XIV. *On certain Properties of square numbers and other quadratic forms, with a Table of odd numbers from* 1 *to* 191, *divided into* 4, 3 *or* 2 *square numbers, the algebraic sum of whose roots (positive or negative) may equal* 1, *by means of which Table all the odd numbers up to* 9503 *may be resolved into not exceeding* 4 *square numbers. By Sir* FREDERICK POLLOCK, *F.R.S., Lord Chief Baron.*

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SOME years ago, in examining the properties of the triangular or trigonal numbers

$$0,\ 1,\ 3,\ 6,\ 10,\ 15,\ \&c.\ \frac{n.(n-1)}{2},$$

I observed that every trigonal number was composed of 4 trigonal numbers, viz. 3 times some prior trigonal number plus the next in the series, either immediately before or after that prior number;

thus
$$45=10+10+10+15$$
$$55=15+15+15+10;$$

or generally, as all numbers are of the form $2n-1$ or of $2n$, all trigonal numbers are of one of the 2 forms, $2n^2-n$, $2n^2+n$,

$$2n^2-n=\frac{n^2-n}{2}\times 3+\frac{n^2+n}{2}$$

and

$$2n^2+n=\frac{n^2+n}{2}\times 3+\frac{n^2-n}{2}.$$

I found also that all the natural numbers in the interval between any two consecutive trigonal numbers, might be composed of 4 trigonal numbers, having the sum of their bases or roots *constant*, viz. the sum of the roots or bases of the 4 trigonal numbers which compose the first of the 2 trigonal numbers.

This will be best explained by an example: the roots or bases are placed over the numbers, and it will be observed their sum is constant in the same interval.

$\frac{8\times 9}{2}=\mathbf{36}$	$\overset{3}{\mathbf{6}}$	$\overset{4}{\mathbf{10}}$	$\overset{4}{\mathbf{10}}$	$\overset{4}{\mathbf{10}}$	$=15$
37	$\overset{3}{6}$	$\overset{3}{6}$	$\overset{4}{10}$	$\overset{5}{15}$	15
38	$\overset{2}{3}$	$\overset{4}{10}$	$\overset{4}{10}$	$\overset{5}{15}$	15
39	$\overset{2}{3}$	$\overset{3}{6}$	$\overset{5}{15}$	$\overset{5}{15}$	15
40	$\overset{2}{3}$	$\overset{3}{6}$	$\overset{4}{10}$	$\overset{6}{21}$	15
41	$\overset{1}{1}$	$\overset{4}{10}$	$\overset{5}{15}$	$\overset{5}{15}$	15
42	$\overset{1}{1}$	$\overset{4}{10}$	$\overset{4}{10}$	$\overset{6}{21}$	15
43	$\overset{1}{1}$	$\overset{3}{6}$	$\overset{5}{15}$	$\overset{6}{21}$	15
44	$\overset{2}{3}$	$\overset{2}{3}$	$\overset{4}{10}$	$\overset{7}{28}$	15
$\frac{9\times 10}{2}=\mathbf{45}$	$\overset{4}{\mathbf{10}}$	$\overset{4}{\mathbf{10}}$	$\overset{4}{\mathbf{10}}$	$\overset{5}{\mathbf{15}}$	17
46	$\overset{3}{6}$	$\overset{4}{10}$	$\overset{5}{15}$	$\overset{5}{15}$	17
47	$\overset{3}{6}$	$\overset{4}{10}$	$\overset{4}{10}$	$\overset{6}{21}$	17
48	$\overset{3}{6}$	$\overset{3}{6}$	$\overset{5}{15}$	$\overset{6}{21}$	17
49	$\overset{2}{3}$	$\overset{4}{10}$	$\overset{5}{15}$	$\overset{6}{21}$	17
50	$\overset{3}{6}$	$\overset{3}{6}$	$\overset{4}{10}$	$\overset{7}{28}$	17
51	$\overset{2}{3}$	$\overset{4}{10}$	$\overset{4}{10}$	$\overset{7}{28}$	17
52	$\overset{2}{3}$	$\overset{3}{6}$	$\overset{5}{15}$	$\overset{7}{28}$	17
53	$\overset{1}{1}$	$\overset{4}{10}$	$\overset{6}{21}$	$\overset{6}{21}$	17
54	$\overset{1}{1}$	$\overset{4}{10}$	$\overset{5}{15}$	$\overset{7}{28}$	17
$\frac{10\times 11}{2}=\mathbf{55}$	$\overset{4}{\mathbf{10}}$	$\overset{5}{\mathbf{15}}$	$\overset{5}{\mathbf{15}}$	$\overset{5}{\mathbf{15}}$	$=19$

From 55 to 66 ($=\overset{5}{15}, \overset{5}{15}, \overset{5}{15}, \overset{6}{21}$) the constant sum of the bases will be 19, and this may be continued without limit.

If the law by which this can be continued were discovered and proved, it would furnish the means of proving FERMAT's theorems of the polygonal numbers; but not being aware of any law by which the series that fills up the intervals could be continued, I turned my attention to the square numbers as containing (apparently) a greater variety of theorems, and as being (certainly) of all quadratic forms that which

is most familiar, and in which calculations or comparisons may be made with the greatest facility.

Very lately I observed the following property of square numbers :—

If any four square numbers a^2, b^2, c^2, d^2 have their roots such that by making one or more positive and the rest negative, the algebraic sum of the roots may equal 1 ; then if the roots whose sum is one less than the others be each increased by 1, and the others be each decreased by 1, the sum of the squares of the roots thus increased and decreased will be equal to $a^2+b^2+c^2+d^2+2$. Let $a+b-c-d=1$, and let c and d become $c+1$, $d+1$, and let a and b become $a-1$, $b-1$, then

$$(a-1)^2+(b-1)^2+(c+1)^2+(d+1)^2=a^2+b^2+c^2+d^2+2\times(-a-b+c+d)+4,$$

but

$$2\times(-a-b+c+d)=-2,$$

therefore the sum of the squares of the new roots$=a^2+b^2+c^2+d^2+2$. If $a-b-c-d=1$, the result is the same, decreasing a and increasing each of b, c and d by 1.

The theorem is more general (as might have been expected).

Theorem A.

For if instead of 1 the algebraic sum of the roots be equal to $2n-1$, and the negative roots be numerically increased by n and the positive roots be decreased by n, the increase in the sum of the squares of the new roots thus formed will be $2n$.

Let $a+b-c-d=2n-1$, then $(a-n)^2+(b-n)^2+(c+n)^2+(d+n)^2=a^2+b^2+c^2+d^2-2an-2bn+2cn+2dn+4n^2$, but $-2an-2bn+2cn+2dn=-(2n-1)\times 2n=-4n^2+2n$, $\therefore$ the sum of the squares of the new roots$=a^2+b^2+c^2+d^2+2n$.

The following table shows the result of different algebraic sums of the roots, with the corresponding increase or decrease of roots and increase of the sum of the squares.

Sum of roots.	Corresponding increase or decrease of roots.	Increase of sum of squares.
1	1	2
3	2	4
5	3	6
7	4	8
9	5	10
11	6	12
&c.	&c.	&c.

There is a similar theorem with respect to the decrease of the sum of the squares.

Theorem B.

If $a+b-c-d=2n+1$ (instead of $2n-1$), then if a and b be each diminished and c and d be increased by n, the sum of the squares of the new roots will be less by $2n$, and $(a-n)^2+(b-n)^2+(c+n)^2+(d+n)^2$ will equal $a^2+b^2+c^2+d^2-2n$.

And a similar table will show the corresponding decrease of the sum of the squares.

Sum of roots.	Increase and decrease.	Decrease of sum of squares.
3	1	2
5	2	4
7	3	6
9	4	8
&c.	&c.	&c.

The use that may be made of these theorems will best appear by an example or two.

51 is composed of 4 square numbers, 25, 16, 9, 1, whose roots are 5, 4, 3, 1.

$$4 + 3 - 5 - 1 = 1$$
$$5 + 3 - 4 - 1 = 3$$
$$5 + 4 - 3 - 1 = 5$$
$$5 + 4 + 1 - 3 = 7$$

Then by Theorem A. square numbers which compose 53, 55, 57, and 59 may be obtained, and by Theorem B. those which compose 49, 47, and 45 by adding to or subtracting from the roots; thus

$$5 + 4 - 3 - 1 = 5 = 2 \times 3 - 1, n = 3$$

3 3 3 3 being deducted or added,

2 1 6 4 become the new roots, the sum of whose squares$=57=51+2\times3$.

$$5 + 4 + 1 - 3 = 7 = 2 \times 3 + 1, n = 3$$

3 3 3 3

2 1 $-$2 6 are the new roots, the sum of whose squares$=45=51-2\times3$.

Again, 5 5 1 0 are roots of squares which compose 51,

$$5 + 5 + 1 \quad 0 = 11 = 2 \times 6 - 1, n = 6$$

6 6 6 6

1 1 $-$5 6 the squares of these new roots$=63=51+12$ $(51+2\times6)$. Also,

$$5 + 5 - 1 \quad 0 = 9 = 2 \times 5 - 1, n = 5$$

5 5 5 5

0 0 6 5 the squares of these new roots $=$ 61 by Theorem A.; by Theorem B. the squares which compose 41 and 43 may be found, and thus the square numbers (not exceeding 4) which compose 51 being given, square numbers not exceeding 4 may be discovered, which compose 41, 43, and all the intermediate odd numbers up to 63.

This method of obtaining the square numbers that compose a succession of odd numbers, suggested that if a method similar to what was observed in the trigonal numbers were adopted as to the square numbers, the series of odd numbers might be

resolved into square numbers. In the trigonal numbers the successive bases of the 4 trigonal numbers into which the terms of the trigonal series are each divisible, are

0	0	0	1
0	1	1	1
1	1	1	2
1	2	2	2
2	2	2	3
2	3	3	3 &c.

If instead of using these numbers as bases or roots of trigonal numbers they be squared and added together, they furnish a series (1, 3, 7, 13, 21, 31, &c.) whose general term is m^2+m+1, or according as m is even or odd, $4n^2\pm 2n+1$: this expression is manifestly divisible into square numbers whose roots will be either $n, n, n, n+1$ or $n-1, n, n, n$, and the sum of the roots will be $4n\pm 1$, and with reference to integral quantities will be a maximum. The terms of the series 1, 3, 7, 13, &c., furnish steps, places, or positions at which the process of increasing the sum of the squares might commence again, and as far as any law of increase is applicable to one term it is applicable to all. I have therefore called this series 1, 3, 7, 13, &c. the gradation-series of this system of resolving the odd numbers into square numbers not exceeding 4.

I have prepared a table in which the odd numbers from 1 to 191, respectively, are divided into square numbers not exceeding 4, the algebraic sum of whose roots may be made equal to 1.

This table of the odd numbers up to 191 is at the end of the paper; the terms of the gradation-series (as they occur) are distinctively denoted, and all the sets of roots of the odd numbers up to 191 are capable of forming 1 as their algebraic sum; and by means of this series any odd number from $4n^2\pm 2n+1$ up to $4n^2\pm 2n+191$ inclusive, may be divided into not exceeding 4 square numbers, whatever be the value of n.

The examination of this series led me to observe a remarkable property of odd numbers with reference to the square numbers (not exceeding 4) into which they may be divided, and which may be stated in the following theorem.

THEOREM C.

Every odd number may be divided into square numbers (not exceeding 4), the algebraic sum of whose roots (positive or negative) will (in some form of the roots) be equal to every odd number from 1 to the greatest possible sum of the roots, or the theorem may be stated in a purely algebraical form thus:

If there be 2 equations,

$$a^2+b^2+c^2+d^2=2n+1$$

and

$$a+b+c+d=2r+1,$$

a, b, c, d being each integral or nil, n and r being positive, and r a maximum, then if

any positive integer r' (not greater than r) be assumed, it will always be possible to satisfy the pair of equations

$$w^2=x^2+y^2+z^2=2n+1$$
$$w+x+y+z=2r'+1$$

by integral values (positive, negative, or nil) of w, x, y, z.

I now propose to show in what manner the table may be used, so as to divide into square numbers (not exceeding 4) any odd number from 1 up to 9503, and any odd number whatever of the form $4n^2\pm2n+2p+1$, where p is not greater than 95 [$95\times2+1=191$].

If $2p+1=a^2+b^2+c^2+d^2$, and if also $a+b+c+d=1$ (a, b, c, d being integral numbers, positive, negative or nil), in other words, if the odd number $2p+1$ be such that the algebraic sum of the roots of the square numbers (not exceeding 4) which compose it may be equal to 1, then it will follow that $m^2+m+1+2p$ may be resolved into square numbers (not exceeding 4) the sum of whose roots will equal $2m+1$, for m^2+m+1 is of the form $4n^2\pm2n+1$; let it equal it, and $m^2+m+1+2p=4n^2\pm2n$, $(a+b+c+d)+a^2+b^2+c^2+d^2=(n\pm a)^2+(n\pm b)^2+(n\pm c)^2+(n\pm d)^2$ (manifestly 4 square numbers), and the sum of the roots$=4n\pm(a+b+c+d)=4n\pm1=2m+1$.

Let the function m^2+m+1 be designated by the notation fm. If every odd number from 1 up to $2m+1$ can be resolved into (not exceeding) 4 square numbers, the algebraic sum of whose roots may equal 1, then every odd number from fm to $fm+2m$ inclusive may be resolved into 4 square numbers, the sum of whose roots may equal $2m+1$; but the next odd number to $fm+2m$ is $f(m+1)$, and since $f(m+1)$ is resolvable into 4 square numbers, the sum of whose roots$=2m+3$; if every odd number from 1 up to $2m+1$ can be resolved into not exceeding 4 square numbers the algebraic sum of whose roots$=1$, then every odd number from 1 up to $f(m+1)+2m$ is resolvable into 4 square numbers, and $f(m+1)+2m=m^2+5m+3$.

In the Table the highest odd number $191=2\times95+1$, therefore $m=95$; and every odd number from 1 up to $95^2+5\times95+3=9503$ may be resolved into not exceeding 4 square numbers, by means of the Table, also every odd number of the form $4n^2\pm2n+2p+1$, whatever be the value of n, provided p be not greater than 95; for example, let it be required to resolve 9301 into 4 square numbers, the next less number of the form m^2+m+1 is $9121=95^2+95+1=4\cdot48^2-2\cdot48+1$, $9301=9120+181$.

181 by the Table is resolvable into

$$1^2+4^2+8^2+10^2$$

and $-1+4+8-10=1$ $9301=(48+1)^2+(48-4)^2+(48-8)^2+(48+10)^2$
$$=49^2+44^2+40^2+58^2;$$

so $4n^2\pm n+181$ is always resolvable into 4 square numbers, whatever be the value of n, and the roots of the square numbers will be $(n\mp1)$, $(n\pm4)$, $(n\pm8)$, $(n\mp10)$.

If the following series of equations be assumed,

$$
\begin{aligned}
1+2d&=2q+1\\
3+2d_1&=2q+1\\
7+2d_2&=2q+1\\
13+2d_3&=2q+1\\
m^2+m+1+2d_{m-1}&=2q+1,
\end{aligned}
$$

and if each of the quantities $2d+1, 2d_1+1, 2d_2+1, 2d_3+1\ldots 2d_{m-1}+1$ can be resolved into 4 square numbers, the algebraic sum of whose roots$=1$, then the given odd number $2q+1$ may be resolved into successive sets of 4 squares, the sum of whose roots will be successively $1, 3, 5, 7\ldots 2m+1$. Hence an odd number, $2q+1$, may be resolved into 4 square numbers, the sum of whose roots shall be equal to $2p+1$, if upon adding 1 to the difference between $2q+1$ and the $(p+1)^{\text{st}}$ term of the gradation-series, the difference so increased can be resolved into 4 square numbers, the algebraic sum of whose roots$=1$. If it be required to resolve 37 into 4 squares, the sum of whose roots shall equal $7=2\times 3+1$, here $p=3$, the $(p+1)^{\text{st}}$ or 4th term of the gradation-series is 13; 13 is of the form $4\cdot 2^2-2\cdot 2+1$, and it equals $2^2+2^2+2^2+1^2$ $(2-1)^2$; the difference between 37 and $14=24$, increased by $1=25$, $25=1^2+2^2+2^2+4^2$, and the roots $+1-2-2+4=1$ and $13+24=(2+2)^2=4^2, 4^2, 1^2, -2^2$

$$
\begin{array}{l}
(2+2)^2\\
(2-1)^2\\
(2-4)^2
\end{array}
$$

and $+4+4+1-2=7$.

If, therefore, every odd number can be resolved into integral square numbers (not exceeding 4) whose algebraic sum will equal 1, then every odd number can be resolved into integral square numbers (not exceeding 4) whose algebraic sum will be 1, 3, 5, &c. [viz. *all the odd numbers up to the maximum*].

I propose (in order not to leave the Theorem C. unproved) to show by the properties of numbers already proved, that every odd number may be resolved into integral square numbers (not exceeding 4) whose algebraic sum will equal 1.

Every odd number may be represented by $2p+1$ (p being any integer): then by FERMAT's theorem of the polygonal numbers (as proved by LEGENDRE, Théorie des nombres), p must either be a trigonal number, or composed of two or three trigonal numbers. If it be a trigonal number, then $p=\frac{q^2+q}{2}$, and $2p+1=q^2+q+1$, which equals $4n^2\pm 2n+1$, which is divisible into $(n\pm 1)^2, n^2, n^2, n^2$, and $n-n\mp n.\pm(n\pm 1)=1$.

If p be composed of 2 trigonal numbers, $p=\frac{q^2+q}{2}+\frac{r^2+r}{2}$, and the sum of any two trigonal numbers is of the form of a^2+a+b^2* and may be assumed equal to a^2+a+b^2,

* If 2 numbers be both odd or both even, they may always be represented by $a+b$ and $a-b$; if one be odd and the other even, they may always be represented by $a+b\pm 1$, $a-b$ or $a+b$, $a-b\pm 1$; and if the 2 numbers be made the bases of trigonal numbers, the sum of the 2 trigonal numbers will always be of the form a^2+a+b^2 or a^2+b+b^2.

therefore $2p+1=2a^2+2a+2b^2+1=(a+1)^2$, a^2, b^2, b^2, and the roots $(a+1), -a, b, -b, =1$. If p be composed of 3 trigonal numbers, then $p=a^2+a+b^2+\frac{m^2+m}{2}$ and $2p+1=2a^2+2a+2b^2+m^2+m+1$, but m^2+m+1 is of the form $4n^2\pm 2n+1$, whose four roots (as already seen) are $n\pm 1$, n, n, n, and if these roots be varied thus,

$$\begin{array}{l} \mp a+n\pm 1 \\ a-n \\ b+n \\ b-n, \end{array}$$

the squares of these four roots will equal $2a^2+2a+2b^2+4n^2\pm 2n+1$, and the algebraic sum of these roots obviously may$=1$. It follows from this, that every possible odd number may be divided into integral square numbers (not exceeding 4), the algebraic sum of whose roots$=1$.

I propose in a future communication to give a different proof of the Theorem C, and instead of proving the Theorem C. by FERMAT's proposition of the trigonal numbers, I shall offer a proof of FERMAT's proposition of the trigonal numbers by the Theorem C; it is obvious that they are so connected that either may be proved from the other.

I am not aware that the theorems A, B, or C, or the method above described of using a gradation-series, have ever been noticed before, and as they appear to add something (however little) to the theory of numbers, I have ventured to present them to the attention of the Royal Society.

NOTE.—Numbers of the form $2n+2$ (*even numbers*) may be resolved into square numbers (not exceeding 4), the algebraic sum of whose roots may always equal 2, and so far they have an analogous property, but they do not possess the analogous property of being resolvable into roots whose algebraic sum will$=2, 4, 6, 8$, &c.

TABLE of odd numbers and of the Roots of the squares (not exceeding 4) into which they may be divided, whose algebraic sum may equal 1.

Odd numbers.	Roots of the Squares into which they may be divided whose algebraic sum may equal 1.	Odd numbers.	Roots of the Squares into which they may be divided whose algebraic sum may equal 1.	Odd numbers.	Roots of the Squares into which they may be divided whose algebraic sum may equal 1.
1	**0 0 0 1**	65	2 3 4 6	129	2 5 6 8
3	**0 1 1 1**	67	1 1 4 7	131	0 5 5 9
5	0 0 1 2	69	1 4 4 6	**133**	**5 6 6 6**
7	**1 1 1 2**	71	1 3 5 6	135	5 5 6 7
9	0 1 2 2	**73**	**4 4 4 5**	137	4 6 6 7
11	0 1 1 3	75	3 4 5 5	139	4 5 7 7
13	**1 2 2 2**	77	3 4 4 6	141	4 5 6 8
15	1 1 2 3	79	3 3 5 6	143	2 3 7 9
17	0 2 2 3	81	2 4 5 6	145	3 6 6 8
19	0 1 3 3	83	0 3 5 7	147	3 5 7 8
21	**2 2 2 3**	85	2 4 4 7	149	2 3 6 10
23	1 2 3 3	87	2 3 5 7	151	3 5 6 9
25	1 2 2 4	89	0 2 6 7	153	2 2 8 9
27	1 1 3 4	**91**	**4 5 5 5**	155	3 4 7 9
29	0 2 3 4	93	4 4 5 6	**157**	**6 6 6 7**
31	**2 3 3 3**	95	3 5 5 6	159	5 6 7 7
33	2 2 3 4	97	3 4 6 6	161	5 6 6 8
35	1 3 3 4	99	3 4 5 7	163	5 5 7 8
37	1 2 4 4	101	0 1 6 8	165	4 6 7 8
39	1 2 3 5	103	2 5 5 7	167	1 2 9 9
41	0 0 4 5	105	2 4 6 7	169	4 6 6 9
43	**3 3 3 4**	107	1 3 4 9	171	4 5 7 9
45	2 3 4 4	109	2 4 5 8	173	1 6 6 10
47	2 3 3 5	**111**	**5 5 5 6**	175	3 6 7 9
49	2 2 4 5	113	4 5 6 6	177	4 5 6 10
51	1 3 4 5	115	4 5 5 7	179	3 5 8 9
53	2 2 3 6	117	4 4 6 7	181	1 4 8 10
55	1 3 3 6	119	3 5 6 7	**183**	**6 7 7 7**
57	**3 4 4 4**	121	2 2 7 8	185	6 6 7 8
59	3 3 4 5	123	3 5 5 8	187	5 7 7 8
61	2 4 4 5	125	3 4 6 8	189	5 6 8 8
63	2 3 5 5	127	1 3 6 9	191	5 6 7 9