XX. *Computation of the Ratio of the Diameter of a Circle to its circumference to 208 places of figures. By* WILLIAM RUTHERFORD, *Esq., of the Royal Military Academy. Communicated by* S. HUNTER CHRISTIE, *Esq., M.A., Sec. R.S. &c. &c.*

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BEFORE the time of MACHIN, the approximation to the ratio of the circumference of a circle to its diameter had been carried as far as seventy-two places of decimals by ABRAHAM SHARP, by means of the series

$$\frac{\pi}{6} = \frac{1}{3}\sqrt{3}\left\{1 - \frac{1}{3}\cdot\frac{1}{3^2} + \frac{1}{5}\cdot\frac{1}{3^4} - \frac{1}{7}\cdot\frac{1}{3^6}, \&c.\right\}.$$

By employing the series arising from the formula,

$$\frac{\pi}{4} = 4\tan^{-1}\frac{1}{5} - \tan^{-1}\frac{1}{239},$$

MACHIN extended the approximation to 100 places. By the same means M. DE LAGNY carried this approximation to 127 places; and in an Oxford manuscript it is extended to 152 places, which, as far as I am aware, is the greatest extent to which the approximation has ever been pushed.

The processes employed in these approximations may be greatly simplified by replacing $\tan^{-1}\frac{1}{239}$ by $\tan^{-1}\frac{1}{70} - \tan^{-1}\frac{1}{99}$, inasmuch as the calculation of the terms of the series involving inverse powers of 70 and 99 may be effected by arithmetical processes of very great facility. By employing the synthetic process of division, the division by 99 (100 − 1) becomes even more simple than that by 9 or 11, since it is effected by adding together two numbers each less than 10.

By means of the formula

$$\frac{\pi}{4} = 4\tan^{-1}\frac{1}{5} - \tan^{-1}\frac{1}{70} + \tan^{-1}\frac{1}{99}*,$$

I have computed the value of $\frac{\pi}{4}$, and thence that of π to 208 places of decimals. Previously to entering upon the calculations I considered whether it would be simpler

* When the calculations for determining the value of π were presented to the Royal Society, it was presumed that the formula $\frac{\pi}{4} = 4\tan^{-1}\frac{1}{5} - \tan^{-1}\frac{1}{70} + \tan^{-1}\frac{1}{99}$ had not before been investigated. I have since found that EULER, in an article entitled "De progressionibus arcuum circularium quorum tangentes secundum certam legem procedunt," obtained the very same formula.—*Novi Commentarii Petropol.*, tom. ix. 1764.

to combine the respective terms of the three series, or to combine only the terms of the two series arising from $\tan^{-1}\frac{1}{70}$ and $\tan^{-1}\frac{1}{99}$, computing the value of $4\tan^{-1}\frac{1}{5}$ separately. The reciprocals of the powers of 5 being terminate decimals, and the results of the several terms in the series for $\tan^{-1}\frac{1}{5}$ circulating in small periods, induced me to compute its value apart from that of the other two.

It is unnecessary to give a lengthened description of the mode of computation, and I shall only briefly state the method of constructing the auxiliary tables which I employed.

The first of these Tables marked (A), comprises the reciprocals of the successive odd powers of 5, and is formed by dividing continuously by 25, or multiplying successively by ·04. Tables (B) and (C) contain the reciprocals of the successive powers of 70 and 99 respectively, the former being obtained by continuous divisions by 70, and the latter by dividing continuously by 99, by the very simple process already adverted to. Tables (D) and (E) contain the values of the several terms of the first series, viz.

$$\frac{1}{5}-\frac{1}{3}\cdot\frac{1}{5^3}+\frac{1}{5}\cdot\frac{1}{5^5}-\frac{1}{7}\cdot\frac{1}{5^7}+\frac{1}{9}\cdot\frac{1}{5^9}-\frac{1}{11}\cdot\frac{1}{5^{11}}+\dots\dots$$

the former table comprising the values of the positive terms, and the latter those of the negative terms, both being derived from the subsidiary table (A). Table (F) is formed in like manner from the subsidiary tables (B) and (C), Part I. comprising the negative terms, and Part II. the positive terms of the series

$$\left(\frac{1}{70}-\frac{1}{99}\right)-\frac{1}{3}\left(\frac{1}{70^3}-\frac{1}{99^3}\right)+\frac{1}{5}\left(\frac{1}{70^5}-\frac{1}{99^5}\right)-\frac{1}{7}\left(\frac{1}{70^7}-\frac{1}{99^7}\right)+\dots$$

And for the readier verification of the summation of the several columns in Tables (D), (E), (F), the sum of each column is written in full in a diagonal position, preserving the local values of the several figures in each sum, from which by a second summation the total sum is finally obtained. The excess of the sum of the positive terms in Table (D) above that of the negative terms in Table (E) is then multiplied by 4 to obtain the value of $4\tan^{-1}\frac{1}{5}$, and the excess of the sum of the positive terms in Table (F) above that of the negative terms gives the value of $\tan^{-1}\frac{1}{70}-\tan^{-1}\frac{1}{99}$. This value is then transferred to Table (D), and subtracted from that of $4\tan^{-1}\frac{1}{5}$, which gives the value of $\frac{\pi}{4}$, and thence, finally, by multiplying by 4, the value of π, or the ratio of the diameter of a circle to the circumference to 208 places of decimals.

In conclusion I have only to remark, that the computations have been very carefully conducted, and that almost every part of the work has been verified by myself or the

independent computations of others. The value of π which I have thus obtained from the formula

$$\frac{\pi}{4} = 4 \tan^{-1}\frac{1}{5} - \tan^{-1}\frac{1}{70} + \tan^{-1}\frac{1}{99}, \text{ is}$$

$\pi = 3{\cdot}14159$	26535	89793	23846	26433	83279	50288	41971
69399	37510	58209	74944	59230	78164	06286	20899
86280	34825	34211	70679	82148	08651	$32\overset{*}{8}23$	06647
09384	46095	50582	23172	53594	08128	48473	78139
20386	33830	21574	73996	00825	93125	91294	01832
80651	744						

which, it is presumed, is accurate to the last, or 208th place of decimals inclusive, the computations having been carried as far as 210 places of figures.

Royal Military Academy,
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* The suspicion about the figure in the 113th place of decimals is now completely removed. I find it to be 8, instead of 7 as it is frequently printed, agreeing with the result as given by VEGA, and also with that in the Oxford Manuscript.