VI. *On the Equation of Differences for an Equation of any Order, and in particular for the Equations of the Orders Two, Three, Four, and Five.* By ARTHUR CAYLEY, *Esq., F.R.S.*

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THE term, *equation of differences*, denotes the equation for the squared differences of the roots of a given equation; the equation of differences afforded a means of determining the number of real roots, and also limits for the real roots, of a given numerical equation, and was upon this account long ago sought for by geometers. In the Philosophical Transactions for 1763, WARING gives, but without demonstration or indication of the mode of obtaining it, the equation of differences for an equation of the fifth order wanting the second term: the result was probably obtained by the method of symmetric functions. This method is employed in the 'Meditationes Algebraicæ' (1782), where the equation of differences is given for the equations of the third and fourth orders wanting the second terms; and in p. 85 the before-mentioned result for the equation of the fifth order wanting the second term, is reproduced. The formulæ for obtaining by this method the equation of differences, are fully developed by LAGRANGE in the 'Traité des Equations Numériques' (1808); and he finds by means of them the equation of differences for the equations of the orders two and three, and for the equation of the fourth order wanting the second term; and in Note III. he gives, after WARING, the result for the equation of the fifth order wanting the second term. It occurred to me that the equation of differences could be most easily calculated by the following method. The coefficients of the equation of differences, *quà* functions of the differences of the roots of the given equation, are leading coefficients of covariants, or (to use a shorter expression) they are "Seminvariants*," that is, each of them is a function of the coefficients which is reduced to zero by one of the two operators which reduce an invariant to zero. In virtue of this property they can be calculated, when their values are known for the particular case in which one of the coefficients of the given equation is zero. To fix the ideas, let the given equation be $(*\between v, 1)^n=0$; then, when the last coefficient or constant term vanishes, the equation breaks up into $v=0$ and into an equation of the degree $(n-1)$, which I call the reduced equation; the equation of differences will break up into two equations, one of which is the equation of differences for the reduced equation, the other is the equation for the squares of the roots of the same reduced equation. This hardly requires a proof; let the roots of the given equation be α, β, γ, δ, &c., those of the equation of differences are $(\alpha-\beta)^2$, $(\alpha-\gamma)^2$, $(\alpha-\delta)^2$, &c., $(\beta-\gamma)^2$, $(\beta-\delta)^2$, $(\gamma-\delta)^2$, &c.; but in putting the constant term equal to zero, we in effect put one of the roots, say α,

* The term "Seminvariant" seems to me preferable to M. BRIOSCHI's term "Peninvariant."

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equal to zero; the roots of the equation of differences thus become β^2, γ^2, δ^2, &c., $(\beta-\gamma)^2$, $(\beta-\delta)^2$, $(\gamma-\delta)^2$, &c. The equation for the squares of the roots can be found without the slightest difficulty; hence if the equation of differences for the reduced equation of the order $(n-1)$ is known, we can, by combining it with the equation for the squares of the roots, form the equation of differences for the given equation with the constant term put equal to zero, and thence by the above-mentioned property of the Seminvariancy of the coefficients, find the equation of differences for the given equation. The present memoir shows the application of the process to equations of the orders two, three, four, and five: part of the calculation for the equation of the fifth order was kindly performed for me by the Rev. R. HARLEY. It is to be noticed that the best course is to apply the method in the first instance to the forms $(a, b, .. \between v, 1)^n=0$, without numerical coefficients (or, as they may be termed, the *denumerate forms*), and to pass from the results so obtained to those which belong to the forms $(a, b, .. \between v, 1)^n=0$, or *standard forms*. The equation of differences, for $(\alpha-\beta)^2$, &c., the coefficients of which are seminvariants, naturally leads to the consideration of a more general equation having for its roots $(\alpha-\beta)^2 (x-\gamma y)^2 (x-\delta y)^2$, &c., the coefficients of which are covariants; and in fact, when, as for equations of the orders two, three, and four, all the covariants are known, such covariant equation can be at once formed from the equation of differences; for equations of the fifth order, however, where the covariants are not calculated beyond a certain degree, only a few of the coefficients of the covariant equation can be thus at once formed. At the conclusion of the memoir, I show how the equation of differences for an equation of the order n can be obtained by the elimination of a single quantity from two equations each of the order $n-1$; and by applying to these two equations the simplification which I have made in BEZOUT's abridged method of elimination, I exhibit the equation of differences for the given equation of the order n, in a compendious form by means of a determinant; the first-mentioned method is, however, that which is best adapted for the actual development of the equation of differences for the equation of a given order.

The equations successively considered are

$$(a, b, c \between v, 1)^2=0,$$
$$(a, b, c, d \between v, 1)^3=0,$$
$$(a, b, c, d, e \between v, 1)^4=0,$$
$$(a, b, c, d, e, f \between v, 1)^5=0.$$

The equation of differences for the quadric, and that for the squares of the roots, are considered to be known, and the other results are derived from them: it will be convenient to write down in the first instance the results for the quadric, the cubic, and the quartic equations, and then explain the process of obtaining them.

For the quadric equation,

Equation of differences is, $0=$

$a^2\times$	
$+1$	$+4\ ac$ $-1\ b^2$

$(\ \ldots\ \between\theta, 1).$

Equation for the squares of the roots is, $0=$

$a^2\times$		
$+1$	$+2\ ac$ $-1\ b^2$	$+1\ c^2$

$(\ \ldots\ \between\theta, 1)^2.$

For the cubic equation,

Equation of differences is, $0=$

$a^4\times$	$a^2\times$		
$+1$	$+6\ ac$ $-2\ b^2$	$+9\ a^2c^2$ $-6\ ab^2c$ $+1\ b^4$	$+27\ a^2d^2$ $-18\ abcd$ $+\ 4\ ac^3$ $+\ 4\ b^3d$ $-\ 1\ b^2c^2$

$(\ \ldots\ \between\theta, 1)^3.$

Equation for the squares of the roots is, $0=$

$a^2\times$			
$+1$	$+2\ ac$ $-1\ b^2$	$+1\ c^2$ $-2\ bd$	$+1\ d^2$

$(\ \ldots\ \between\theta, 1)^3.$

For the quartic equation,

Equation of differences is, $0=$

$a^6\times$	$a^4\times$	$a^2\times$					
$+1$	$+8\ ac$	$+\ 8\ a^3e$	$+16\ a^4ce$	$-112\ a^4e^2$	$-192\ a^3ce^2$	$+256\ a^3e^3$	
	$-3\ b^2$	$-\ 2\ a^2bd$	$+26\ a^4d^2$	$+\ 56\ a^3bde$	$+216\ a^3d^2e$	$-192\ a^2bde^2$	
		$+22\ a^2c^2$	$-\ 6\ a^3b^2e$	$+\ 24\ a^3c^2e$	$+\ 72\ a^2b^2e^2$	$-128\ a^2c^2e^2$	
		$-16\ ab^2c$	$-30\ a^3bcd$	$+\ 48\ a^3cd^2$	$-120\ a^2bcde$	$+144\ a^2cd^2e$	
		$+\ 3\ b^4$	$+28\ a^3c^3$	$-\ 32\ a^2b^2ce$	$-\ 54\ a^2bd^3$	$-\ 27\ a^2d^4$	
			$+\ 8\ a^2b^3d$	$-\ 25\ a^2b^2d^2$	$+\ 32\ a^2c^3e$	$+144\ ab^2ce^2$	
			$-24\ a^2b^2c^2$	$-\ 54\ a^2bc^2d$	$+\ 18\ a^2c^2d^2$	$-\ 6\ ab^2d^2e$	$)(\theta, 1)^6.$
			$+\ 8\ ab^4c$	$+\ 17\ a^2c^4$	$+\ 18\ ab^3de$	$-\ 80\ abc^2de$	
			$-\ 1\ b^6$	$+\ 6\ ab^4e$	$-\ 6\ ab^2c^2e$	$+\ 18\ abcd^3$	
				$+\ 38\ ab^3cd$	$+\ 42\ ab^2cd^2$	$+\ 16\ ac^4e$	
				$-\ 12\ ab^2c^3$	$-\ 26\ abc^3d$	$-\ 4\ ac^3d^2$	
				$-\ 6\ b^5d$	$+\ 4\ ac^5$	$-\ 27\ b^4e^2$	
				$+\ 2\ b^4c^2$	$-\ 9\ b^4d^2$	$+\ 18\ b^3cde$	
					$+\ 6\ b^3c^2d$	$-\ 4\ b^3d^3$	
					$-\ 1\ b^2c^4$	$-\ 4\ b^2c^3e$	
						$+\ 1\ b^2c^2d^2$	

Equation for squares of the roots is,

$a^2\times$					
$+1$	$+2\ ac$	$+2\ ae$	$+2\ ce$	$+1\ e^2$	$)(\theta, 1)^4.$
	$-1\ b^2$	$-2\ bd$	$-1\ d^2$		
		$+1\ c^2$			

The multiplication of the equation of differences and the equation for the squares of the roots of the quadric equation, gives the equation, $0=$

$a^4\times$	$a^2\times$			
$+1$	$+6\ ac$	$+9\ a^2c^2$	$+4\ ac^3$	$)(\theta, 1)^3,$
	$-2\ b^2$	$-6\ ab^2c$	$-1\ b^2c^2$	
		$+1\ b^4$		

where all the coefficients except the last are reduced to zero by the operator

$$3a\partial_b+2b\partial_c+c\partial_d,$$

and they are consequently (without any alteration) coefficients of the equation of differences of the cubic equation: the last coefficient is not reduced to zero by the operator, and requires therefore to be completed by the adjunction of the terms in d (the series, here and in every other case, is of course a finite one, the number of terms might easily be calculated *à priori*). Let the value be $L_0+L_1d+L_2d^2+$ &c., we have $L_0=+4ac^3-1b^2c^2$; and putting for shortness $\nabla'=3a\partial_b+2b\partial_c$, the operator which reduces this to zero is $\nabla'+c\partial_d$; we ought therefore to have

$$0=\ \nabla'L_0+\ d\ \Big|\ \nabla'L_1+\ d^2\ \Big|\ \nabla'L_2+\ldots$$
$$\quad +\ cL_1\ \Big|\ 2cL_2\ \Big|\ 3cL_3$$

and consequently

$$L_1 = -\frac{1}{c}\nabla' L_0,\ L_2 = -\frac{1}{2c}\nabla' L_1,\ L_3 = -\frac{1}{3c}\nabla' L_2,\quad \&c.,$$

giving

$$L_1 = -\frac{1}{c}\{(-6+24=)18abc^2 - 4b^3c\} = -18abc + 4b^3,$$

$$L_2 = -\frac{1}{2c}\{-54a^2c + (36-36=)0\, ab^2\} = +27a^2,$$

$$L_3 = 0,\ \&c.,$$

and consequently for the last coefficient the value above written down; it will be presently seen how in more complicated cases the calculations should be arranged.

Again, multiplying together the equation of differences and the equation for the squares of the roots of the cubic equation, we obtain an equation which it is not necessary to write down, as it can be at once formed by putting $e=0$ in the equation of differences for the quartic equation. And from the equation so obtained, by the adjunction of the terms in e, we find the equation of differences for the quartic equation, viz. each coefficient is of the form $L_0 + L_1 e + L_2 e^2 +$ &c., where L_0 is known, and such coefficient is reduced to zero by the operator

$$4a\partial_b + 3b\partial_c + 2c\partial_d + d\partial_e;$$

or putting for shortness $\nabla' = 4a\partial_b + 3b\partial_c + 2c\partial_d$, the operator $\nabla' + d\partial_e$. We have therefore

$$L_1 = -\frac{1}{d}\nabla' L_0,\quad L_2 = -\frac{1}{2d}\nabla' L_1,\quad L_3 = -\frac{1}{3d}\nabla' L_2,\quad \&c.$$

It is to be observed that the last coefficient of the equation of differences is the discriminant, and that the above method of calculating the coefficients of the equation of differences, as applied to the last coefficient, is nothing else than the method of calculating the discriminant given in my Fourth Memoir on Quantics.

The multiplication of the equation of differences, and the equation for the squares of the roots of the quartic equation, gives, in like manner, the equation of differences for the quintic equation, except as to the terms involving f; and these are obtained as above, viz. each coefficient is of the form $L_0 + L_1 f + L_2 f^2 +$ &c., where L_0 is known; such coefficient is reduced to zero by the operator

$$5a\partial_b + 4b\partial_c + 3c\partial_d + 2d\partial_e + e\partial_f;$$

or putting for shortness $\nabla' = 5a\partial_b + 4b\partial_c + 3c\partial_d + 2d\partial_e$, the operator $\nabla' + e\partial_f$. We have, therefore,

$$L_1 = -\frac{1}{e}\nabla' L_0,\quad L_2 = -\frac{1}{2e}\nabla' L_1,\quad L_3 = -\frac{1}{3e}\nabla' L_2,\ \&c.,$$

which give L_1, L_2, &c. by means of L_0. For the calculation of $\nabla' L$ (where L is any one of the coefficients L_0, L_1, &c.), it is proper to separate the terms involving

the different powers of e, and write $L=M_0+M_1e+M_2e^2+$ &c., $\nabla'=\nabla''+2d\partial_e$, where $\nabla''=5a\partial_b+4b\partial_c+3c\partial_d$. We have then

$$\nabla L = \begin{array}{c|c|c} \nabla''M_0 + \; e & \nabla''M_1 + \; e^2 & \nabla''M_2 + \text{ \&c.,} \\ +2d\,M_1 & +4dM_2 & +6dM_3, \end{array}$$

or, what is the same thing,

$$\frac{1}{e}\nabla L = \begin{array}{c|c} \nabla''M_1 + \; e & \nabla''M_2 + \cdots \\ +4dM_2 & +4dM_3\,; \end{array}$$

and as an equation which should be satisfied identically, and which would therefore serve as a verification,

$$\nabla''M_0+2dM_1=0\,;$$

but as a verification was obtained by other means, the equations of this kind were not used for the purpose. It may be interesting to give the actual calculation of one of the coefficients, say of coefficient θ^2 (which, with coefficient θ, was calculated by Mr. Harley).

Calculation of coefficient θ^2 in equation of differences for the quintic equation

$$(a, b, c, d, e, f \between v, 1)^5 = 0.$$

Calculation of L_0.

	$2ae$	$-2bd$	$+c^2$	$2ce$	$-d^2$	e^2		
	Coeff. θ^0.			Coeff. θ^1.		Coeff. θ^2.		In eq. of diff. for quartic.
a^4e^4	+512					−112	=	+400
a^3bde^3	−384	−512				+ 56		−840
$a^3c^2e^3$	−256		+256	−384		+ 24		−360
$a^3cd^2e^2$	+288			+432	+192	+ 48		+960
a^3d^4e	− 54				−216			−270
$a^2b^2ce^3$	+288			+144		− 32		+400
$a^2b^2d^2e^2$	− 12	+384			− 72	− 25		+275
$a^2bc^2de^2$	−160	+256	−192	−240		− 54		−390
a^2bcd^3e	+ 36	−288		−108	+120			−240
abd^5		+ 54			+ 54			+108
$a^2c^4e^2$	+ 32		−128	+ 64		+ 17		− 15
$a^2c^3d^2e$	− 8		+144	+ 36	− 32			+140
$a^2c^2d^4$			− 27		− 18			− 45
ab^4e^3	− 54					+ 6		− 48
ab^3cde^2	+ 36	−288		+ 36		+ 38		−178
ab^3d^3e	− 8	+ 12			− 18			− 14
$ab^2c^3e^2$	− 8		+144	− 12		− 12		+112
$ab^2c^2d^2e$	+ 2	+160	− 6	+ 84	+ 6			+246
ab^2cd^4		− 36			− 42			− 78
abc^4de		− 32	− 80	− 52				−164
abc^3d^3		+ 8	+ 18		+ 26			+ 52
ac^6e			+ 16	+ 8				+ 24
ac^5d^2			− 4		− 4			− 8
b^5de^2		+ 54				− 6		+ 48
$b^4c^2e^2$			− 27			+ 2		− 25
b^4cd^2e		− 36		− 18				− 54
b^4d^4		+ 8			+ 9			+ 17
b^3c^3de		+ 8	+ 18	+ 12				+ 38
$b^3c^2d^3$		− 2	− 4		− 6			− 12
b^2c^5e			− 4	− 2				− 6
$b^2c^4d^2$			+ 1		+ 1			+ 2

$L_0 = A + Be + Ce^2 + De^3 + Ee^4$, suppose, then,

−B=		−C=		−D=		−E=	
a^2d^3	+270	a^3cd^2	−960	a^3bd	+840	a^4	−400
a^2bcd^3	+240	$a^2b^2d^2$	−275	a^3c^2	+360		
$a^2c^3d^2$	−140	a^2bc^2d	+390	a^2b^2c	−400		
ab^3d^3	+ 14	a^2c^4	+ 15	ab^4	+ 48		
$ab^2c^2d^2$	−246	ab^3cd	+178				
abc^4d	+164	ab^2c^3	−112				
ac^6	− 24	b^5d	− 48				
b^4cd^2	+ 54	b^4c^2	+ 25				
b^3c^3d	− 38						
b^2c^5	+ 6						

Calculation of L_1.

Terms of $L_1 f$ not involving e.

$-f(\nabla''B+4dC)$ viz.	$5a\partial_b$	$+4b\partial_c$	$+3c\partial_d$	$4d$		
		$-B$		$-C$		
a^3cd^3f	+1200		+3240	−3840	=	+600
$a^2b^2d^3f$	+ 210	+ 960		−1100		+ 70
$a^2bc^2d^2f$	−2460	−1680	+2160	+1560		−420
a^2c^4df	+ 820		− 840	+ 60		+ 40
ab^3cd^2f	+1080	−1968	+ 126	+ 712		− 50
ab^2c^3df	− 570	+2624	−1476	− 448		+130
abc^5f	+ 60	− 576	+ 492			− 24
b^5d^2f		+ 216		− 192		+ 24
b^4c^2df		− 456	+ 324	+ 100		− 32
b^3c^4f		+ 120	− 114			+ 6

Terms of $L_1 f$ involving e^1.

$-ef(\nabla''C+6dD)$ viz.	$5a\partial_b$	$+4b\partial_c$	$+3c\partial_d$	$6d$		
		$-C$		$-D$		
a^3bd^2ef	−2750	−3840		+5040	=	−1550
a^3c^2def	+1950		−5760	+2160		−1650
a^2b^2cdef	+2670	+3120	−1650	−2400		+1740
a^2bc^3ef	−1120	+ 240	+1170			+ 290
ab^4def	−1200	+ 712		+ 288		− 200
ab^3c^2ef	+ 500	−1344	+ 534			− 310
b^5cef		+ 200	− 144			+ 56

Terms of $L_1 f$ involving e^2.

$-e^2f(\nabla''D+8dE)$ viz.	$5a\partial_b$	$4b\partial_c$	$3c\partial_d$	$8d$		
		$-D$		$-E$		
a^4de^2f	+4200			−3200	=	+1000
a^3bce^2f	−4000	+2880	+2520			+1400
$a^2b^3e^2f$	+ 960	−1600				− 640

Calculation of L_2.

$L_1=A+Be+Ce^2$, suppose, where

$-\frac{1}{2}B=$		$-\frac{1}{2}C=$	
a^3bd^2	+775	a^4d	−500
a^3c^2d	+825	a^3bc	−700
a^2b^2cd	−870	a^2b^3	+320
a^2bc^3	−145		
ab^4d	+100		
ab^3c^2	+155		
b^5c	− 28		

Terms of $L_2 f^2$ not involving e.

$-f^2(\nabla''\tfrac{1}{2}B+4d\tfrac{1}{2}C)$ viz.	$5a\partial_b$	$4b\partial_c$	$3c\partial_d$	$4d$		
		$-\tfrac{1}{2}B$		$-\tfrac{1}{2}C$		
$a^4d^2f^2$	+3875			−2000	=	+1875
a^3bcdf^2	−8700	+6600	+4650	−2800		− 250
$a^3c^3f^2$	− 725		+2475			+1750
$a^2b^3df^2$	+2000	−3480		+1280		− 200
$a^2b^2c^2f^2$	+2325	−1740	−2610			−2025
ab^4cf^2	− 700	+1240	+ 300			+ 840
ab^6f^2		− 112				− 112

Terms of $L_2 f^2$ involving e^1.

$-ef^2\nabla''\tfrac{1}{2}C$ viz.	$5a\partial_b$	$4b\partial_c$	$3c\partial_d$		
		$-\tfrac{1}{2}C$			
a^4cef^2	−3500		−1500	=	−5000
$a^3b^2ef^2$	+4800	−2800			+2000

Calculation of L_3 (gives $L_3=0$).

$$L_2=A+Be,$$

$-\tfrac{1}{3}B=$	
a^4c	$+1666\tfrac{2}{3}$
a^3b^2	$-\ 666\tfrac{2}{3}$

Terms of $L_3 f^3$.

$-f^3\nabla''\tfrac{1}{3}B$ viz.	$5a\partial_b$	$4b\partial_c$	$3c\partial_d$		
		$-\tfrac{1}{3}B$			
a^4bf^3	$-6666\tfrac{2}{3}$	$+6666\tfrac{2}{3}$		=	0

And the required coefficient of θ^2 is

$$L_0+L_1f+L_2f^2.$$

All the coefficients were calculated in this manner, except the last coefficient, which was deduced from the known value of the discriminant for the standard form. And we have thus the complete expression of the equation of differences for the general quintic equation under the denumerate form $(a, b, c, d, e, f \between v, 1)^5=0$, viz.—

Equation of differences for $(a, b, c, d, e, f \between v, 1)^5=0$ is

θ^{10}	θ^{9}	θ^{8}	θ^{7}	θ^{6}	θ^{5}	θ^{4}
$a^8\times$	$a^6\times$	$a^4\times$	$a^2\times$			
$+1$	$+10\ ac$	$+10\ a^3e$	$+50\ a^4ce$	$+200\ a^6df$	$+625\ a^6f^2$	$+1750\ a^5cf^2$
	$-\ 4\ b^2$	$-\ 4\ a^2bd$	$+25\ a^4d^2$	$-\ 95\ a^6e^2$	$-250\ a^5bef$	$-\ 950\ a^5def$
		$+39\ a^2c^2$	$-20\ a^3b^2e$	$-120\ a^5bcf$	$+400\ a^5cdf$	$+\ 40\ a^5e^3$
		$-30\ ab^2c$	$-50\ a^3bcd$	$+\ 36\ a^5bde$	$-360\ a^5ce^2$	$-\ 130\ a^4bcef$
		$+\ 6\ b^4$	$+80\ a^3c^3$	$+124\ a^5c^2e$	$+260\ a^5d^2e$	$-\ 700\ a^4b^2f^2$
			$+16\ a^2b^3d$	$+\ 92\ a^5cd^2$	$-110\ a^4b^2df$	$+\ 142\ a^4bde^2$
			$-81\ a^2b^2c^2$	$+\ 32\ a^4b^3f$	$+169\ a^4b^2e^2$	$+\ 380\ a^4bd^2f$
			$+30\ ab^4c$	$-\ 98\ a^4b^2ce$	$-240\ a^4bc^2f$	$-\ 522\ a^4c^2e^2$
			$-\ 4\ b^6$	$-\ 44\ a^4b^2d^2$	$-104\ a^4bcde$	$+\ 240\ a^4c^2df$
				$-160\ a^4bc^2d$	$-104\ a^4bd^3$	$+\ 708\ a^4cd^2e$
				$+\ 95\ a^4c^4$	$+196\ a^4c^3e$	$-\ 53\ a^4d^4$
				$+\ 18\ a^3b^4e$	$+118\ a^4c^2d^2$	$+\ 128\ a^3b^3ef$
				$+116\ a^3b^3cd$	$+150\ a^3b^3cf$	$+\ 388\ a^3b^2ce^2$
				$-104\ a^3b^2c^3$	$-\ 10\ a^3b^3de$	$-\ 394\ a^3b^2cdf$
				$-\ 20\ a^2b^5d$	$-180\ a^3b^2c^2e$	$-\ 378\ a^3b^2d^2e$
				$+\ 45\ a^2b^4c^2$	$+\ 20\ a^3b^2cd^2$	$-\ 144\ a^3bc^3f$
				$-\ 10\ ab^6c$	$-220\ a^3bc^3d$	$-\ 480\ a^3bc^2de$
				$+\ 1\ b^8$	$+\ 66\ a^3c^5$	$-\ 156\ a^3bcd^3$
					$-\ 24\ a^2b^5f$	$+\ 194\ a^3c^4e$
					$+\ 66\ a^2b^4ce$	$+\ 52\ a^3c^3d^2$
					$-\ 3\ a^2b^4d^2$	$+\ 66\ a^2b^4df$
					$+192\ a^2b^3c^2d$	$-\ 84\ a^2b^4e^2$
					$-\ 66\ a^2b^2c^4$	$+\ 194\ a^2b^3c^2f$
					$-\ 8\ ab^6e$	$+\ 330\ a^2b^3cde$
					$-\ 66\ ab^5cd$	$+\ 92\ a^2b^3d^2$
					$+\ 24\ ab^4c^3$	$-\ 152\ a^2b^2c^3e$
					$+\ 8\ b^7d$	$+\ 174\ a^2b^2c^2d^2$
					$-\ 3\ b^6c^2$	$-\ 140\ a^2bc^4d$
						$+\ 25\ a^2c^6$
						$-\ 70\ ab^5cf$
						$-\ 42\ ab^5de$
						$+\ 32\ ab^4c^2e$
						$-\ 144\ ab^4cd^2$
						$+\ 100\ ab^3c^3d$
						$-\ 18\ ab^2c^5$
						$+\ 8\ b^7f$
						$-\ 2\ b^6ce$
						$+\ 22\ b^6d^2$
						$-\ 16\ b^5c^2d$
						$+\ 3\ b^4c^4$

$(\mathfrak{X}\theta, 1)^{10}=0$, viz. the function in ϑ is

θ^3.	θ^2.	θ.	θ^0.
$-3750\ a^5ef^2$	$-5000\ a^4cef^2$	$-6250\ a^4df^3$	$+3125\ a^4f^4$
$+1500\ a^4bdf^2$	$+1875\ a^4d^2f^2$	$+5000\ a^4e^2f^2$	$-2500\ a^3bef^3$
$+1500\ a^4be^2f$	$+1000\ a^4de^2f$	$+3750\ a^3bcf^3$	$-3750\ a^3cdf^3$
$+2500\ a^4c^2f^2$	$+400\ a^4e^4$	$-250\ a^3bdef^2$	$+2000\ a^3ce^2f^2$
$-2150\ a^4cdef$	$+2000\ a^3b^2ef^2$	$-2000\ a^3be^3f$	$+2250\ a^3d^2ef^2$
$-80\ a^4ce^3$	$-250\ a^3bcdf^2$	$-3750\ a^3c^2ef^2$	$-1600\ a^3de^3f$
$+700\ a^4d^3f$	$+1400\ a^3bce^2f$	$+3000\ a^3cd^2f^2$	$+256\ a^3e^5$
$+570\ a^4d^2e^2$	$-1550\ a^3bd^2ef$	$+200\ a^3cde^2f$	$+2000\ a^2b^2df^3$
$-40\ a^3b^2def$	$-840\ a^3bde^3$	$+320\ a^3ce^4$	$-50\ a^2b^2e^2f^2$
$-2450\ a^3b^2cf^2$	$+1750\ a^3c^3f^2$	$-450\ a^3d^3ef$	$+2250\ a^2bc^2f^3$
$-118\ a^3b^2e^3$	$-1650\ a^3c^2def$	$-40\ a^3d^2e^3$	$-2050\ a^2bcdef^2$
$+290\ a^3bc^2ef$	$-360\ a^3c^2e^3$	$-1000\ a^2b^3f^3$	$+160\ a^2bce^3f$
$-400\ a^3bcd^2f$	$+600\ a^3cd^3f$	$+1950\ a^2b^2cef^2$	$-900\ a^2bd^3f^2$
$-158\ a^3bcde^2$	$+960\ a^3cd^2e^2$	$-1150\ a^2b^2d^2f^2$	$+1020\ a^2bd^2e^2f$
$-596\ a^3bd^3e$	$-270\ a^3d^4e$	$+1170\ a^2b^2de^2f$	$-192\ a^2bde^4$
$+80\ a^3c^3df$	$-200\ a^2b^3df^2$	$+72\ a^2b^2e^4$	$-900\ a^2c^3ef^2$
$-308\ a^3c^3e^2$	$-640\ a^2b^3e^2f$	$-2100\ a^2bc^2df^2$	$+825\ a^2c^2d^2f^2$
$+612\ a^3c^2d^2e$	$-2025\ a^2b^2c^2f^2$	$+1380\ a^2bc^2e^2f$	$+560\ a^2c^2de^2f$
$-102\ a^3cd^4$	$+1740\ a^2b^2cdef$	$-550\ a^2bcd^2ef$	$-128\ a^2c^2e^4$
$+490\ a^2b^4f^2$	$+400\ a^2b^2ce^3$	$-504\ a^2bcde^3$	$-630\ a^2cd^3ef$
$+180\ a^2b^3cef$	$+70\ a^2b^2d^3f$	$+180\ a^2bd^4f$	$+144\ a^2cd^2e^3$
$+112\ a^2b^3d^2f$	$+275\ a^2b^2d^2e^2$	$+138\ a^2bd^3e^2$	$+108\ a^2d^5f$
$+92\ a^2b^3de^2$	$+290\ a^2bc^3ef$	$+675\ a^2c^4f^2$	$-27\ a^2d^4e^2$
$+86\ a^2b^2c^2df$	$-420\ a^2bc^2d^2f$	$-330\ a^2c^3def$	$-1600\ ab^3cf^3$
$+388\ a^2b^2c^2e^2$	$-390\ a^2bc^2de^2$	$-224\ a^2c^3e^3$	$+160\ ab^3def^2$
$+150\ a^2b^2cd^2e$	$-240\ a^2bcd^3e$	$+60\ a^2c^2d^3f$	$-36\ ab^3e^3f$
$+160\ a^2b^2d^4$	$+108\ a^2bd^5$	$+434\ a^2c^2d^2e^2$	$+1020\ ab^2c^2ef^2$
$-48\ a^2bc^4f$	$+40\ a^2c^4df$	$-198\ a^2cd^4e$	$+560\ ab^2cd^2f^2$
$-504\ a^2bc^3de$	$-15\ a^2c^4e^2$	$+27\ a^2d^6$	$-746\ ab^2cde^2f$
$+106\ a^2c^5e$	$+140\ a^2c^3d^2e$	$-240\ ab^4ef^2$	$+144\ ab^2ce^4$
$-7\ a^2c^4d^2$	$-45\ a^2c^2d^4$	$+1320\ ab^3cdf^2$	$+24\ ab^2d^3ef$
$-68\ ab^5ef$	$+840\ ab^4cf^2$	$-1230\ ab^3ce^2f$	$-6\ ab^2d^2e^3$
$-86\ ab^4cdf$	$-200\ ab^4def$	$-12\ ab^3d^2ef$	$-630\ abc^3df^2$
$-178\ ab^4ce^2$	$-48\ ab^4e^3$	$+18\ ab^3de^3$	$+24\ abc^3e^2f$
$-54\ ab^4d^2e$	$-310\ ab^3c^2ef$	$-450\ ab^2c^3f^2$	$+356\ abc^2d^2ef$
$+234\ ab^3c^2de$	$-50\ ab^3cd^2f$	$+594\ ab^2c^2def$	$-80\ abc^2de^3$
$+34\ ab^3c^3f$	$-178\ ab^3cde^2$	$+282\ ab^2c^2e^3$	$-72\ abcd^4f$
$-148\ ab^3cd^3$	$-14\ ab^3d^3e$	$-154\ ab^2cd^3f$	$+18\ abcd^3e^2$
$-56\ ab^2c^4e$	$+130\ ab^2c^3df$	$-114\ ab^2cd^2e^2$	$+108\ ac^5f^2$
$+112\ ab^2c^3d^2$	$+112\ ab^2c^3e^2$	$+6\ ab^2d^4e$	$-72\ ac^4def$
$-34\ abc^5d$	$+246\ ab^2c^2d^2e$	$-72\ abc^4ef$	$+16\ ac^4e^3$
$+4\ ac^7$	$-78\ ab^2cd^4$	$+24\ abc^3d^2f$	$+16\ ac^3d^3f$
$+16\ b^6df$	$-24\ abc^5f$	$-186\ abc^3de^2$	$-4\ ac^3d^2e^2$
$+26\ b^6e^2$	$-164\ abc^4de$	$+116\ abc^2d^3e$	$+256\ b^5f^3$
$-6\ b^5c^2f$	$+52\ abc^3d^3$	$-18\ abcd^5$	$-192\ b^4cef^2$
$-30\ b^5cde$	$+24\ ac^6e$	$+36\ ac^5e^2$	$-128\ b^4d^2f^2$
$+28\ b^5d^3$	$-8\ ac^5d^2$	$-24\ ac^4d^2e$	$+144\ b^4de^2f$
$+8\ b^4c^3e$	$-112\ b^6f^2$	$+4\ ac^3d^4$	$-27\ b^4e^4$
$-24\ b^4c^2d^2$	$+56\ b^5cef$	$-192\ b^5df^2$	$+144\ b^3c^2df^2$
$+8\ b^3c^4d$	$+24\ b^5d^2f$	$+216\ b^5e^2f$	$-6\ b^3c^2e^2f$
$-1\ b^2c^6$	$+48\ b^5de^2$	$+72\ b^4c^2f^2$	$-80\ b^3cd^2ef$
	$-32\ b^4c^2df$	$-120\ b^4cdef$	$+18\ b^3cde^3$
	$-25\ b^4c^2e^2$	$-54\ b^4ce^3$	$+16\ b^3d^4f$
	$-54\ b^4cd^2e$	$+32\ b^4d^3f$	$-4\ b^3d^3e^2$
	$+17\ b^4d^4$	$+18\ b^4d^2e^2$	$-27\ b^2c^4f^2$
	$+6\ b^3c^4f$	$+18\ b^3c^3ef$	$+18\ b^2c^3def$
	$+38\ b^3c^3de$	$-6\ b^3c^2d^2f$	$-4\ b^2c^3e^3$
	$-12\ b^3c^2d^3$	$+42\ b^3c^2de^2$	$-4\ b^2c^2d^3f$
	$-6\ b^2c^5e$	$-26\ b^3cd^3e$	$+1\ b^2c^2d^2e^2$
	$+2\ b^2c^4d^2$	$+4\ b^3d^5$	
		$-9\ b^2c^4e^2$	
		$+6\ b^2c^3d^2e$	
		$-1\ b^2c^2d^4$	

$(\mathfrak{X}\theta, 1)^{10}$

It may be remarked, that if ω is an imaginary cube root of unity, then the roots of the equation $(1, 1, 1, 1, 1, 1\between v, 1)^5=0$ are $-1, \omega, \omega^2, -\omega, -\omega^2$; the differences of the roots are

$$\begin{array}{rcccccccccc} & -1-\omega, & -1-\omega^2, & -1+\omega, & -1+\omega^2, & \omega-\omega^2, & 2\omega, & \omega+\omega^2, & \omega^2+\omega, & 2\omega^2, & -\omega+\omega^2 \\ = & \omega^2, & \omega, & -1+\omega, & -1+\omega^2, & \omega-\omega^2, & 2\omega, & -1, & -1, & 2\omega^2, & -\omega+\omega^2, \end{array}$$

and the squares of the differences are

$$\omega, \quad \omega^2, \quad -3\omega, \quad -3\omega^2, \quad -3, \quad 4\omega^2, \quad 1, \quad 1, \quad 4\omega, \quad -3,$$

from which the equation of differences is found to be

$$(\theta^2+\theta+1)(\theta^2-3\theta+9)(\theta^2+4\theta+16)(\theta^2-2\theta+1)(\theta^2+6\theta+9)=0;$$

or multiplying out, it is

$$(1, 6, 21, 46, 108, 546, 493, -1410, -567, -540, +1296\between\theta, 1)^{10}=0;$$

which is what the preceding expression of the equation of differences becomes upon writing therein $a=b=c=d=e=f=1$. Moreover, upon passing (as will presently be done) to the standard form, and then writing $a=b=c=d=e=f=1$, all the coefficients (except the first coefficient, which is equal to unity) should become equal to zero; these two tests afford a complete verification of the result.

The following corrections have to be made in WARING's result, as given by himself and LAGRANGE (WARING, Phil. Trans. 1763):—

WARING, Meditationes Algebraicæ, p. 85:—

for $+169\,q^3s$ *read* $+196\,q^3s$ (in coefficient w^5).

LAGRANGE, Equations Numériques, p. 108:—

for $+1200$ CE *read* $+200$ CE (in d)
for -169 B^{3}D *read* -196 B^{3}D (in e)
for -25 B^6 *read* $+25$ B^6 (in f)
for $+27$ C^{4}D^2 *read* -27 C^{4}D^2 (in k).

It may be noticed, that if in the coefficients of the several powers of θ (as they are written down in the columns, without regarding the power of a which multiplies the entire column), we attend only to the terms independent of a, we have the series

$$\begin{array}{llllllll} 1, & -4b^2, & +6b^4, & -4b^6, & +1b^8, & +8b^7d, & +\ 8b^7f, & \&c. \\ & & & & & -3b^6c^2, & -\ 2b^6ce & \\ & & & & & & +22b^6d^2 & \\ & & & & & & -16b^5c^2d & \\ & & & & & & +3b^4c^4, & \end{array}$$

the law of the first terms of which, up to the term $+1b^8$, is obvious; but the term $+1b^8$, which is the last term of this initial series, is also the first term of a terminal series, the terms of which are deduced from the coefficients in the equation of differences for the

quartic equation $(a, b, c, d, e \between v, 1)^4=0$, viz. these coefficients are

$a^6\times$	$a^4\times$	$a^2\times$	&c.
$+1$	$+8ac$	$+\ 8a^3e$	
	$-3b^2$	$-2a^2bd$	
		$+22a^2c^2$	
		$-16ab^2c$	
		$+3b^4$;	

and by writing b, c, d, e, f in the place of a, b, c, d, e respectively, and multiplying by b^2, we have the above-mentioned series,

$$+1b^8, \quad \begin{matrix} +8b^7d, \\ -3b^6c^2. \end{matrix} \quad \&c.$$

It is easy to see, *à priori*, in the case of an equation of any order, that this property holds good.

Passing now to the standard forms,—

For the quadric $(a, b, c \between v, 1)^2=0$, the equation of differences is, $0=$

$a^2\times$	$4\times$
$+1$	$+1\ ac$
	$-1\ b^2$

$\between \theta, 1)$.

For the cubic equation $(a, b, c, d \between v, 1)^3=0$, the equation of differences is, $0=$

$a^4\times$	$18\,a^2\times$	$81\times$	$27\times$
$+1$	$+1\ ac$	$+1\ a^2c^2$	$+1\ a^2d^2$
	$-1\ b^2$	$-2\ ab^2c$	$-6\ abcd$
		$+1\ b^4$	$+4\ ac^3$
			$+4\ b^3d$
			$-3\ b^2c^2$

$\between \theta, 1)^3$.

For the quartic equation $(a, b, c, d, e \between v, 1)^4=0$, the equation of differences is, $0=$

$a^6\times$	$48a^4\times$	$8a^2\times$	$32\times$	$16\times$	$1152\times$	$256\times$
$+1$	$+1\ ac$	$+\ 1\ a^3e$	$3\ a^4ce$	$-\ 7\ a^4e^2$	$-\ 1\ a^3ce^2$	$+\ 1\ a^3e^3$
	$-1\ b^2$	$-\ 4\ a^2bd$	$+\ 13\ a^4d^2$	$+\ 56\ a^3bde$	$+\ 3\ a^3d^2e$	$-\ 12\ a^2bde^2$
		$+\ 99\ a^2c^2$	$-\ 3\ a^3b^2e$	$+\ 54\ a^3c^2e$	$+\ 1\ a^2b^2e^2$	$-\ 18\ a^2c^2e^2$
		$-192\ ab^2c$	$-\ 90\ a^3bcd$	$+\ 288\ a^3cd^2$	$-10\ a^2bcde$	$+\ 54\ a^2cd^2e$
		$+\ 96\ b^4$	$+189\ a^3c^3$	$-\ 192\ a^2b^2ce$	$-12\ a^2bd^3$	$-\ 27\ a^2d^4$
			$+\ 64\ a^2b^3d$	$-\ 400\ a^2b^2d^2$	$+\ 6\ a^2c^3e$	$+\ 54\ ab^2ce^2$
			$-432\ a^2b^2c^2$	$-1944\ a^2bc^2d$	$+\ 9\ a^2c^2d^2$	$-\ 6\ ab^2d^2e$
			$+384\ ab^4c$	$+1377\ a^2c^4$	$+\ 4\ ab^3de$	$-180\ abc^2de$
			$-128\ b^6$	$+\ 96\ ab^4e$	$-\ 3\ ab^2c^2e$	$+108\ abcd^3$
				$+3648\ ab^3cd$	$+56\ ab^2cd^2$	$+\ 81\ ac^4e$
				$-2592\ ab^2c^3$	$-78\ abc^3d$	$-\ 54\ ac^3d^2$
				$-1536\ b^5d$	$+27\ ac^5$	$-\ 27\ b^4e^2$
				$+1152\ b^4c^2$	$-32\ b^4d^2$	$+108\ b^3cde$
					$+48\ b^3c^2d$	$-\ 64\ b^3d^3$
					$-18\ b^2c^2$	$-\ 54\ b^2c^3e$
						$+\ 36\ b^2c^2d^2$.

$\between \theta, 1)^6$.

For the quintic equation $(a, b, c, d, e, f \between v, 1)^5 = 0$, the equation of differences is,

θ^{10}	θ^{9}	θ^{8}	θ^{7}	θ^{6}	θ^{5}	θ^{4}
$a^8\times$	$100\,a^6\times$	$50\,a^4\times$	$2500\,a^2\times$	$125\times$	$625\times$	$2500\times$
$+1$	$+1\ ac$	$+1\ a^3e$	$+1\ a^4ce$	$+16\ a^6df$	$+1\ a^6f^2$	$+7\ a^5cf^2$
	$-1\ b^2$	$-4\ a^2bd$	$+1\ a^4d^2$	$-19\ a^6e^2$	$-10\ a^5bef$	$-19\ a^5def$
		$+78\ a^2c^2$	$-1\ a^3b^2e$	$-48\ a^5bcf$	$+64\ a^5cdf$	$+2\ a^5e^3$
		$-150\ ab^2c$	$-10\ a^3bcd$	$+72\ a^5bde$	$-144\ a^5ce^2$	$-13\ a^4bcef$
		$+75\ b^4$	$+32\ a^3c^3$	$+496\ a^5c^2e$	$+208\ a^5d^2e$	$-7\ a^4b^2f^2$
			$+8\ a^2b^3d$	$+736\ a^5cd^2$	$-44\ a^4b^2df$	$+71\ a^4bde^2$
			$-81\ a^2b^2c^2$	$+32\ a^4b^3f$	$+169\ a^4b^2e^2$	$+76\ a^4bd^2f$
			$+75\ ab^4c$	$-980\ a^4b^2ce$	$-192\ a^4bc^2f$	$-522\ a^4c^2e^2$
			$-25\ b^6$	$-880\ a^4b^2d^2$	$-416\ a^4bcde$	$+96\ a^4c^2df$
				$-6400\ a^4bc^2d$	$-832\ a^4bd^3$	$+1416\ a^4cd^2e$
				$+7600\ a^4c^4$	$+1568\ a^4c^3e$	$-212\ a^4d^4$
				$+450\ a^3b^4e$	$+1888\ a^4c^2d^2$	$+32\ a^3b^3ef$
				$+11600\ a^3b^3cd$	$+300\ a^3b^3cf$	$+970\ a^3b^2ce^2$
				$-20800\ a^3b^2c^3$	$-100\ a^3b^3de$	$-394\ a^3b^2cdf$
				$-5000\ a^2b^5d$	$-3600\ a^3b^2c^2e$	$-1890\ a^3b^2d^2e$
				$+22500\ a^2b^4c^2$	$+800\ a^3b^2cd^2$	$-288\ a^3bc^3f$
				$-12500\ ab^6c$	$-17600\ a^3bc^3d$	$-4800\ a^3bc^2de$
				$+3125\ b^8$	$+10560\ a^3c^5$	$-3120\ a^3bcd^3$
					$-120\ a^2b^5f$	$+3880\ a^3c^4e$
					$+3300\ a^2b^4ce$	$+2080\ a^3c^3d^2$
					$-300\ a^2b^4d^2$	$+165\ a^2b^4df$
					$+38400\ a^2b^3c^2d$	$-525\ a^2b^4e^2$
					$-26400\ a^2b^2c^4$	$+970\ a^2b^3c^2f$
					$-1000\ ab^6e$	$+8250\ a^2b^3cde$
					$+33000\ ab^5cd$	$+4600\ a^2b^3d^3$
					$+24000\ ab^4c^3$	$-7600\ a^2b^2c^3e$
					$+10000\ b^7d$	$+17400\ a^2b^2c^2d^2$
					$-3750\ b^6c^2$	$-28000\ a^2bc^4d$
						$+10000\ a^2c^6$
						$-875\ ab^5cf$
						$-2625\ ab^5de$
						$+4000\ ab^4c^2e$
						$-36000\ ab^4cd^2$
						$+50000\ ab^3c^3d$
						$-18000\ ab^2c^5$
						$+250\ b^7f$
						$-625\ b^6ce$
						$+13750\ b^6d^2$
						$-20000\ b^5c^2d$
						$+7500\ b^4c^4$

$0=(\mathfrak{X}\theta,1)^{10}=0$, viz. the function in ϑ is

θ^3.	θ^2.	θ.	θ^0.
$1250\times$	$62500\times$	$62500\times$	$3125\times$
$-\ 3\ a^5ef^2$	$-\ 4\ a^4cef^2$	$-\ 1\ a^4df^3$	$+\ 1\ a^4f^4$
$+\ 12\ a^4bdf^2$	$+\ 3\ a^4d^2f^2$	$+\ 2\ a^4e^2f^2$	$-\ 20\ a^3bef^3$
$+\ 30\ a^4be^2f$	$+\ 4\ a^4de^2f$	$+\ 3\ a^3bcf^3$	$-\ 120\ a^3cdf^3$
$+\ 40\ a^4c^2f^2$	$+\ 4\ a^4e^4$	$-\ 1\ a^3bdef^2$	$+\ 160\ a^3ce^2f^2$
$-\ 172\ a^4cdef$	$+\ 4\ a^3b^2ef^2$	$-\ 20\ a^3be^3f$	$+\ 360\ a^3d^2ef^2$
$-\ 16\ a^4ce^3$	$-\ 2\ a^3bcdf^2$	$-\ 30\ a^3c^2ef^2$	$-\ 640\ a^3de^3f$
$+\ 112\ a^4d^3f$	$+\ 28\ a^3bce^2f$	$+\ 48\ a^3cd^2f^2$	$+\ 256\ a^3e^5$
$+\ 228\ a^4d^2e^2$	$-\ 62\ a^3bd^2ef$	$+\ 8\ a^3cde^2f$	$+\ 160\ a^2b^2df^3$
$-\ 8\ a^3b^2def$	$-\ 84\ a^3bde^3$	$+\ 32\ a^3ce^4$	$-\ 10\ a^2b^2e^2f^2$
$-\ 98\ a^3b^2cf^2$	$+\ 28\ a^3c^3f^2$	$-\ 36\ a^3d^3ef$	$+\ 360\ a^2bc^2f^3$
$-\ 59\ a^3b^2e^3$	$-\ 132\ a^3c^2def$	$-\ 8\ a^3d^2e^3$	$-\ 1640\ a^2bcdef^2$
$+\ 116\ a^3bc^2ef$	$-\ 72\ a^3c^2e^3$	$-\ 2\ a^2b^3f^3$	$+\ 320\ a^2bce^3f$
$-\ 320\ a^3bcd^2f$	$+\ 96\ a^3cd^3f$	$+\ 39\ a^2b^2cef^2$	$-\ 1440\ a^2bd^3f^2$
$-\ 316\ a^3bcde^2$	$+\ 384\ a^3cd^2e^2$	$-\ 46\ a^2b^2d^2f^2$	$+\ 4080\ a^2bd^2e^2f$
$-\ 2384\ a^3bd^3e$	$-\ 216\ a^3d^4e$	$+\ 117\ a^2b^2de^2f$	$-\ 1920\ a^2bde^4$
$+\ 128\ a^3c^3df$	$-\ 4\ a^2b^3df^2$	$+\ 18\ a^2b^2e^4$	$-\ 1440\ a^2c^3ef^2$
$-\ 1232\ a^3c^3e^2$	$-\ 32\ a^2b^3e^2f$	$-\ 168\ a^2bc^2df^2$	$+\ 2640\ a^2c^2d^2f^2$
$+\ 4896\ a^3c^2d^2e$	$-\ 81\ a^2b^2c^2f^2$	$+\ 276\ a^2bc^2e^2f$	$+\ 4480\ a^2c^2de^2f$
$-\ 1632\ a^3cd^4$	$+\ 348\ a^2b^2cdef$	$-\ 220\ a^2bcd^2ef$	$-\ 2560\ a^2c^2e^4$
$+\ 49\ a^2b^4f^2$	$+\ 200\ a^2b^2ce^3$	$-\ 504\ a^2bcde^3$	$-10080\ a^2cd^3ef$
$+\ 180\ a^2b^3cef$	$+\ 28\ a^2b^2d^3f$	$+\ 144\ a^2bd^4f$	$+\ 5760\ a^2cd^2e^3$
$+\ 224\ a^2b^3d^2f$	$+\ 275\ a^2b^2d^2e^2$	$+\ 276\ a^2bd^3e^2$	$+\ 3456\ a^2d^5f$
$+\ 460\ a^2b^3de^2$	$+\ 116\ a^2bc^3ef$	$+\ 108\ a^2c^4f^2$	$-\ 2160\ a^2d^4e^2$
$+\ 344\ a^2b^2c^2df$	$-\ 336\ a^2bc^2d^2f$	$-\ 264\ a^2c^3def$	$-\ 640\ ab^3cf^3$
$+\ 3880\ a^2b^2c^2e^2$	$-\ 780\ a^2bc^2de^2$	$-\ 448\ a^2c^3e^3$	$+\ 320\ ab^3def^2$
$+\ 3000\ a^2b^2cd^2e$	$-\ 960\ a^2bcd^3e$	$+\ 96\ a^2c^2d^3f$	$-\ 180\ ab^3e^3f$
$+\ 6400\ a^2b^2d^4$	$+\ 864\ a^2bd^5$	$+1736\ a^2c^2d^2e^2$	$+\ 4080\ ab^2c^2ef^2$
$-\ 384\ a^2bc^4f$	$+\ 64\ a^2c^4df$	$-1584\ a^2cd^4e$	$+\ 4480\ ab^2cd^2f^2$
$-20160\ a^2bc^3de$	$-\ 60\ a^2c^4e^2$	$+\ 432\ a^2d^6$	$-14920\ ab^2cde^2f$
$+\ 8480\ a^2c^5e$	$+1120\ a^2c^3d^2e$	$-\ 12\ ab^4ef^2$	$+\ 7200\ ab^2ce^4$
$-\ 1120\ a^2c^4d^2$	$-\ 720\ a^2c^2d^4$	$+\ 264\ ab^3cdf^2$	$+\ 960\ ab^2d^3ef$
$-\ 170\ ab^5ef$	$+\ 84\ ab^4cf^2$	$-\ 615\ ab^3ce^2f$	$-\ 600\ ab^2d^2e^3$
$-\ 860\ ab^4cdf$	$-\ 100\ ab^4def$	$-\ 12\ ab^3d^2ef$	$-10080\ abc^3df^2$
$-\ 4450\ ab^4ce^2$	$-\ 60\ ab^4e^3$	$+\ 45\ ab^3de^3$	$+\ 960\ abc^3e^2f$
$-\ 2700\ ab^4d^2e$	$-\ 310\ ab^3c^2ef$	$-\ 180\ ab^2c^3f^2$	$+28480\ abc^2d^2ef$
$+23400\ ab^3c^2de$	$-\ 100\ ab^3cd^2f$	$+1188\ ab^2c^2def$	$-16000\ abc^2de^3$
$+\ 680\ ab^3c^3f$	$-\ 890\ ab^3cde^2$	$+1410\ ab^2c^2e^3$	$-11520\ abcd^4f$
$-29600\ ab^3cd^3$	$-\ 140\ ab^3d^3e$	$-\ 616\ ab^2cd^3f$	$+\ 7200\ abcd^3e^2$
$-11200\ ab^2c^4e$	$+\ 520\ ab^2c^3df$	$-1140\ ab^2cd^2e^2$	$+\ 3456\ ac^5f^2$
$+44800\ ab^2c^3d^2$	$+1120\ ab^2c^3e^2$	$+\ 120\ ab^2d^4e$	$-11520\ ac^4def$
$-27200\ abc^5d$	$+4920\ ab^2c^2d^2e$	$-\ 288\ abc^4ef$	$+\ 6400\ ac^4e^3$
$+\ 6400\ ac^7$	$-3120\ ab^2cd^4$	$+\ 192\ abc^3d^2f$	$+\ 5120\ ac^3d^3f$
$+\ 400\ b^6df$	$-\ 192\ abc^5f$	$-3720\ abc^3de^2$	$-\ 3200\ ac^3d^2e^2$
$+\ 1625\ b^6e^2$	$-6560\ abc^4de$	$+4640\ abc^2d^3e$	$+\ 256\ b^5f^3$
$-\ 300\ b^5c^2f$	$+4160\ abc^3d^3$	$-1440\ abcd^5$	$-\ 1920\ b^4cef^2$
$-\ 7500\ b^5cde$	$+1920\ ac^6e$	$+1440\ ac^5e^2$	$-\ 2560\ b^4d^2f^2$
$+14000\ b^5d^3$	$-1280\ ac^5d^2$	$-1920\ ac^4d^2e$	$+\ 7200\ b^4de^2f$
$+\ 4000\ b^4c^3e$	$-\ 28\ b^6f^2$	$+\ 640\ ac^3d^4$	$-\ 3375\ b^4e^4$
$-24000\ b^4c^2d^2$	$+\ 140\ b^5cef$	$-\ 96\ b^5df^2$	$+\ 5760\ b^3c^2df^2$
$+16000\ b^3c^4d$	$+\ 120\ b^5d^2f$	$+\ 270\ b^5e^2f$	$-\ 600\ b^3c^2e^2f$
$-\ 4000\ b^2c^6$	$+\ 600\ b^5de^2$	$+\ 72\ b^4c^2f^2$	$-16000\ b^3cd^2ef$
	$-\ 320\ b^4c^2df$	$-\ 600\ b^4cdef$	$+\ 9000\ b^3cde^3$
	$-\ 625\ b^4c^2e^2$	$-\ 675\ b^4ce^3$	$+\ 6400\ b^3d^4f$
	$-2700\ b^4cd^2e$	$+\ 320\ b^4d^3f$	$-\ 4000\ b^3d^3e^2$
	$+1700\ b^4d^4$	$+\ 450\ b^4d^2e^2$	$-\ 2160\ b^2c^4f^2$
	$+\ 120\ b^3c^4f$	$+\ 180\ b^3c^3ef$	$+\ 7200\ b^2c^3def$
	$+3800\ b^3c^3de$	$-\ 120\ b^3c^2d^2f$	$-\ 4000\ b^2c^3e^3$
	$-2400\ b^3c^2d^3$	$+2100\ b^3c^2de^2$	$-\ 3200\ b^2c^2d^3f$
	$-1200\ b^2c^5e$	$-2600\ b^3cd^3e$	$+\ 2000\ b^2c^2d^2e^2$
	$+\ 800\ b^2c^4d^2$	$+\ 800\ b^3d^5$	
		$-\ 900\ b^2c^4e^2$	
		$+1200\ b^2c^3d^2e$	
		$-\ 400\ b^2c^2d^4$	

$(\mathfrak{X}\theta,1)^{10}$

The coefficients in the preceding equations of differences are functions of the seminvariants of the quantics to which they belong; for instance, in the case of the quartic, the coefficient of θ^4 is

$$8a^2\{a^2(ae-4bd+3c^2)+96(ac-b^2)^2\},$$

that of θ^3 is

$$32\{-13a^3(ace-ad^2-b^2e+2bcd-c^3)+16a^2(ac-b^2)(ae-4bd+3c^2)+128(ac-b^2)^3\},$$

and so for the other coefficients; and by replacing each seminvariant by the covariant to which it belongs, we pass from the solution of the original problem of finding the equation for $\theta=(\alpha-\beta)^2$, to that of the problem of finding the equation for

$$\theta=\frac{(\alpha-\beta)^2}{(x-\alpha y)^2(x-\beta y)^2}.$$

The results are as follows:—

For the quadric $(a, b, c\between x, y)^2$, the equation in θ is, $0=$

$$(\text{U}^2,\ 4\square\between\theta, 1),$$

where U is the quadric, $\square$ the discriminant.

For the cubic $(a, b, c, d\between x, y)^3$, the equation in θ is, $0=$

$$(\text{U}^4,\ 18\text{U}^2\text{H},\ 81\text{H}^2,\ 27\square\between\theta, 1)^3,$$

where U is the cubic, H the Hessian, $\square$ the discriminant.

For the quartic $(a, b, c, d, e\between x, y)^4$, the equation in θ is, $0=$

$$\left(\begin{array}{l} \text{U}^6, \\ 48\text{U}^4\text{H}, \\ 8\text{U}^2(\text{U}^2\text{I}+96\text{H}^2), \\ 32(-13\text{U}^3\text{J}+16\text{U}^2\text{HI}+128\text{H}^3), \\ 16(-\ 7\text{U}^2\text{I}^2-288\text{UHJ}+384\text{H}^2\text{I}), \\ 1152(-3\text{UIJ}+2\text{HI}^2), \\ 256(\ \text{I}^3\ -27\text{J}^2), \end{array}\between\theta, 1\right)^6,$$

where U is the quartic, H the Hessian, I and J the quadrinvariant and the cubinvariant respectively.

For the quintic $(a, b, c, d, e, f\between x, y)^5$, the equation in θ, as far as it can be expressed in terms of known covariants, is, $0=$

$$\left(\begin{array}{l} \text{U}^8, \\ 100\text{U}^6(\text{Tab. No. 15}), \\ 50\text{U}^4[\text{U}^2(\text{Tab. No. 14})+75(\text{Tab. No. 15})^2], \\ \cdot \\ \cdot \\ \cdot \\ 3125\ \text{Discriminant}(=\text{Tab. No. 26}) \end{array}\between\theta, 1\right)^{10},$$

where the Tables referred to are those in my Second Memoir on Quantics.

The form of the preceding results may be modified by writing $\theta=\vartheta\div\mathrm{U}^2$; we have thus the equations for

$$\vartheta=a^2(\alpha-\beta)^2(x-\gamma y)^2(x-\delta y)^2\ldots;$$

thus for example, in the case of the cubic $(a, b, c, d\between x, y)^3$, the equation for $\vartheta[=a^2(\alpha-\beta)^2(x-\gamma y)^2]$ is

$$0=(1,\ 18\mathrm{H},\ 81\mathrm{H}^2,\ 27\square\mathrm{U}^2\between\vartheta,\ 1)^3.$$

This equation may be written

$$(\vartheta+9\mathrm{H})^2\vartheta+27\square\mathrm{U}^2=0;$$

or putting $v=\sqrt{\vartheta}$, we have

$$v^3+9\mathrm{H}v+\mathrm{U}\sqrt{-27\square}=0,$$

an equation the roots of which are

$$a(\alpha-\beta)(x-\gamma y),\qquad a(\beta-\gamma)(x-\alpha y),\qquad a(\gamma-\alpha)(x-\beta y),$$

and which leads to the formula, given in my Fifth Memoir on Quantics, for the solution of a cubic equation. But this decomposition of the equation in ϑ is peculiar to the cubic.

The equation of differences for an equation of any order may be found by the following entirely distinct method. Let the proposed equation $(*\between v, 1)^n=0$, be for shortness represented by $\phi v=0$, and let x, y be any two distinct roots; we have not only $\phi x=0$, $\phi y=0$, but also $\phi x+\phi y=0$, $\frac{\phi x-\phi y}{x-y}=0$. Writing $\theta=(x-y)^2$, $s=x+y$, we have

$$x=\tfrac{1}{2}(s+\sqrt{\theta}),\qquad y=\tfrac{1}{2}(s-\sqrt{\theta}),$$

values which are to be substituted for x, y in the equations

$$\phi x+\phi y=0,\qquad \frac{\phi x-\phi y}{x-y}=0.$$

We have thus two equations rational in s and θ, and the elimination between them of the quantity s leads to the required equation in θ. But it is proper to modify the form of the system; in fact the two equations are, as regards s, the first of them of the degree n, the second of the degree $\overline{n-1}$; but if we write

$$n(\phi x+\phi y)-\frac{(x+y)(\phi x-\phi y)}{x-y}=0,\qquad \frac{\phi x-\phi y}{x-y}=0,$$

then each of the equations will be of the same degree $n-1$ in s.

For instance, let $\phi v=(a, b, c, d\between v, 1)^3$, then $x=\frac{1}{2}(s+\sqrt{\theta})$, $y=\frac{1}{2}(s+\sqrt{\theta})$; the equations $\phi x+\phi y=0$, $\frac{\phi x-\phi y}{x-y}=0$, are

$$s^3a+3s^2 2b+3s(4c+a\theta)+8d+6b\theta=0,$$
$$3s^2a+3s\,4b+\ 12c+a\theta\qquad\qquad=0;$$

and multiplying the first equation by 3 and the second by $-s$, adding and dividing by 2, we have an equation

$$s^2 3b+s(12c+4a\theta)+12d+9b\theta=0.$$

The second equation and this equation may be written

$$(3a,\ 12b\qquad 12c+a\theta\ \between s,\ 1)^2=0,$$
$$(3b,\ 12c+4a\theta,\ 12d+9b\theta\between s,\ 1)^2=0,$$

and the elimination of s from these equations gives the required equation in θ. The result may be obtained under either of the two forms,

$$\{a^2\theta^2+(15ac-27b^2)\theta-36(bd-c^2)\}\{a^2\theta+3\ (ac-b^2)\}+3\{2ab\theta+3(ad-bc)\}^2=0$$

and

$$\{4a^2\theta^2+(24ac-27b^2)\theta-36(bd-c^2)\}\{a^2\theta+12(ac-b^2)\}+3\{\ ab\theta+6(ad-bc)\}^2=0,$$

the expansions of which respectively coincide with the before-mentioned result.

In the case of the quartic equation $\phi v=(a,\ b,\ c,\ d\between v,\ 1)^4=0$, we have

$$s^4a+4s^32b+6s^2(\ 4c+a\theta)+4s(8d+6b\theta)+16e+24c\theta+a\theta^2=0,$$
$$4s^3a+6s^24b+4s\,(12c+a\theta)+\ 32d+8b\theta\qquad\qquad=0,$$

from which we derive another cubic equation; and the two cubic equations are

$$(4a,\ \ 24b\qquad,\ 48c+\ 4a\theta,\ \ 32d+\ 8b\theta\qquad\between s,\ 1)^3=0,$$
$$(4b,\ \ 24c+10a\theta,\ \ 48d+44b\theta,\ \ 32e+48c\theta+2a\theta^2\between s,\ 1)^3=0,$$

from which, if s be eliminated, we have the equation in θ.

Similarly, for the quintic equation $\phi v=(a,\ b,\ c,\ d,\ e,\ f\between v,\ 1)^5=0$, we have

$$s^5a+\ 5s^42b+10s^3(\ 4c+a\theta)+10s^2(\ 8d+6b\theta)+5s(16e+24c\theta+a\theta^2)+32f+80d\theta+10b\theta^2=0,$$
$$5s^4a+10s^34b+10s^2(12c+a\theta)+\ 5s\,(32d+8b\theta)+\ \ (80e+40c\theta+a\theta^2)\qquad=0,$$

from which we derive another quartic equation; the two equations are

$$(5a,\ \ 40b\qquad,\ 120c+\ 10a\theta,\ 160d+\ 40b\theta\qquad,\ 80e+\ 40c\theta+\ \ a\theta^2\between s,\ 1)^4=0,$$
$$(5b,\ \ 40c+20a\theta,\ 120d+130b\theta,\ 160e+280c\theta+12a\theta^2,\ 80f+200d\theta+25b\theta^2\between s,\ 1)^4=0,$$

from which, if s be eliminated, we have the equation in θ.

But to apply BEZOUT'S method to the two equations each of the order $n-1$, which result from the equation of the nth order $\phi v=(*\between v,\ 1)^n=0$. The process is as follows:—Suppose, in general, that s is to be eliminated from the two equations

$$\mathrm{F}s=0,\quad \mathrm{G}s=0,$$

each of the order $n-1$; it is only necessary to form the expression

$$\frac{\mathrm{F}s\mathrm{G}s'-\mathrm{F}s'\mathrm{G}s}{s-s'},$$

which will be a function of s, s' of the form

$$\left(\begin{array}{l|} a_{0,0},\ a_{0,1}\ \ldots\ a_{0,n-2} \\ a_{1,0}, \\ \vdots \\ a_{n-2,0} \end{array}\between s,\ 1)^{n-2}(s',\ 1)^{n-2}$$

where the coefficients are such that $a_{l,m}=a_{m,l}$; and by equating to zero the determinant formed with these coefficients, we have the result of the elimination.

In the present case, writing for a moment $\phi\frac{1}{2}(s+\sqrt{\theta})=\mathrm{A}$, $\phi\frac{1}{2}(s-\sqrt{\theta})=\mathrm{B}$, and in like manner $\phi\frac{1}{2}(s'+\sqrt{\theta})=\mathrm{A}'$, $\phi\frac{1}{2}(s'=\sqrt{\theta})=\mathrm{B}'$, we have

$$\mathrm{F}s=n(\mathrm{A}+\mathrm{B})-\frac{s(\mathrm{A}-\mathrm{B})}{\sqrt{\theta}},\quad \mathrm{G}s=\frac{\mathrm{A}-\mathrm{B}}{\sqrt{\theta}},$$

$$\mathrm{F}s'=n(\mathrm{A}'+\mathrm{B}')-\frac{s'(\mathrm{A}'-\mathrm{B}')}{\sqrt{\theta}},\ \mathrm{G}s'=\frac{\mathrm{A}'-\mathrm{B}'}{\sqrt{\theta}},$$

and therefore

$$\mathrm{F}s\mathrm{G}s'-\mathrm{F}s'\mathrm{G}s=n\frac{(\mathrm{A}+\mathrm{B})(\mathrm{A}'-\mathrm{B}')-(\mathrm{A}-\mathrm{B})(\mathrm{A}'+\mathrm{B}')}{\sqrt{\theta}}-\frac{(s-s')(\mathrm{A}-\mathrm{B})(\mathrm{A}'-\mathrm{B}')}{\theta};$$

or reducing and dividing by $s-s'$,

$$-\frac{\mathrm{F}s\mathrm{G}s'-\mathrm{F}s'\mathrm{G}s}{s-s'}=2n\frac{\mathrm{AB}'-\mathrm{A}'\mathrm{B}}{\sqrt{\theta}(s-s')}+\frac{(\mathrm{A}-\mathrm{B})\mathrm{A}'-\mathrm{B}'}{\theta}.$$

Hence, substituting for A, B, A′, B′ these values, we have the expression

$$2n\frac{\phi\frac{1}{2}(s+\sqrt{\theta})\phi\frac{1}{2}(s'-\sqrt{\theta})-\phi\frac{1}{2}(s-\sqrt{\theta})\phi\frac{1}{2}(s'+\sqrt{\theta})}{(s-s')\sqrt{\theta}}$$
$$+\frac{\{\phi\frac{1}{2}(s+\sqrt{\theta})-\phi\frac{1}{2}(s-\sqrt{\theta})\}\{\phi\frac{1}{2}(s'+\sqrt{\theta})-\phi\frac{1}{2}(s'-\sqrt{\theta})\}}{\theta},$$

which is of the form

$$\left(\begin{array}{cccc} a_{0,0}, & a_{0,1} & \ldots & a_{0,n-2} \\ a_{1,0} & & & \cdot \\ \cdot & & & \\ \cdot & & & \\ a_{n-2,0} & & \cdot & \cdot \end{array}\right\rangle\!\!\left(s,1\right)^{n-2}(s',1)^{n-2}$$

and equating to zero the determinant formed with the coefficients, we have an equation in θ which is the equation of differences of the given equation $\phi v=0$. For instance, if the given equation is $\phi v=(a,b,c,d\between v,1)^3=0$, then we have

$$8\phi\tfrac{1}{2}(s+\sqrt{\theta})=(a,2b+a\sqrt{\theta},4c+4b\sqrt{\theta}+a\theta,8d+12c\sqrt{\theta}+6b\theta+a\theta\sqrt{\theta}\between s,1)^3$$
$$=(\mathrm{A},\mathrm{B},\mathrm{C},\mathrm{D}\between s,1)^3,$$

$$8\phi\tfrac{1}{2}(s-\sqrt{\theta})=(a,2b-a\sqrt{\theta},4c-4b\sqrt{\theta}+a\theta,8d-12c\sqrt{\theta}+6b\theta-a\theta\sqrt{\theta}\between s,1)^3$$
$$=(\mathrm{A}',\mathrm{B}',\mathrm{C}',\mathrm{D}'\between s,1)^3;$$

and the function in s, s' is

$$\frac{6}{\sqrt{\theta}}\left\{\begin{array}{l} 3(\mathrm{AB}'-\mathrm{A}'\mathrm{B})s^2s'^2 \\ +3(\mathrm{AC}'-\mathrm{A}'\mathrm{C})ss'(s+s') \\ +\ (\mathrm{AD}'-\mathrm{A}'\mathrm{D})(s^2+ss'+s'^2) \\ +9(\mathrm{BC}'-\mathrm{B}'\mathrm{C})ss' \\ +3(\mathrm{BD}'-\mathrm{B}'\mathrm{D})(s+s') \\ +3(\mathrm{CD}'-\mathrm{C}'\mathrm{D}) \end{array}\right\}$$
$$+\frac{1}{\theta}(\mathrm{A}-\mathrm{A}',\mathrm{B}-\mathrm{B}',\mathrm{C}-\mathrm{C}',\mathrm{D}-\mathrm{D}'\between s,1)^3(\mathrm{A}-\mathrm{A}',\mathrm{B}-\mathrm{B}',\mathrm{C}-\mathrm{C}',\mathrm{D}-\mathrm{D}'\between s',1)^3,$$

which is equal to

$$12\left\{\begin{array}{l}-3a^2\,s^2s'^2\\ -12ab\;ss'(s+s')\\ +(-12ac-a^2\theta)(s^2+ss'+s'^2)\\ +(36ac-72b^2+9a^2\theta)ss'\\ +(24ad-72bc+12ab\theta)(s+s')\\ +96bd-144c^2+(-48ac+72b^2)\theta-3a^2\theta^2\end{array}\right\}$$

$$+4(3as^2+12bs+12c+a\theta)(3as'^2+12bs'+12c+a\theta);$$

or reducing and dividing by 32, this is

$$\begin{array}{l}\{9(ac-b^2)+3a^2\theta\}ss'\\ +\{9(ad-bc)+6ab\theta\}(s+s')\\ +36(bd-c^2)+(-15ac+27b^2)\theta-a^2\theta^2,\end{array}$$

the terms in s^2, s'^2 disappearing, as they should do. Writing this under the form

$$\left(\begin{array}{ll}9(ac-b^2)+3a^2\theta, & 9(ad-bc)+6ab\theta\\ 9(ad-bc)+6ab\theta, & 36(bd-c^2)+(-15ac+27b^2)\theta-a^2\theta^2\end{array}\right)\!\!\between\!(s,1)(s',1)$$

and equating the determinant to zero, we have the required equation in θ: the form is precisely that which is obtained by the ordinary process of applying BEZOUT's method to the two equations

$$(3a,\ 12b,\ \ 12c+\ a\theta\between s,1)^2=0,$$
$$(3b,\ 12c+4a\theta,\ 12d+8b\theta\between s,1)^2=0,$$

being in fact the before-mentioned equation

$$\big(a^2\theta^2+(15ac-27b^2)\theta-36(bd-c^2)\big)\big(a^2\theta+3(ac-b^2)\big)+3\big(2ab\theta+3(ad-bc)\big)^2=0.$$

But, as already remarked, this elimination process is less convenient for the complete development of the result, than the method first explained in the present memoir.