XI. *Analysis of the Roots of Equations. By the Rev.* R. Murphy, *M.A., Fellow of Caius College, Honorary Member of various Philosophical Societies. Communicated by* J. W. Lubbock, *Esq. F.R.S.*

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1. The object of this memoir is to show how the constituent parts of the roots of algebraical equations may be determined, by considering the conditions under which they vanish, and conversely to show the signification of each such constituent part.

2. In equations of degrees higher than the second the same constituent part of the root is found in several places governed by the same radical sign, but affected with the different corresponding roots of unity as multipliers.

3. The root of every equation, of which the coefficients are rational, contains a rational part, for the sum of the roots could not otherwise be rational.

This rational part, as such, is insusceptible of change in the different roots of the same equation, consequently its value is the coefficient of the second term (with a changed sign) divided by the number of roots, or index of the first term.

4. The supposed evanescence of any of the other constituent parts implies that a relation exists between the roots; if such a relation be expressed by equating a function of the roots to zero, that constituent part will be the product of all such functions, and a numerical factor.

5. The joint evanescence of various constituent parts implies the co-existence of various relations between the roots, and thus an interpretation may be given to each of the constituent parts, riveting the expression of the root in the memory, and beautifully converting the solution of a problem into a condensed enunciation of various theorems.

For simplicity these principles are first applied to equations of lower degrees.

6. Let us take for example the general quadratic equation

$$x^2 + a\,x + b,$$

the two roots of which are represented by x_1, x_2 in the formulæ,

$$x_1 = -\frac{a}{2} + \sqrt{\alpha}$$

$$x_2 = -\frac{a}{2} - \sqrt{\alpha}.$$

To find α suppose $\alpha = 0$, a relation is then established between the quantities x_1 and x_2, viz.

$$x_1 - x_2 = 0,$$

$x_1 - x_2$ vanishing with α is a factor of it; and since the roots must be symmetrically involved in α, the other factor is $x_2 - x_1$, whence

$$\alpha = k\,(x_1 - x_2)^2,$$

where k is simply a number.

Put for $-\frac{a}{2}$ its expression in terms of the roots, and we thus have

$$x_1 = \frac{x_1 + x_2}{2} + \frac{x_1 - x_2}{2} \cdot \sqrt{4\,k},$$

$$x_2 = \frac{x_1 + x_2}{2} - \frac{x_1 - x_2}{2} \sqrt{4\,k},$$

whence

$$x_1 - x_2 = (x_1 - x_2)\ \sqrt{4\,k}, \text{ or } k = \frac{1}{4};$$

then

$$\alpha = \frac{1}{4}\,(x_1 - x_2)^2,$$

which is a symmetrical function, and therefore easily expressed by the coefficients.

7. From this it follows that $\alpha = 0$ is the condition that the equation may have two equal roots; but if the proposed quadratic be represented by $\varphi = 0$, and its derived equation by $\varphi' = 0$, which is the same as $2\,x + a = 0$, the condition for two equal roots is obtained by eliminating x between these equations, which by the theory of elimination gives as the sought condition

$$\varphi'\,(x_1) \cdot \varphi'\,(x_2) = 0;$$

we have thus

$$\alpha = k'\,\varphi'\,(x_1) \cdot \varphi'\,(x_2),$$

k' being numerical.

Now $\varphi\,(x) = (x - x_1)\,(x - x_2)$, therefore $\varphi'\,(x) = (x - x_1) + (x - x_2)$, whence

$$\varphi'\,(x_1) = x_1 - x_2, \quad \varphi'\,(x_2) = x_2 - x_1,$$

consequently $k' = -\frac{1}{4}$; and converting the sum and product of x_1, x_2 into the coefficients, we obtain

$$x_1 = -\frac{a}{2} + \sqrt{\left(\frac{a^2}{4} - b\right)} \qquad x_2 = -\frac{a}{2} - \sqrt{\left(\frac{a^2}{4} - b\right)}.$$

8. The constituent parts in the roots which have been the objects of investigation were $-\frac{a}{2}$ and α, and with respect to their evanescence we have the following theorem.

The vanishing of that part of the root of a quadratic which is under the radical sign implies the existence of two equal roots.

The vanishing of the other part which is unaffected by that sign signifies that the roots are equal, but with contrary signs.

By the aid of this theorem we shall be able to find two of the three constituent parts of the roots of a cubic equation.

9. Let us now extend the same views to equations of the third degree, and let $x_1\, x_2\, x_3$ be the three roots of the cubic $x^3 + a\,x^2 + b\,x + c = 0$. Put

$$x_1 = -\frac{a}{3} + \sqrt[3]{\alpha} + \sqrt[3]{\beta}$$

$$x_2 = -\frac{a}{3} + \theta\sqrt[3]{\alpha} + \theta^2\sqrt[3]{\beta}$$

$$x_3 = -\frac{a}{3} + \theta^2\sqrt[3]{\alpha} + \theta\sqrt[3]{\beta}.$$

The numbers θ, θ^2 are the imaginary cube roots of unity, and we may observe that the formulæ for x_3, x_1 differ only from that for x_2 in having θ^2, θ^3 respectively instead of θ.

10. The quantities α, β, which are obviously similarly involved, are the roots of a quadratic, and of the forms

$$\alpha = \alpha' + \sqrt{\alpha''}$$
$$\beta = \alpha' - \sqrt{\alpha''};$$

the three quantities $-\frac{a}{3}$, α' and α'' are the constituent parts of the roots of the cubic in the sense in which those words have been used; the first is the same as $\frac{x_1 + x_2 + x_3}{3}$, and the other two can be found from the conditions of their evanescence as follows.

11. Suppose $\alpha'' = 0$, the theorem of art. 8. gives us $\alpha = \beta$, whence we find $x_2 = x_3$; now $x_2 - x_3$ vanishing with α'' is a factor of it; and the other symmetrical factors are $x_3 - x_2$, $x_1 - x_3$, $x_3 - x_1$, $x_1 - x_2$, $x_2 - x_1$, k being a number, we must therefore have

$$\alpha'' = k\,(x_1 - x_2)^2\,(x_1 - x_3)^2\,(x_2 - x_3)^2.$$

12. To find α' in like manner, suppose $\alpha' = 0$ the theorem of art. 8 before referred to, in this case makes $\alpha = -\beta$, and the three roots of the cubic accordingly are changed to the following:

$$x_1 = -\frac{a}{3}$$

$$x_2 = -\frac{a}{3} + (\theta - \theta^2)\sqrt[3]{\alpha}$$

$$x_3 = -\frac{a}{3} - (\theta - \theta^2)\sqrt[3]{\alpha};$$

whence we readily find that $2\,x_1 = x_2 + x_3$, therefore $2\,x_1 - x_2 - x_3$ is a factor of α', and the other symmetrical factors are $2\,x_2 - x_1 - x_3$, and $2\,x_3 - x_1 - x_2$; hence if k' be a numerical factor

$$\alpha' = k'\,(2\,x_1 - x_2 - x_3)\,(2\,x_2 - x_1 - x_3)\,(2\,x_3 - x_1 - x^2);$$

these symmetrical functions can be expressed by the given coefficients of the equation.

13. The constants k, k' may be easily found in various ways, perhaps the simplest means is to suppose $\alpha' = 0$ to find k, and $\alpha'' = 0$ to find k'.

If we put $x_3 = 0$ and $x_1 = 2\,x_1$, we have $\alpha' = 0$, and $\alpha'' = 4\,k\,x_1^6$, hence $\alpha = 2\,x_1^3\,\sqrt{k}$, $\beta = -\,2\,x_1^3\,\sqrt{k}$, and substituting in the formula for x_2, we have (since $a = -\,3\,x_1$)

$$2\,x_1 = x_1 + x_1\,\theta\,\sqrt[3]{2\,\sqrt{k}} - x_1\,\theta^2\sqrt[3]{2\,\sqrt{k}}$$

or

$$1 = \theta\,(1 - \theta)\,\sqrt[3]{2\,\sqrt{k}},$$

therefore

$$2\,\sqrt{k} = \frac{1}{(-\,\theta)^3} = \frac{1}{3\,(\theta^2 - \theta)}$$

and

$$4\,k = \frac{1}{9\,(\theta - 2 + \theta^2)},$$

but $1 + \theta + \theta^2 = 0$; therefore

$$k = \frac{-\,1}{2^2\,.\,3^3}.$$

14. In like manner to find k' suppose $x_2 = x_3 = 0$, which makes $\alpha'' = 0$, and $\alpha' + 2\,k'\,.\,x_1^3$, and $\alpha = \beta = \alpha'$, therefore

$$x_1 = \frac{x_1}{3} + 2\,x_1\,\sqrt[3]{2\,k'}$$

or

$$\frac{1}{3} = \sqrt[3]{2\,k'},$$

whence

$$k' = \frac{1}{2\,.\,3^3}$$

15. The preceding analysis furnishes the following formula,

$$\begin{aligned} 3\,x_1 = (x_1 + x_2 + x_3) &+ \sqrt[3]{\Big\{\tfrac{1}{2}\,(2\,x_1 - x_2 - x_3)\,(2\,x_2 - x_1 - x_3)\,(2\,x_3 - x_1 - x_2)} \\ &\qquad\qquad + \tfrac{3}{2}\,\sqrt{-\,3\,(x_1 - x_2)^2\,(x_1 - x_3)^2\,(x_2 - x_3)^2}\,\Big\} \\ &+ \sqrt[3]{\Big\{\tfrac{1}{2}\,(2\,x_1 - x_2 - x_3)\,(2\,x_2 - x_1 - x_3)\,(2\,x_3 - x_1 - x_2)} \\ &\qquad\qquad - \tfrac{3}{2}\,\sqrt{-\,3\,(x_1 - x_2)^2\,(x_1 - x_3)^2\,(x_2 - x_3)^2}\,\Big\} \end{aligned}$$

the corresponding formulæ for $3\,x_2$, $3\,x_3$ being obtained by writing θ and θ^2 before the cubic radical signs.

In consequence of the negative multiplier $-\,3$ under the sign of square root it is visible that this formula is not arithmetically applicable when the three roots are real and unequal, which is usually termed the irreducible case.

16. The cubic surds in the formula above given are actually extractible, which verifies the solution.

To this end let $2x_1 - x_2 - x_3 = A$, and $x_2 - x_3 = B$, then

$$2x_2 - x_1 - x_3 = \frac{1}{2}(3B - A)$$

$$2x_3 - x_1 - x_2 = \frac{1}{2}\cdot(-3B - A)$$

therefore

$$\frac{1}{2}(2x_1 - x_2 - x_3)(2x_2 - x_1 - x_3)(2x_3 - x_1 - x_2) = \frac{1}{8}(A^3 - 9AB^2).$$

Again

$$x_1 - x_2 = \frac{1}{2}(A - B)$$

$$x_1 - x_3 = \frac{1}{2}(A + B)$$

therefore

$$\frac{3\sqrt{-3}}{2}\cdot(x_1 - x_2)(x_1 - x_3)(x_2 - x_3) = \frac{3\sqrt{-3}}{8}(A^2B - B^3)$$

the total quantity under the first cubic surd thus becomes

$$\frac{1}{8}\{A^3 + 3A^2.(B\sqrt{-3}) + 3A.(B\sqrt{-3})^2 + (B\sqrt{-3})^3\},$$

the cube root of which is $\frac{1}{2}\{A + B\sqrt{-3}\}$, and the actual root of the second surd is similarly $\frac{1}{2}\{A - B\sqrt{-3}\}$. But

$$\frac{1}{2}(A + B\sqrt{-3}) = x_1 - x_2\cdot\frac{1-\sqrt{-3}}{2} - x_3\cdot\frac{1+\sqrt{-3}}{2}$$
$$= x_1 + x^2\theta^2 + x^3\theta.$$

And

$$\frac{1}{2}(A - B\sqrt{-3}) = x_1 + x_2\theta + x_3\theta;$$

the formulæ of this solution become then those of VAUDERMONDE, viz.

$$3x_1 = (x_1 + x_2 + x_3) + (x_1 + x_2\theta^2 + x_3\theta) + (x_1 + x_2\theta + x_3\theta^2)$$
$$3x_2 = (x_1 + x_2 + x_3) + (x_1\theta + x_2 + x_3\theta^2) + (x_1\theta^2 + x_2 + x_3\theta)$$
$$3x_3 = (x_1 + x_2 + x_3) + (x_1\theta^2 + x_2\theta + x_3) + (x_1\theta + x_2\theta^2 + x_3),$$

which contain a complete verification.

17. The two constituent parts α' α'' of the roots of a cubic equation have been resolved into factors by observing the relations established between the roots by their evanescence; another mode exists for forming the same quantities by elimination between the proposed equation and its first and second derived equations.

Let the given equation $x^3 + ax^2 + bx + c = 0$ be represented by $\phi(x) = 0$, and the first derived, viz. $3x^2 + 2ax + b = 0$ by $\phi'(x) = 0$, the second derived $6x + 2a = 0$ by $\phi''(x) = 0$.

18. When $\alpha'' = 0$ we have seen that the equation $\varphi(x) = 0$ has two equal roots. But when $\varphi(x) = 0$ has equal roots, is expressed by making the result of eliminating x between $\varphi(x) = 0$ and $\varphi'(x) = 0$ to vanish, and by the theory of elimination this result is $\varphi'(x_1) . \varphi'(x_2) . \varphi'(x_3)$; hence we have ($h$ being a number),

$$\alpha'' = h\, \varphi'(x_1) . \varphi'(x_2) . \varphi'(x_3).$$

19. Suppose now the joint evanescence of α' and α'', the equation has then three equal roots $x_1 = x_2 = x_3$, the system of equations $\alpha' = 0$ $\alpha'' = 0$, the first of three dimensions relative to the roots, the second of six, are therefore the two conditions necessary for the existence of three equal roots.

The results of the elimination of x between $\varphi(x) = 0$ $\varphi''(x) = 0$ and $\varphi(x) = 0$ $\varphi'(x) = 0$, give also the two conditions for three equal roots of the same dimensions as the above, with which this system is identical, we have thus, h' being numerical,

$$\alpha' = h'\, \varphi''(x_1) . \varphi''(x_2) . \varphi''(x_3).$$

20. Since

$$\varphi(x) = (x - x_1)(x - x_2)(x - x_3)$$
$$\varphi'(x) = (x - x_2)(x - x_3) + (x - x_1)(x - x_3) + (x - x_3)(x - x_2)$$
$$\varphi''(x) = 2(x - x_3) + 2(x - x_2) + 2(x - x_1);$$

therefore

$$\varphi'(x_1) . \varphi'(x_2) . \varphi'(x_3) = -(x_1 - x_2)^2 (x_1 - x_3)^2 (x_2 - x_3)^2$$
$$\varphi''(x_1) . \varphi''(x_2) . \varphi''(x_3) = 8(2x_1 - x_2 - x_3)(2x_2 - x_1 - x_3)(2x_3 - x_1 - x_2);$$

the values of α' α'' are therefore conformable to those before found, and

$$h = -k = \frac{1}{2^2 . 3^3} \qquad h' = \frac{k'}{8} = \frac{1}{2^4 . 3^3}.$$

21. In the elimination of a quantity between two equations into which that quantity enters rationally, it is in general indifferent which of the two equations is selected that its roots may be substituted for such quantity, for the dimensions of the product is the same whether we substitute the n roots of an equation of n dimensions in one of m, or the m roots of the latter in the former of n dimensions, and take the products; these products are not only of the same dimensions but imply the coexistence of the same system of equations, and can only differ from each other by numerical multipliers, when the numerical coefficient in the one differs from that of the other.

In the present instance, if ξ_1 ξ_2 be the roots of the equation $\varphi' = 0$, and X that of $\varphi'' = 0$, the values of α' α'' may also be expressed in the following form,

$$\alpha'' = \mathrm{H}\, \varphi(\xi_1) . \varphi(\xi_2)$$
$$\alpha'' = \mathrm{K} . \varphi(\mathrm{X})$$

the factors H, K being numbers.

22. Now if we observe that by the equation $\varphi'(\xi_1) = 0$,

$$\xi_1^2 = -\frac{2a}{3} . \xi_1 - \frac{b}{3}$$

and therefore

$$\xi_1^3 = -\frac{2a}{3}.\xi_1^2 - \frac{b}{3}.\xi_1 = \left(\frac{4a^2}{9} - \frac{b}{3}\right).\xi_1 + \frac{2ab}{9}$$

it will follow that $\varphi(\xi_1)$ or $\xi_1^3 + a\xi_1^2 + b\xi_1 + c$, is the same as $A\xi_1 + B$, putting

$$A = -\left(\frac{2a^2}{9} - \frac{2b}{3}\right)$$

$$B = c - \frac{ab}{9}.$$

Hence

$$\varphi(\xi_1).\varphi(\xi_2) = (A\xi_1 + B)(A\xi_2 + B);$$

and since $\xi_1 \xi_2 = \frac{b}{3}$ and $\xi_1 + \xi_2 = -\frac{2a}{3}$, therefore

$$\alpha'' = H\left(A^2.\frac{b}{3} - \frac{2a}{3}.AB + B^2\right)$$

$$= H\left(c^2 - \frac{2ab}{3}.c + \frac{4a^3}{27}.c + \frac{4b^3}{27} - \frac{a^2b^2}{27}\right).$$

Again since $X = -\frac{a}{3}$ therefore

$$\alpha' = K\left(c - \frac{ab}{3} + \frac{2a^3}{27}\right).$$

Suppose a and b to vanish, then $\sqrt{\alpha'} = c\sqrt{H}$, $\alpha'' = K.c$; therefore

$$x_1 = \sqrt[3]{c\{K + \sqrt{H}\}} + \sqrt[3]{c\{K - \sqrt{H}\}} = \sqrt[3]{-c}: \text{ hence}$$

$$k + \sqrt{H} = -1\; K - \sqrt{H} = 0;$$

therefore

$$K = -\frac{1}{2}, H = \frac{1}{4};$$

thus the transformation is affected from symmetrical functions to given coefficients.

23. Expressions for α', α'' having been found by various methods, we have also the following properties with respect to their evanescence,

$$x_1 = -\frac{a}{3} + \sqrt[3]{(\alpha' + \sqrt{\alpha''})} + \sqrt[3]{(\alpha' - \sqrt{\alpha''})}.$$

When the quantity (α'') under the quadratic and cubic radicals vanishes, the proposed equation has two equal roots.

When the quantity (α') under the cubic but not under the quadratic surd vanishes, two of the three differences of the roots are equal, or one root is one half the sum of the other two.

When both these quantities (α', α'') vanish conjointly, the given equation has three equal roots.

Conversely when the proposed has two equal roots the equation of condition is $\alpha'' = 0$, when it has three equal roots the two equations of condition are $\alpha' = 0$, $\alpha'' = 0$.

When the quantity $\left(-\frac{a}{3}\right)$ in the expression for the root which is unaffected by surds vanishes, one root is equal to the sum of the other two with changed signs.

When $-\frac{a}{3}$ and α'' vanish conjointly, two of the roots are each one half of the third with a changed sign.

When $-\frac{a}{3}$ and α' vanish jointly, the equation has two roots equal, but with contrary signs, and the third root is zero.

When $-\frac{a}{3}$, α', α'' all vanish simultaneously, the three roots are equal to each other and to zero.

24. *Biquadratic Equations.*—Let x_1, x_2, x_3, x_4 be the four roots of the equation

$$x^4 + a\,x^3 + b\,x^2 + c\,x + d,$$

the number 4 being composite allows the subdivision of the sums of the roots taken two and two into three pairs, which have rational sums, thus

$$x_1 + x_2 + \frac{a}{2} = -\left(x_3 + x_4 + \frac{a}{2}\right), \left(x_1 + x_3 + \frac{a}{2}\right) = -\left(x_2 + x_4 + \frac{a}{2}\right),$$

$$x_1 + x_4 + \frac{a}{2} = -\left(x_2 + x_3 + \frac{a}{2}\right),$$

therefore the equation, of which the roots are

$$x_1 + x_2 + \frac{a}{2},\ x_1 + x_3 + \frac{a}{2},\ x_1 + x_4 + \frac{a}{2},\ -\left(x_3 + x_4 + \frac{a}{2}\right),\ \&\text{c.}$$

is one of six dimensions, but without terms involving the odd powers of the unknown quantity, and therefore these quantities are the square roots of the roots of a cubic equation. Put therefore

$$x_1 + x_2 + \frac{a}{2} = 2\sqrt{\alpha}$$

$$x_1 + x_3 + \frac{a}{2} = 2\sqrt{\beta}$$

$$x_1 + x_4 + \frac{a}{2} = 2\sqrt{\gamma},$$

therefore

$$2\,x_1 + (x_1 + x_2 + x_3 + x_4) + \frac{3\,a}{2} = 2\,(\sqrt{\alpha} + \sqrt{\beta} + \sqrt{\gamma}),$$

or

$$x_1 = -\frac{a}{4} + \sqrt{\alpha} + \sqrt{\beta} + \sqrt{\gamma},$$

the quantities α, β, γ being the roots of a cubic equation, are of the form

$$\alpha = \alpha' + \sqrt[3]{\beta'} + \sqrt[3]{\beta''}$$

$$\beta = \alpha' + \theta\sqrt[3]{\beta'} + \theta^2\sqrt[3]{\beta''}$$

$$\gamma = \alpha' + \theta^2\sqrt[3]{\beta'} + \theta\sqrt[3]{\beta''},$$

when θ, θ^2 are the imaginary cube roots of unity, and β', β'' being the roots of a quadratic are of the forms

$$\beta' = \alpha'' + \sqrt{\alpha'''}$$
$$\beta'' = \alpha'' - \sqrt{\alpha'''}.$$

The three other roots of the biquadratic are

$$x_2 = 2\sqrt{\alpha} - \left(x_1 + \frac{a}{2}\right) = -\frac{a}{4} + \sqrt{\alpha} - \sqrt{\beta} - \sqrt{\gamma}$$

$$x_3 = 2\sqrt{\beta} - \left(x_1 + \frac{a}{2}\right) = -\frac{a}{4} - \sqrt{\alpha} + \sqrt{\beta} - \sqrt{\gamma}$$

$$x_4 = 2\sqrt{\gamma} - \left(x_1 + \frac{a}{2}\right) = -\frac{a}{4} - \sqrt{\alpha} - \sqrt{\beta} + \sqrt{\gamma}.$$

The constituent or essentially different parts of the roots are $\frac{a}{4}$, α', α'', α''', which we proceed to analyse by the conditions of their evanescence.

25. Suppose $\alpha''' = 0$, then by art. 23 two of the roots of the cubic are equal, or $\beta = \gamma$, from whence we have $x_3 = x_4$, therefore $x_3 - x_4$ is a factor of α''', and forming all the other symmetrical factors, we have

$$\alpha''' = k\,(x_1 - x_2)^2\,(x_1 - x_3)^2\,(x_1 - x_4)^2\,(x_2 - x_3)^2\,(x_2 - x_4)^2\,(x_3 - x_4)^2,$$

k being a numerical multiplier.

26. Next suppose $\alpha'' = 0$, then by the properties of the roots of the cubic already demonstrated we have $2\,\alpha = \beta + \gamma$, or $\alpha - \beta = \gamma - \alpha$, whence

$$(\sqrt{\alpha} + \sqrt{\beta})\,(\sqrt{\alpha} - \sqrt{\beta}) = (\sqrt{\gamma} + \sqrt{\alpha})\,(\sqrt{\gamma} - \sqrt{\alpha}),$$

therefore

$$(x_1 - x_4)\,(x_2 - x_3) = (x_1 - x_3)\,(x_4 - x_2)\,;$$

one factor of α'' is found thus to be

$$(x_1 - x_4)\,(x_2 - x_3) + (x_1 - x_3)\,(x_2 - x_4).$$

The two remaining symmetrical factors are

$$(x_1 - x_2)\,(x_3 - x_4) + (x_1 - x_4)\,(x_3 - x_2)$$
$$(x_1 - x_3)\,(x_4 - x_2) + (x_1 - x_2)\,(x_4 - x_3)\,;$$

and α'' is the product of all three multiplied by a numerical factor k'.

27. Again, suppose $\alpha' = 0$, then by the article above referred to $\alpha + \beta + \gamma = 0$; but

$$\beta + \gamma = (x_1 - x_2)^2 + (x_3 - x_4)^2$$
$$\alpha + \gamma = (x_1 - x_3)^2 + (x_2 - x_4)^2$$
$$\alpha + \beta = (x_1 - x_4)^2 + (x_2 - x_3)^2,$$

the sum of which being *per se* symmetrical, shows that α' has no other but a numerical factor; therefore

$$\alpha' = k''\,\{(x_1 - x_2)^2 + (x_1 - x_3)^2 + (x_1 - x_4)^2 + (x_2 - x_3)^2 + (x_2 - x_4)^2 + (x_3 - x_4)^2\}.$$

28. The numbers k, k', k'' may be found by the values already given for the roots of a cubic, by which we have

$$\alpha = \alpha' + \sqrt[3]{(\alpha'' + \sqrt{\alpha'''})} + \sqrt[3]{(\alpha'' - \sqrt{\alpha'''})}$$

and

$$\alpha' = \frac{\alpha + \beta + \gamma}{3}$$

therefore

$$\text{K}'' = \frac{1}{2 \, . \, 3}$$

$$\alpha'' = \frac{1}{2 \, . \, 3^3} \, . \, (2 \, \alpha - \beta - \gamma) \, (2 \, \beta - \alpha - \gamma) \, (2 \, \gamma - \alpha - \beta)$$

and the factors into which k' is multiplied are the same as

$$\frac{1}{2^3} \, . \, (2 \, \alpha - \beta - \gamma) \, . \, (2 \, \beta - \alpha - \gamma) \, . \, (2 \, \gamma - \alpha - \beta)$$

therefore

$$k' = \frac{1}{2^4 \, 3^3}.$$

Lastly

$$\alpha''' = - \frac{1}{2^2 . 3^3} \, . \, (\alpha - \beta)^2 \, (\alpha - \gamma)^2 \, (\beta - \gamma)^2.$$

But since

$$\alpha + \gamma = (x_1 - x_3)^2 + (x_2 - x_4)^2$$

and

$$\beta + \gamma = (x_1 - x_4)^2 + (x_2 - x_3)^2$$

therefore

$$\begin{aligned} \alpha - \beta &= 2 \, (x_1 \, x_4 - x_1 \, x_3 + x_2 \, x_3 - x_2 \, x_4) \\ &= 2 \, (x_1 - x_2) \, (x_4 - x_3). \end{aligned}$$

Similarly $\alpha - \gamma$ and $\beta - \gamma$ are expressed, and comparing the expression for α''' thus arising with that found before, we have

$$- 2^2 \, . \, 3^3 \, \alpha''' = 2^6 \, . \, \frac{\alpha'''}{k} \text{ or } k = - \frac{2^4}{3^3}.$$

29. Let us next seek the same quantities α', α'', α''', by the theory of elimination.

When $\alpha''' = 0$, the proposed equation which it will be convenient to express by $\varphi \, (x) = 0$ has two equal roots, the condition for which is also obtained by eliminating x between $\varphi \, (x) = 0$ and the first derived $\varphi' \, (x) = 0$, the function of the coefficients arising from this elimination is of the same dimensions, and expresses the same condition as the constituent quantity α''', and therefore only differs from it by a numerical multiplier.

This quantity in a symmetrical form relative to the roots is therefore

$$\alpha''' = h \, . \, \varphi' \, (x_1) \, . \, \varphi' \, (x_2) \, . \, \varphi' \, (x_3) \, . \, \varphi' \, (x_4)$$

and since

$$\varphi' \, (x_1) = (x_1 - x_2) \, (x_1 - x_3) \, (x_1 - x_4)$$

we have the same result as by the former method and $h = k$.

30. When α'' vanishes jointly with α''', then since $\alpha = \beta = \gamma$ we have also

$x_2 = x_3 = x_4$, therefore the equation $\phi(x) = 0$ has three equal roots, and since $\alpha''' = 0$ denotes the existence of two equal roots, therefore $\alpha'' = 0$ is the additional condition for a third, or the system of equations $\alpha'' = 0$, $\alpha''' = 0$, are equivalent to the system $\phi(x) = 0$, $\phi'(x) = 0$, $\phi''(x) = 0$.

Now $\frac{\phi''(x)}{2} = 6x^2 + 3ax + b = z + b$ for abridgment, we shall next form an equation in z indicative of two equal roots, and eliminating z by the equation $z + b = 0$ we shall obtain α''.

When $x_1 = x_2$, then $6x_1^2 + 3ax_1 = 6x_1^2 - 3x_1(2x_1 + x_3 + x_4)$

$$= -3(x_1 x_3 + x_1 x_4)$$
$$= -3(x_1 x_3 + x_2 x_4)$$

the equation

$$z + 3(x_1 x_3 + x_2 x_4) = 0$$

expresses that $x_1 = x_2$, and forming a cubic equation by taking all the symmetrical simple equations, a condition in z for the existence of any pair of equal roots is obtained, viz.

$$F(z) = \{z + 3(x_1 x_3 + x_2 x_4)\} . \{z + 3(x_1 x_2 + x_3 x_4)\} . \{z + 3(x_1 x_4 + x_2 x_3)\} = 0$$

or

$$F(z) = z^3 + 3bz^2 + 9(ac - 4d)z + 27\{d(a^2 - 4b) + e^2\} = 0;$$

and if z be eliminated between this equation, and $z + b = 0$, the result multiplied by a constant will be α'' or

$$\alpha'' = h' F(-b).$$

Now one factor of

$$F(-b) = 2(x_1 x_3 + x_2 x_4) - (x_1 x_2 + x_1 x_4 + x_2 x_3 + x_3 x_4)$$
$$= (x_1 - x_2)(x_3 - x_4) + (x_1 - x_4)(x_3 - x_2);$$

the other two being symmetrical with it gives the same value of α'' as before, and $h' = k'$.

31. If $\alpha' = 0$, $\alpha'' = 0$, $\alpha''' = 0$ simultaneously, then $x_1 = x_2 = x_3 = x_4$. Now α' is a function of two dimensions, as is that arising by eliminating x between $\phi''(x) = 0$, $\phi'''(x) = 0$, and gives the same condition, but $\phi'''(x) = 6(4x + a)$, therefore $\phi''\left(\frac{-a}{4}\right)$ is the result of this elimination, therefore $\alpha' = h''\phi''\left(\frac{-a}{4}\right)$. Now

$$\phi''(x) = 2(6x^2 + 3ax + b)$$

therefore

$$\phi''\left(\frac{-a}{4}\right) = 2b - \frac{3a^2}{4},$$

and our former value of α' is

$$k''\{\Sigma 3x_1^2 - \Sigma 2x_1 x_2\} = k''(3a^2 - 8b)$$

hence

$$h'' = -4k''.$$

32. Collecting the results of the last three articles they give us

$$\alpha' = -\frac{2}{3} \cdot \phi''\left(\frac{-a}{4}\right)$$

$$\alpha'' = \frac{1}{2^4 \cdot 3^3} \cdot \mathrm{F}(-b)$$

$$\alpha''' = -\frac{2^4}{3^3} \cdot \phi'(x_1) \cdot \phi'(x_2) \cdot \phi'(x_3) \cdot \phi'(x_4),$$

all of which may be expressed easily in terms of the coefficients.

33. *Theorems deduced.*—The root of the biquadratic $x^4 + a x^3 + b x^2 + c x + d = 0$ being expressed by

$$x_1 = -\frac{a}{4} + \sqrt{\{\alpha' + \sqrt[3]{(\alpha'' + \sqrt{\alpha'''})} + \sqrt[3]{(\alpha'' - \sqrt{\alpha'''})}\}}$$
$$+ \sqrt{\{\alpha' + \theta \sqrt[3]{(\alpha'' + \sqrt{\alpha'''})} + \theta^2 \sqrt[3]{(\alpha'' - \sqrt{\alpha'''})}\}}$$
$$+ \sqrt{\{\alpha' + \theta^2 \sqrt[3]{(\alpha'' + \sqrt{\alpha'''})} + \theta \sqrt[3]{(\alpha'' - \sqrt{\alpha'''})}\}},$$

where θ, θ^2 are the imaginary cube roots of unity, then the condition $\alpha''' = 0$ denotes the existence of two equal roots in the proposed equation.

The condition $\alpha'' = 0$ denotes the following relation of the roots

$$(x_1 - x_4)(x_2 - x_3) + (x_1 - x_3)(x_2 - x_4) = 0.$$

The system of coexisting conditions $\alpha'' = 0$, $\alpha''' = 0$ are necessary and sufficient for the existence of three equal roots.

The condition $\alpha' = 0$ denotes the following relation of the roots,

$$\Sigma (x_1 - x_2)^2 = 0.$$

The simultaneous system of conditions $\alpha' = 0$, $\alpha'' = 0$, $\alpha''' = 0$ essentially and sufficiently express the coexistence of four equal roots.

The rational part of the root $-\frac{a}{4}$ only vanishes with the sum of the roots.

34. We now proceed to determine the constituent parts of the roots of equations of the fifth degree by the conditions of their evanescence.

θ, θ^2 represent the imaginary cube roots of unity.

ω, ω^2, ω^3, ω^4 the imaginary fifth roots of unity.

x_1, x_2, x_3, x_4, x_5 are the five roots of the proposed equation of the fifth degree, viz.

$$x^5 + a x^4 + b x^3 + c x^2 + d x + e = 0.$$

$$x_1 = -\frac{a}{5} + \sqrt[5]{\alpha} + \sqrt[5]{\beta} + \sqrt[5]{\gamma} + \sqrt[5]{\delta}$$

$$x_2 = -\frac{a}{5} + \omega \sqrt[5]{\alpha} + \omega^2 \sqrt[5]{\beta} + \omega^3 \sqrt[5]{\gamma} + \omega^4 \sqrt[5]{\delta}$$

$$x_3 = -\frac{a}{5} + \omega^2 \sqrt[5]{\alpha} + \omega^4 \sqrt[5]{\beta} + \omega \sqrt[5]{\gamma} + \omega^3 \sqrt[5]{\delta}$$

$$x_4 = -\frac{a}{5} + \omega^3 \sqrt[5]{\alpha} + \omega \sqrt[5]{\beta} + \omega^4 \sqrt[5]{\gamma} + \omega^2 \sqrt[5]{\delta}$$

$$x_5 = -\frac{a}{5} + \omega^4 \sqrt[5]{\alpha} + \omega^3 \sqrt[5]{\beta} + \omega^2 \sqrt[5]{\gamma} + \omega \sqrt[5]{\delta};$$

the formulæ for x_3, x_4, x_5, x_1 are derived from the formula for x_2, by writing successively ω^2, ω^3, ω^4, ω^5 instead of ω.

$$\alpha = \alpha' + \sqrt{\beta'} + \sqrt{\gamma'} + \sqrt{\delta'}$$
$$\beta = \alpha' + \sqrt{\beta'} - \sqrt{\gamma'} - \sqrt{\delta'}$$
$$\gamma = \alpha' - \sqrt{\beta'} + \sqrt{\gamma'} - \sqrt{\delta'}$$
$$\delta = \alpha' - \sqrt{\beta'} - \sqrt{\gamma'} + \sqrt{\delta'},$$

such being the forms of the roots of a biquadratic.

Again, the expressions for β', γ', δ' as roots of a cubic, are

$$\beta' = \alpha'' + \sqrt[3]{\beta''} + \sqrt[3]{\gamma''}$$
$$\gamma' = \alpha'' + \theta \sqrt[3]{\beta''} + \theta^2 \sqrt[3]{\gamma''}$$
$$\delta' = \alpha'' + \theta^2 \sqrt[3]{\beta''} + \theta \sqrt[3]{\gamma''}.$$

Lastly, β'', γ'' as roots of a quadratic, are expressed by the following formulæ:

$$\beta'' = \alpha''' + \sqrt{\alpha^{\text{IV}}}$$
$$\gamma'' = \alpha''' - \sqrt{\alpha^{\text{IV}}}.$$

The quantities $-\frac{a}{5}$, α', α'', α''', α^{IV} are the constituent or essentially distinct parts of the roots x_1, x_2, x_3, x_4, x_5, and the analysis of their formation is to be sought by observing all the conditions under which each may vanish.

35. If for $-\frac{a}{5}$ we put $\frac{x_1 + x_2 + x_3 + x_4 + x_5}{5}$, it is obvious that the system of five equations for the roots is equivalent to one of only four, viz.

$$5 \sqrt[5]{\alpha} = x_1 + \omega^4 x_2 + \omega^3 x_3 + \omega^2 x_4 + \omega x_5$$
$$5 \sqrt[5]{\beta} = x_1 + \omega^3 x_2 + \omega x_3 + \omega^4 x_4 + \omega^2 x_5$$
$$5 \sqrt[5]{\gamma} = x_1 + \omega^2 x_2 + \omega^4 x_3 + \omega x_4 + \omega^3 x_5$$
$$5 \sqrt[5]{\delta} = x_1 + \omega x_2 + \omega^2 x_3 + \omega^3 x_4 + \omega^4 x_5$$

the right hand members of which equations differ from each other only by the particular fifth root of unity in use, which considered as ω in the first, will be ω^2, ω^3, ω^4 respectively in the second, third, and fourth.

36. Suppose $\alpha^{\text{IV}} = 0$, then by art. 33 two roots of the biquadratic must be equal.

First let $\gamma = \delta$, which can only happen under the five following relations,

$$\sqrt[5]{\gamma} = \sqrt[5]{\delta},\ \sqrt[5]{\gamma} = \omega \sqrt[5]{\delta},\ \sqrt[5]{\gamma} = \omega^2 \sqrt[5]{\delta},\ \sqrt[5]{\gamma} = \omega^3 \sqrt[5]{\delta},\ \sqrt[5]{\gamma} = \omega^4 \sqrt[5]{\delta},$$

which furnish five factors, linear functions of the differences of the roots, and since

these differences may be taken either way, we have also from the same equations five other factors equal to the former and with contrary signs.

In this manner ten factors of α^{IV} may be found by equating any pair of the four quantities α, β, γ, δ, and the number of pairs being six, the whole number of factors of α^{IV} is sixty, these factors are very easily formed, and here we present the first thirty factors, the remaining thirty being formed merely by changing the signs of these, or, which is the same, inverting the order of the differences of the roots in each factor.

37. For greater clearness we shall subdivide these thirty factors into five groups, from the first of which x_1 is excluded, from the second x_2, and so on; each subdivision contains six factors, four of which are of one form, and two of a different form; they are as follow:

$$\left.\begin{array}{lr}
(x_2 - x_4) + \omega (x_3 - x_4) + \omega^2 (x_3 - x_5) \quad \ldots\ldots & (1.) \\
(x_3 - x_2) + \omega (x_5 - x_2) + \omega^2 (x_5 - x_4) \quad \ldots\ldots & (2.) \\
(x_4 - x_5) + \omega (x_2 - x_5) + \omega^2 (x_2 - x_3) \quad \ldots\ldots & (3.) \\
(x_5 - x_3) + \omega (x_4 - x_3) + \omega^2 (x_4 - x_2) \quad \ldots\ldots & (4.) \\
(x_2 - x_5) + (\omega^2 + \omega^3) (x_3 - x_4) \quad \ldots\ldots & (5.) \\
(x_3 - x_4) + (\omega^2 + \omega^3) (x_5 - x_2) \quad \ldots\ldots & (6.)
\end{array}\right\} (A.)$$

$$\left.\begin{array}{lr}
(x_1 - x_4) + \omega (x_5 - x_4) + \omega^2 (x_5 - x_3) \quad \ldots\ldots & (1.) \\
(x_3 - x_5) + \omega (x_4 - x_5) + \omega^2 (x_4 - x_1) \quad \ldots\ldots & (2.) \\
(x_4 - x_3) + \omega (x_1 - x_3) + \omega^2 (x_1 - x_5) \quad \ldots\ldots & (3.) \\
(x_5 - x_1) + \omega (x_3 - x_1) + \omega^2 (x_3 - x_4) \quad \ldots\ldots & (4.) \\
(x_1 - x_3) + (\omega^2 + \omega^3) (x_5 - x_4) \quad \ldots\ldots & (5.) \\
(x_5 - x_4) + (\omega^2 + \omega^3) (x_3 - x_1) \quad \ldots\ldots & (6.)
\end{array}\right\} (B.)$$

$$\left.\begin{array}{lr}
(x_1 - x_2) + \omega (x_4 - x_2) + \omega^2 (x_4 - x_5) \quad \ldots\ldots & (1.) \\
(x_2 - x_5) + \omega (x_1 - x_5) + \omega^2 (x_1 - x_4) \quad \ldots\ldots & (2.) \\
(x_4 - x_1) + \omega (x_5 - x_1) + \omega^2 (x_5 - x_2) \quad \ldots\ldots & (3.) \\
(x_5 - x_4) + \omega (x_2 - x_4) + \omega^2 (x_2 - x_1) \quad \ldots\ldots & (4.) \\
(x_1 - x_5) + (\omega^2 + \omega^3) (x_4 - x_2) \quad \ldots\ldots & (5.) \\
(x_4 - x_2) + (\omega^2 + \omega^3) (x_5 - x_1) \quad \ldots\ldots & (6.)
\end{array}\right\} (C.)$$

$$\left.\begin{array}{lr}
(x_1 - x_5) + \omega (x_3 - x_5) + \omega^2 (x_3 - x_2) \quad \ldots\ldots & (1.) \\
(x_2 - x_3) + \omega (x_5 - x_3) + \omega^2 (x_5 - x_1) \quad \ldots\ldots & (2.) \\
(x_3 - x_1) + \omega (x_2 - x_1) + \omega^2 (x_2 - x_5) \quad \ldots\ldots & (3.) \\
(x_5 - x_2) + \omega (x_1 - x_2) + \omega^2 (x_1 - x_3) \quad \ldots\ldots & (4.) \\
(x_1 - x_2) + (\omega^2 + \omega^3) (x_3 - x_5) \quad \ldots\ldots & (5.) \\
(x_3 - x_5) + (\omega^2 + \omega^3) (x_2 - x_1) \quad \ldots\ldots & (6.)
\end{array}\right\} (D.)$$

$$\left.\begin{array}{lr}(x_1 - x_3) + \omega (x_2 - x_3) + \omega^2 (x_2 - x_4) & (1.)\\ (x_2 - x_1) + \omega (x_4 - x_1) + \omega^2 (x_4 - x_3) & (2.)\\ (x_3 - x_4) + \omega (x_1 - x_4) + \omega^2 (x_1 - x_2) & (3.)\\ (x_4 - x_2) + \omega (x_3 - x_2) + \omega^2 (x_3 - x_1) & (4.)\\ (x_1 - x_4) + (\omega^2 + \omega^3) (x_2 - x_3) & (5.)\\ (x_2 - x_3) + (\omega^2 + \omega^3) (x_4 - x_1) & (6.)\end{array}\right\}(E.)$$

The other thirty factors of α^{IV} differ from these only in having ω^4 instead of ω, thus changing the sign of (1.), (A.), to obtain the first of these factors, and writing it in an inverted order it gives

$$\omega^2 \{(x_5 - x_3) + \omega^4 (x_4 - x_3) + \omega^3 (x_4 - x_2)\}$$

differing from (4.), (A.) in having ω^4 for ω, for the numerical factor ω^2 may be rejected since $(\omega^2)^{30} = 1$.

The quantity α^{IV} is the product of these sixty factors and a numerical constant k.

38. Formation of the factors of α'''.

The factors in the group A of the preceding article denote for abridgment by their number placed as a subindex to A, and so for all the others, thus by B_3 is meant the third factor in the group B.

The quantity α''' is composed of the three factors of ten dimensions each. Every such factor is the sum of two parts, each decomposable into ten simple factors.

In the first pair of these simple factors x_1 does not enter, in the second pair x_2 is excluded, and so on.

These three compound factors are found as follows: First,

$$A_1 = \frac{5}{\omega(\omega - 1)} (\sqrt[5]{\gamma} - \sqrt[5]{\delta}) \qquad A_2 = -\frac{5}{\omega(\omega - 1)} (\sqrt[5]{\beta} - \sqrt[5]{\delta})$$

$$B_2 = \frac{5}{\omega^3(\omega - 1)} (\sqrt[5]{\gamma} - \omega\sqrt[5]{\delta}) \qquad B_3 = -\frac{5}{\omega^4(\omega - 1)} (\sqrt[5]{\beta} - \omega^2\sqrt[5]{\delta})$$

$$C_3 = \frac{5}{(\omega - 1)} (\sqrt[5]{\gamma} - \omega^2\sqrt[5]{\delta}) \qquad C_4 = -\frac{5}{\omega^2(\omega - 1)} (\sqrt[5]{\beta} - \omega^4\sqrt[5]{\delta})$$

$$D_4 = \frac{5}{\omega^2(\omega - 1)} (\sqrt[5]{\gamma} - \omega^3\sqrt[5]{\delta}) \qquad D_1 = -\frac{5}{(\omega - 1)} (\sqrt[5]{\beta} - \omega\sqrt[5]{\delta})$$

$$E_1 = \frac{5}{\omega^4(\omega - 1)} (\sqrt[5]{\gamma} - \omega^4\sqrt[5]{\delta}) \qquad E_2 = -\frac{5}{\omega^3(\omega - 1)} (\sqrt[5]{\beta} - \omega^3\sqrt[5]{\delta})$$

$$A_3 = \frac{5}{\omega(\omega - 1)} (\sqrt[5]{\alpha} - \sqrt[5]{\gamma}) \qquad A_4 = -\frac{5}{\omega(\omega - 1)} (\sqrt[5]{\alpha} - \sqrt[5]{\beta})$$

$$B_4 = \frac{5}{(\omega - 1)} (\sqrt[5]{\alpha} - \omega^2\sqrt[5]{\gamma}) \qquad B_1 = -\frac{5}{(\omega - 1)} (\sqrt[5]{\alpha} - \omega\sqrt[5]{\beta})$$

$$C_1 = \frac{5}{\omega^4(\omega - 1)} (\sqrt[5]{\alpha} - \omega^4\sqrt[5]{\gamma}) \qquad C_2 = -\frac{5}{\omega^4(\omega - 1)} (\sqrt[5]{\alpha} - \omega^2\sqrt[5]{\beta})$$

$$D_2 = \frac{5}{\omega^3(\omega - 1)} (\sqrt[5]{\alpha} - \omega\sqrt[5]{\gamma}) \qquad D_3 = -\frac{5}{\omega^3(\omega - 1)} (\sqrt[5]{\alpha} - \omega^3\sqrt[5]{\beta})$$

$$\mathrm{E}_3 = \frac{5}{\omega^2(\omega-1)}(\sqrt[5]{\alpha} - \omega^3\sqrt[5]{\gamma}) \qquad \mathrm{E}_4 = -\frac{5}{\omega^2(\omega-1)}(\sqrt[5]{\alpha} - \omega^4\sqrt[5]{\beta})$$

$$\mathrm{A}_5 = \frac{5}{\omega^2(\omega-1)}(\sqrt[5]{\beta} - \sqrt[5]{\gamma}) \qquad \mathrm{A}_6 = \frac{5}{\omega^2(\omega-1)}(\sqrt[5]{\alpha} - \sqrt[5]{\delta})$$

$$\mathrm{B}_5 = -\frac{5}{\omega-1}(\sqrt[5]{\beta} - \omega\sqrt[5]{\gamma}) \qquad \mathrm{B}_6 = -\frac{5}{\omega(\omega-1)}(\sqrt[5]{\alpha} - \omega^3\sqrt[5]{\delta})$$

$$\mathrm{C}_5 = \frac{5}{\omega^3(\omega-1)}(\sqrt[5]{\beta} - \omega^2\sqrt[5]{\gamma}) \qquad \mathrm{C}_6 = -\frac{5}{\omega-1}(\sqrt[5]{\alpha} - \omega\sqrt[5]{\delta})$$

$$\mathrm{D}_5 = -\frac{5}{\omega(\omega-1)}(\sqrt[5]{\beta} - \omega^3\sqrt[5]{\gamma}) \qquad \mathrm{D}_6 = \frac{5}{\omega^4(\omega-1)}(\sqrt[5]{\alpha} - \omega^4\sqrt[5]{\delta})$$

$$\mathrm{E}_5 = \frac{5}{\omega^4(\omega-1)}(\sqrt[5]{\beta} - \omega^4\sqrt[5]{\gamma}) \qquad \mathrm{E}_6 = \frac{5}{\omega^3(\omega-1)}(\sqrt[5]{\alpha} - \omega^2\sqrt[5]{\delta})$$

Therefore

$$\left.\begin{array}{l} \gamma - \delta = h\,.\,\mathrm{A}_1\,\mathrm{B}_2\,\mathrm{C}_3\,\mathrm{D}_4\,\mathrm{E}_1 \\ \beta - \delta = -\,h\,.\,\mathrm{A}_2\,\mathrm{B}_3\,\mathrm{C}_4\,\mathrm{D}_1\,\mathrm{E}_2 \\ \alpha - \gamma = h\,.\,\mathrm{A}_3\,\mathrm{B}_4\,\mathrm{C}_1\,\mathrm{D}_2\,\mathrm{E}_3 \\ \alpha - \beta = -\,h\,.\,\mathrm{A}_4\,\mathrm{B}_1\,\mathrm{C}_2\,\mathrm{D}_3\,\mathrm{E}_4 \\ \beta - \gamma = h\,.\,\mathrm{A}_5\,\mathrm{B}_5\,\mathrm{C}_5\,\mathrm{D}_5\,\mathrm{E}_5 \\ \alpha - \delta = h.\mathrm{A}_6.\mathrm{B}_6.\mathrm{C}_6.\mathrm{D}_6.\mathrm{E}_6 \end{array}\right\}, \; h = \left(\frac{\omega-1}{5}\right)^5.$$

39. Now the conditions necessary for the evanescence of α''' are by art. 33.

$$(\alpha-\delta)(\beta-\gamma) + (\alpha-\gamma)(\beta-\delta) = 0$$
$$(\alpha-\gamma)(\delta-\beta) + (\alpha-\beta)(\delta-\gamma) = 0$$
$$(\alpha-\beta)(\gamma-\delta) + (\alpha-\delta)(\gamma-\beta) = 0.$$

Substitute the values of $\alpha-\delta$, $\beta-\gamma$, &c. above found, and including h^2 in the numerical multiplier k of the whole, we shall have

$$\begin{aligned} \alpha''' = k'\,(&\mathrm{A}_5\mathrm{A}_6.\mathrm{B}_5\mathrm{B}_6.\mathrm{C}_5\mathrm{C}_6.\mathrm{D}_5\mathrm{D}_6.\mathrm{E}_5\mathrm{E}_6 - \mathrm{A}_2\mathrm{A}_3.\mathrm{B}_3\mathrm{B}_4.\mathrm{C}_4\mathrm{C}_1.\mathrm{D}_1\mathrm{D}_2.\mathrm{E}_2\mathrm{E}_3) \\ \times\,(&\mathrm{A}_5\mathrm{A}_6.\mathrm{B}_5\mathrm{B}_6.\mathrm{C}_5\mathrm{C}_6.\mathrm{D}_5\mathrm{D}_6.\mathrm{E}_5\mathrm{E}_6 + \mathrm{A}_4\mathrm{A}_1.\mathrm{B}_1\mathrm{B}_2.\mathrm{C}_2\mathrm{C}_3.\mathrm{D}_3\mathrm{D}_4.\mathrm{E}_4\mathrm{E}_1) \\ \times\,(&\mathrm{A}_2\mathrm{A}_3.\mathrm{B}_3\mathrm{B}_4.\mathrm{C}_4\mathrm{C}_1.\mathrm{D}_1\mathrm{D}_2.\mathrm{E}_2\mathrm{E}_3 + \mathrm{A}_4\mathrm{A}_1.\mathrm{B}_1\mathrm{B}_2.\mathrm{C}_2\mathrm{C}_3.\mathrm{D}_3\mathrm{D}_4.\mathrm{E}_4\mathrm{E}_1). \end{aligned}$$

40. The condition that α'' may vanish is $\Sigma(\alpha-\beta)^2 = 0$ by art. 33. Hence

$$\begin{aligned} \alpha'' = k''\,(&\mathrm{A}_1^2\mathrm{B}_2^2\mathrm{C}_3^2\mathrm{D}_4^2\mathrm{E}_1^2 + \mathrm{A}_2^2\mathrm{B}_3^2\mathrm{C}_4^2\mathrm{D}_1^2\mathrm{E}_2^2 + \mathrm{A}_5^2\mathrm{B}_5^2\mathrm{C}_5^2\mathrm{D}_5^2\mathrm{E}_5^2 \\ &+ \mathrm{A}_3^2\mathrm{B}_4^2\mathrm{C}_1^2\mathrm{D}_2^2\mathrm{E}_3^2 + \mathrm{A}_4^2\mathrm{B}_1^2\mathrm{C}_2^2\mathrm{D}_3^2\mathrm{E}_4^2 + \mathrm{A}_6^2\mathrm{B}_6^2\mathrm{C}_6^2\mathrm{D}_6^2\mathrm{E}_6^2), \end{aligned}$$

k'' being a numerical quantity.

41. The terms which compose α' are indicated below by the indices which enter them placed between brackets.

$$[5] = 4\,\Sigma\,x_1^5$$
$$[4,1] = -\,5\,\Sigma\,x_1^4\,x_2$$
$$[3,2] = -\,10\,\Sigma\,x_1^3\,x_2^2$$

$$
\begin{aligned}
[3, 1, 1] = &- 20\, x_1^3 (x_2 x_3 + x_2 x_4 + x_3 x_5 + x_4 x_5 - 4\, x_2 x_5 - 4\, x_3 x_4) \\
&- 20\, x_2^3 (x_1 x_4 + x_1 x_5 + x_3 x_4 + x_3 x_5 - 4\, x_1 x_3 - 4\, x_4 x_5) \\
&- 20\, x_3^3 (x_1 x_2 + x_1 x_4 + x_2 x_5 + x_4 x_5 - 4\, x_1 x_5 - 4\, x_2 x_4) \\
&- 20\, x_4^3 (x_1 x_3 + x_1 x_5 + x_2 x_3 + x_2 x_5 - 4\, x_1 x_2 - 4\, x_3 x_5) \\
&- 20\, x_5^3 (x_1 x_2 + x_1 x_3 + x_2 x_4 + x_3 x_4 - 4\, x_1 x_4 - 4\, x_2 x_3)
\end{aligned}
$$

$$
\begin{aligned}
[2, 2, 1] = - 30\, \{ & x_1^2 x_2^2 (x_3 + x_5 - 4\, x_4) + x_1^2 x_3^2 (x_4 + x_5 - 4\, x_2) \\
& + x_1^2 x_4^2 (x_2 + x_3 - 4\, x_5) + x_1^2 x_5^2 (x_2 + x_4 - 4\, x_3) \\
& + x_2^2 x_3^2 (x_1 + x_4 - 4\, x_5) + x_2^2 x_4^2 (x_1 + x_5 - 4\, x_3) \\
& + x_2^2 x_5^2 (x_3 + x_4 - 4\, x_1) + x_3^2 x_4^2 (x_2 + x_5 - 4\, x_1) \\
& + x_3^2 x_5^2 (x_1 + x_2 - 4\, x_4) + x_4^2 x_5^2 (x_1 + x_3 - 4\, x_2) \}
\end{aligned}
$$

$$[2, 1, 1, 1] = - 60\, \Sigma\, x_1^2 x_2 x_3 x_4$$

$$[1, 1, 1, 1, 1] = 480\, x_1 x_2 x_3 x_4 x_5.$$

The quantity α' is the sum of all these multiplied by a constant k'''.

42. It remains to give the value of the constants k, k', k'', k''', which may be easily found by comparison of the above values with the constituent parts of the roots of a biquadratic; they are as follows:

$$k = - \left(\frac{\omega - 1}{5}\right)^{60} \cdot \frac{2^4}{3^3}$$

$$k' = \left(\frac{\omega - 1}{5}\right)^{30} \cdot \frac{1}{2^4 . 3^3}$$

$$k'' = \left(\frac{\omega - 1}{5}\right)^{10} \cdot \frac{1}{2 . 3}$$

$$k''' = \left(\frac{1}{5}\right)^{5} \cdot \frac{1}{4};$$

and the term uninfluenced by surds is $\frac{x_1 + x_2 + x_3 + x_4 + x_5}{5}$.

43. All the constituent parts except the last-mentioned vanish when all the roots are equal, but for their separate evanescence the condition for the equality of roots is insufficient, the requisite conditions are easily seen from the factors of such parts already given; they are of two classes, arising from the different relations of the couples $\{(\omega, \omega^2), (\omega, \omega^3), (\omega^3, \omega^4), (\omega^3, \omega^4)\}$, and of the two couples $\{(\omega^2, \omega^3), (\omega, \omega^4)\}$, each member of which contains the same imaginary part with the other, which is not the case with the former couples.

44. I cannot at present, from the pressure of other engagements, pursue these investigations further; if any additional light to the analyst is furnished by the preceding imperfect reflections on a subject so often treated, the author's object will in a great measure be attained; they at least tend to show the imperfection of our

knowledge with respect to the conditions resulting from elimination, where more than two equations are concerned, and they exhibit in the higher powers relations between pairs of roots which have not been as yet expressed, even by the differential calculus.

London, April 1, 1837.