III. *Of such Ellipsoids consisting of homogeneous matter as are capable of having the resultant of the attraction of the mass upon a particle in the surface, and a centrifugal force caused by revolving about one of the axes, made perpendicular to the surface.* By James Ivory, *K.H. M.A. F.R.S. L. & E. Instit. Reg. Sc. Paris. Corresp. et Reg. Sc. Gottin. Corresp.*

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1. In the *Conn. des Temps* for 1837 it is announced that a homogeneous ellipsoid with three unequal axes, and consisting of particles that attract one another according to the law of nature, may be in equilibrium when it revolves with a proper velocity about the least axis. Lagrange has considered this problem in its utmost generality. The illustrious Geometer found the true equations from which the solution must be derived: but he inferred from them that a homogeneous planet cannot be in equilibrium unless it have a figure of revolution. Nevertheless M. Jacobi has proved that an equilibrium is possible in some ellipsoids of which the three axes have a certain relation to one another. The same thing is demonstrated by M. Liousville in 23rd cahier of the *Journal de l'École Polytechnique*. M. de Pontécoulant has also touched on the subject*. M. Jacobi has thus detected an inadvertence into which those had fallen who preceded him in this research. He has shown that the equations which, according to Lagrange, are capable of solution only in figures of revolution, may be solved in a certain class of ellipsoids with three unequal axes. But the transcendent equations of M. Jacobi, although fit for numerical computation on particular suppositions, leave unexplored the points of the problem which it is most interesting to know.

It is easy to find a property characteristical of all spheroids with which an equilibrium is possible on the supposition of a centrifugal force. From any point in the surface of the ellipsoid draw a perpendicular to the least axis, and likewise a line at right angles to the surface: if the plane passing through these two lines contain the resultant of the attractions of all the particles of the spheroid upon the point in the surface, the equilibrium will be possible; otherwise not. This will be evident, if it be considered that the resultant of the centrifugal force and the attraction of the mass must be a force perpendicular to the surface of the ellipsoid, which requires that the directions of the three forces shall be contained in one plane. This determination obviously comprehends all spheroids of revolution; but, on account of the

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complicated nature of the attractive force, it is difficult to deduce from it whether an equilibrium be possible, or not, in spheroids with three unequal axes.

The problem is unconnected with the physical conditions of equilibrium: it is purely a geometrical question respecting a property of certain ellipsoids.

2. Let the three semi-axes of an ellipsoid be represented by

$$k,\ k\sqrt{1+\lambda^2},\quad k\sqrt{1+\lambda'^2},$$

λ being supposed greater than λ'; and put x, y, z respectively parallel to the axes, for the coordinates drawn from a point in the surface to the principal sections of the solid: from the same point draw the line ϱ within the ellipsoid at right angles to its surface; and ϱ being limited by the principal section perpendicular to k, the axis of rotation, put p and q for the coordinates of the end of it in that plane, p being parallel to y, and q to z: from the condition that ϱ is perpendicular to the surface of the ellipsoid, it is easy to deduce the values of p and q, viz.

$$p = y\cdot\frac{\lambda^2}{1+\lambda^2},\quad q = z\cdot\frac{\lambda'^2}{1+\lambda'^2}.$$

Again, from the same point in the surface, draw the line ϱ' in the direction of the resultant of the attraction of the whole mass of the ellipsoid; and let r and s, respectively parallel to y and z, represent the coordinates of the foot of ϱ' in the same principal section as before: then ϱ' will be the diagonal of a parallelopiped of which the three sides are x, $y-r$, $z-s$; and the only three forces acting parallel to the sides of the parallelopiped and equivalent to the single force in the direction of the diagonal, will be proportional to the sides, x, $y-r$, $z-s$. Now from the nature of the ellipsoid, the attractive forces perpendicular to the principal sections, are proportional to the coordinates x, y, z; and may be represented by $\mathrm{A}\,x$, $\mathrm{B}\,y$, $\mathrm{C}\,z$: and, as these forces have their resultant in the direction of ϱ', it follows from what has been said, that they will be proportional to x, $y-r$, $z-s$. In consequence we have these equations,

$$\mathrm{A} = \frac{y}{y-r}\cdot\mathrm{B},\quad \mathrm{A} = \frac{z}{z-s}\cdot\mathrm{C};$$

$$r = y\,.\left(1-\frac{\mathrm{B}}{\mathrm{A}}\right),\quad s = z\,.\left(1-\frac{\mathrm{C}}{\mathrm{A}}\right):$$

and, by combining the values of r and s with those of p and q before found, we obtain

$$\frac{r-p}{s-q} = \frac{y}{z}\cdot\frac{\mathrm{B}-\dfrac{\mathrm{A}}{1+\lambda^2}}{\mathrm{C}-\dfrac{\mathrm{A}}{1+\lambda'^2}}.$$

Let σ denote the third side of the triangle which has ϱ and ϱ' for its other sides: then σ will represent the only force which, together with the attractive force ϱ', will produce a resultant in the direction of ϱ at right angles to the surface of the ellipsoid. Now σ cannot stand for a centrifugal force unless, in every position, it be invariably

parallel to the line $\sqrt{y^2 + z^2}$ drawn from the point in the surface of the ellipsoid at right angles to the axis of rotation; and this condition requires that the triangle of which the sides are σ, $r - p$, $s - q$, shall be similar to the triangle formed by the parallel lines $\sqrt{y^2 + z^2}$, y, z. From the similarity of the triangles, we deduce

$$\frac{r-p}{s-q} = \frac{y}{z}:$$

and hence, in consequence of the last formula, we finally obtain,

$$\mathrm{B} - \frac{\mathrm{A}}{1+\lambda^2} = \mathrm{C} - \frac{\mathrm{A}}{1+\lambda'^2} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (1.)$$

Every ellipsoid which verifies this formula is capable of an equilibrium when it is made to revolve with a proper angular velocity about the least axis; for the line ϱ' representing the attraction upon a point in the surface, the line σ will represent a centrifugal force, both in quantity and direction; and the resultant of these two forces will be perpendicular to the surface of the ellipsoid.

The equation (1.) results immediately from the investigation of LAGRANGE, who concluded that it admits of solution only in spheroids of revolution, that is, when $\lambda = \lambda'$ and $\mathrm{B} = \mathrm{C}$. By expressing the functions A, B, C in elliptic integrals, M. JACOBI has found that the equation may be solved when the three axes have a certain relation. It is therefore demonstrated in general, that a certain class of ellipsoids with three unequal axes is susceptible of an equilibrium on the supposition of a centrifugal force; but it still remains to investigate the precise limits within which this extension of the problem is possible, and to determine the ellipsoid when the centrifugal force is given.

3. In order to solve the problem in the view now taken of it, we must have recourse to the equations of LAGRANGE, which contain all the necessary conditions. Let f denote the intensity of the centrifugal force at the distance equal to unit from the axis of rotation; the same force urging the point in the surface of the ellipsoid at the distance $\sqrt{y^2 + z^2}$ from the axis, will be equal to $f\sqrt{y^2 + z^2}$, the components of which in the directions of y and z are respectively fy and fz. Now

$$\mathrm{A}\,x, \ \mathrm{B}\,y, \ \mathrm{C}\,z,$$

are the attractions of the mass of the ellipsoid; wherefore the total forces urging the point in the surface are

$$\mathrm{A}\,x, \ (\mathrm{B} - f)\,y, \ (\mathrm{C} - f)\,z.$$

These forces must have their resultant in the direction of ϱ perpendicular to the surface of the ellipsoid; and as they are parallel to the sides of a parallelopiped, of which ϱ is the diagonal, they will be proportional to those sides, that is, to

$$x, \quad y - p = \frac{y}{1+\lambda^2}, \quad z - q = \frac{z}{1+\lambda'^2}.$$

We thus obtain these two equations,

$$\frac{B-f}{A} = \frac{1}{1+\lambda^2}, \quad \frac{C-f}{A} = \frac{1}{1+\lambda'^2};$$

and from these we deduce

$$\left.\begin{array}{l} f = B - \dfrac{A}{1+\lambda^2}, \\ f = C - \dfrac{A}{1+\lambda'^2}, \end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2.)$$

which coincide with the equations of LAGRANGE.

It is next requisite to substitute for the symbols A, B, C, what they stand for. The values given in the *Mécanique Céleste* are in a convenient form for this purpose, viz.

$$d\,F = \frac{x^2\,d\,x}{\sqrt{(1+\lambda^2 x^2)\,.\,(1+\lambda'^2 x^2)}},$$

$$A = \frac{3\,M}{k^3}\int_0^1 d\,F, \quad B = \frac{3\,M}{k^3}\int_0^1 \frac{d\,F}{1+\lambda^2 x^2}, \quad C = \frac{3\,M}{k^3}\int_0^1 \frac{d\,F}{1+\lambda'^2 x^2}.$$

In these expressions M is the mass of the ellipsoid; therefore if we put ϱ for the density, we shall have

$$\frac{M}{k^3} = \frac{4\,\pi\,\rho}{3}\cdot(1+\lambda^2)^{\frac{1}{2}}(1+\lambda'^2)^{\frac{1}{2}}.$$

These several values being substituted in the equations (2.), the result will be

$$\left.\begin{array}{l} q = \dfrac{f}{\frac{4\,\pi}{3}\,\varrho}, \\ q = \sqrt{\dfrac{1+\lambda'^2}{1+\lambda^2}}\cdot\displaystyle\int_0^1 \frac{\lambda^2\,.\,3\,x^2\,d\,x\,(1-x^2)}{(1+\lambda^2 x^2)^{\frac{3}{2}}\,(1+\lambda'^2 x^2)^{\frac{1}{2}}}, \\ q = \sqrt{\dfrac{1+\lambda^2}{1+\lambda'^2}}\cdot\displaystyle\int_0^1 \frac{\lambda'^2\,.\,3\,x^2\,d\,x\,(1-x^2)}{(1+\lambda^2 x^2)^{\frac{1}{2}}\,(1+\lambda'^2 x^2)^{\frac{3}{2}}}. \end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (3.)$$

Here q stands for the proportion of the intensities of the centrifugal and attractive forces; it depends only on the kind of matter of which the spheroid is formed, and the velocity of rotation.

4. The equations (3.) comprehend all ellipsoids that are susceptible of an equilibrium on the supposition of a centrifugal force. To begin with the more simple case of the spheroid of revolution, let $\lambda = \lambda' = l$; and the two equations will coincide in one, viz.

$$q = \int_0^1 \frac{l^2\,.\,3\,x^2\,(1-x^2)\,d\,x}{(1+l^2 x^2)^2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4.)$$

which expresses the relation between q and l, in a spheroid of revolution having its semi-axes equal to k and $k\sqrt{1+l^2}$.

From the equation (4.) we learn that q will be known when l is given, or that every spheroid of a determinate form requires an appropriate velocity of rotation.

The inspection of the same equation is sufficient to show that q is positive for all values of l^2; and as it vanishes both when l^2 is zero and infinitely great, it must pass at least once from increasing to decreasing, or it will admit of at least one maximum value. By differentiating with regard to l we obtain

$$\frac{d\,q}{2\,l\,d\,l} = \int_0^1 \frac{3\,x^2\,(1-x^2)\,(1-l^2\,x^2)}{(1+l^2\,x^2)^3}; \quad \dots \dots \dots \quad (5.)$$

from which formula we learn that $\frac{d\,q}{2\,l\,d\,l}$ is positive between the limits $l^2 = 0$ and $l^2 = 1$; that it will consist of a positive and a negative part when l^2 is greater than 1; and the positive part decreasing while the negative part increases, that it will ultimately be negative when l^2 is infinitely great. It follows therefore that $\frac{d\,q}{2\,l\,d\,l}$ can be only once equal to zero, and consequently that q can have only one maximum value, while l^2 increases from 0 to ∞. Applying to the equations (4.) and (5.) the known method of integration, we get

$$q = \frac{3\,(3+l^2)}{2\,l^3} \text{ arc tan } l - \frac{9}{2\,l^2},$$

$$\frac{d\,q}{2\,l\,d\,l} = \frac{3\,(9+l^2)}{2\,l^4} \text{ arc tan } l - \frac{3\,l\,(9+7\,l^2)}{1+l^2},$$

of which expressions the first will verify the other. To determine the maximum of q, we have

$$\frac{d\,q}{2\,l\,d\,l} = 0$$

$$\text{arc tan } l = \frac{9\,l + 7\,l^3}{(1+l^2)\,(9+l^2)};$$

and the only value of l in this last equation is

$$l = 2{\cdot}5293.$$

By substituting this value of l we obtain 0·3370 for the maximum of q. With respect to spheroids of revolution it thus appears that an equilibrium is impossible when q or $\frac{f}{\frac{4\,\pi}{3}\,\rho}$ is greater than 0·3370: in the extreme case, when q is equal to 0·3370, there is only one form of equilibrium, the axes of the spheroid being

$$k \text{ and } k\sqrt{1+(2{\cdot}5293)^2} = 2{\cdot}7197\,k;$$

but when q is less than 0·3370 there are two different forms of equilibrium, the equatorial radius of one being less, and of the other greater, than 2·7197 k, k being the semi-axis of rotation.

The number of the forms of equilibrium in spheroids of revolution is purely a mathematical deduction from the expression of q; and as this has been known since the time of MACLAURIN, the discussion of it was all that was wanted for perfecting this part of the theory.

5. Returning now to the general equations of the problem, let

$$\varphi = \int_0^1 3\, x^2\, d\, x \,.\, \frac{\sqrt{(1+\lambda^2)\,.\,(1+\lambda'^2)}}{\sqrt{(1+\lambda^2 x^2)\,.\,(1+\lambda'^2 x^2)}};$$

and it will be found that the two equations (3.) are thus expressed:

$$q = \frac{d\,\varphi}{d\,\lambda}\,\lambda, \quad q = \frac{d\,\varphi}{d\,\lambda'}\,\lambda'.$$

Further, put $p = \lambda\,\lambda'$, $\tau^2 = (\lambda - \lambda')^2$, and we shall have

$$\varphi = \int_0^1 3\, x^2\, d\, x \,.\, \frac{\sqrt{(1+p)^2 + \tau^2}}{\sqrt{(1+p\,x^2)^2 + \tau^2\, x^2}};$$

and the two values of q in the partial differentials of φ relatively to λ and λ' being expressed in the partial differentials relatively to p and τ^2, we shall obtain

$$q = \frac{d\,\phi}{d\,\lambda}\,\lambda = \frac{d\,\phi}{d\,p}\,p + \frac{d\,\phi}{\tau\, d\,\tau}\,(\lambda - \lambda')\,\lambda,$$

$$q = \frac{d\,\varphi}{d\,\lambda'}\,\lambda' = \frac{d\,\varphi}{d\,p}\,p - \frac{d\,\varphi}{\tau\, d\,\tau}\,(\lambda - \lambda')\,\lambda'.$$

These two values of q coalesce in one when $\lambda - \lambda' = 0$, that is, in spheroids of revolution; and we thus fall again upon the same equation that has already been discussed. In all other cases the two values cannot subsist together, unless

$$\left.\begin{aligned} q &= \frac{d\,\phi}{d\,p}\,p, \\ 0 &= \frac{d\,\phi}{\tau\, d\,\tau}, \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots \quad (6.)$$

which equations apply exclusively to ellipsoids with three unequal axes, and solve the problem with regard to that class. The latter of the equations (6.) expresses the relation that the two quantities p and τ^2 must have to one another in every ellipsoid with three unequal axes which is susceptible of an equilibrium. The fluxional operation indicated being performed in the same equation, the result will be,

$$0 = \int_0^1 \frac{x^2\,(1-x^2)\,(1-p^2\,x^2)\,d\,x}{\left((1+p\,x^2)^2 + \tau^2\,x^2\right)^{\frac{3}{2}}}, \quad \ldots\ldots\ldots \quad (7.)$$

which is no other than a transformation of the equation (1.), and is equivalent to other transformations of the same equation found by M. Jacobi and M. Liousville.

The formula (7.) cannot be verified unless p, or $\lambda\,\lambda'$, be greater than 1; for if p were equal to 1, or less than 1, the integral would be positive. This agrees with the limitation of M. Jacobi.

If any value be assigned to τ^2, it is evident that a corresponding value of p may be found which will verify the formula (7.): for, if p be made to increase continually above 1, the integral, which is positive at first, will finally be negative; and it must be zero, in passing from one of these states to the other. This proves that there does exist an infinite number of ellipsoids not of revolution, which are susceptible of an equilibrium.

Let V stand for the integral in the equation (7.); and supposing that p and τ^2 vary so as always to satisfy that equation, we shall have

$$\frac{d\,\mathrm{V}}{d\,p}\,d\,p + \frac{d\,\mathrm{V}}{\tau\,d\,\tau}\,\tau\,d\,\tau = 0.$$

Now, τ^2 representing any positive quantity, we may conceive it to increase from zero to be infinitely great; in which case it follows from the nature of the function V, that during the whole increase $\frac{d\,\mathrm{V}}{\tau\,d\,\tau}\,\tau\,d\,\tau$ will be negative: wherefore the other term $\frac{d\,\mathrm{V}}{d\,p}\,d\,p$ will be positive; which requires that p decrease continually. Since p decreases when τ^2 increases, the greatest value of p will answer to the least value of τ^2, that is, to zero; and hence, by making $\tau^2 = 0$ in the formula (7.), we shall obtain this equation, viz.

$$0 = \int_0^1 \frac{x^2\,(1 - x^2)\,(1 - p^2\,x^2)\,d\,x}{(1 + p\,x^2)^3},$$

for finding the greatest value of p.

It is obvious that there is only one value of p that will verify the equation just found; for the integral can pass only once from being positive to be negative while p increases from 1 to be infinitely great. Let $p = l^2$, $l\,x = z$; and the equation will be changed into this which follows,

$$0 = \int_0^l \frac{d\,z\,\left(l^2\,z^2 - (1 + l^4)\,z^4 + l^2\,z^6\right)}{(1 + z^2)^3}:$$

of which the integral is,

$$0 = l^2\,z + \frac{(1 + l^2)^2}{8} \cdot \frac{3\,z + 5\,z^3}{(1 + z^2)^2} - \frac{3 + 14\,l^2 + 3\,l^4}{8} \int_0^l \frac{d\,z}{1 + z^2}:$$

and hence, by making $z = l$, we deduce

$$\text{arc tan}\; l = \frac{3\,l + 13\,l^3}{3 + 14\,l^2 + 3\,l^4}.$$

The only solution of this equation is $l = 1{\cdot}3934$; and $1{\cdot}9414$ is therefore the greatest value of $p = l^2$. Thus, in all the ellipsoids susceptible of an equilibrium by revolving about the least axis, $\lambda\,\lambda' = p$ is contained between the limits $1{\cdot}9414$ and 1, while $(\lambda - \lambda')^2 = \tau^2$, increases from zero to be infinitely great.

An elliptical spheroid formed of a homogeneous fluid, can be in equilibrium by the action of a centrifugal force, only when it revolves about the least axis. What has been said determines completely the series of ellipsoids with which an equilibrium is possible, when the three axes are unequal. Representing these axes by

$$k, \quad k\sqrt{1 + \lambda^2}, \quad k\sqrt{1 + \lambda'^2},$$

it has been shown that $\lambda\,\lambda'$ must be contained between the limits $1{\cdot}9414$ and 1, while $(\lambda - \lambda')^2$ varies from zero to be infinitely great. One limit is when $\lambda = \lambda'$, being a spheroid of revolution of which the axes are

$$k \text{ and } k\sqrt{2{\cdot}9414} = k \times 1{\cdot}7150.$$

Supposing λ and λ' to vary from this extreme, when the first increases, the other will decrease; so that, when λ is infinitely great, λ' will be zero; which proves that the other extreme limit is a cylinder extending indefinitely on either side of the base, which is a circle having k for its radius.

6. It remains to consider the value of q. In the first of the equations (6.), let the operation indicated be performed, and the result will be

$$q = \int_0^1 \frac{p \cdot 3x^2(1-x^2)\,dx \cdot \{(1+p)(1+px^2) + p\tau^2 x^2\}}{\left((1+p)^2+\tau^2\right)^{\frac{1}{2}} \cdot \left((1+px^2)^2+\tau^2 x^2\right)^{\frac{3}{2}}};$$

and from this we obtain the value of q in the extreme case when $\tau^2 = 0$, or when λ and λ' are equal, viz.

$$q = \int_0^1 \frac{p \cdot 3x^2(1-x^2)\,dx}{(1+px^2)^2},$$

which is no other than the determination of q in a spheroid of revolution having its axes equal to

$$k \text{ and } k\sqrt{2{\cdot}9414} = k \times 1{\cdot}7150.$$

In the other extreme case, when τ^2 is infinitely great, q is zero.

It has been shown that for every given value of τ^2, there is only one value of p, and only one ellipsoid; and when τ^2 and p are both ascertained, the foregoing expression proves that q is fully determined. Thus there is an appropriate value of q to every ellipsoid susceptible of an equilibrium.

In the formula for q, one of the two quantities, τ^2 and p, increases when the other decreases; and hence it may be surmised that more than one ellipsoid may answer to a given value of q. Some calculation is necessary to elucidate this point. For the sake of abridging expressions, put

$$\begin{aligned} \mathrm{P} &= \sqrt{(1+p)^2+\tau^2} \\ \mathrm{Q} &= \sqrt{(1+px^2)^2+\tau^2 x^2} \\ \mathrm{M} &= p\,(1+p)\,(1+px^2) + p^2\tau^2 x^2 \\ du &= 3x^2\,(1-x^2)\,dx, \end{aligned}$$

the variation of du being between the limits $x = 0$ and $x = 1$: then, the foregoing value of q will be thus written:

$$q = \int \frac{du \cdot \mathrm{M}}{\mathrm{P}\,\mathrm{Q}^3};$$

and, q being considered a function of p and τ^2, the fluxion with respect to p will be

$$\frac{dq}{dp} = \int \frac{du}{\mathrm{P}\,\mathrm{Q}^3} \cdot \left\{\frac{d\mathrm{M}}{dp} - \frac{(1+p)\,\mathrm{M}}{\mathrm{P}^2} - \frac{3x^2(1+px^2)\,\mathrm{M}}{\mathrm{Q}^2}\right\}:$$

it will be found that

$$\frac{(1+p)\,\mathrm{M}}{\mathrm{P}^2} = p\,(1+px^2) - \frac{\tau^2 p\,(1-p^2x^2)}{\mathrm{P}^2},$$

$$\frac{3x^2(1+px^2)\,\mathrm{M}}{\mathrm{Q}^2} = 3px^2\,(1+p) - \frac{3\tau^2 p\,x^4\,(1-p^2x^2)}{\mathrm{Q}^2};$$

wherefore

$$\frac{d q}{d p} = \left(\frac{1}{\mathrm{P}} + \frac{\tau^2 p}{\mathrm{P}^3}\right) . \int \frac{d u \,(1 - p^2 x^2)}{\mathrm{Q}^3} + \frac{p}{\mathrm{P}} \int d u \,(1 - x^2)$$
$$+ \frac{p \,\tau^2}{\mathrm{P}} . \int d u \,. \frac{2 \,\mathrm{Q}^2 x^2 + 3 \,x^4 - 3 \,p^2 x^6}{\mathrm{Q}^5} ;$$

in consequence of the formula (7.) the first term is zero, so that we have

$$\frac{d q}{d p} = \frac{p}{\mathrm{P}} \int d u \,(1 - x^2) + \frac{p \,\tau^2}{\mathrm{P}} \int d u \,. \frac{2 \,x^2 + (3 + 4 \,p) \,x^4 - p^2 x^6 + 2 \,\tau^2 x^4}{\mathrm{Q}^5}.$$

And because $3 + 4 \,p$ is always greater than p^2, it follows that $\frac{d q}{d p}$ is essentially positive.

Again, by taking the fluxion relatively to τ^2, we have

$$\frac{d q}{\tau \,d \tau} = \int \frac{d u}{\mathrm{P} \,\mathrm{Q}^3} . \left\{\frac{p^2 x^2 \,\mathrm{P}^2 - \mathrm{M}}{\mathrm{P}^2} + \frac{p^2 x^2 \,\mathrm{Q}^2 - 3 \,x^2 \,\mathrm{M}}{\mathrm{Q}^2}\right\} ;$$

that is,

$$\frac{d q}{\tau \,d \tau} = - \frac{p + p^2}{\mathrm{P}^3} \int d u \,. \frac{1 - p^2 x^2}{\mathrm{Q}^3}$$
$$- \frac{p}{\mathrm{P}} \int d u \,. \frac{(3 + 2 \,p) \,x^2 + (3 \,p + p^2) \,x^4 - p^3 x^6 + 2 \,p \,\tau^2 x^2}{\mathrm{Q}^5} :$$

Of this value the first term is zero by the formula (7.); and attending to the limits of p and of the integral, the second term is essentially negative.

Now we have

$$d q = \frac{d q}{d p} . \,d p + \frac{d q}{\tau \,d \tau} . \,\tau \,d \tau :$$

if we suppose τ^2 to increase, p will decrease; and according to what has been shown the two parts of $d \,q$ will be negative. Wherefore, while τ^2 increases from zero to be infinitely great, q will decrease continually from its first value to zero; and for every possible value of q there will be only one value of τ^2, and consequently only one ellipsoid susceptible of an equilibrium.

It would be superfluous to pursue this investigation further, and a mere waste of labour to seek the easiest formulas for solving a problem which, it appears from what has been shown, can have no application in the theory of the figure of the planets. It is extremely probable that no such figures as those required for the equilibrium of ellipsoids with three unequal axes, will be found to exist in nature. It seems difficult to admit that any circumstances, or the action of any forces we are acquainted with, could induce upon a mass of fluid a figure adjusted with such mathematical nicety to the attraction of the mass and the centrifugal force. If the existence of such a figure can be supposed, would it be permanent? Would not the least action of the other bodies of the system upon it be sufficient to destroy the exact confor-

mation on which the equilibrium depends, and leave the fluid to adjust its figure solely by the attraction and the centrifugal force of its particles? The discovery of JACOBI makes no change in the usual theory of the figure of the planets; but it is valuable, as it completes a mathematical speculation, and finally settles what relates to the figure of ellipsoids susceptible of an equilibrium.