XII. *Remarks on the Theory of the Dispersion of Light, as connected with Polarization. By the Rev.* BADEN POWELL, *M.A. F.R.S. F.G.S., Savilian Professor of Geometry in the University of Oxford.*

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Introductory Observations.

(1.) IN the course of four successive papers, I have laid before the Royal Society the comparison of observations of the refractive indices for the standard rays in various media, with the results calculated from the formula of theory, as deduced upon the most improved views of the hypothesis of undulations; the cases discussed including the greatest range of data which experiment has yet furnished.

The degree of accordance thus exhibited between observation and theory, even in the most extreme case, I believe will now be considered sufficient to warrant the conclusion, that the theory at least affords a very satisfactory approximation to the expression and explanation of the actual law of nature; and this in the instance of a class of phenomena which it had long been the reproach of that theory to be supposed incapable of accounting for, if they were not absolutely contradictory to it. At any rate, if, in the instances referred to, the discrepancies should by any be thought still too great, or if it should be contended that other cases may yet arise in which theory may be put to a severer test, yet, with so strong a presumption in its favour, the only fair inference would be, that further examination was required of the principles on which any extension or modification of the theory might be pursued.

(2.) The investigation, then, having advanced thus far, it seemed desirable, as a sequel to my former papers, to devote the present to some remarks connected with the *theory* which has been thus applied.

The facts of interference, on which the undulatory theory was originally based by Dr. YOUNG, obliged us to adopt *some* idea of an *alternating* motion, as well as a motion of *translation*, in our conception of light. And this, with all the accessions it has received, especially from the investigations of FRESNEL, has at the present day been connected by the labours of M. CAUCHY and others with general dynamical principles, which regulate the propagation of vibratory motions through an elastic medium.

(3.) More precisely, from such dynamical principles certain differential equations of motion have been deduced; the integration of which gives the well-known expression for a wave, involving the relation between the velocity and the wave-length which explains the dispersion. The *direct* and *complete* integration of these forms, effected

by M. CAUCHY, and simplified by Mr. TOVEY and Mr. KELLAND, involves certain conditions, viz. *the evanescence of certain terms*; the interpretation of which implies peculiar views of the constitution of the ether.

Whether a *direct* and *complete* solution, *without* the introduction of these conditions, can be attained, does not appear. But Mr. TOVEY has been the first (as far as I am aware) to point out (in one of his valuable series of papers in the London and Edinburgh Journal of Science) that, *without* these conditions, a certain form of the wave-function *is a particular solution* of the equations; and this form is precisely that expressing *elliptically polarized light.*

(4.) Upon a careful examination of that most important and ingenious investigation, it appeared to me that there were one or two questions connected with the subject still requiring to be cleared up.

If the *absence* of the conditions in question be *essential* to the case of elliptically and circularly polarized light, it follows that *all the preceding investigations* (which depend on the fulfilment of those conditions) *are applicable only to unpolarized and plane-polarized light,* and consequently the *general* integration is limited in a most material part of its application; a defect which is only remedied by the supplementary investigation of Mr. TOVEY, in which, for this case, a *particular* solution is assigned.

It seemed, then, necessary to show explicitly that the non-fulfilment of the conditions, that is, the *non-evanescence* of the terms in question, is *essential* for elliptically polarized light, as their *evanescence* is for common light; and thus to exhibit distinctly the relation between the cases of elliptically polarized, and those of plane-polarized and unpolarized light: and again, to remove, if possible, the obscurity and discrepancy of opinion in which the physical interpretation of those conditions, with regard to the supposed constitution of the etherial medium, appeared to be involved.

With this object, it seemed necessary to take up the subject from its first principles; and, in pursuing the inquiry, I gladly acknowledge the valuable suggestions I have derived in the course of some correspondence with Mr. LUBBOCK. I have been thus led to combine the elucidation of the points just referred to, in a connected view with preceding results, by a simplified method; and to trace clearly the relation between the changes in the conditions supposed with regard to the ether, the resulting state of polarization in the ray, and the expression for the dispersion.

Analytical Investigation.

(5.) For the purpose of the ensuing investigation it will be necessary to premise a brief statement of the formulas adopted from the undulatory theory. On the ordinary principles of that theory, the form

$$u = \Sigma \{\alpha \sin (n t - k x)\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1.)$$

is the sum of a number of functions, each expressing a single wave, whose aggregate constitutes a ray of light; (u) is the displacement of the vibrating molecule of ether;

(α) the amplitude of vibration; (t) the time from the beginning of the disturbance; (x) the distance along the ray from the molecule first agitated; while $\left(\frac{n}{k}\right)$ is the velocity of propagation, which further involves the relation

$$\frac{1}{\mu} = \frac{n}{k} \quad \text{and } k = \frac{2\pi}{\lambda}, \quad \ldots \ldots \quad (2.)$$

where (μ) is the refractive index, and (λ) the wave-length in the medium.

(6.) The investigations of FRESNEL require us to suppose the vibrations performed in planes at right angles to the direction of the ray: or, in general, referring the whole motions to three rectangular axes x, y, z, of which we may suppose x to coincide with the direction of the ray, and naming the displacements in each of those directions respectively ξ, η, ζ, we must suppose $\xi = 0$, while each of the others are functions of x and t of the form above, viz.

$$\left.\begin{aligned} \eta &= \Sigma \{\alpha \sin (n t - k x)\} \\ \zeta &= \Sigma \{\beta \sin (n t - k x)\} \end{aligned}\right\} \quad \ldots \ldots \quad (3.)$$

In polarized light the amplitudes α β are in FRESNEL's notation

$$\alpha = \cos i \quad \text{and } \beta = \sin i,$$

where i is the angle formed by the direction of the vibration with that of polarization; whence also

$$\alpha^2 + \beta^2 = 1. \quad \ldots \ldots \quad (4.)$$

In plane polarized light either $i = 0$, or $i = \frac{\pi}{2}$; hence one of the expressions (3.) disappears.

For elliptically polarized light both are retained; but one vibration is retarded by a quantity (b), or we must take

$$\left.\begin{aligned} \eta &= \Sigma \{\alpha \sin (n t - k x)\} \\ \zeta &= \Sigma \{\beta \sin (n t - k x + b)\}, \end{aligned}\right\} \quad \ldots \ldots \quad (5.)$$

expressions which will easily be found to give, on substitution of the value of η in that of ζ, the equation to the ellipse described by the etherial molecules, which becomes that of a circle if $b = \frac{\pi}{2}$ and $\alpha = \beta$. If $\beta = 0$ the formulas are reduced to those for common light, and α and β are not restricted by the condition (4.).

(7.) From (2.) we have $\frac{1}{\mu} = \frac{n}{2\pi}\lambda$; . . . . (6.)

hence we may observe; 1st. If $\frac{n}{k}$ be constant there is no dispersion. In this case it is easily found that the formulas (3.) are solutions of the well-known equation of vibratory motion,

$$\frac{d^2 u}{d t^2} = -\frac{n^2}{k^2}\frac{d^2 u}{d x^2}.$$

2ndly. If n be an independent constant these expressions give the index in the simple inverse ratio of the wave-length, which is manifestly not the law of the unequal refrangibility of the primary rays, as appears at once from what is termed the irrationality of dispersion. But the application of the formulas to the ordinary phenomena is independent of this, and will be equally valid if μ be dependent on λ, precisely or very nearly in the relation expressed by the formula which has been compared with experiment in my previous papers, and which thus explains the unequal refrangibility.

Now it has been shown by the distinguished mathematicians already mentioned, that, under certain conditions, the expressions (3.) can be derived from equations of motion for the vibrations of an elastic medium on dynamical principles, which also involve the desired relation between μ and λ. It will be necessary to state and explain these equations, which may be most shortly done as follows:

(8.) Let the coordinates in space of any molecules $m\ m'$ be respectively

$$\left.\begin{array}{llll} m \ldots\ldots & x & y & z \\ m' \ldots\ldots & x + \Delta x & y + \Delta y & z + \Delta z \end{array}\right\} \quad \ldots\ldots\ldots \quad (7.)$$

and the distance between them

$$r = \sqrt{(\Delta x^2 + \Delta y^2 + \Delta z^2)} \quad \ldots\ldots\ldots \quad (8.)$$

Let the force which maintains the system of molecules as an elastic medium be any function of the distance as $f(r)$: then it will be easily seen that we have for the forces in the direction of the three axes, in *equilibrium*,

$$\left.\begin{array}{l} \Sigma f(r) \dfrac{\Delta x}{r} = 0 \\[2ex] \Sigma f(r) \dfrac{\Delta y}{r} = 0 \\[2ex] \Sigma f(r) \dfrac{\Delta z}{r} = 0 \end{array}\right\} \quad \ldots\ldots\ldots \quad (9.)$$

Let the system be *disturbed*, and after a time (t) let the displacements in the direction of the three axes be respectively

$$\left.\begin{array}{llll} m \ldots\ldots & \xi & \eta & \zeta \\ m' \ldots\ldots & \xi + \Delta \xi & \eta + \Delta \eta & \zeta + \Delta \zeta \end{array}\right\} \quad \ldots\ldots\ldots \quad (10.)$$

also the distance becomes

$$r + \Delta r = \sqrt{\{(\Delta x + \Delta \xi)^2 + (\Delta y + \Delta \eta)^2 + (\Delta z + \Delta \zeta)^2\}} \quad \ldots\ldots \quad (11.)$$

In this condition it will be easily seen that we have for the forces in the direction of the three axes,

$$\left.\begin{array}{l} \dfrac{d^2 \xi}{d t^2} = \Sigma \left\{ f(r + \Delta r) \dfrac{\Delta x + \Delta \xi}{r + \Delta r} \right\} \\[2ex] \dfrac{d^2 \eta}{d t^2} = \Sigma \left\{ f(r + \Delta r) \dfrac{\Delta y + \Delta \eta}{r + \Delta r} \right\} \\[2ex] \dfrac{d^2 \zeta}{d t^2} = \Sigma \left\{ f(r + \Delta r) \dfrac{\Delta r + \Delta \zeta}{r + \Delta r} \right\} \end{array}\right\} \quad \ldots\ldots\ldots \quad (12.)$$

On expanding the value of $\frac{f(r+\Delta r)}{r+\Delta r}$, neglecting the squares, substituting in the above equations (12.), observing that they will involve the expressions (9.) for equilibrium, which vanish, and writing for abridgement,

$$\frac{f(r+\Delta r)}{r+\Delta r} = \varphi(r) = \frac{f(r)}{r} + \psi(r)\, r\,\Delta r.$$

while we have

$$r\,\Delta r = \Delta x\,\Delta \xi + \Delta y\,\Delta \eta + \Delta z\,\Delta \zeta;$$

these equations of motion (12.) will ultimately become

$$\left.\begin{aligned}
\frac{d^2\xi}{dt^2} &= \Sigma\left\{\varphi(r)\,\Delta\xi + \psi(r)\,\Delta x\,(\Delta x\,\Delta\xi + \Delta y\,\Delta\eta + \Delta z\,\Delta\zeta)\right\}\\
\frac{d^2\eta}{dt^2} &= \Sigma\left\{\varphi(r)\,\Delta\eta + \psi(r)\,\Delta y\,(\Delta x\,\Delta\xi + \Delta y\,\Delta\eta + \Delta z\,\Delta\zeta)\right\}\\
\frac{d^2\zeta}{dt^2} &= \Sigma\left\{\varphi(r)\,\Delta\zeta + \psi(r)\,\Delta z\,(\Delta x\,\Delta\xi + \Delta y\,\Delta\eta + \Delta z\,\Delta\zeta)\right\}.
\end{aligned}\right\} \quad (13.)$$

These equations form the common basis of all the investigations of the subject as originally pursued by MM. NAVIER and CAUCHY. They apply generally to all elastic media, or systems of molecules, affected only by their mutual attractions and repulsions, and slightly disturbed.

(9.) In applying this hypothesis to the case of *light*, agreeably to what has been already observed (§. 6.), we have

$$\Delta\xi = 0 \quad \text{and} \quad \frac{d^2\xi}{dt^2} = 0,$$

so that the equations (13.) are reduced to

$$\left.\begin{aligned}
\frac{d^2\eta}{dt^2} &= \Sigma\left\{\varphi(r)\,\Delta\eta + \psi(r)\,\Delta y\,[\Delta y\,\Delta\eta + \Delta z\,\Delta\zeta]\right\}\\
\frac{d^2\zeta}{dt^2} &= \Sigma\left\{\varphi(r)\,\Delta\zeta + \psi(r)\,\Delta z\,[\Delta y\,\Delta\eta + \Delta z\,\Delta\zeta]\right\}
\end{aligned}\right\} \quad . \quad . \quad . \quad (14.)$$

The *general direct* integration of these forms, and even of the preceding (13.), has been effected by the writers already referred to; *but only upon the suppositions*

$$\Sigma\{\Delta y\,\Delta z\} = 0, \quad \text{and } \Sigma\{\sin k\,\Delta x\} = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad (15.)$$

the import of which will be considered in the sequel. Mr. TOVEY by a skilful analysis gives a *particular solution* corresponding to the case of elliptically polarized light, *without introducing those conditions*. The investigation, however, appears susceptible of being simplified; and we shall here pursue a method at once attaining this object, and enabling us more clearly to trace out some interesting results, as follows:

Taking the finite differences of the expressions (5.), we have

$$\Delta\eta = \alpha\left[2\sin^2\left(\frac{k\,\Delta x}{2}\right)\sin(n\,t - k\,x) - \sin k\,\Delta x\cos(n\,t - k\,x)\right] \quad . \quad . \quad (16.)$$

$$\Delta \zeta = \left\{ \begin{array}{l} \beta \cos b \left[2 \sin^2 \left(\frac{k \Delta x}{2} \right) \sin (n t - k x) - \sin k \Delta x \cos (n t - k x) \right] \\ + \beta \sin b \left[\sin k \Delta x \sin (n t - k x) - 2 \sin^2 \left(\frac{k \Delta x}{2} \right) \cos (n t - k x) \right]. \end{array} \right\} \quad (17.)$$

Also, differentiating them, we find,

$$\frac{d^2 \eta}{d t^2} = - n^2 \alpha \sin (n t - k x) \quad . \quad . \quad . \quad (18.)$$

$$\frac{d^2 \zeta}{d t^2} = - n^2 \beta \cos b \sin (n t - k x) - n^2 \beta \sin b \cos (n t - k x) \quad . \quad . \quad . \quad (19.)$$

On arranging the terms in (14.) we may write for abridgement,

$$\left. \begin{array}{r} p = \varphi (r) + \psi (r) \Delta y^2 \\ p' = \varphi (r) + \psi (r) \Delta z^2 \\ q = \psi (r) \Delta y \Delta z \\ 2 \theta = k \Delta x \end{array} \right\} \quad . \quad . \quad . \quad (20.)$$

and those forms (14.) will thus become,

$$\frac{d^2 \eta}{d t^2} = \Sigma \left[p \Delta \eta + q \Delta \zeta \right] \quad . \quad . \quad . \quad (21.)$$

$$\frac{d^2 \zeta}{d t^2} = \Sigma \left[p' \Delta \zeta + q \Delta \eta \right] \quad . \quad . \quad . \quad (22.)$$

On substituting in these forms the values of $\Delta \eta$ and $\Delta \zeta$ above, viz. (16.) (17.), we shall have,

$$\frac{d^2 \eta}{d t^2} = \Sigma \left\{ \begin{array}{l} \left. \begin{array}{l} q \, \{\beta \sin b\} \quad \sin 2 \theta \\ \left. \begin{array}{l} + q \, \{\beta \cos b \\ + p \, \{\alpha \end{array} \right\} 2 \sin^2 \theta \end{array} \right\} \sin (n t - k x) \\ \left. \begin{array}{l} \left. \begin{array}{l} - p \, \{\alpha \\ - q \, \{\beta \cos b \end{array} \right\} \sin 2 \theta \\ - q \, \{\beta \sin b\} \quad 2 \sin^2 \theta \end{array} \right\} \cos (n t - k x) \end{array} \right\} \quad . \quad . \quad . \quad (23.)$$

$$\frac{d^2 \zeta}{d t^2} = \Sigma \left\{ \begin{array}{l} \left. \begin{array}{l} p' \, \{\beta \sin b\} \sin 2 \theta \\ \left. \begin{array}{l} + p' \, \{\beta \cos b \\ + q \, \{\alpha \end{array} \right\} 2 \sin^2 \theta \end{array} \right\} \sin (n t - k x) \\ \left. \begin{array}{l} \left. \begin{array}{l} - q \, \{\alpha \\ - p' \, \{\beta \cos b \end{array} \right\} \sin 2 \theta \\ - p' \, \{\beta \sin b\} 2 \sin^2 \theta \end{array} \right\} \cos (n t - k x) \end{array} \right\} \quad . \quad . \quad . \quad (24.)$$

On comparing these expressions with those for the same functions (18.) (19.) which must be identical, and equating in each the terms which form the coefficients respectively of $\sin (n t - k x)$ and of $\cos (n t - k x)$ (since they must hold good for all

values of those variable terms), we shall have the following equations :

$$-n^2\alpha = \Sigma \left\{ \begin{array}{l} +\,q\{\beta \sin b\} \quad \sin 2\theta \\ \left.\begin{array}{l} +\,q\{\beta \cos b \\ +\,p\{\alpha \end{array}\right\} 2\sin^2\theta \end{array} \right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (25.)$$

$$0 = \Sigma \left\{ \begin{array}{l} \left.\begin{array}{l} -\,p\,\{\alpha \\ -\,q\,\{\beta \cos b \end{array}\right\} \sin 2\theta \\ -\,q\,\{\beta \sin b\}\ 2\sin^2\theta \end{array} \right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (26.)$$

$$-n^2\beta \cos b = \Sigma \left\{ \begin{array}{l} +\,p'\,\{\beta \sin b\} \quad \sin 2\theta \\ \left.\begin{array}{l} +\,p'\,\{\beta \cos b \\ +\,q\ \{\alpha \end{array}\right\} 2\sin^2\theta \end{array} \right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (27.)$$

$$-n^2\beta \sin b = \Sigma \left\{ \begin{array}{l} \left.\begin{array}{l} -\,q\ \{\alpha \\ -\,p'\,\{\beta \cos b \end{array}\right\} \sin 2\theta \\ -\,p'\,\{\beta \sin b\,\}\ 2\sin^2\theta \end{array} \right\} \quad . \quad . \quad . \quad . \quad . \quad (28.)$$

From the two last forms (27.) (28.) we obtain by multiplication and addition,

$$-n^2\beta = \Sigma \left\{ \begin{array}{l} \left.\begin{array}{l} p'\,\{\beta \cos 2b \\ q\ \{\alpha \cos b \end{array}\right\} 2\sin^2\theta \\ -\,q\ \{\alpha \sin b\ \} \quad \sin 2\theta \end{array} \right\} \quad . \quad . \quad . \quad . \quad . \quad (29.)$$

And in like manner from (25.) and (29.) we deduce

$$-n^2(\alpha^2+\beta^2) = \Sigma \left\{ \left.\begin{array}{l} p'\,\{\beta^2 \cos 2b \\ +\,p\ \{\alpha^2 \\ +\,q\ \{2\,\alpha\,\beta \cos b \end{array}\right\} 2\sin^2\theta \right\} \quad . \quad . \quad . \quad . \quad (30.)$$

In the case of elliptically polarized light we have

$$\alpha^2+\beta^2 = 1\,;$$

and it is thus obvious that *when the quantities involved give the value of* n^2 *as here expressed,* viz.

$$-n^2 = \Sigma \left\{ \left.\begin{array}{l} p'\,\{\beta^2 \cos 2b\} \\ +\,p\ \{\alpha^2\} \\ +\,q\ \{2\,\alpha\beta \cos b\} \end{array}\right\} 2\sin^2\theta \right\} \quad . \quad . \quad . \quad . \quad (31.)$$

and when this value is substituted in (18.) (19.), *and consequently in* (6.), *the formula* (5.) *for elliptically polarized light is a particular solution of the differential equations of motion of the system of molecules* (14.).

Deductions and Remarks.

(10.) It is evident that in (31.) expressing the whole coefficient by a single letter, we have the abridged form

$$n^2 = \Sigma\,\{H'^2 \sin^2\theta\}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (32.)$$

which, on dividing by k^2, restoring the value of θ (20.) and of k (2.), will give for the velocity $\left(\text{or}\frac{n}{k}\right)$ (on multiplying both numerator and denominator by $\left(\frac{\Delta x}{2}\right)$ on the second side of the equation, and including all the coefficients in a new single term,) the formula

$$\frac{1}{\mu^2} = \Sigma\left\{\mathrm{H}^2 \frac{\sin^2\left(\frac{\pi\,\Delta x}{\lambda}\right)}{\left(\frac{\pi\,\Delta x}{\lambda}\right)^2}\right\}, \quad \text{(33.)}$$

the same which has been deduced for unpolarized and plane polarized light.

From this formula, on the supposition of a common coefficient, I made my approximate computations in Nos. I. and II. of my former Researches. On expanding the sine and dividing by the arc there results a series of even powers of λ, which, in a generalized form, is that from which Sir W. R. HAMILTON deduced his elegant formula for interpolation, by which I performed my calculations in No. III. From a similar series, also, M. CAUCHY has deduced the elaborate method of calculation which he has applied to FRAUNHOFER'S indices in his *Nouveaux Exercices de Mathématiques*, and also in his lithographed memoir on Interpolation: whilst the same series obtained by Mr. KELLAND, and modified by the introduction of the value of λ *in vacuo* (that term being in the above formula the wave-length *in the medium*), or

$$\mathrm{P} = \frac{1}{\mu^2} + \left(\frac{\mu}{\lambda}\right)^2 \mathrm{Q} - \left(\frac{\mu}{\lambda}\right)^4 \mathrm{L} \quad \text{(34.)}$$

is the formula which I employed and explained in No. IV. Precisely equivalent results are also deduced by Mr. TOVEY.

(11.) The term on which the dispersion depends, viz.

$$\frac{\sin\left(\frac{\pi\,\Delta x}{\lambda}\right)}{\left(\frac{\pi\,\Delta x}{\lambda}\right)}, \quad \text{(35.)}$$

manifestly approaches to unity as Δx diminishes in comparison with λ, the application of which was originally pointed out by Mr. AIRY. The dispersion is insensible in free space, and large in dense media. Hence Mr. KELLAND has inferred that the ether is in a state of *greater* density in free space, and *less* in dense media. We shall recur to this point in the sequel.

(12.) Experiment does not show that the state of a ray as to polarization, produces any difference in the magnitude of its refractive index. Hence it follows that in the formula (30.) *the values of n^2 must be constant for all values of b, as well as of α and β.* Hence for any particular value of b the several sums of terms involved must be supposed to vary in magnitude, so that, joined with the condition (4.), the whole expression shall always remain constant, and equal to that in the case where b vanishes, and where α and β are no longer restricted to the condition (4.).

The examination of the cases thus arising, is closely connected with the question of the evanescence of the terms before alluded to (15.). These relations therefore we shall now proceed to trace. But it will be desirable first to point out more clearly the nature and meaning of those conditions.

(13.) It is evident from the value of r (8.) that it does not change its sign along with a change of sign in Δx, Δy, or Δz; the same is true therefore of $\varphi(r)$ and $\psi(r)$. It is also evident that Δy^2, Δz^2, do not change the signs. But the product $\Delta y \, \Delta z$, will change signs as each value of Δy corresponds to a negative and a positive value of Δz.

Hence the terms

$$\begin{aligned}
&\psi(r) \sin k \Delta x \\
&\psi(r) \Delta y^2 \sin k \Delta x \\
&\psi(r) \Delta z^2 \sin k \Delta x \\
&\psi(r) \Delta y \Delta z \sin k \Delta x \\
&\psi(r) \Delta y \Delta z \sin^2\left(\frac{k \Delta x}{2}\right)
\end{aligned}$$

will have values with opposite signs. But,

$$\begin{aligned}
&\psi(r) \sin^2\left(\frac{k \Delta x}{2}\right) \\
&\psi(r) \Delta y^2 \sin^2\left(\frac{k \Delta x}{2}\right) \\
&\psi(r) \Delta z^2 \sin^2\left(\frac{k \Delta x}{2}\right)
\end{aligned}$$

will have all their values with the same sign. If then we consider the sums of a number of such terms respectively, those of the first set may become $= 0$ if the positive and negative sums be equal. In those of the second set this cannot happen.

Hence resuming our abridged notation we may have either

$$\left.\begin{aligned}
\Sigma(p \sin 2\theta) &= 0 \\
\Sigma(p' \sin 2\theta) &= 0 \\
\Sigma(q \sin 2\theta) &= 0
\end{aligned}\right\} \quad . \quad . \quad . \quad (36.) \qquad \text{or} \qquad \left.\begin{aligned}
\Sigma(p \sin 2\theta) &= s \\
\Sigma(p' \sin 2\theta) &= s_{,} \\
\Sigma(q \sin 2\theta) &= s_{,,}
\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (38.)$$

$$\Sigma(q \sin^2\theta) = 0 \quad . \quad . \quad . \quad . \quad (37.) \qquad\qquad \Sigma(q \sin^2\theta) = s_{,,,} \quad . \quad . \quad . \quad . \quad . \quad (39.)$$

but always

$$\left.\begin{aligned}
\Sigma(p \sin^2\theta) &= s' \\
\Sigma(p' \sin^2\theta) &= s''
\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (40.)$$

The conditions which give these values respectively, are dependent on the general supposition relative to the arrangement of the molecules. If the distribution of the molecules in space be uniform, the values (36.) (37.) obtain. If not uniform, the

values (38.) (39.) may obtain; but in particular cases, or for particular directions of the ray, the greater number of terms of one sign may compensate for the greater magnitudes of those of the other; or the conditions (36.) (37.) may hold good.

(14.) We now resume the consideration of the formulas before deduced.

From (28.) we have

$$\sin b = \frac{\alpha \Sigma (q \sin 2\theta) + \beta \cos b \Sigma (p' \sin 2\theta)}{n^2 \beta - 2\beta \Sigma (p' \sin^2 \theta)}. \qquad (41.)$$

Now if the conditions (36.) obtain, that is, if we have

$$\Sigma (q \sin 2\theta) = 0, \text{ and } \Sigma (p' \sin 2\theta) = 0; \qquad (42.)$$

then (41.) will give

$$\sin b = 0; \text{ or } b = 0. \qquad (43.)$$

But if the polarization be elliptical, b must have a finite value; or in this case, consequently, the conditions (38.) must obtain; or *the non-evanescence of those terms is essential to the investigation for elliptically polarized light.*

(15.) If the polarization be *circular,* we have

$$\alpha = \beta, \text{ and } b = \frac{\pi}{2} \cos b = 0, \sin b = 1;$$

thus the form (26.) becomes

$$0 = \left\{ \begin{array}{c} \Sigma (p \sin 2\theta) \\ - 2 \Sigma (q \sin^2 \theta). \end{array} \right\} \qquad (45.)$$

But the condition of *the non-evanescence of the terms holds good* for the same reason as in the last case. . . . . (46.)

(16.) For plane polarized light let $\beta = 0$, then the form (26.) becomes

$$0 = \Sigma (p \sin 2\theta); \qquad (47.)$$

or for plane polarization it is essential to suppose the terms evanescent, or the conditions (36.) (37.) *to be fulfilled.* . . . . (48.)

(17.) For unpolarized light we have

$$b = 0 \quad \sin b = 0 \quad \cos b = 1;$$

but α and β are arbitrary. Thus the formula (26.) becomes

$$0 = \left\{ \begin{array}{c} \alpha \Sigma (p \sin 2\theta) \\ + \beta \Sigma (q \sin 2\theta). \end{array} \right\}. \qquad (49.)$$

But since this must be true for all values of α and β which are independent of p, q and θ, it follows that each term separately must be $= 0$; *or for unpolarized light it is essential to suppose the terms evanescent, or the conditions* (36.) (37.) *to be fulfilled.* . . . . (50.)

(18.) We here take the axis (x) as the direction of the ray, which consequently may

have any position whatever in respect to particular lines of direction which may subsist among the molecules of the etherial medium.

Now, supposing the arrangement of the molecules to be perfectly uniform, since the direction of the ray (x) must pass through m, it follows that the sums of the distances of all those on each side of the line x, whether in the plane of y or of z, will be equal, or the hypothesis (36.) (37.), or (48.) (50.) exactly fulfilled, *for all positions of* x in the medium.

If the arrangement be *not* uniform, the same hypothesis in general may be *approximately* fulfilled in proportion as we suppose a *greater number of such molecules taken into account*, or the summation extended to a greater number of terms, that is, a greater number of molecules *within the sphere of the influence of the propagation of motion* from the molecule first agitated. But if we suppose any portion of the ether be so circumstanced that the sphere of influence is more limited, then the conditions (36.) (37.) will not be fulfilled even approximately, or the hypothesis (38.) (39.) will obtain; though they would approach more or less towards fulfilment in different positions of the ray. The diminution of the sphere of influence may arise either from a decrease in the force, or an increase in the distances of the molecules.

(19.) Here also we may remark, in connexion with what was observed before (§. 11.), that at the bounding surface of vacuum and a medium, or generally of two media of different densities, we can hardly suppose the change of density in the ether to take place abruptly; but must, from all analogy, imagine a thin stratum on either side, within which there is a gradual alteration in the arrangement of the molecules, and this more considerable in proportion to the difference of refractive powers of the media.

Thus, even supposing the molecules uniformly distributed in the two media or portions of space, it is evident that within this stratum they will not be uniformly arranged. And hence, though on either side the conditions (36.) (37.) should be fulfilled, *within the stratum* the conditions (38.) (39.) would take effect.

Conclusion.

(20.) The general conclusions which I conceive have been obtained are as follows:

When light is elliptically or circularly polarized, that is, when one of the two component vibrations is retarded behind the other, then, in the differential equations of motion, the opposite terms do not destroy each other in the summation; that is, the arrangement of the molecules is not uniform.

When light is plane-polarized, or unpolarized, that is, when there is no retardation, or the phases of the component vibrations are simultaneous, then the opposite sums destroy each other; that is, either the arrangement is uniform, or the sphere of the influence of the force so great, that the conditions are fulfilled very nearly.

Since both kinds of light can be propagated indifferently through ordinary media, it follows that the sphere of influence of the force, or number of molecules taken into

account, does not here depend on the arrangement of the molecules of ether in the medium, but on the retardation of one of the vibrations behind the other, or the absence of it, originally impressed on the ray in the respective cases.

These inferences appear of importance in connexion with the consideration of the causes on which polarization depends, and the laws of elasticity of the ether in crystallized bodies, as also the state of the ether at the bounding surfaces of media of different density, and the changes which may be effected in the state of polarization of a ray: and in what has here been laid down, I trust that the whole subject (without entering into specific controversy) will be found relieved from some degree of difficulty and objection.

NOTE TO ART. 6.

If in equation (5.) $b = \frac{\pi}{2}$ (without supposing $\alpha = \beta$) the vibration resulting is still elliptical. In this case the expressions may be put into the form

$$\left.\begin{aligned} \eta &= \Sigma \{\alpha \sin (n t - k x)\} \\ \zeta &= \Sigma \{\beta \cos (n t - k x)\} \end{aligned}\right\} \quad \ldots\ldots\ldots \quad (52.)$$

This form is that adopted by Professor MACCULLAGH* to express the elliptical vibrations of the two rays in quartz, and by means of which he connects the laws of M. BIOT, and the theory of Mr. AIRY, with certain differential equations of vibratory motion, provided the quantities α, β, n, and k, are so assumed as to fulfil these conditions, viz. that if A and B be the squares of the velocities of the ordinary and extraordinary rays in common double refraction, in quartz we replace them respectively by

$$A - k\frac{\beta}{\alpha}C \quad \text{and } B - k\frac{\alpha}{\beta}C$$

where C is a new constant determined from BIOT's observations. The differential equations referred to are these:

$$\left.\begin{aligned} \frac{d^2\eta}{dt^2} &= A\frac{d^2\eta}{dx^2} + C\frac{d^3\zeta}{dx^3} \\ \frac{d^2\zeta}{dt^2} &= B\frac{d^2\zeta}{dx^2} - C\frac{d^3\eta}{dx^3} \end{aligned}\right\} \quad \ldots\ldots\ldots \quad (53.)$$

But on taking the partial differential coefficients of the expressions (52.) it is easily seen (from the particular forms of those functions) that the equations (53.) are reducible to,

$$\left.\begin{aligned} \frac{d^2\eta}{dt^2} &= \left[A - k\frac{\beta}{\alpha}C\right]\frac{d^2\eta}{dx^2} \\ \frac{d^2\zeta}{dt^2} &= \left[B + k\frac{\alpha}{\beta}C\right]\frac{d^2\zeta}{dx^2} \end{aligned}\right\}$$

which are of the well-known form (§. 7.) for vibratory motion.

* Memoirs of the Royal Irish Academy, 1836.