would give four separate and continuous lines upon the surface of our globe, upon any point of which the needle being placed, it would be horizontal. In other words, there may be four such lines as that which has been denominated the magnetic equator.

7. Observation, however, seems to indicate only one single branch of this intersection; though it must be confessed that the greater part of the observations, and the mode of determination of the position of the equator, are far from satisfactory. The great difficulty of procuring good instruments, and the almost equally great difficulty in making a correct observation with any instrument whatever, at places which would give results free from suspicion of foreign and local sources of error; the extremely small number of observations actually attempted, and the very hypothetical character of the formula by which the equator is determined from observations made on either side at the distance of a few degrees; all these reasons, and others, render the delineation of this line, as laid down by M. Morlet, very far from satisfactory. The four branches may indeed be easily conceived to lie so near to one another, that of points which have been actually observed or inferred from observation and theoretical reductions, some might be in one and others in other branches of the fourfold system of lines; and hence that the spherical polygon traced through these might not be in reality composed of chords of any one single branch of the system.

8. It therefore becomes necessary to examine the curve more minutely as to the number and circumstances of the branches of which it is actually composed on any hypothesis which is consistent with the other phenomena to be accounted for. Since, however, the general equation in terms of x and y is altogether unfitted for our purpose, and the equation between the radius vector and polar angle offers no simplification in the form of the expression, I was led to attempt it by examining the relation which subsists between the angles made by radiants from the poles to points in the curve, with the line joining the poles. This also proved almost equally useless in respect to the purpose I had in view; but upon trying the radiants themselves the object was completely attained.

Resuming equation (59.), and recollecting that

$$y^2 + (x + a)^2 = r_{\prime}^2$$

$$y^2 + (x - a)^2 = r_{\prime\prime}^2,$$

and putting also for abbreviation (and at the same time retaining the homogeneity of the equations into which these terms enter,)

$$\overline{a+g_{,}}\,.\,\overline{a+g_{,,}} = k^2$$
$$\overline{a-g_{,}}\,.\,\overline{a-g_{,,}} = h^2,$$

we have (59.) converted into

$$(r_{,}^2 - h^2)\, r_{,,}^3 - (r_{,,}^2 - k^2)\, r_{,}^3 = 0\,; \qquad (61.)$$

or, arranging them in reference to $r_{,,}$, it is

$$r_{,,}^3 - \frac{r_{,}^3}{r_{,}^2 - h^2}\,.\,r_{,,}^2 = -\frac{k^2\, r_{,}^3}{r_{,}^2 - h^2}. \qquad (62.)$$

For $r_{,,}$ write $r + \frac{r_{,}^3}{3\,(r_{,}^2 - h^2)}$, and the equation (62.) becomes

$$r^3 - \frac{r_{,}^6\,.\,r}{3\,(r_{,}^2 - h^2)^2} = \frac{2\, r_{,}^9}{27\,(r_{,}^2 - h^2)^3} - \frac{k^2\, r_{,}^3}{r_{,}^2 - h^2}, \qquad (63.)$$

which is a cubic equation wanting the second term, and which for accommodation to the usual notation may be written for the moment thus:

$$r^3 - 3\, c\, r = 2\, d.$$

Then we have

$$c^3 + d^2 = \frac{27\, k^2\, r_{,}^6\, \{-4\, r_{,}^6 + 27\, k^2\, (r_{,}^2 - h^2)^2\}}{4 \times 27^2\, (r_{,}^2 - h^2)^4}. \qquad (64.)$$

Now in the case before us, putting $g_{,} = -(a + a_{,})$ and $g_{,,} = (a + a_{,,})$, which, since the poles of the terrestrial magnet being either *within* or *upon the surface* of the earth is always the case in nature, we have

$$\left.\begin{array}{l} h^2 = \overline{a - g_{,}}\,.\,\overline{a - g_{,,}} = -\,a_{,,}\,(2\,a + a_{,}) \\ k^2 = \overline{a + g_{,}}\,.\,\overline{a + g_{,,}} = -\,a_{,}\,(2\,a + a_{,,}) \end{array}\right\}, \qquad (65.)$$

and which equations, since a, $a_{,}$, and $a_{,,}$ are essentially $+$, are themselves essentially $-$. These values of h^2 and k^2 inserted in (30.) render the whole value of $c^3 + d^2$ essentially $+$, whatever the value of $r_{,}$ may be. There is hence one pair of symmetrical branches indicated by this method also, as in the former. But in addition to this we learn at once that there is *only one* such pair; since when the root is given by CARDAN's formula, (which is the case here,) that root is the only real one*.

9. The conclusion is now established, that there is only one sheet of the tangential surface compatible with the actual condition of the terrestrial magnet, and hence only one line of intersection between it and the earth's surface; or, in other words, *the magnetic equator is one isolated and continuous line on the surface of the earth.*

10. Also, since from other considerations it can be shown that the two poles are

* LAGRANGE's test might have been employed instead of this; but as that appears to be something more laborious, I have preferred the present one.

not in the same diameter of the earth, nor equally distant from the centre in any chord of the earth, the two axes of revolution of the sphere and tangential surface do not coincide; and hence their common intersection is not a plane, nor its trace on the sphere, that is, the magnetic equator, a circle of any magnitude whatever. It is therefore a curve of double curvature. The conclusion, therefore, deduced by BIOT from HUMBOLDT's observations, and the conclusion deduced by MORLET from the discussion of all the observations he could collect from authentic sources, are quite consistent in this respect with the hypothesis of the duality of the poles.

11. The discussion of any further cases of this problem need not be given here. In a geometrical point of view the discussion would be interesting, and under that aspect this paper would be incomplete without them; but as they have no bearing upon the main object of the present research, and are moreover so perfectly analogous to those we have just given as to offer not the slightest difficulty by the same method which has been here employed, any further notice of them would be altogether superfluous, and irrelevant to the purpose we have in view.

12. It is to be remarked, however, that these results are true only on the hypothesis of the forces in the poles being related by the equation $F_{,} \pm F_{,,} = 0$. Under any other condition than this the highest power of x would not disappear from equation (59.); and hence the equation would be of an *even* order, and hence the branches of the magnetic equator would be *two* at least, and always an *even* number. The *apparent singleness* of branches furnished by observation is a strong argument in favour of the duality of the poles and equality of their intensities; but as the method by which the magnetic equator has been laid down is far from satisfactory, too much reliance should not be placed on this argument, decisive as it otherwise would certainly be.

13. The equation of the tangential surface and the equation of the sphere would completely define that line considered in reference to rectangular coordinates; that is, in the usual manner of considering the equations of lines situated in space. In the form of the equation of the locus of N on a meridian plane marked (60.), we have only to write $y^2 + z^2$ instead of y^2, and the result is the equation of the surface referred to the axis of x and any other two axes at right angles to it and to one another. The coordinates of the centre of the sphere referred to these axes, and the radius of the sphere, being also given, we have its equation in the usual form, viz. $r^2 = (x - a_{,})^2 + (y - b_{,})^2 + (z - c_{,})^2$. By the transformation of coordinates we can change the axes of reference to any given axes, as, for instance, to the polar axis and the intersection of the equator by two rectangular meridians. In the next place, to adapt the expression to the usual mode of denoting spherical position, (latitude and longitude,) we must transform this into a polar equation, and put the radius vector of the resulting equation constant and equal to the terrestrial radius. The equation thus obtained will be one between the latitude and longitude of the points which constitute the magnetic equator.

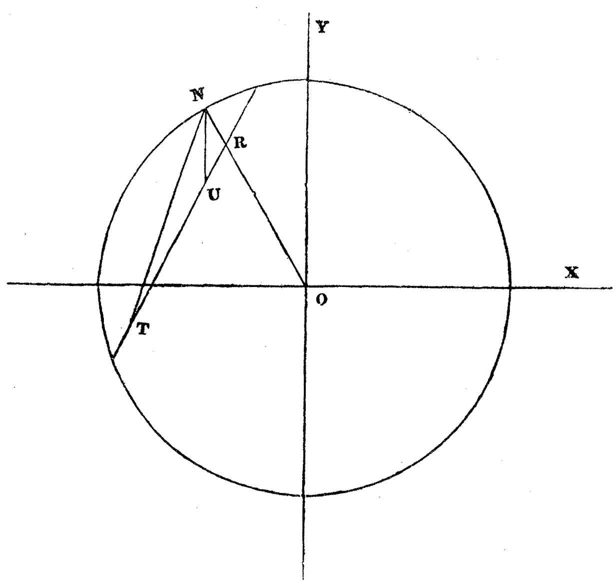
The state of the physical problem is not at present such as to render any further mathematical details respecting this curve necessary in this place.

XVIII.—*On the Points of the Earth's Surface at which the Needle takes a position vertical to the Horizon.*

As our hypothesis is that of two poles, or resultant centres of force, the freely-suspended magnetic needle will always lie in the plane which passes through its own centre and the centres of magnetic force. The dipping-needle lies, therefore, wholly in the plane passing through the place of observation and the true magnetic poles. But when the needle is vertical to the horizon, it passes through the centre of the earth; and hence the plane of the magnetic meridian also passes through the centre, and makes with the sphere a section, which is a great circle. Also, as this plane then passes through three points not in a right line, it is unique; or, in other words, there is only one circle of the sphere in which the needle can be placed to be capable of taking a vertical direction, and that is a great circle.

It is also obvious, from the expressions already given for the inclination of the tangent to the radiants from the poles to points in the magnetic curve, that there are only isolated points in that circumference in which the phenomenon of verticity can take place; and it is our business in this section to inquire into their possible number, and the method of determining their actual number and their respective positions.

XIX.—Let O be the centre of the great circle in which we have just shown all the possible vertical needles must lie, and T U the magnetic poles, and N one of these points. Take the centre of the circle as origin of coordinates.



Let T U be denoted by the coordinates $a_{\prime}\ b_{\prime}$ and $a_{\prime\prime}\ b_{\prime\prime}$ respectively; then the equation of the magnetic axis is

$$(x-a_{\prime})(b_{\prime\prime}-b_{\prime})=(y-b_{\prime})(a_{\prime\prime}-a_{\prime}) \quad (66.)$$

Also the equation of O N is

$$xy'-x'y=0, \quad . \quad . \quad . \quad (67.)$$

where $x'\,y'$ are the coordinates of N.

The intersection of these gives the coordinates of R, the point where the tangent to the magnetic curve at N, and whose poles are T and U, intersects the magnetic axis.

From (66.) we have

$$y = \frac{b_{\prime\prime} - b_{\prime}}{a_{\prime\prime} - a_{\prime}} x - \frac{b_{\prime\prime} a_{\prime} - b_{\prime} a_{\prime\prime}}{a_{\prime\prime} - a_{\prime}}, \qquad (68.)$$

and from (69.) we have

$$y = \frac{y'}{x_{\prime}} x, \qquad (69.)$$

which two equations, (68.) (69.), equated, give, after simple reduction,

$$\left.\begin{aligned} x &= \frac{(a_{\prime} b_{\prime\prime} - a_{\prime\prime} b_{\prime}) x_{\prime}}{(b_{\prime\prime} - b_{\prime}) x' - (a_{\prime\prime} - a_{\prime}) y'}, \\ \text{and hence also} \quad y &= \frac{(a_{\prime} b_{\prime\prime} - a_{\prime\prime} b_{\prime}) y'}{(b_{\prime\prime} - b_{\prime}) x' - (a_{\prime\prime} - a_{\prime}) y'}. \end{aligned}\right\} \qquad (70.)$$

Again, we have,

$$\left.\begin{aligned} \mathrm{T\,R}^2 &= \left\{ a_{\prime} - \frac{(a_{\prime} b_{\prime\prime} - a_{\prime\prime} b_{\prime}) x'}{(b_{\prime\prime} - b_{\prime}) x' - (a_{\prime\prime} - a_{\prime}) y'} \right\}^2 + \left\{ b_{\prime} - \frac{(a_{\prime} b_{\prime\prime} - a_{\prime\prime} b_{\prime}) y'}{(b_{\prime\prime} - b_{\prime}) x' - (a_{\prime\prime} - a_{\prime}) y'} \right\}^2 \\ &= \frac{\{(a_{\prime\prime} - a_{\prime})^2 + (b_{\prime\prime} - b_{\prime})^2\} (b_{\prime} x' - a_{\prime} y')^2}{\{(b_{\prime\prime} - b_{\prime}) x' - (a_{\prime\prime} - a_{\prime}) y'\}^2}. \end{aligned}\right\} \qquad (71.)$$

And in the same manner we obtain

$$\mathrm{R\,U}^2 = \frac{\{(a_{\prime\prime} - a_{\prime})^2 + (b_{\prime\prime} - b_{\prime})^2\} (b_{\prime\prime} x' - a_{\prime\prime} y')}{\{(b_{\prime\prime} - b_{\prime}) x' - (a_{\prime\prime} - a_{\prime}) y'\}^2} \qquad (72.)$$

Also

$$\mathrm{T\,N}^2 = (a_{\prime} - x')^2 + (b_{\prime} - y')^2 \qquad (73.)$$

$$\mathrm{U\,N}^2 = (a_{\prime\prime} - x')^2 + (b_{\prime\prime} - y')^2. \qquad (74.)$$

But by the property of the magnetic curve, expressed in equation (44.), we have

$$\frac{\mathrm{R\,U}}{\mathrm{R\,T}} = \frac{\mathrm{U\,N}^3}{\mathrm{T\,N}^3},$$

or

$$\frac{b_{\prime} x' - a_{\prime} y'}{b_{\prime\prime} x' - a_{\prime\prime} y'} = \left\{ \frac{(a_{\prime} - x')^2 + (b_{\prime} - y')^2}{(a_{\prime\prime} - x')^2 + (b_{\prime\prime} - y')^2} \right\}^{\frac{3}{2}} \qquad (75.)$$

This is the equation of the curve of contact of the tangent from O to the magnetic curve, whose poles are T and U, with the curve; and in rectangular coordinates is, like the magnetic curve itself, of the eighth order when freed from fractions and radicals.

Having now eliminated $x\,y$ from the preceding equations in which they appeared, we may drop the distinguishing accents from $x'\,y'$ in (75.) and reduce the fractions and radicals. We thus obtain

$$(b_{\prime} x - a_{\prime} y)^2 \{(a_{\prime\prime} - x)^2 + (b_{\prime\prime} - y)^2\}^3 = (b_{\prime\prime} x - a_{\prime\prime} y)^2 \{(a_{\prime} - x)^2 + (b_{\prime} - y)^2\}^3 . \qquad (76.)$$

And, as the intersection of this curve with the circle gives the points concerning which this inquiry is instituted, we may write the circle at once, its radius being r,

$$x^2 + y^2 = r^2. \qquad (77.)$$

The determination of x and y from these two equations will require the solution of an equation of the tenth degree. For putting (76.) under the form

$$\begin{aligned}&(b_{,}x - a_{,}y)^2 [a_{,,}^2 + b_{,,}^2 + x^2 + y^2 - 2(a_{,,}x + b_{,,}y]^3 \\ &\quad = (b_{,,}x - a_{,,}y)[a_{,}^2 + b_{,}^2 + x^2 + y^2 - 2(a_{,}x + b_{,}y)]^3,\end{aligned}$$

we have it converted at once, by means of (77.), into

$$\left.\begin{aligned}&(b_{,}x - a_{,}y)^2 [a_{,,}^2 + b_{,,}^2 + r^2 - 2(a_{,,}x + b_{,,}y)]^3 \\ &= (b_{,,}x - a_{,,}y)[a_{,}^2 + b_{,}^2 + r^2 - 2(a_{,}x + b_{,}y)]^3.\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (78.)$$

Hence by means of (77.), which is of the second, and (78.), which is of the fifth, degree, we obtain an equation of the tenth, from which to determine x or y, and hence to find the points at which the needle will be vertical. Still the reduction is extremely laborious, and hence, also, our means of determining how many of the roots are real, and how many are imaginary; that is to say, how many of its roots are compatible, *simultaneously or separately*, with the values to which $a_{,}$ $b_{,}$ and $a_{,,}$ $b_{,,}$ are limited by the physical which must be appended to the algebraical conditions from which the equations (77.) and (78.) were formed. Those conditions are $r^2 > a_{,}^2 + b_{,}^2$, and $r^2 > a_{,,}^2 + b_{,,}^2$, that is, of the poles being *within* the earth; but whether the perpendicular from the centre of the earth upon the magnetic axis intersects that axis *between* the poles or not, cannot be, *à priori*, nor yet from any knowledge furnished by experiment, at present determined. As, however, all the cases that can arise from all possible positions of the two poles are included in the above formulæ, and as they evidently cannot *simultaneously* exist, we are entitled to infer that all the roots are not simultaneously real. In the absence, however, of these considerations, we learn that there are not more than ten points on the surface of the earth at which the needle can be vertical; and that whatever may be the number of them, it is at all events even, viz. 2, 4, 6, 8, or 10. We also learn that how many soever of these be real, they are all in one plane passing through the centre of the earth; and with respect to the great circle in which it cuts the terrestrial surface, taken as the axis of spherical coordinates, *all magnetic phenomena on the surface of the earth are symmetrically disposed.* How far this is verified, within the limits of errors of observation and of local interference with the full development of the effects of the magnetic force, has not yet been inquired into. Indeed, till this plane has been determined it would be impossible to conduct the inquiry in a direct manner; and as only one of these points is yet actually assigned with any close degree of approximation, we are not yet in a condition to enter upon the inquiry. Still, these facts combined with observations relative to other phenomena, especially respecting dip and intensity, (the variation for obvious geometrical reasons included,) accurately made, might furnish important aid in a tentative determination of the plane itself; the method of proceeding in which must be sufficiently obvious to those inquirers to whose minds the geometry of coordinates is familiar.

In conclusion of the present paper I shall, though I have not been able to decompose the equation which results from (77.) and (78.) into factors, yet hazard the conjecture of its being even in its *literal* form capable of such a resolution; so that the component equations are, when viewed simultaneously, the one essentially imaginary with the values which render the other real. Of course these must be into factors of even degrees,—probably two of the fourth and one of the second. Several instances of this kind are well known to geometers; a very remarkable one of which is the expression given by M. BRET for the determination of the foci of a line of the second order, in GERGONNE'S *Annales des Mathématiques*, tom. viii. I offer it, however, only as a conjecture, which future researches may show, after all, to be too hastily made.

Royal Military Academy, Woolwich,
December 16, 1834.