X. *Investigation of an Extensive Class of Partial Differential Equations of the Second Order, in which the Equation of* Laplace's *Functions is included.*

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Theorem. If u be a function of x and y satisfying the equation

$$\frac{d^2u}{dxdy}+\alpha_n e^{\phi}u=0,$$

where
$$\frac{d^2\phi}{dxdy}+ce^{\phi}=0,$$

then the solution will be

$$u=\mathrm{D}^{-n}v_n$$

where
$$\mathrm{D}=e^{-\phi}\frac{d}{dy},$$

where
$$v_n=\int e^{-\frac{\beta n}{\Delta\beta n}\phi}\chi y dy+\psi x,$$

χy and ψx arbitrary functions of y and x,

and
$$\beta_n=\frac{\Delta\alpha_{n-1}\,.\,\Delta\alpha_{n-2}\,.\,.\,.\,.\,.}{(\Delta\alpha_{n-1}-c)(\Delta\alpha_{n-2}-c)\,.\,.\,.\,.\,.}$$

where
$$\Delta\alpha_r=\alpha_{r+1}-\alpha_r,$$

and α_n is a function of n vanishing for $n=0$ and for $n=-1$.

I will proceed to demonstrate this curious theorem as briefly as possible.

According to the notation, we may write the given equation

$$\mathrm{D}\frac{du}{dx}+\alpha_n u=0.$$

Then if
$$v_n=\mathrm{D}^n u$$

we have
$$v_{n+1}=\mathrm{D}^{n+1}u=\mathrm{D}v_n.$$

Suppose that
$$\mathrm{D}\frac{dv_n}{dx}+\beta_n z\mathrm{D}v_n=\mathrm{D}^n\left\{\mathrm{D}\frac{du}{dx}+\alpha_n u\right\}. \qquad (\alpha.)$$

where z is a function of x and y to be determined, also β_n a function of u.

Writing $n+1$ for n in equation (α.), we ought to have

$$\mathrm{D}\frac{dv_{n+1}}{dx}+\beta_{n+1}z\mathrm{D}v_{n+1}=\mathrm{D}^{n+1}\left\{\mathrm{D}\frac{du}{dx}+\alpha_{n+1}u\right\} \qquad (\beta.)$$

This circumstance will serve to determine z and β_n as follows: we have identically

$$D^{n+1}\left\{D\frac{du}{dx}+\alpha_{n+1}u\right\}=D^{n+1}\left\{D\frac{du}{dx}+\alpha_n u\right\}+\Delta\alpha_n D^{n+1}u$$

$$\dot{=}D^2\frac{dv_u}{dx}+\beta_n D\left\{zDv_n\right\}+\Delta\alpha_n\,.\,Dv_n \text{ by } (\alpha)$$

$$=D^2\frac{dv_n}{dx}+\beta_n D\left\{zv_{n+1}\right\}+\Delta\alpha_n\,.\,v_{n+1}.$$

But
$$\frac{dv_{n+1}}{dx}=\frac{d}{dx}Dv_n=\frac{d}{dx}\left(e^{-\phi}\frac{dv_n}{dy}\right)$$

$$=-\frac{d\phi}{dx}e^{-\phi}\frac{dv_n}{dy}+e^{-\phi}\frac{d}{dy}\cdot\frac{dv_n}{dx}$$

$$=-\frac{d\phi}{dx}Dv_n+D\frac{dv_n}{dx};$$

$$\therefore\quad D^2\frac{dv_n}{dx}=D\left\{\frac{dv_{n+1}}{dx}+\frac{d\phi}{dx}v_{n+1}\right\}.$$

Hence
$$D\frac{dv_{n+1}}{dx}+D\left(\frac{d\phi}{dx}v_{n+1}\right)+\beta_n D(zv_{n+1})+\Delta\alpha_n v_{n+1}$$

ought to be identical with $D\frac{dv_{n+1}}{dx}+\beta_{n+1}zDv_{n+1}$, and hence the conditions

$$\frac{d\phi}{dx}+\beta_n z=\beta_{n+1}z$$

$$D\left\{\frac{d\phi}{dx}+\beta_n z\right\}=-\Delta\alpha_n.$$

Eliminating z, we have

$$D\frac{d\phi}{dx}=-\frac{\Delta\alpha_n\Delta\beta_n}{\beta_{n+1}}=-c,$$

or
$$\frac{d^2\phi}{dxdy}+ce^{\phi}=0,$$

and
$$c\beta_{n+1}=\Delta\alpha_n\{\beta_{n+1}-\beta_n\},$$

or
$$\beta_{n+1}=\frac{\Delta\alpha_n}{\Delta\alpha_n-c}\cdot\beta_n;$$

$$\therefore\quad \beta_n=\frac{\Delta\alpha_{n-1}\,.\,\Delta\alpha_{n-2}\,.\,.\,.\,.}{(\Delta\alpha_{n-1}-c)(\Delta\alpha_{n-2}-c)\,.\,.\,.\,.}:$$

by these determinations we establish the formula (β.) as a consequence of (α.), and therefore if the formula (α.) be true for any value of n, it will be (subject to the above conditions) true for the next superior value.

Now, when $n=0$, $v_0=D^0u=u$, and provided α_0 and α_{-1} are each $=0$, α_0 and $\Delta\alpha_{-1}$ will be each 0, and $\therefore$ α_0 and β_0 each $=0$, and the equation (α.) reduces to $D\frac{dv_0}{dx}$

$=D\frac{du}{dx}$, and is therefore true for $u=0$. Under these restrictions it will therefore be true for any positive integral value of n. Now the symbol D represents $e^{-\phi}\frac{d}{dy}$, and therefore if $U=0$, $D^nU=0$, so that we have

$$D\frac{dv_n}{dx}+\beta_n z Dv_n=0,$$

or

$$e^{-\phi}\frac{d^2v_n}{dxdy}+\frac{\beta_n}{\Delta\beta_n}\cdot\frac{d\phi}{dx}\cdot e^{-\phi}\frac{dv_n}{dy}=0,$$

or

$$\frac{\frac{d}{dx}\cdot\frac{dv_n}{dy}}{\frac{dv_n}{dy}}=-\frac{\beta_n}{\Delta\beta_n}\cdot\frac{d\phi}{dx};$$

$\therefore$ integrating with respect to x,

$$\frac{dv_n}{dy}=e^{-\frac{\beta_n}{\Delta\beta_n}\phi}\chi y$$

$$v_n=\int e^{-\frac{\beta_n}{\Delta\beta_n}\phi}\chi y dy+\psi x,$$

and

$$u=D^{-n}v_n=\int e^{\phi}\int e^{\phi}\ldots\ldots v_n dy dy\ldots.$$

the integral sign repeated n times. The theorem is therefore demonstrated.

It may be easily shown that the equation of LAPLACE's coefficients is included in the class here considered.

The equation of LAPLACE by a proper choice of independent variables assumes the form

$$\frac{d^2u}{dxdy}+\frac{n\,.\,n+1}{4\cos^2\frac{y-x}{2}}\cdot u=0.$$

Hence with reference to the preceding investigation,

$$D=\cos^2\frac{y-x}{2}\cdot\frac{d}{dy}\text{ and }\alpha_n=\frac{n\,.\,\overline{n+1}}{4}.$$

Hence $\qquad e^{-\phi}=\cos^2\frac{y-x}{2};$

$\therefore \qquad \frac{d\phi}{dx}=-\tan\frac{y-x}{2}$

$$\frac{d^2\phi}{dxdy}+\frac{1}{2}e^{\phi}=0.$$

Hence $\qquad c=\frac{1}{2}$. Also $\Delta\alpha_n=\frac{n+1}{2}$; $\qquad \Delta\alpha_n-c=\frac{n}{2}$;

$\therefore \qquad \beta_n=n$ and $\Delta\beta_n=1$.

Inserting these values in the final formula, we have

$$v_n=\int\cos^{2n}\frac{y-x}{2}\,\chi y dy+\psi x,$$

and

$$u=\int\cos^{-2}\frac{y-x}{2}\int\cos^{-2}\frac{y-x}{2}\ldots\ldots v_n dydy\ldots\ldots n \text{ times},$$

which agrees with Mr. HARGREAVE's solution.