XVI. *Algebraical Researches, containing a disquisition on* Newton's *Rule for the Discovery of Imaginary Roots, and an allied Rule applicable to a particular class of Equations, together with a complete invariantive determination of the character of the Roots of the General Equation of the fifth Degree, &c.*

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Turns them to shapes and gives to airy nothing
A local habitation and a name.

(1) This memoir in its present form is of the nature of a trilogy; it is divided into three parts, of which each has its action complete within itself, but the same general cycle of ideas pervades all three, and weaves them into a sort of complex unity. In the first is established the validity of Newton's rule for finding an inferior limit to the number of imaginary roots of algebraical equations as far as the fifth degree inclusive. In the second is obtained a rule for assigning a like limit applicable to equations of the form $\Sigma(ax+b)^m=0$, m being any positive integer, and the coefficients a, b real. In the third are determined the absolute invariantive criteria for fixing unequivocally the character of the roots of an equation of the fifth degree, that is to say, for ascertaining the exact number of real and imaginary roots which it contains. This last part has been added since the original paper was presented to the Society. It has grown out of a foot-note appended to the second, itself an independent offshoot from the first part, but may be studied in a great measure independently of what precedes, and constitutes, in the author's opinion, by far the most valuable portion of the memoir, containing as it does a complete solution of one of the most interesting and fruitful algebraical questions which has ever yet engaged the attention of mathematicians [1]. I propose in a subsequent addition to the memoir to resume and extend some of the investigations which incidentally arise in this part. The foot-notes are numbered and lettered for facility of reference, and will be found in many instances of equal value with the matter in the text, to which they serve as a kind of free running accompaniment and commentary.

[1] I owe my thanks to my eminent friend Professor De Morgan for bringing under my notice, in a marked manner, the original question from which all the rest has proceeded. As all roads are said to lead to Rome, so I find, in my own case at least, that all algebraical inquiries sooner or later end at that Capitol of Modern Algebra over whose shining portal is inscribed "Theory of Invariants."

PART I.—ON NEWTON'S RULE FOR THE DISCOVERY OF IMAGINARY ROOTS.

(2) In the 'Arithmetica Universalis,' in the first chapter on equations, NEWTON has given a rule for discovering an inferior limit to the number of imaginary roots in an equation of any degree, without proof or indication of the method by which he arrived at it, or the evidence upon which it rests([2]). MACLAURIN, in vol. xxxiv. p. 104, and vol. xxxvi. p. 59 of the Philosophical Transactions, CAMPBELL([3]) in vol. xxxviii. p. 515 of the same, and other authors of reputation have sought in vain for a demonstration of this marvellous and mysterious rule([4]). Unwilling to rest my belief in it on mere empirical evidence, I

([2]) It appears to be the prevalent belief among mathematicians who have considered the question, that NEWTON was not in possession of other than empirical evidence in support of his rule.

([3]) CAMPBELL'S memoir is rather on an analogous rule to NEWTON'S than on the rule itself, to which he refers only by way of comparison with his own. In it the same singular error of reasoning is committed as in the notes of the French edition of the 'Arithmetica,' viz. of assuming, without a shadow of proof, that if each of a set of criteria indicates the existence of some imaginary roots, a succession of sets of such criteria must indicate the existence of at least as many distinct imaginary pairs of roots as there are such sets (see par. at foot of p. 528, Phil. Trans., vol. xxxv.)—much as if, supposing a number of dogs to be making a point in the same field, the existence could be assumed of as many birds as pointers.

([4]) Mr. ARCHIBALD SMITH has obligingly called my attention to WARING'S treatment of the question of NEWTON'S rule in the 'Meditationes Analyticæ.' On superficial examination the reader might be induced to suppose that in part 9, p. 68, ed. 1782, WARING had deduced a proof of the rule from the preceding propositions; but on looking into the case will find that there is not the slightest vestige of proof, the rule being stated, but without any demonstration whatever being either adduced or alleged. In fact, on turning to the preface of this (the last) edition of the 'Meditationes,' the reader will find at p. 11 an explicit avowal of the demonstration being wanting. After referring in order to CAMPBELL'S, MACLAURIN'S, and NEWTON'S rules, as well as his own, for discovering the existence of impossible roots, he adds these words:

"At omnes hæ regulæ prædictæ perraro invenerunt verum numerum impossibilium radicum in æquationibus multarum dimensionum *et adhuc demonstratione egent*; vulgares enim demonstrationes solummodo probant impossibiles radices in data æquatione contineri, non vero quod *saltem tot sunt quot invenit regula*."

"Vera resolutio problematis est perdifficilis et valde laboriosa; cognitum est radices ex possibilitate per æqualitatem transire ad impossibilitatem; ergo in generali resolutione hujusce problematis necesse est invenire casum in quo radices datæ æquationis evadunt æquales; resolutio autem hujus casus valde laboriosa est; et consequenter resolutio generalis prædicti problematis magis *erit* laboriosa."

Written in Latin, and when the proper language of algebra was yet unformed, it is frequently a work of much labour to follow WARING'S demonstrations and deductions, and to distinguish his assertions from his proofs. I find he agrees with the opinion expressed by myself, that NEWTON'S rule will *not* "pene," as stated by NEWTON, but only "perraro," give the true number of imaginary roots. Like myself, too, in the body of the memoir WARING has given theorems of probability in connexion with rules of this kind, but without any clue to his method of arriving at them. Their correctness may legitimately be doubted.

[Since the above was sent to press, I have been enabled to ascertain that the great name of EULER is to be added to the long list of those who have fallen into error in their treatment of this question: see Institutiones Calculi Differentialis, vol. ii. cap. xiii. He says (p. 555, edition of Prony), "videndum est utrum hæc duo criteria (meaning NEWTON'S criteria of imaginariness) sint contigua necne; priori casu numerus radicum imaginarium non augebitur; posteriori vero *quia criteria litteras prorsus diversas involvunt*, unumquodque binas radices imaginarias monstrabit."

The force of the supposed argument is contained in the words in italics. It is sufficiently met by the question, why or how the conclusion follows from them? Moreover the letters of two non-contiguous criteria are *not* necessarily *prorsus diversæ*; for two criteria with but a single other intervening between them will contain one letter in common.]

have investigated and obtained a demonstration of its truth as far as the fifth degree inclusive, which, although presenting only a small instalment of the desired result, I am induced to offer for insertion in the Transactions in the hope of exciting renewed attention to a subject so intimately bound up with the fundamental principles of algebra.

Before commencing the inquiry I ought to state that, in addition to the rule for detecting the existence of a certain number of imaginary roots, NEWTON has given a remarkable subsidiary method for dividing this number into two parts, representing respectively how many of the positive and how many of the negative roots indicated by DESCARTES'S rule are, so to say, absorbed, and thereby obtains two distinct limits to the number of positive and the number of negative roots separately: of the grounds of this method, as far as I am aware, no one has even attempted an explanation, nor do I propose here to enter upon it; the rule, as I treat it, may be stated, not in NEWTON'S own words, but most simply as follows:—

If the literal parts of the coefficients of an equation affected with the usual binomial coefficients be a, b, c, d, e . . . h, k, l, *and if we form the successive criteria* b^2-ac; c^2-bd; d^2-ce; . . . ; k^2-hl, *or, which is the same thing differently expressed, if we write down the determinants*(5) *of all the successive quadratic derivatives of the given equation, then as many sequences as there are of negative signs in the arithmetical values of these criteria, so many pairs of imaginary roots at least there will be in the given equation.* If we choose to consider a^2 and l^2 also as criteria, appearing at the beginning and end of the series, then we may vary the expression of the rule by saying that there will be at least as many imaginary roots as there are variations of sign in the complete series so formed.

It will, however, be found more convenient for our present purpose to confine the designation of criteria to the determinants above alluded to.

(3) I shall deal with the homogeneous equation $f(x, y)=0$ so that the question of the reality of the roots is that of the reality of the ratios $\frac{x}{y}$ or $\frac{y}{x}$. It is obvious, from known principles, that f cannot have fewer imaginary roots than exist in $\frac{d}{dx}f$ or $\frac{d}{dy}f$(6), or, more generally, than in $\left(\frac{d}{dx}+\lambda\frac{d}{dy}\right)f$; from which it immediately follows(7) that if f have all its roots real, and the quadratic derivatives of f be called Q_1, Q_2, ... Q_{n-1}, and the coeffi-

(5) To avoid the possibility of misapprehension, I state here once for all, that in the *discriminant* of a form of any degree I suppose the sign to be so taken as to render *positive* the term which is a power of the product of the first and last coefficients; and it may be well to remember that with this definition the number of real roots in any equation $\equiv 0$ or 1 to modulus 4 when the discriminant is positive, and $\equiv 2$ or 3 when the discriminant is negative; whereas the Determinant of a Quadratic form is to be taken in the same sense as that in which it is used by GAUSS, and is the same for such form as the Discriminant with the sign changed.

(6) This rule I find merges in the following more general and symmetrical one. Let f, ϕ be any two quantics in x, y; call the Jacobian of f, ϕ J; then the difference between the number of real roots in f and the like number in ϕ, taken positively and augmented by unity, cannot exceed the number of real roots in J. When ϕ is made equal to y, this theorem recurs to the familiar one alluded to in the text.

(7) By operating upon f successively with *any* $(n-2)$ distinct factors each of the form $\left(\frac{d}{dx}+\lambda_i\frac{d}{dy}\right)$.

cients of any function F of two degrees lower than f, whose roots are also *all* real, be $p_1, p_2, \ldots. p_{n-1}$, the quadratic function $p_1Q_1+p_2Q_2+\ldots+p_{n-1}Q_{n-1}$ must have its roots real, *i. e.* its discriminant must be positive: a particular consequence of this is, that by causing F to consist successively of the single terms x^{n-2}, $x^{n-3}y$, ... xy^{n-3}, y^{n-2}, we see that the determinants of $Q_1, Q_2, \ldots Q_{n-1}$ must each of them be positive; or, in other words, if any of the Newtonian criteria of an equation are negative, it must have *some* imaginary roots, which is all that MACLAURIN, CAMPBELL, and others have succeeded in proving.

(4) The labour of proof of the cases hereinafter considered will be much lightened by the following rule of induction, viz., granting NEWTON's rule to be true for the degree $n-1$, it must be true for all those cases appertaining to the degree n in which the series of the signs of the criteria does not commence with $-+$ and end with $+-$: to prove this, we have only to remember that f must have at least as many imaginary roots as $\frac{df}{dx}$ or $\frac{df}{dy}$, and that the criterion-series corresponding to $\frac{df}{dx}$ and to $\frac{df}{dy}$ will be found by cutting off from the series of f one term to the right and left respectively(8). If, now, the series for f begins with $++$ or $--$ or $+-$, the number of negative *sequences* is the same as when the left-hand sign is removed; so that it is only necessary to prove that the number of imaginary roots in f is not less than the number of negative sequences in $\frac{df}{dx}$; but this, by hypothesis, is not greater than the number of pairs of imaginary roots in $\frac{df}{dx}$, and, *à fortiori*, not greater than the number of such in f. In like manner, if the two *last* criteria of f are not $+-$, it may be shown that the truth of the rule for such form of f is implied in what is supposed to be known to be true for $\frac{df}{dy}$.

We may therefore limit our attention, as we ascend in the scale of proof, to those forms of f in which the criterion-series begins with $-+$ and ends with $+-$. Accordingly, since the rule is a truism for $n=2$, it is at once proved, by virtue of the above considerations, for $n=3$(9).

(8) For
$$\frac{d}{dx}(a, b, \ldots k, l \between x, y)^n = n(a, b, \ldots k \between x, y)^{n-1},$$
and
$$\frac{d}{dy}(a, b, \ldots k, l \between x, y)^n = n(\ b, \ldots k, l \between x, y)^{n-1}.$$

(9) The theorem for the case of cubic equations may be also proved directly as follows:

Writing the equation $ax^3+3bx^2y+3cxy^2+dy^3=0$, the two criteria are $L=b^2-ac$, $M=c^2-bd$; and the discriminant is $a^2d^2+4ac^3+4db^3-3b^2c^2-6abcd=\Delta$.

1. Let L and M be of opposite signs, so that one and only one of them is negative. Then
$$\Delta=(ad-bc)^2-4(b^2-ac)(c^2-bd)=(ad-bc)^2-4LM,$$
and is therefore positive.

2. Let L and M be both negative. The equation may evidently, by writing x and y for $a^{\frac{1}{3}}x$, $d^{\frac{1}{3}}y$, be brought under the form
$$x^3+3\varepsilon x^2y+3\eta xy^2+y^3=0,$$
with the conditions $\varepsilon^2<\eta$, $\eta^2<\varepsilon$; from which we may deduce that ε and η are both positive, and $\varepsilon\eta<1$ and >0.

If all the criteria are zero, it is evident that, whatever n may be, all the roots are real. In every other case we shall find that *zero* may be made positive or negative at will. Thus in the case before us, if the two criteria are $0+$ or $0-$, there will be a pair of imaginary roots, as the first may be read as $-+$ and the second as $+-$.

To prove this, we have only to observe that in either case $\frac{df}{dx}$ will have two equal roots; so that f will be of the form $(ax+by)^3+cy^3$, which obviously, for any real values of a, b, c, has only one real root.

(5) We may now pass to the case of $n=4$, and excluding for the moment the consideration of *zeros*, limit our attention to the criterion series $-+-$.

Let $ax^4+4bx^3y+6cx^2y^2+4dxy^3+ey^4=0$ be the equation for which the signs of the criteria b^2-ac, c^2-bd, d^2-ce are $-+-$. Call these criteria L, M, N respectively. It has to be proved that all four roots are imaginary, since there are two distinct negative sequences, each sequence consisting of a single $-$. Let x become $x+\varepsilon y$ (10), where ε is an infinitesimal quantity, and transformed into one between u and y; then we have obviously,

$$\delta a=0,\quad \delta b=a\varepsilon,\quad \delta c=2b\varepsilon,\quad \delta d=3c\varepsilon,\quad \delta e=4d\varepsilon,$$

$$\delta \mathrm{L}=2b\delta b-a\delta c=0,\quad \delta \mathrm{M}=2c\delta c-b\delta d-d\delta b=(bc-ad)\varepsilon,$$

$$\delta^2\mathrm{M}=(b\delta c+c\delta b-a\delta d)\varepsilon=2(b^2-ac)\varepsilon^2=2\mathrm{L}\varepsilon^2;$$

so that $\delta^2\mathrm{M}$ is essentially negative, since L is so.

Hence, by continually augmenting x by an infinitesimal variation, we may, leaving L unaltered, so choose the sign of ε as to decrease M: nor can this process stop when $bc-ad$ becomes zero, by reason that $\delta^2\mathrm{M}$ is *negative*. Hence we may reduce M to zero. Now,

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Also we have

$$\begin{aligned}\Delta&=1+4(\varepsilon^3+\eta^3)-6\varepsilon\eta-3\varepsilon^2\eta^2\\&>1+4(\varepsilon+\eta)\varepsilon\eta-6\varepsilon\eta-3\varepsilon^2\eta^2\\&>1-6\varepsilon\eta+8(\varepsilon\eta)^{\frac{3}{2}}-3\varepsilon^2\eta^2;\end{aligned}$$

or, writing $\varepsilon\eta=q^2$,

$$\begin{aligned}\Delta&>1-6q^2+8q^3-3q^4,\\&>(1-q)^3(1+3q);\end{aligned}$$

but $1>q>0$. Hence Δ is positive.

Hence in either case two of the roots of the cubic are impossible. Or the same thing may be shown more immediately from the identities

$$a^2\Delta=(a^2d+2b^3-3abc)^2+4(ac-b^2)^3,$$
$$d^2\Delta=(ad^2+2c^3-3bcd)^2+4(bd-c^2)^3,$$

so that Δ must be positive, and therefore two roots imaginary, if either $bd>c^2$ or $ca>b^2$. It may be noticed that the square and cube in these identities are semi-invariants, being in the first of them unaffected by the change of x into $x+hy$, and in the second by the change of y into $y+hx$.

(10) This method of infinitesimal substitution is that which I applied in my memoir "On the Theory of Forms," in the Cambridge and Dublin Mathematical Journal, to obtain the partial differential equations to every possible species of invariants (including covariants and contravariants) of forms, or systems of forms, with a single set or various sets of variables, proceeding upon the pregnant principle that every finite linear substitution may be regarded as the result of an indefinite number of *simple* and *separate* infinitesimal variations impressed upon the variables. M. Aronhold has erroneously ascribed to others the priority of the publication of these equations.

in the course of this reduction, either N retains its sign or changes it; and if the latter is the case, N must have passed through zero. If when M becomes zero N is still negative, the criteria of the linearly transformed equation become $-0-$; and it may be noticed that its first, middle, and last coefficients must have the same sign, by virtue of the negativity of the two last criteria, and the second and fourth the same signs, by virtue of the zero middle criterion; consequently the equation will take the form

$$(\lambda^2+e^4)x^4\pm4e^3\varepsilon x^3y+be^2\varepsilon^2x^2y^2\pm4e\varepsilon^3xy^3+(\mu^2+\varepsilon^4)y^4=0,$$

or

$$\lambda^2x^4+\mu^2y^4+(ex\pm\varepsilon y)^4=0,$$

which obviously has all its roots impossible. This being true of the transformed equation, will also, on the suppositions made, be equally so of the original equation.

Let us next suppose that N changes its sign either at the instant when, or before M becomes zero. If M and N both become zero together, so that the criteria of the transformed equation bear the signs $-$ 0 0, calling the transformed equation $F=0$, $\frac{dF}{dy}$ will have all its roots equal, and F will therefore be of the form $(ax+by)^4+kx^4$, with the condition $(a^3b)^2-(a^4+k)(a^2b^2)<0$.

Hence k is positive, and consequently $F=0$ has all its roots imaginary; and the same, as before, must hold good of the original equation $f=0$.

It remains then only to consider the case when N becomes zero before M vanishes. When this is the case, as soon as N is reduced to zero, in lieu of the substitution of $x+\varepsilon y$ for x, we must leave x unaltered, and continue substituting $y+\varepsilon x$ for y. We thus start from the sequence $-+0$; N will then always remain zero, and we must either come to the series $-$ 0 0, which we know, from what has been shown above, corresponds to four imaginary roots, or to the sequence $0+0$, which I shall proceed to consider.

Since the first and last coefficients must have the same sign, we may, by giving either variable a proper multiple([11]), make these two coefficients alike, and with the first,

([11]) ([a]) The form $(1, e, e^2, e, 1)(x, y)^4$ may be regarded as a new and, for many purposes, useful canonical form of a binary quartic. It may be made to comprise within its sphere of representation all forms corresponding to two or four imaginary factors, but excludes the case of four real factors. The ordinary canonical form $(1, 0, 6m, 0, 1)(x, y)^4$ comprises within its spheres of representation those forms for which the factors are all real or all imaginary, but, so far as real transformations are concerned, excludes the case of two real and two imaginary factors [that case is met by the form $1, 0, 6m, 0, -1)(x, y)^4$], as may easily be established either by decomposing the form first named into its factors, or by the consideration that its discriminant Δ is $(1-9m^2)^2$, and is therefore always positive; whereas if a form which it is used to represent have two real and two unreal factors, its discriminant is negative. If now the determinant of transformation be D, and the discriminant corresponding thereto be called Δ', we have $\Delta'=D^6\Delta$, showing that D^2 is negative, and the transformation therefore unreal.

([b]) The *reality* of m for each of these cases (usually assumed without proof) may be demonstrated as follows: Calling the cubic invariant and the discriminant of any cubic form T, D, we shall have, using the *ordinary* canonical form, $\frac{(m-m^3)^2}{(1-9m^2)^2}=\frac{T^2}{D}$, showing that when D is positive, which is the case of four real or unreal factors, there will

second, and third, as well as the third, fourth, and fifth coefficients form geometrical series; hence it is obvious that the transformed equation may be reduced to one or the other of the two following forms, viz.

$$x^4+4ex^3y+6e^2x^2y^2-4exy^3+y^4=0, \quad \ldots \quad (a)$$

or

$$x^4+4ex^3y+6e^2x^2y^2+4exy^3+y^4=0, \quad \ldots \quad (b)$$

with the condition in the latter case that e^4-e^2 is positive, *i. e.* $e^2>1$.

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be one real value of m, and when D is negative, a real value of im. The former case possesses over the latter a striking distinction, which is that *all* the roots of m will be real; for, as I have shown elsewhere, if m is one root the complete system of roots will be $\pm m$, $\pm\frac{1-2m}{1+3m}$, $\pm\frac{1+2m}{1-3m}$: in the latter case the reality of the two values $\pm im$ does not seem necessarily to imply the reality of the other 4 values of the system.

([c]) Analogy suggests the establishment of an analogous canonical form or forms for ternary cubics, of which, as is well known and is even dimly foreshadowed in NEWTON'S Enumeration of Lines of the Third Order, the theory runs closely parallel to that of binary quartics. This will be effected by assuming the form

$$F(x, y, z)=\Sigma x^3+3e\Sigma x^2y+6gxyz,$$

and assuming g so as to make the discriminants of

$$\frac{dF}{dx}, \frac{dF}{dy}, \frac{dF}{dz}$$

all zero. This gives rise to a quadratic equation in g, of which the roots are $g=e$, $g=2e^2-e$. When $g=e$, I find

$$S=e(1-e)^3, \quad T=(1-e)^4(1+4e-8e^2), \quad \Delta=T^2+64\,S^3=(1+8e)(1-e)^8.$$

When $g=2e^2-e$, I find $\Delta=(1-e)^i(1-4e)^j(1+2e)^k$, where i, j, k are integers to be determined. These forms will, I think, be found important in the future perspective discussion of curves of the third degree. Whilst I yield to no one in admiration of the surpassing genius with which NEWTON has handled these curves, I cannot withhold the expression of my opinion that every theory of forms in which invariants are ignored must labour under an inherent imperfection, and that NEWTON, from want of acquaintance with the indelible characters which their invariants stamp upon curves, has in the parallel which he has drawn between the generation by shadows of all conics from a common type, and of all cubic curves from a limited number of forms, either himself fallen into error of conception, or at least used language which could scarcely fail to lead others into such error. For no species whatever of cubic curve can be formed for which an infinite number of individuals cannot be found which defy linear or perspective transformation into each other; whereas all conics proper may be propagated as shadows from a single individual. It should be noticed in connexion with this subject, that the *indelible* characters of quartic binary, and cubic ternary forms are two in number, viz. the value of $\frac{s^3}{t^2}$ (where s, t are the two fundamental invariants in either case) and the *sign* of t. The indelibility of the sign of s being implied in the invariability of the value of $\frac{s^3}{t^2}$, does not constitute a distinct character. Of course all symmetrical invariants have an invariable sign; but this is not the case with skew invariants, as *ex. gr.* M. HERMITE'S octodecimal invariant of a binary quintic, which will change its sign with that of the determinant of transformation.

([d]) Whilst upon this subject of invariants, I may allow myself to make a remark bearing upon what will be noticed further on in the text about a case of equality between roots not necessarily being a mark of transition from real to imaginary roots. If a, b, c, d being the roots of a binary quartic we form a secondary cubic, of which the roots are $(a-b)(c-d)$, $(a-c)(d-b)$, $(a-d)(b-c)$, it may be easily shown that two of these quantities become equal, or, in other words, the roots of the original equation mark out a harmonic group of points when t (the cubinvariant) is zero. Notwithstanding which a change of sign in t will not command a change of character in the above three roots of the secondary (nor consequently of the original equation), because it is not an odd but an even power of t, viz. t^2, which enters into the discriminant of the secondary.

It must be remembered that we know, from the form of the criteria-series to the derivatives in respect to either x or y (indifferently), that the equation must have *some* imaginary roots; and the question therefore lies between its having two or four. If the discriminant is negative, the former will be the case, if positive, the latter. I shall show that in each equation the discriminant is positive.

Let s, t represent in general the quartic invariants, then we have to show that s^3-27t^2 is positive.

In case (a), $s=1+4e^2+3e^4 = (1+e^2)(1+3e^2)$

$$t=\begin{vmatrix} 1 & e & e^2 \\ e & e^2-e \\ e^2-e & 1 \end{vmatrix} \begin{aligned} &=e^2-e^4-e^4-e^2-e^6-e^2 \\ &=-e^2-2e^4-e^6 \\ &=-e^2(1+e^2)^2, \end{aligned}$$

so that

$$s^3-27t^2=(1-e^2)^3\{(1+3e^2)^3-27e^4(1+e^2)\}=(1+e^2)^3(1+9e^2),$$

and is positive.

In case (b),

$$s=(1-4e^2+3e^4)=(1-e^2)(1-3e^2)$$

$$t=\begin{vmatrix} 1 & e & e^2 \\ e & e^2 & e \\ e^2 & e & 1 \end{vmatrix} \begin{aligned} &=e^2+e^4+e^4-e^2-e^6-e^2 \\ &=-e^2+2e^4-e^6=-e^2(1-e^2)^2, \end{aligned}$$

and

$$\begin{aligned} s^3-27t^2&=(1-e^2)^3\big((1-3e^2)^3-27e^4(1-e^2)\big) \\ &=(1-e^2)^3(1-9e^2). \end{aligned}$$

The above can only be negative when e^2 lies between 1 and $\frac{1}{9}$; but in the case supposed $e>1$. Hence the discriminant is positive, and the roots are all imaginary([12]). Thus, then, the theorem is established for $n=4$, as well as for the cases where the criteria are zero (as will have been observed in the course of the demonstration), as for those where they are *plus* or *minus*; and it should be observed that the demonstration proceeds upon our being able to show that the quartic, in the case where it resists reduction to the case of the cubic, viz. where the criteria are negative at the two extremes and positive in the middle, may by real linear transformations be changed into a form where either the middle criterion is zero and the two extremes negative, or the two extremes zero, and the middle one positive.

([12]) The reader conversant only with ordinary algebra may easily verify this result. For writing $\frac{x}{y}+\frac{y}{x}=z$, the equation becomes $z^2+4ez+6e^2-2=0$, and this will have its roots impossible unless $4e^2>6e^2-2$, or $2e^2-2$ negative, which it cannot be, since $e^2>1$, and consequently $x:y$ has all its roots impossible. Moreover the same conclusion would (as before shown) hold good unless e^2 lay between 1 and $\frac{1}{9}$; for on making $z=2$, the function above written in z becomes $2+8e+6e^2$, or $2(1+e)(1+3e)$; and making $z=-2$, it becomes $2-8e+6e^2$, or $2(1-e)(1-3e)$, which two quantities evidently have both positive signs unless e lies between 1 and $\frac{1}{3}$, or between -1 and $-\frac{1}{3}$; so that the first and third Sturmian functions are (except on that supposition) respectively positive and negative for $z=2$, and also for $z=-2$, showing that no root of z can lie between 2 and -2, and consequently that all the roots of $x:y$ remain impossible.

Observation.—To make the foregoing demonstration quite exact, it should be noticed that when the criteria L, M, N have been brought to the form $-+0$, and the series of substitutions of $y+\varepsilon x$ for y has set in, we have

$$N=0, \quad \delta N=0, \quad \delta M=(cd-be)\varepsilon, \quad \delta^2 M=N\varepsilon=0, \quad \delta^3 M=0.$$

Consequently if $cd-be$ should become *zero*, we can no longer go on decreasing M. But as soon as $cd-be=0$, since we have also $d^2=ce$, b, c, d, e come to be in geometrical progression, and the transformed equation takes the form

$$ax^4+4\omega x^3y+6\omega^2x^2y^2+4\omega^3xy^3+\omega^4x^4=0,$$

with the condition $\omega^2-a\omega^2$ negative, or $a>1$. Hence we have $q^2x^4+(x+\omega y)^4=0$, which obviously has all its roots impossible (13).

(6) We may now pass on to equations of the fifth degree, in which the case resisting induction will be that where the criterion-series bears the signs

$$- + + -.$$

Let the criteria be called L, M, N, P, so that writing the equation

$$ax^5+5bx^4y+10cx^3y^2+10dx^2y^3+5exy^4+fy^5=0,$$

$$L=b^2-ac, \quad M=c^2-bd, \quad N=d^2-ce, \quad P=e^2-df,$$

and writing for x, $x+\varepsilon y$, we have, as before,

$$\delta L=0, \quad \delta M=(bc-ad)\varepsilon, \quad \delta^2 M=L\varepsilon^2,$$

so that M may be continually diminished.

If M becomes zero before either N or P changes its sign, the criterion-series for the transformed equation becomes $-\ 0 + -$, and for its derivative in respect to x, the series is $0 + -$, which proves the existence of four imaginary roots in the transformed, and consequently also in the given equation. In like manner, if N becomes zero before M or P have changed their signs, the criterion-series becomes $- + 0 -$, which obviously leads to the same result. So likewise the same inference may be drawn if L and M, or M and N, or L, M, N become zeros all at the same time, and we have only to consider the case when, L and M retaining their signs, N becomes zero. At this moment the order of the substitutions must be reversed, and for y must be written $y+\varepsilon x$; we shall then have

$$P=0, \quad \delta P=0, \quad \delta N=(de-cf)\varepsilon \ldots\ldots;$$

(13) From the first and third criteria it follows that in the form $(a, b, c, d, e \between x, y)^4$, a, c, e have the same sign and may be regarded as all positive; so that writing $a-\frac{b^2}{c}=h^2$, $e-\frac{d^2}{c}=k^2$, the form becomes $h^2x^2+F+k^2y^2$, where

$$F=\frac{b^2}{c}x^4+4bx^3y+bcx^2y^2+4dxy^3+\frac{d^2}{c}y^4,$$

and consequently the given form will have all its roots imaginary when this is true for F, so that we might have proceeded at once to deal with the forms marked (*a*), (*b*) at p. 585; but as the method of homographic transformation by infinitesimal substitutions appears to be necessary in passing to the corresponding forms in the case of the fifth degree, and as in treating that case reference is made to what appears above, I have thought that no object would be gained by altering the text.

and reasoning as in the preceding case for $n=4$ (with the sole difference, that if δN vanishes by virtue of $de-cf$ vanishing, we should have P$=0$, N$=0$, and the criterion-series $-\ +\ 0\ 0$, which at once indicates the existence of four imaginary roots), we see that there remains only to consider the case where the criterion-series takes the form $0\ +\ +\ 0$. It is scarcely necessary to observe that all the criteria can never vanish simultaneously; for that would indicate the equality of all the roots in the transformed, and therefore in the given equation, whose own criteria, contrary to hypothesis, would also be all zero. The zero values of the two extreme criteria indicates that the three first and the three last literal parts of the coefficients are in geometrical progression, from which it will immediately be seen that the equation to be considered may be thrown (by substituting in lieu of x and y suitable multiples of x and y, which will not affect the characters of the criteria) into the convenient form

$$x^5+5\varepsilon x^4y+10\varepsilon^2x^3y^2+10\eta^2x^2y^3+5\eta xy^4+y^5=0,$$

with the two conditions $\varepsilon^4-\varepsilon\eta^2$ *positive*, $\eta^4-\eta\varepsilon^2$ *positive.*

The form of the criterion-series, apocopated from either end, shows that two of the roots must be imaginary; and consequently, in order to establish the existence of two imaginary pairs of roots, it is only necessary to show that the discriminant of the above equation, subject to the above conditions, must remain always positive. That discriminant I proceed to determine; but as a guide to the form under which it is to be expressed, the following observation is important. Let us take the more general form

$$ax^5+bx^4y+cx^3y^2+dx^2y^3+exy^4+fy^5=0,$$

where

$$a=1,\quad b=\lambda\varepsilon,\quad c=\mu\varepsilon^2,\quad d=\mu\eta^2,\quad e=\lambda\eta,\quad f=1,$$

λ, μ being any numerical quantities.

The discriminant will evidently be a symmetrical function of e and ε.

Let $a^pb^qc^rd^se^t$ be the literal part of any term in the discriminant. By the *law of weight* we must have

$$q+2r+3s+4t=5\times4=20.$$

But in the equation before us, $a^pb^qc^rd^se^t$ (to a numerical factor *près*) is $\varepsilon^{q+2r}\eta^{2s+t}$, and

$$(q+2r)-(2s+t)=(q+2r+3s+4t)-5(s+t)$$
$$=5(4-s+t).$$

Hence the difference between the indices of ε and η in each term is a multiple of 5, and consequently, since the discriminant is a symmetrical function in ε and η, it will be a rational integral function of $\varepsilon^5+\eta^5$ and $\varepsilon\eta$. Moreover, as no such term as c^4d^4 can figure in the discriminant, which, as we know, must in all cases contain one or the other of the two final and of the two initial coefficients, we see that no term can be of higher than the 14th degree in ε, η, nor yet so high, for the only terms that could be of that degree would be bc^3d^3e; but making a and f each zero in the original form, it becomes obvious

that all the terms free from a and f contain b^2e^2 as a factor([14]). Hence, in fact, the discriminant will be only of the twelfth degree in ε, η, and being therefore of only the second degree in $\varepsilon^5+\eta^5$, will admit of comparatively easy treatment.

(7) Before proceeding to the calculation of this discriminant, it will be useful to investigate, as a Lemma ancillary to the subsequent discussion, under what conditions four of the roots of the supposed equation will become imaginary when $\varepsilon=\eta$.

In this case writing $\frac{x}{y}+\frac{y}{x}=z$, the equation

$$\frac{1}{x+1}(1,\ \varepsilon,\ \varepsilon^2,\ \varepsilon^2,\ \varepsilon,\ 1\between x,\ y)^5=0$$

becomes

$$z^2-2-z+1+5\varepsilon(z-1)+10\varepsilon^2=z^2+(5\varepsilon-1)z+10\varepsilon^2-5\varepsilon-1=0,$$

or say $fz=0$.

The determinant of $f(z)$ is thus $(5\varepsilon-1)^2-40\varepsilon^2+20\varepsilon+y$, i. e. $5(1-\varepsilon)(1+3\varepsilon)$; and all the roots of z, and consequently of (x, y), will be impossible, unless z lies between 1 and $-\frac{1}{3}$.

Now
$$f(2)=1+5\varepsilon+10\varepsilon^2,$$
$$f'(2)=3+5\varepsilon;$$

so that when z has any real roots, *i. e.* when ε lies between 1 and $-\frac{1}{3}$, $f(2)$, $f'(2)$ are both positive, and the Sturmian functions are of the signs $+++$.

Again,
$$f(-2)=5-15\varepsilon+10\varepsilon^2=5(1-\varepsilon)(1-2\varepsilon),$$
$$f'(-2)=-5+5\varepsilon;$$

so that, on the same supposition as before, the Sturmian functions are $\pm-+$, viz.

$$+-+ \text{ when } \tfrac{1}{2}>\varepsilon>-\tfrac{1}{3},$$
$$--+ \text{ when } 1>\varepsilon>\tfrac{1}{2}.$$

In the former case two real roots, in the latter one real root of z lies between 2, -2. Hence in the former case no real roots of z lie between the limits ∞, 2, and the limits -2, $-\infty$, and in the latter case one real root lies between those limits. Hence x, y will have four imaginary roots, unless ε lies between 1 and $\frac{1}{2}$, and two such roots in every other case.

Thus the discriminant of $(1,\ \varepsilon,\ \varepsilon^2,\ \eta^2,\ \eta,\ 1\between x,\ y)^5$, when $\varepsilon=\eta$, is negative when ε lies between 1 and $\frac{1}{2}$, but for every other value of ε is positive, save that it vanishes when

$$\varepsilon=1, \text{ or } \varepsilon=\tfrac{1}{2}\,(^{15}), \text{ or } \varepsilon=-\tfrac{1}{3}.$$

(8) I now proceed to calculate the discriminant of the form

$$x^5+5\varepsilon x^4y+10\varepsilon^2x^3y^2+10\eta^2x^2y^3+5\eta xy^4+y^5$$

([14]) For the discriminant of $xy\phi(x, y)$ = the discriminant of $\phi(x, y)$ multiplied by the square of the product of the resultant of (x, ϕ) and of (y, ϕ).

([15]) When $\varepsilon=\frac{1}{2}$ the discriminant of $f(z)$ does not vanish, but $z=-2$ satisfies the equation in z, and consequently $\frac{x}{y}$ has two equal roots -1, so that the discriminant of the original equation vanishes.

for general values of ε, η. This will be accomplished most expeditiously by taking the resultant of the two derivatives of the above form, say U and V, where

$$U=x^4+4\varepsilon x^3y+6\varepsilon^2x^2y^2+4\eta^2xy^3+\eta y^4,$$
$$V=\varepsilon x^4+4\varepsilon^2x^3y+6\eta^2x^2y^2+4\eta xy^3+y^4;$$

so that

$$\varepsilon U-V\ \ =6(\varepsilon^3-\eta^2)x^2y^2+4(\varepsilon\eta^2-\eta)xy^3+(\varepsilon\eta-1)y^4=y^2P,$$
$$-U+\eta V=(\varepsilon\eta-1)x^4+4(\eta\varepsilon^2-\varepsilon)x^3y+6(\eta^3-\varepsilon^2)x^2y^2=x^2Q.$$

Hence

$$\textit{Resultant of } (U, V)=\frac{1}{(\varepsilon\eta-1)^4}\times \textit{Resultant of } (y^2P, x^2Q)=\textit{Resultant of } (P, Q);$$

where

$$P=6(\varepsilon^3-\eta^2)x^2+4(\varepsilon\eta^2-\eta)xy+(\varepsilon\eta-1)y^2,$$
$$Q=(\varepsilon\eta-1)x^2+4(\eta\varepsilon^2-\varepsilon)xy+6(\eta^3-\varepsilon^2)y^2.$$

Hence, calling Δ the discriminant of the original form, we obtain by the well-known formula for the resultant of two binary quadratics, writing for the moment

$$P=(B,\ 4\eta A,\ A\between x, y)^2,\quad Q=(A,\ 4\varepsilon A,\ B'\between x, y)^2,$$

$$\Delta=(4\varepsilon A^2-4\eta AB')(4\eta A^2-4\varepsilon AB)+(A^2-BB')^2$$
$$=(1-16\varepsilon\eta)A^4+16(\varepsilon^2B+\eta^2B')A^3-16\varepsilon\eta BB'A^2-2BB'A^2+B^2B'^2.$$

Hence writing $\varepsilon\eta=q$, $\varepsilon^5+\eta^5=S$,

$$\Delta=(1-16q)(q-1)^4+96(S-2q^2)(q-1)^3-72(8q+1)(q^3+q^2-S)(q-1)^2$$
$$+36^2(q^3+q^2-S)^2.$$

Let $S-q^2-q^3=\sigma$, $q-1=p$, so that

$$S-2q^2=\sigma-q^2+q^3=\sigma+(p+1)^2p.$$

Then

$$\Delta=36^2\sigma^2+72(8p+9)p^2\sigma+96p^3\sigma+96(p+1)^2p^4-(16p+15)p$$
$$=1296\sigma^2+(648p^2+672p^3)\sigma+96p^6+176p^5+81p^4,$$
$$=\tfrac{1}{9}\{108\sigma+27p^2+28p^3)^2+729p^4+1584p^5+864p^6-(27p^2+28p^3)^2\},$$

or

$$9\Delta=(108\sigma+27p^2+28p^3)^2+72p^5+80p^6.$$

(9) Hence we see at once that Δ can be negative only when p lies between 0 and $-\frac{9}{10}$, *i. e.* when $\varepsilon\eta$ (which is $p+1$) lies between 1 and $\frac{1}{10}$. Accordingly when Δ is negative, ε and η must be both positive or both negative. The latter supposition may easily be disproved as follows: treating the equation $\Delta=0$ as a quadratic equation in σ, in order that Δ may be capable of becoming negative, its discriminant in respect to σ must be negative, and its value when $\sigma=-\infty$ is positive. Now

$$S=\varepsilon^5+\eta^5,\quad p+1=\varepsilon\eta,\quad \sigma=S-(p+1)^2-(p+1)^3;$$

so that when ε and η are real we have

$$S>2(p+1)^{\frac{5}{2}}\,(^{16}),\ \text{i. e. } \sigma>-(p+1)^2+2(p+1)^{\frac{5}{2}}-(p+1)^3$$

[16] It is of course understood that $(p+1)^{\frac{5}{2}}$ is to be taken *positive*.

when ε, η are both positive, and

$$S<-2(p+1)^{\frac{5}{2}}\text{(16 bis)}, \text{ i. e. } \sigma<(p+1)^2-(p+1)^3-2(p+1)^{\frac{5}{2}}$$

when ε, η are both negative.

If now we substitute $(p+1)^2+(p+1)^3-2(p+1)^{\frac{5}{2}}$ for σ in Δ, I say that the resulting value will be positive whatever positive value be given to $(p+1)$; in fact, if we write $p=\nu^2-1$, and make $\sigma=-\nu^4+2\nu^5-\nu^6$, so that Δ becomes a function of the twelfth degree in ν, this function is what the discriminant of the equation in x, y becomes when we have $\varepsilon=\eta=\nu$; but in the antecedent Lemma it has been shown that this discriminant is only negative when the two equal quantities ε or η, or, which is the same thing, when ν lies between 1 and $\frac{1}{2}$; hence Δ is positive when ν is negative, and consequently when

$$\sigma=(p+1)^2+(p+1)^3-2(p+1)^{\frac{5}{2}}.$$

Thus Δ, a quadratic function in σ, and its discriminant are respectively $+$ and $-$ for this value of σ, as well as for $\sigma=-\infty$. Hence no real root of σ lies between such value of σ and $-\infty$, and consequently Δ must be always positive when ε and η are both negative. Hence, if Δ is negative, we must have $1>\varepsilon\eta>\frac{1}{10}$; $\varepsilon>0$; $\eta>0$. But our *criteria* give

$$\varepsilon^4-\varepsilon\eta^2>0, \quad \eta^4-\eta\varepsilon^2>0,$$

which, when $\varepsilon>0$, $\eta>0$, imply $\varepsilon^3>\eta^2$, $\eta^3>\varepsilon^2$, and consequently $\varepsilon\eta>1$, which is in contradiction to the inequality $1>\varepsilon\eta$. Hence when these criteria are satisfied the determinant is necessarily *positive*, and all the roots are imaginary, which completes the proof of NEWTON's rule for equations of the fifth degree.

(10) It follows as a corollary to the Lemma employed in the preceding investigation, that if in Δ we write $\sigma=-(\nu^2-\nu^3)^2$ and $p=\nu^2-1$, and distinguish this particular value by the symbol (Δ), then (Δ) ought to break up into the product of odd powers of $\nu-1$, $\nu-\frac{1}{2}$ of some even power of $(\nu+\frac{1}{3})$, and of a factor incapable of changing its sign, and remaining always positive. This may be easily verified; for dividing (Δ) by $(\nu-1)^4$, we obtain

$$1296\nu^8\left(648(\nu+1)^2+24(\nu^2-1)(\nu+1)^2\right)\nu^4+96(\nu^2-1)^2(\nu+1)^4+176(\nu^2-1)(\nu+1)^4+81(\nu+1)^4;$$

and collecting the terms $1296\nu^8-648\nu^6(\nu+1)^2+81(\nu+1)^4$ whose sum contains the factor $(\nu-1)$, we have

$$\begin{aligned}\frac{(\Delta)}{(\nu-1)^5}= {} & 648(\nu^7+\nu^6+\nu^5+\nu^4+\nu^3+\nu^2+\nu+1)\\ & -1296(\nu^6+\nu^5+\nu^4+\nu^3+\nu^2+\nu+1)\\ & -648(\nu^5+\nu^4+\nu^3+\nu^2+\nu+1)\\ & +81(\nu^3+5\nu^2+11\nu+15)\\ & -24(\nu^7+3\nu^6+3\nu^5+\nu^4)\\ & +96(\nu^7+5\nu^6+9\nu^5+5\nu^4-5\nu^3-9\nu^2-5\nu-1)\\ & +176(\nu^5+5\nu^4+10\nu^3+10\nu^2+5\nu+1)\\ = {} & 720\nu^7-240\nu^6-328\nu^5+40\nu^4+65\nu^3+5\nu^2-5\nu-1.\end{aligned}$$

Hence

$$\begin{aligned}(\Delta)&=(\nu-1)^5(2\nu-1)^3\{90\nu^4+105\nu^3+49\nu^2+11\nu+1\}\\ &=(\nu-1)^5(2\nu-1)^3(3\nu+1)^2\{10\nu^2+5\nu+1\};\end{aligned}$$

(16 bis) It is of course understood that $(p+1)^{\frac{5}{2}}$ is to be taken *positive*.

showing, agreeably with what was seen in the Lemma, that the discriminant of

$$(1, \varepsilon, \varepsilon^2, \varepsilon^2, \varepsilon, 1 \between x, y)^5$$

vanishes then, and then only, when

$$\varepsilon=1, \text{ or } \varepsilon=\tfrac{1}{2}, \text{ or } \varepsilon=-\tfrac{1}{3},$$

but does not *change its sign*, except as ε passes through the limits 1 and $\frac{1}{2}$, and only within those limits can become negative(17).

(11) Although the theory of the possibility of the roots of $(1, \varepsilon, \varepsilon^2, \eta^2, \eta, 1 \between x, y)^5=0$ has now been completely investigated, so far as is necessary for the proof of NEWTON'S theorem applied to equations of the fifth degree, it will be found that the labour will not be ill spent of considering more closely the real nature of the criteria which separate the case of one pair from that of two pairs of impossible roots in the above equation. NEWTON'S *criteria* being constructed so as to cover every possible case for equations of every degree, will always be found to fit loosely, so to speak, upon each case treated *per se*; so that more precise conditions can be assigned in each particular case than those which are furnished by his rule. So, *ex. gr.*, it may be remembered that in the equation $(1, e, e^2, e, 1 \between x, y)^4=0$, NEWTON'S rule implies only that when $e>1$, the roots are all impossible; but we have found further that unless $1>e>\frac{1}{3}$ (a much closer condition), the same thing takes place.

It is obvious from what has been demonstrated above, that if we treat p and σ, which are respectively $\varepsilon\eta-1$ and $\varepsilon^5+\eta^5-\varepsilon^2\eta^2-\varepsilon^3\eta^3$, as the abscissa and ordinate of a variable point in a plane, the curve $\Delta=0$, i. e. $(108\sigma+27p^2+28p^3)^2+72p^5+80p^6=0$ will be the line of demarcation between those values of ε, η which correspond to one pair, and those which correspond to two pairs of imaginary roots.

For all values of ε, η corresponding to internal points of the curve Δ there will be two imaginary and three distinct real roots; for all such as correspond to external points there will be four imaginary roots, and for points *on* the curve two imaginary and two equal roots.

The curve Δ is a curve of the 6th degree whose form will presently be discussed. But there is an important remark to be made in the first instance. Not all the points

(17) In *general* the case of equal roots of an equation is the state of transition of two real roots into imaginary, or *vice versâ*. But we see by the above instance that this is not necessarily the case *always*, for Δ vanishes on making $\varepsilon=-\frac{1}{3}$, and two roots become equal without any change in the nature of the roots when ε passes from being greater to being less than $-\frac{1}{3}$. In such case, however, there is a sort of unstable equilibrium in the form of the equation, by which I mean that the effect of any general infinitesimal change performed upon the coefficients of the equation would be either to cause the real roots in the neighbourhood of $\varepsilon=-\frac{1}{3}$ to disappear by the factor $(\varepsilon+\frac{1}{3})^2$ becoming superseded by a quadratic function of ε with impossible roots, or else a region in the neighbourhood of $\varepsilon=-\frac{1}{3}$ would reappear, for which the equation would acquire two real roots, owing to $(\varepsilon+\frac{1}{3})^2$ becoming superseded by a quadratic function of ε with real roots, in which case there would be two values in the neighbourhood of $\varepsilon=-\frac{1}{3}$, for *each* of which there would be a pair of equal roots in the equation. The above is probably the first instance distinctly noticed of this singular obliteration of the usual effect upon real and imaginary roots of a passage through equality, owing to the appearance of a square factor in the discriminant.

within the curve Δ will correspond to *real* values of ε, η. In order that these quantities may be real, we must have

$$\varepsilon^5+\eta^5>2(\varepsilon\eta)^{\frac{5}{2}},$$

$$\text{i. e. } \sigma+q^2+q^3>2q^{\frac{5}{2}}, \text{ where } q=p+1,$$

or

$$\sigma^2+2(q^2+q^3)\sigma+q^4-4q^5+q^6>0.$$

Writing this inequality under the form $R>0$, we see that the curve $R=0$ will represent a second sextic curve intersecting the former. Δ may be called the curve of the discriminant or *discriminatrix*, and will be a close curve, and R the curve of equal parameters or *equatrix*, and will consist of a single infinite branch. All points on the latter correspond to equal values of ε, η, those on one side of it to real values of ε, η, and those on the other side of it to conjugate values of the form $\lambda+i\mu$, $\lambda-i\mu$ respectively. Thus the area confined within the curve Δ will be divided into two portions by the equatrix, and it is impossible to shut one's eyes to the inquiry as to the meaning of the variable point lying in that portion which gives conjugate values to ε, η. It becomes clear by analogy that some kind of distinction must be capable of being drawn between the nature of the roots of the equation $(1, \varepsilon, \varepsilon^2, \eta^2, \eta, 1 \between x, y)^5=0$ when ε, η are conjugate, in some sense similar or parallel to that which we know to exist between them when ε, η are real; and obviously this inference cannot be confined to equations of the particular form and degree of that above written; in a word, equations whose coefficients are not real but conjugate, must have roots of two kinds, one analogous to the real, the other to the imaginary roots of equations with real coefficients. This inference will be justified in the sequel; but in the meanwhile it will be desirable to complete the investigation of the special equation under consideration, by a discussion of the forms and relations of the two curves Δ and R. These curves we know *à priori*, from what has been already demonstrated, can only meet in the three points corresponding to

$$\varepsilon=\eta=1, \quad \varepsilon=\eta=\tfrac{1}{2}, \quad \varepsilon=\eta=-\tfrac{1}{3};$$

and since $p=\varepsilon\eta-1$, the abscissæ of these three points will be 0, $-\frac{1}{4}$, $-\frac{8}{9}$.

Moreover the 3rd point will be distinguished from the other two by the circumstance that Δ does not change its sign as p passes through the value $-\frac{8}{9}$. Consequently the two curves must touch each other at this point.

Since when $\Delta=0$ p lies between 0 and $-\frac{9}{10}$, the curve Δ is confined to the negative side of the axis of σ. It is also confined to the negative side of the axis of p.

For between the limits $p=0$, $p=-\frac{9}{10}$,

$$648p^2+672p^3, \text{ i. e. } 24(27p^2+28p^3) \text{ is obviously positive,}$$

and

$$96p^6+176p^5+81p^4=\frac{p^4}{6}\{(24p+22)^2+2\} \text{ is always positive.}$$

Hence the two values of σ are both negative throughout the extent of the curve Δ.

Thus $\varepsilon^5+\eta^5-\varepsilon^2\eta^2-\varepsilon^3\eta^3$ being negative, $\varepsilon^3-\eta^2$ and $\eta^3-\varepsilon^2$ have the same signs when ε, η

are *real*, as should be the case; for in order that Δ may be capable of vanishing, $\varepsilon(\varepsilon^3-\eta^2)$ and $\eta(\eta^3-\varepsilon^2)$ must, by NEWTON's rule, be *both* negative, which could not be the case if either ε or η were negative; so that $\varepsilon^3-\eta^2$ and $\eta^3-\varepsilon^2$ must have the same signs, in fact each must be negative.

The curve Δ under consideration has a multiple point of the 4th order of multiplicity at the origin, where it is touched by the axis of p. Its distance from the axis for the extreme value of p, viz. $p=-\frac{9}{10}$, is $\frac{27}{2000}$.

It has three real maxima and minima, two belonging to its upper portion and one to the lower portion at the points, for which p has the *approximate* values $-\frac{9}{16}$, $-\frac{15}{22}$, and $-\frac{7}{8}$ (18).

The curve R, i. e. $\sigma=\left((p+1)\pm(p+1)^{\frac{3}{2}}\right)^2$, has the values 0 and -4 at the origin, a cusp at its extremity corresponding to $p=-1$, where both of its branches meet and touch the axis of p, and a negative maximum in its upper branch at the point where $p=-\frac{5}{9}$.

At all points within the curve R, ε and η are conjugate, and for the points outside real. Its lower branch will meet and touch the lower portion of Δ at the point where $p=-\frac{8}{9}$, and its upper branch will intersect and pass out of the upper branch of Δ at the point where $p=-\frac{3}{4}$. The only part of the area Δ therefore which corresponds to real values of ε, η, is that which is included between the upper segment of Δ and the upper branch of R, and extends only from $p=0$ to $p=-\frac{3}{4}$, i. e. from $\varepsilon\eta=1$ to $\varepsilon\eta=\frac{3}{4}$. Hence we may easily find an inferior limit to the values of ε and η when the equation (ε, η) has two real roots; for we have in that case ε, η, $\eta^2-\varepsilon^3$, $\varepsilon^2-\eta^3$ all positive. Hence

$$\eta^5>\varepsilon^3\eta^3>q^3, \quad \eta^5<\varepsilon^2\eta^2<q^2.$$

Consequently ε, η must each of them always lie between $q^{\frac{2}{5}}$, $q^{\frac{3}{5}}$; and since the least value of q is $\frac{1}{4}$, ε, η must each be always greater than $(\frac{1}{4})^{\frac{3}{5}}$, i. e. than $\cdot 33499$ (19).

(18) The large numbers which enter into Δ may be usefully reduced, and the equation $\Delta=0$ made more manageable, by aid of the simple substitutions $\sigma=-\frac{27v}{64}$, $p=-\frac{9u}{4}$. The equation $\Delta=0$ then becomes

$$(v-3u^2+7u^3)^2=2u^5-5u^6,$$

whose maxima and minima will be given by the equation

$$(v-3u^2+7u^3)(-6u+21u^2)=5u^4-15u^5;$$

which, making $1-3u=\omega$, becomes

$$270\omega^3-46\omega^2-9\omega+1=0,$$

whose roots are all real, and are one just a little greater than $-\frac{1}{6}$, another a little less than $\frac{1}{4}$, and the third a very little less than $\frac{1}{11}$ respectively; whence $p=\frac{3}{4}(\omega-1)$ will have the approximate values given in the text.

(19) $\varepsilon:\eta$ will have a maximum value, which can be found by writing $\delta\varepsilon:\delta\eta::\varepsilon:\eta$; and consequently, remembering that $q=p+1$, $S=\varepsilon^5+\eta^5$, $\sigma=S-q^2-q^3$,

$$\delta S:\delta q::5S:2q,$$

and therefore

$$\delta\sigma:\delta p::5\sigma+q^2-q^3:2q::5\sigma+p(p+1)^2:2(p+1).$$

Substituting the values of $\delta\sigma:\delta p$ in $\delta\Delta=0$, and combining the result with the equation $\Delta=0$, p and σ may be found by the solution of a numerical equation of the 5th degree, and then ε and η may be found by the solution

There is a third curve not undeserving of notice, of only the 3rd degree, which embodies the joint effect of the two middle criteria (the two extremes being supposed to be each zero) in the two cases where NEWTON'S rule will prove all the roots of the equation under consideration to be impossible. These criteria are $c_1=\varepsilon^4-\varepsilon\eta^3$, $c_2=\eta^4-\eta\varepsilon^3$. But

$$c_1\eta^4+c_2\varepsilon^4=q(2q^3-\mathrm{S})=q(2q^3-q^3-q^2-\sigma)=q(q^3-q^2-\sigma),$$

which for all values of q on the positive side of the line $p=-1$ (i. e. $q=0$) will have the same sign as $q^3-q^2-\sigma$, which we may call K[20]; and K positive will evidently imply that c_1, c_2 are one or both of them positive. The whole plane will be divided by the curve K into an *upper* region (commencing at $\sigma=\infty$), for which K is negative, and a lower region, in which K is positive. For any point of the curve K, $\sigma=q^3-q^2$, which within the limits of q with which we are concerned, viz. those within which Δ lies, is negative; for any point of the curve R, the smaller absolute value of σ is

$$-q^3-q^2+2q^{\frac{5}{2}}=q^3-q^2+2(q^{\frac{5}{2}}-q^3),$$

which $<q^3-q^2$ within the limits in question. So that, remembering that each of these values of σ is negative, we see that the portion of the area Δ corresponding to real values of ε, η will be completely above the curve K, *i. e.* in the negative region of K, and that accordingly Δ for *real values* of ε, η can never vanish when K is positive, as should be the case. This remark does not, however, apply to the conjugate region of Δ; for the curve K will *pass through*[21] the lower or conjugate portion of the area Δ.

(12) I may now say a few words on the signification of that portion of Δ in which ε and η are conjugate imaginary quantities.

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of a quadratic and the extraction of 5th roots. To find the maxima and minima values of ε and η themselves exactly would lead to the solution of an equation of a degree quite unmanageable.

But we may first find the greatest maximum and least minimum values of S, i. e. $\varepsilon^5+\eta^5$, by making $\delta\sigma=(2q+3q^2)\delta q$ in $\delta\Delta=0$, which leads to an equation (I forget whether) of the 3rd or 5th degree (it is one of the two): calling this maximum and minimum m, μ respectively, and naming ϱ (which of course must exceed unity) the greatest quotient of $\frac{\varepsilon}{\eta}$ or $\frac{\eta}{\varepsilon}$, we shall have

$$\sqrt[5]{\frac{\varrho^5}{1+\varrho^5}m}>\varepsilon\,;\ \eta>\sqrt[5]{\frac{1}{1+\varrho^5}\mu}.$$

These limits will be tolerably near to the absolute maximum and minimum values of ε or η. It may be noticed that we know, from what has gone before, that ϱ can never exceed $\left(\frac{1}{q}\right)^{\frac{1}{5}}$; and consequently ϱ^5 cannot exceed 4, since q is always $>\frac{1}{4}$.

(20) I call K the Indicatrix, as exhibiting the joint effect of the *indicia* or criteria of the Rule.

(21) This may easily be verified; for at the point $p=-\frac{3}{4}$ it will be found that the ordinate in K and the lower ordinate in Δ are equal, and at the point $p=-\frac{9}{10}$ the lower ordinate in Δ is $-\frac{27}{2000}$, and in K is $-\frac{18}{2000}$; which shows that the curve K entering the area Δ when at the lower half of the curve, at a point where $p=-\frac{3}{4}$, must pass through its upper contour in order to cut the line $p=-\frac{9}{10}$ as it does above the point where Δ is touched by that line.

The curve K has its negative maximum at the point $q=\frac{2}{3}$, i. e. $p=-\frac{1}{3}$. It passes through the origin, and begins with sweeping under the curve Δ, which it enters exactly under the point where R quits Δ, and passes

In general, let

$$(a+i\alpha,\ b+i\beta,\ c+i\gamma,\ \ldots..,\ c-i\gamma,\ b-i\beta,\ a-i\alpha \between x,\ y)^n=0$$

be an equation in which all the coefficients, reckoning simultaneously from the two ends, are conjugate to one another, and the central coefficient, if there is one, which can only be when n is even, *real.*

Let $\frac{x}{y}=p+iq$ satisfy this equation. Then evidently $\frac{y}{x}=p-iq$ will also satisfy it; or, which is the same thing, $\frac{x}{y}=\frac{p+iq}{p^2+q^2}$ will satisfy it.

Now either this root will be identical with the former one, or a distinct root; in the former case we must have $p^2+q^2=1$, and the root will be of the form $\cos\alpha+i\sin\alpha$; in the second case p^2+q^2 will differ from unity, and there will be a pair of imaginary roots of the form $\varrho(\cos\alpha+i\sin\alpha)$, $\frac{1}{\varrho}(\cos\alpha+i\sin\alpha)$, in which the real parts ϱ, $\frac{1}{\varrho}$ are reciprocal to one another, and the directive parts $e^{-i\alpha}$ identical. Moreover, if we write the given equation under the form $U+iV=0$, and suppose, as can always be done, that U and V have been divested of any algebraical common factor, it may easily be shown that the equation so prepared, and which may be called a Conjugate Equation *proper*, can have no real roots and no *pairs of imaginary roots* in the sense in which that term is employed in the theory of equations with real coefficients; but the distinction between *simple* or solitary and *twin* or associated roots reappears in the theory of conjugate equations, under a different form. It will of course be understood that the class of simple roots for which the modulus is unity is quite as general as that of twin roots, for each of which the modulus may be anything different from unity, just as in the ordinary theory the case of real is quite as general as that of imaginary roots, although the former may be represented by points on a fixed straight line, whilst the points representing the latter may be anywhere in the plane, this liberty of displacement being balanced, so to say, by the constraint of coupling. The general geometrical representation of the roots of a real equation is a system of points in a line, and a system of pairs of points at equal distances on opposite sides of the line. So the general geometrical representation of the roots of a conjugate equation will be system of points in the circumference of a circle to

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through Δ at a point very close indeed to the horizontal extremity of Δ. It may be noticed that when $p=-\frac{3}{4}$, the smaller ordinates of R and Δ are each $-\frac{1}{64}$, the ordinate of K and the larger ordinate of Δ being each $-\frac{3}{64}$.

I have found the points of contact of K with Δ by actually substituting q^3-q^2, i. e. $p(p+1)^2$ for σ in $\Delta=0$. This gives the equation

$$2064p^4+7352p^3+9823p^2+5832p+1296=0,$$

one factor of which is $4p+3$, dividing out which we have

$$516p^3+1451p^2+1368p+432=0.$$

The Newtonian criterion applied to the three first coefficients of the above gives $-1362\frac{5}{9}$, showing that two of the roots are impossible; the remaining real root I find to be $\cdot 8946$, &c. It does not appear to be a rational number.

radius unity, and of points situated in pairs in the same radii at reciprocal distances from the centre. In a word, in each case we may say that the roots can be geometrically represented by points on a circle, and pairs of points electrical images of each other in respect to the circle, but the radius of the circle in the one case will be infinity, in the other unity. Conjugate like real equations will have all their invariants of an even degree real, and those of an odd degree will be pure imaginaries, or real quantities affected with the multiplier i. Their morphological derivatives (covariants, contravariants, &c.) will be also conjugate forms. The whole doctrine of equations, as regards the separation of real from imaginary roots, and the determination of the limits within which the former lie, will reproduce itself with suitable modifications in the theory of conjugate equations, in which simple, on the one hand, and coupled or twin roots, on the other, will correspond respectively as analogues to the real and imaginary roots of the ordinary theory. Thus the following theorem may be demonstrated without difficulty, viz., in any conjugate equation the number of coupled roots is congruent to 0 in respect to the modulus 4 when the discriminant is positive, and to 2 in respect to the same modulus when the discriminant is negative([22]). We see now how to interpret the

([22]) ([a]) A very simple linear transformation shows the immediate connexion between the solitary and associated roots of conjugate with the real and paired imaginary roots of ordinary equations. For if $f(x, y)=0$ be a conjugate equation, writing

$$y=v+iu, \quad x=v-iu,$$

$f(x, y)$ becomes $F(u, v)$, a real form in u, v.

When u, v are real, we have

$$\frac{y}{x}=\frac{v+iu}{v-iu}=\cos\left(\tan^{-1}\frac{v}{u}\right)+i\sin\left(\tan^{-1}\frac{v}{u}\right);$$

when $\frac{v}{u}=c\pm i\gamma$, the two values correspond to

$$\frac{y}{x}=\frac{c+i\gamma+i}{c+i\gamma-i}, \quad \left(\frac{y}{x}\right)'=\frac{c-i\gamma+i}{c-i\gamma-i}.$$

Thus

$$\frac{y}{x}:\left(\frac{y}{x}\right)'::c^2+(\gamma+i)^2:c^2+(\gamma-i)^2;$$

also

$$\frac{y}{x}\times\left(\frac{y}{x}\right)'=\frac{c^2-1+\gamma^2+2ci}{c^2-1+\gamma^2-2ci},$$

of which the modulus is obviously unity.

([b]) Now it is known that if t be the number of real, and τ of imaginary roots in the real form, $(u, v)^n$, its discriminant, bears the sign $(-)^{\frac{t(t-1)}{2}}$. Hence the sign of the discriminant of the conjugate form $(x, y)^n$ (since the determinant of $v+iu$, $v-iu$ is $2i$) will be $(-)^q$, where

$$q=\frac{n(n-1)}{2}+\frac{t(t-1)}{2}=\frac{(t+\tau)(t-1+\tau)+t(t-1)}{2}=t(t-1)+t\tau+\frac{\tau(\tau-1)}{2}.$$

Hence since τ and $t(t-1)$ are both even, $(-)^q=(-)^{\frac{\tau(\tau-1)}{2}}$, and the sign of the discriminant of a conjugate form is $+$ or $-$ according as the number of imaginary roots does or does not contain 4 as a factor.

It must be remembered that the sign of the discriminant is not in general the same as that of the *zeta* or squared product of differences of the roots. The sign of the *zeta* for real equations follows precisely the same law as the sign of the *discriminant* for conjugate ones.

effect of the variable point whose coordinates are $\varepsilon^5+\eta^5$ and $\varepsilon\eta$ lying within the area Δ, in that portion of it for which ε, η became imaginary; viz. it is that in such case the equation (ε, η), which then becomes of a conjugate form, will have three simple and two twin roots; and thus the unity of the interpretation is restored if we choose, as we very well may, to extend the use of these terms to the real roots and the paired imaginary roots of ordinary equations. We may neglect the curve of reality R altogether, and affirm that all over the area Δ, ε, η will have such values as will give rise to three simple and two coupled roots.

(13) That part of the theorem of NEWTON which had received a demonstration from MACLAURIN and CAMPBELL in the generalized form in which I have enunciated it in this paper, may be easily extended to the case of conjugate equations. It will, as applied to them, read thus: If the $(n-1)$ quadratic derivatives of a conjugate form of the nth degree, all whose roots are simple, be multiplied respectively by the coefficients of any other conjugate form, all whose roots are also *simple*, of the degree $(n-2)$, and the sum of these products be taken as a new quadratic form, the discriminant of this latter must be positive, or, which is the same thing, its determinant must be negative.

(14) So much for the case of $n=5$. If we were to proceed to the consideration of equations of the 6th degree, *two* cases of resistance would present themselves in the demonstration of NEWTON'S rule, viz. one in which the signs of the criteria are $-+++-$, the other $-+-+-$. In the latter it would only be necessary to show that the discriminant is necessarily negative, since we know from the derivatives that the equation must have four imaginary roots, and the choice would lie between the alternatives of there being four or six. In the former case the derivatives only indicate the necessary existence of two real roots, and it would become requisite to prove that there must be four or six—an alternative which depends not on the sign of one function of the coefficients, but on the nature of the signs of two such functions given by STURM'S or any equivalent theorem. It would thus become requisite to prove that two functions of the coefficients, say L, M, could not *both* be negative; and this might be shown by demonstrating the existence of two quantities, L′, M′, other functions of the coefficients incapable of assuming any but the positive sign such that $L'L+M'M$ would be necessarily positive.

PART II.—ON THE LIMIT TO THE NUMBER OF REAL ROOTS IN EQUATIONS OF THE FORM $\Sigma(ax+b)^n$.

(15) I shall now proceed to the consideration of a theorem relating to a particular class of ordinary equations, which occurred to me in the course of and in connexion with the preceding investigations. The theorem itself, but unaccompanied by proof, has appeared in the 'Comptes Rendus' of the Academy for the month of March 1864.

Both as regards its nature and the processes involved in the proof, it stands in close relation to NEWTON'S rule, my study of which in fact led me to its discovery. It will therefore take its place most appropriately in this paper.

Certain preliminary properties of circulation introducing some new notions of polarity must be first established, by way of Lemmas to the proof in question.

By a *type* let us understand a succession of symbols of any subject matter whatever susceptible of receiving the signs $+-$, or any suchlike indications of opposite polarity.

Let $a, b, c, \ldots l, k, l$ be any such type, where the *elements* $a, b, c, \ldots$ may be regarded either as points in a line or rays in a pencil affected respectively with the signs of $+$ and $-$.

Then by a *per-rotatory* circulation of such type, I mean the act of passing from the first element to the second, from the second to the third, &c., from the last but one to the last, and from the last to the first.

By a *trans-rotatory* circulation of the same, I mean the act of passing from the first to the second, the second to the third, &c., from the last but one to the last, and from the last to the first, *with its sign reversed.*

A type considered subject to per-rotatory circulation may be termed a Per-rotatory Type; one subject to the other sort of circulation, a Trans-rotatory Type.

If a, b, c, d, e be a per-rotatory type, its direct *phases* are

$$\begin{array}{ccccc} a, & b, & c, & d, & e, \\ b, & c, & d, & e, & a, \\ c, & d, & e, & a, & b, \\ d, & e, & a, & b, & c, \\ e, & a, & b, & c, & d, \end{array}$$

and its retrograde phases

$$\begin{array}{ccccc} a, & e, & d, & c, & b, \\ e, & d, & c, & b, & a, \\ d, & c, & b, & a, & e, \\ c, & b, & a, & e, & d, \\ b, & a, & e, & d, & c. \end{array}$$

If, on the other hand, a, b, c, d, e be a trans-rotatory type, its direct *phases* will be

$$\begin{array}{ccccc} a, & b, & c, & d, & \bar{e}, \\ b, & c, & d, & \bar{e}, & \bar{a}, \\ c, & d, & \bar{e}, & \bar{a}, & b, \\ d, & \bar{e}, & \bar{a}, & \bar{b}, & \bar{c}, \\ \bar{e}, & \bar{a}, & \bar{b}, & \bar{c}, & \bar{d}, \end{array}$$

and its retrograde phases

$$\begin{array}{ccccc} a, & \bar{e}, & \bar{d}, & \bar{c}, & \bar{b}, \\ \bar{e}, & \bar{d}, & \bar{c}, & \bar{b}, & \bar{a}, \\ \bar{d}, & \bar{c}, & \bar{b}, & \bar{a}, & e, \\ \bar{c}, & \bar{b}, & \bar{a}, & e, & d, \\ \bar{b}, & \bar{a}, & e, & d, & c, \end{array}$$

where the sign (—) is, for greater convenience of writing, placed over instead of before the elements which it affects; and so on in general a type of n elements, whether per-rotatory or trans-rotatory, will admit of n direct and n retrograde phases.

If we count the number of variations of sign in the circulations of any phase of a per-rotatory type, this number will be the same for all the phases, and will be an even number; this even number may be termed the variation-index of the type.

So, again, if whatever be the original signs of the element in a trans-rotatory type, we count the number of variations in the circulation of any of its phases, this number also will be constant and will be odd, and this odd number may then be termed the variation-index of the type.

(16) Let any phase be taken of a per-rotatory type, and out of such phase let any element be *suppressed*; then we obtain a type one degree lower in the elements, which, if we please, we may consider as a trans-rotatory type, and such trans-rotatory type may be termed a derivative of the original per-rotatory one.

In like manner any phase being taken of a trans-rotatory type, one element may be suppressed, and the reduced type treated as a per-rotatory one, and termed a derivative of the original trans-rotatory one.

We may now enunciate the following important general proposition, viz.

Any trans-rotatory type or any per-rotatory type whose variation-index is different from zero being given, a per-rotatory derivative of the one and a trans-rotatory derivative of the other may be found such that the variation-index of the derived types in either case shall be less by a unit than the variation-index of the types from which they are derived.

Case (1). Let the given type be per-rotatory. Then by hypothesis, since it has some variations, we may find a phase of it beginning with + and ending with —, by which I mean beginning with an element that is positive and ending with one that is negative. This gives rise to two sub-cases.

$$\text{T, the phase in question, will be } + \ldots\ldots + -$$
$$\Theta\text{, the phase in question, will be } + \ldots\ldots - -.$$

In either sub-case let the last sign be suppressed, and the result treated as a trans-rotatory type; then T, Θ become respectively T′, Θ', where

$$\text{T}' \text{ is } + \ldots\ldots +$$

and

$$\Theta' \text{ is } + \ldots\ldots -$$

and evidently the variation-index of T — variation-index of T′= number of changes of sign in $+-+$ less changes of sign in $+-=2-1=1$; and again variation-index of $\Theta-$ variation-index of $\Theta'=$ number of changes of sign in $--+$ less changes of sign in $--=1-0=1$. Hence the theorem is proved for the case where the given type is per-rotatory.

Case (2). Let the given type be *trans-rotatory*.

Then, again, there must either be a phase of the form P, or one of the form Φ, where

P represents a *continual succession* of signs of the same name as $++\ldots+$ or $--\ldots-$, and Φ represents a succession beginning with one sign as $+$ and ending with one or more signs $-$, or else beginning with $-$ and ending with a succession of signs $+$. Essentially, then, as a change of signs throughout a whole succession does not affect the variation-index, we may suppose

$$\mathrm{P}=+\ldots\ldots++,$$
$$\Phi=-\ldots-+\ldots+,$$

the signs intervening between the two expressed signs $-$ in Φ being filled up in any manner whatever, and those between the two signs $+$ with signs exclusively $+$.

Let now that phase of Φ be taken which commences with the first sign of the final succession of $+$. Then Φ becomes

$$(\Phi)=+\ldots++\ldots.+,$$

which is of the form

$$+\ldots..++,$$

so that P is only a particular case of (Φ). If the last sign in (Φ) be suppressed and the result treated as a per-rotatory type be called $(\Phi)'$, so that $(\Phi)'=+\ldots\ldots+$, we have variation-index in $(\Phi)-$ variation-index in $(\Phi)'=$ changes of sign in $-+$ less changes of sign in $++=1-0=1$.

Hence the proposition is established for both cases.

(17) The theorem to which this Lemma-proposition is to be applied concerns equations of the form

$$\varepsilon_1 u_1^m+\varepsilon_2 u_2^m+0\ldots+\varepsilon_n u_n^m=0,$$

where $u_1, u_2, \ldots, u_n$ are any linear functions of x, y; m is any positive integer, and $\varepsilon_1, \varepsilon_2, \ldots \varepsilon_n$ are each respectively and separately, either *plus unity* or *minus unity*.

Such an equation for convenience of reference may be termed a superlinear equation, and the function equated to zero a superlinear function.

Every superlinear function may be conceived as having attached to it a pencil of rays constructed in a manner about to be explained.

1. We may conceive the function to be prepared in such a manner, that supposing $ax+by$ to be any one of the n linear elements u, every b shall be positive. If m is even, this can be effected by writing when required for $ax+by$, $-ax-by$ without further change. If m is odd, we may write when required $-ax-by$ in place of $ax+by$, changing at the same time the factor ε, which appertains to $(ax-by)^m$ from $+1$ to -1, or *vice versâ*, from -1 to $+1$.

Now take in a plane any two axes of coordinates $O\xi$, $O\eta$, and consider a, b as the ξ and η coordinates of a point. All the n points thus obtained, on account of every b being positive, will lie on the same side of the axis $O\eta$, and thus the entire n linear functions will be represented by a pencil of n rays, the two extreme rays of which make an angle less than two right angles with each other; but each term of the superlinear function contains, besides $(ax+by)^n$, a definite multiple $+1$, or -1, and we must accordingly, to

completely express such term, conceive every ray affected with a distinct sign $+$ or $-$. A pencil thus drawn with its rays so polarized will give a complete representation of any given superlinear function, and may be called its type-pencil([23]).

I am now able to state the following proposition:

(18) *The number of real roots in a superlinear equation cannot exceed the variation-index of its type pencil, regarded as a per-rotatory type, if the degree of the equation be even, and as a trans-rotatory type if the degree of the equation be odd.* I prove this inductively as follows.

1. Suppose the theorem to be true when the variation-index of the type-pencil is not greater than the even number ν, and consider an equation of the odd degree $(2i+1)$, for which the type-pencil viewed as trans-rotatory has the variation-index $\nu+1$.

Let a *phase* of this type be taken, say corresponding to the rays $\varrho_n, \varrho_{n-1} \ldots \varrho_2, \varrho_1$, such that the per-rotatory type obtained by striking out the term ϱ_1 has the variation-index ν (as we know may be done by virtue of the Lemma).

Take for new axes $O\xi'$, $O\eta'$, when $O\xi'$ coincides with ϱ_1; then it is clear that the pencil $\varrho_n, \varrho_{n-1} \ldots \varrho_2, \varrho_1$ will still serve as a type-pencil to the given function, the only change being that some of the rays, namely those that did lie on one side of ϱ_1, have been inverted in direction and changed in sign (corresponding to a change in the coefficient a, b, accompanied with a change in the sign of the corresponding ε), whilst the rays on the other side of ϱ_1 have been left unaltered.

The points (a_1, b_1), $(a_2, b_2) \ldots (a_n, b_n)$ corresponding to the rays $\varrho_1, \varrho_2, \ldots \varrho_n$ will, with respect to the new axes, change their values, becoming converted into $(\alpha_1, 0)$, (α_2, β_2), $(\alpha_3, \beta_3), \ldots (\alpha_n, \beta_n)$, where $\beta_2, \beta_3, \ldots \beta_n$ will still all be positive, the angle between ϱ_1 and ϱ_n being the same as between the two extreme rays in the original figure of the type-pencil, and the superlinear equation may now be written in the form

$$F(u, v)=\varepsilon_1(\alpha_1 u)^{2i+1}+\varepsilon_2(\alpha_2 u+\beta_2 v)^{2i+1}+\varepsilon_3(\alpha_3 u+\beta_3 v)^{2i+1}+\varepsilon_n(\alpha_n u+\beta_n v)^{2i+1}=0,$$

where u, v are real linear functions of x, y.

([23]) Let a circle be imagined pierced by a pencil containing any number of rays protracted in both directions, say in the opposite points a, α; b, β; c, γ; d, δ; and let these points, taken in order of natural succession from left to right, or right to left, be $a, b, c, d, \alpha, \beta, \gamma, \delta$. Then, commencing with any point c, a *complete* circulation will be represented by the succession of transits

$$c \text{ to } d, \quad d \text{ to } \alpha, \quad \alpha \text{ to } \beta, \quad \beta \text{ to } \gamma, \quad \gamma \text{ to } \delta, \quad \delta \text{ to } a, \quad a \text{ to } b, \quad b \text{ to } c.$$

But whether $\alpha, \beta, \gamma, \delta$ bear respectively the same signs or signs contrary to those of a, b, c, d, the transit between any two points β to γ will be of the same nature, as regards continuance or change of sign, as the transit from b to c, and thus we see that the complete cycle or total revolution above indicated is only a reduplication of, and may be fully designated by the hemicyclic succession c to d, d to α, α to β, β to γ, for which the number of variations therefore will be the same as for any similar succession obtained by commencing with any other element in the original system of points instead of c. If the opposite points bear like signs, the above succession of transits may be indicated by the order c, d, a, b, c; if they bear contrary signs by the order $c, d, \bar{a}, \bar{b}, \bar{c}$, and thus it is that the idea arises of the two kinds of so-called circulation, but which are in fact only more or less disguised species of semicirculation.

Let the derivative of this function be taken in regard to v, and we have

$$\frac{1}{2i+1}\mathrm{F}'(u,v)=\beta_2\varepsilon_2(\alpha_2u+\beta_2v)^{2i}+\beta_3\varepsilon_3(\alpha_3u+\beta_3v)^{2i}\ldots+\beta_n\varepsilon_n(\alpha_nu+\beta_nv)^{2i},$$

where $\beta_2\varepsilon_2$, $\beta_3\varepsilon_3\ldots\beta_n\varepsilon_n$ have the same signs as $\varepsilon_2, \varepsilon_3, \ldots \varepsilon_n$ respectively.

Now the pencil-type of $\mathrm{F}'(u, v)$ will be the per-rotatory type $\varrho_n, \varrho_{n-1}, \ldots \varrho_2$, of which by construction the variation-index is ν. Hence by hypothesis $\mathrm{F}'(u, v)$ has not more than ν real roots, i. e. at least $2i-\nu$ imaginary roots. Hence $\mathrm{F}(u, v)$ has at least that number of imaginary roots, i. e. at most $(2i+1)-(2i-\nu)$, i. e. $\nu+1$ real roots. Hence if the theorem is true for ν an even number, it is true for $\nu+1$.

In like manner let us proceed to show that when it is true for ν an odd number, it would remain true for $\nu+1$.

The reasoning will be precisely similar to that followed in the antecedent case. We must find a phase of the *per-rotatory* type $\varrho_n, \varrho_{n-1}, \ldots \varrho_2, \varrho_1$ having the variation-index ν such that the trans-rotatory reduced type $\varrho_n, \varrho_{n-1}, \ldots \varrho_2$ shall have the variation-index $\nu-1$; the new pencil will still continue to be a type-pencil of the given superlinear function, the change of direction in the bunch of rays one on side of ϱ_1 being now unaccompanied with change of sign, such change corresponding to $\varepsilon(ax+by)^{2i}$ becoming changed into $\varepsilon(-ax-by)^{2i}$ without ε undergoing a change of sign.

As before, the axes of coordinates are transformed from ξ, η into ξ', η', and we obtain

$$\mathrm{F}(u, v)=\varepsilon_1(\alpha_1u)^{2i}+\varepsilon_2(\alpha_2u+\beta_2v)^{2i}+\ldots+\varepsilon_n(\alpha_nu+\beta_nv)^{2n+1},$$

$$\frac{1}{2i}\mathrm{F}'(u, v)=\beta_2\varepsilon_2(\alpha_2u+\beta_2v)^{2i-1}+\ldots+\beta_n\varepsilon_n(\alpha_nu+\beta_nv)^{2i}.$$

for which the type-pencil is the trans-rotatory type $\varrho_n, \varrho_{n-1}, \ldots \varrho_2$, of which by construction the variation-index is $\nu-1$, so that its number of imaginary roots is $2i-(\nu-1)$, and consequently the number of real roots of $\mathrm{F}(u, v)$ will be $\nu+1$.

Thus, then, if the theorem be true for ν, whether ν be even or odd, it will be true for $\nu+1$.

But when $\nu=0$, the superlinear function becomes a sum of even powers of linear functions of x, y, all taken with the same sign, of which the number of roots is evidently 0. Hence, being true for this case, the proposition is true universally.

It will be noticed that the algebraical part (as distinguished from the purely polar-tactic part of the above demonstration) depends on the same principle of which such abundant use has been made in the former part of this dissertation, viz. that the number of imaginary roots in any ordinary algebraical equation in x cannot be increased when we operate any homographic substitution upon x, and take the derivative of the equation thus transformed in lieu of the original(24).

(24) For greater clearness I present in an inverted order of arrangement a summary of the foregoing argument.

By an ith derivative of $f(x, y)$ is meant any derived form

$$\left(\lambda_1\frac{d}{dx}+\mu_1\frac{d}{dy}\right)\left(\lambda_2\frac{d}{dx}+\mu_2\frac{d}{dy}\right)\ldots\left(\lambda_i\frac{d}{dx}+\mu_i\frac{d}{dy}\right)f(x, y),$$

(19) The proposition above established leads immediately to the theorem and corollary following, viz.

THEOREM. If $c_1, c_2, \dots c_n$ be a series of ascending or descending magnitudes, and m any positive integer, the equation

$$\lambda_1(x+c_1)^m+\lambda_2(x+c_2)^m+\dots+\lambda_n(x+c_n)^m=0$$

cannot have more real roots than there are changes of sign in the sequence $\lambda_1, \lambda_2, \dots \lambda_n, (-)^m\lambda_1$.

For obviously $(1, c_1)$, $(1, c_2), \dots (1, c_n)$ will be points corresponding to rays within a semirevolution, and therefore forming a type-pencil.

Corollary. If the above equation be transformed by any real homographic substitution into the form

$$\mu_1(y+\gamma_1)^m+\mu_2(y+\gamma_2)^m+\dots+\mu_n(y+\gamma_n)^m=0,$$

where $\gamma_1, \gamma_2, \dots \gamma_n$ are taken in ascending or descending order, the number of changes of sign in the series $\mu_1, \mu_2, \dots \mu_n, (-)^m\mu$ is *invariable*(25); for the effect of any such formation will be to leave the type-pencil unaltered except in its *phase.*

(20) If we look to the undeveloped form of the superlinear function

$$S=\varepsilon_1u_1^m+\varepsilon_2u_2^m+\dots+\varepsilon_nu_n^m,$$

and are supposed to possess no knowledge of the coefficients which enter into the linear elements u, we may still draw some general inferences as to the limit of the number of real roots in $S=0$. Thus if the number of positive units ε is j, and of the negative units k, and j is not greater than k, it is obvious that, whatever may be the form of the type-pencil to S, its variation-index cannot be more than $2j$ when m is even, nor more than $2j+1$ when m is odd; for the arrangement the most favourable to the largeness of the number of the real roots is that where every two rays with the signs belong-

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the λ, μ quantities being any real quantities whatever. Then I say—

1. If T is the type-pencil (per-rotatory or trans-rotatory) of any superlinear form F, every derivative of T of the contrary name is the type-pencil of some first derivative of F, as shown in art. (18).

2. A derivative of T of contrary name may be found such that its variation-index shall be less by a unit than that of T itself, as shown in art. (16).

3. Hence if i is the variation-index of the type-pencil of F, an ith derivative of F may be found such that its variation-index shall be zero, and consequently having no real roots.

Hence, finally, since the number of real roots of any rational integral homogeneous function in x, y cannot exceed by more than i the number of the real roots in any of its ith derivatives, F cannot have more real roots than there are units in the variation-index of its type-pencil.

The subtle point of the argument, it will be noticed, lies in forming the conception of the variation-index to a trans-rotatory pencil, in which the singular phenomenon occurs of a reversal of *relative polarity* in passing from the last ray to the first, whereas in a per-rotatory pencil any ray indifferently may be regarded as the initial ray, no such reversal in that case taking place.

(25) It may be noticed that, contrariwise, the limit to the number of real roots given by NEWTON's criteria is *not* an invariant; it fluctuates with the homographic transformations operated upon the equation; and a question suggests itself as to the maximum value the number of imaginaries indicated by the rule can attain. I presume this maximum is not in all cases necessarily the actual number of the imaginary roots possessed by the equation.

ing to the j group of ε are separated by one or more of the rays with a contrary sign to themselves. Thus it appears that when only the units $\varepsilon_1, \varepsilon_2, \ldots \varepsilon_n$ are given, we may impose a maximum upon the number of real roots in the superlinear equation; this limit may be called the *absolute maximum*, being the double of the inferior number of like signs in the series $\varepsilon_1, \varepsilon_2, \ldots \varepsilon_n$ when the degree is even, and one more than such double when the degree is odd [26].

The *specific maximum*, on the other hand, will depend on the form of the type-pencil, and cannot be ascertained until the coefficients of the linear elements are given. It can never exceed, but may be less than the absolute maximum. It may, indeed, be easily proved that *in general* the specific maximum will be less than the absolute maximum. Thus, by way of example, suppose the degree to be even, and the inferior number of like signs to be 2; the absolute maximum number of real roots will be four, but the specific maximum will more generally be only two. For let the number of linear terms in the superlinear function be $2+n$, n being 2 or any greater number; and first, to fix the ideas, suppose $n=2$. The type-pencil, which is to be read per-rotatorily, consists of four rays, say a, b, c, d, following each other in uninterrupted circular order, of which two are to bear positive and two negative signs. If the two negative signs fall on a, c or on b, d, the variation-index will be 4, but in the other four cases of incidence such index will be only 2. Consequently the chance is 2 to 1 [27] that the specific maximum, which may be 4, is not greater than 2; and consequently the chance that there will be four real roots in the equation will be only a chance (too difficult to be calculated, but which is a function of the degree of the equation) *of* the chance $\frac{1}{3}$ that there will be as many as four real roots in the equation $u_1^n+u_2^n-u_3^n-u_4^n=0$, where u_1, u_2, u_3, u_4 are

[26] [a] If a superlinear form of an odd degree contains an odd number of terms, say $2k+1$, the greatest value of the *inferior* number of like signs is k, and the extreme limit to the number of real roots will be $2k+1$.

If it contain an even number of terms, say $2k$, the greatest value of the inferior index is k; but for this particular case it will readily be seen that a limit may be assigned to the variation-index closer than that given by the rule in the text; in fact the variation-index cannot in that case exceed $2k-1$, which will therefore be the extreme limit to the number of real roots. Now suppose the canonizant of an odd-degreed function of x, y to have all its roots real, then it may be expressed by a superlinear form of which the number of terms will be $2i+1$ or $2i$, according as the degree is $4i+1$ or $4i-1$. In the one case the number of real roots cannot exceed $2i+1$, in the other $2i-1$. Hence the following somewhat curious theorem:

[b] *If the canonizant of an odd-degreed quantic in* x, y, *of the degree* $4i\pm1$, *has* no *imaginary roots, the quantic itself must have at least* i *pairs of imaginary roots.* From the fact that when the roots of the canonizant of a quintic are all real there must be one pair at least of imaginary roots, we can infer that when the discriminant of a quintic is positive and that of its canonizant is negative, the equation has one real and four imaginary roots. This observation has led to a long train of reflections, which will be found embodied in the 3rd part of the memoir.

[27] This, in fact, is identical in substance with the noted problem of determining the chance that two straight lines drawn on a black board will cross. Mr. Cayley, of whom it may be so truly said, whether the matter he takes in hand be great or small, "nihil tetigit quod non ornavit," suggests the following independent proof of this. Taking unity as the length of the contour, fixing the extremity of one of the lines, and calling s the distance of its other end from it measured on the contour, the chance of the second line crossing this is easily seen to be $2s(1-s)$, which, integrated between $s=0$, $s=1$, gives $\frac{1}{3}$, as before obtained.

unknown linear functions of x: thus we are entitled to say that *in general* the number of real roots in such an equation is *not* the maximum four, but a less number. This remark is of importance, as showing that on this subject it is possible to speak with scientific certainty, and on other than empirical grounds, of what may *in general* be expected to take place. Thus we find NEWTON declaring twice over in the chapter quoted, that *in general* his rule will give not merely the maximum, but the actual number of the imaginary roots in an equation. I am strongly inclined to doubt the truth of this assertion; but it is important to be satisfied by analogy that such an assertion may rest on a scientific and demonstrative basis, and not on the utterly fallacious foundation of arithmetical empiricism([28]).

([28]) A few additional words on this question of probability may not be unacceptable. In order to meet the case of the degree of the superlinear form or equation being odd as well as even, let it be supposed known under the form

$$\Sigma_n^1 \lambda_i (x+c_i)^m,$$

the values of the quantities c_1 being supposed to be left wholly indeterminate, and only the signs of the quantities λ to be given. Let ω be the inferior number of like signs in the λ series, meaning thereby that the number of signs of one sort is ω, and of the other sort ω, or more than ω.

Let the probability of the specific maximum of real roots being $2k$ when m is even, be represented by p_{2k}, and of its being $2k+1$ when m is odd by π_{2k+1}; also let s_{2k}, σ_{2k+1} represent the number of cases when ω and n are given which correspond to the specific maximum being $2k$, $2k+1$ respectively. Suppose $\omega=1$, then obviously, when m is even, we have $s_2=n$, $p_2=1$. But when n is odd $\sigma_1=2$ (for when either extreme element alone is negative the trans-rotatory cycle has the variation-index unity), and $\sigma_3=n-2$, so that

$$\pi_1=\frac{2}{n}, \quad \pi_3=\frac{n-2}{n}.$$

Again, suppose $\omega=2$, m being even; then obviously s_2 is the number of contiguous duads in a cycle of n elements, and s_4 is the remaining number of duads; hence

$$s_2=n,\ s_4=n\frac{n-1}{2}-n=n\frac{n-3}{2};$$

so that

$$p_2=\frac{2}{n-1}, \quad p_4=\frac{n-3}{n-1}.$$

2nd. Suppose $\omega=2$, m being odd, so that σ_1, σ_3, σ_5 will have to be separately estimated. To fix the ideas, let the λ series be termed a, b, c, d, e, f, g, in which two of the elements are supposed of one sign, say negative, and the rest of the opposite sign, say positive; then the only dispositions of sign which correspond to the specific maximum being 1 are those in which a, b or else f, g are both negative. Hence $\sigma_1=2$. Again, the dispositions of sign which make the specific maximum equal to 3 are those in which a, g are both negative, those in which a and c, d, e, or f are negative, those in which g and e, d, c, or b are negative, and, finally, those in which any two contiguous elements except the a and g are negative. Hence $\sigma_3=1+2(n-3)+(n-3)=3n-8$; and it should be observed that this result cannot be prejudiced in its generality by the supposition of any of the components of σ_3 becoming negative, since $\omega=2$ implies that n is at least 4. Hence, finally,

$$\sigma_5=\frac{n^2-n}{2}-(3n-8)-2=\frac{n^2-7n+12}{2}=\frac{(n-3)(n-4)}{2};$$

so that

$$\pi_1=\frac{4}{n^2-n}, \quad \pi_3=\frac{6n-20}{n^2-n}, \quad \pi_5=\frac{n^2-7n+16}{n^2-n}.$$

This example serves to show how much more difficult is the computation of the respective probabilities when m is odd than when m is even, owing to the break of continuity in the cycle of readings on passing from the last to the first term.

NOTES TO SECTION II.

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On the probability of the specific superior limit to the number of real roots in a superlinear equation equalling any assigned integer.

(21) The question comes to that of determining the probability of a per-rotatory or trans-rotatory pencil with a definite number of rays of each kind possessing a given variation-index.

Since the foot note below was written, a method has occurred to me of obtaining the probability in question in general terms, as follows.

1. For a *per-rotatory* pencil of μ positive and ν negative rays. Let $[\mu, \nu, g]$ be the probability of the rays being so disposed as to give rise to $2g$ variations of sign in making a complete revolution. Then there will be g distinct groups of positive, and g of negative rays. The number of partitions with permutations of the parcels *inter se* of μ elements in g parcels is $\frac{(\mu-1)(\mu-2)..(\mu-g+1)}{1.2..(g-1)}$, and of ν elements into g parcels is $\frac{(\nu-1)(\nu-2)..(\nu-g+1)}{1.2..(g-1)}$.

If we combine each parcel with each in every possible way, and then imagine the combined parcels let into a circle containing $m+n$ places and shifted round in the circle through a complete revolution, we shall obtain

$$(\mu+\nu)\times\frac{(\mu-1)(\mu-2)..(\mu-g+1)1}{1.2..(g-1)}\cdot\frac{(\nu-1)(\nu-2)..(\nu-g+1)}{1.2..(g-1)}$$

arrangements; but on examination it will be found that every arrangement so produced will be repeated g times; moreover it is obvious that no other arrangement giving rise to g groups of each sort can be found. Hence the true number of distinct groupings of the sort in question is

$$\frac{(\mu+\nu)}{g}\cdot\frac{(\mu-1)(\mu-2)..(\mu-g+1)}{1.2..(g-1)}\cdot\frac{(\nu-1)(\nu-2)..(\nu-g+1)}{1.2..(g-1)}.$$

It seems hardly worth while to pursue this subject in greater detail. I will only notice that when m is even the chance of the specific maximum attaining the absolute maximum, *i. e.* becoming 2ω, will depend on the proportion of the ways in which in a cycle of n elements ω of them may be marked with a distinctive sign in such a way that no two of such signs shall come together. Accordingly I find by a computation of no great difficulty (understanding πx to mean $1.2.3\ldots x$),

$$s_{2\omega}=\frac{n\pi(n-\omega-1)}{\pi\omega\pi(n-2\omega)},$$

and hence, since the total number of combinations of n elements ω and ω together is $\frac{\pi(n)}{\pi\omega\pi(n-\omega)}$, I deduce

$$p_{2\omega}=\frac{\pi(n-\omega)\pi(n-\omega-1)}{\pi(n-1)\pi(n-2\omega)}.$$

Thus when n has its minimum value, viz. 2ω, $p_{2\omega}=\frac{\pi\omega\pi(\omega-1)}{\pi(\omega-1)}$, and becomes very small as ω increases. When again n increases towards infinity $p_{2\omega}$ approaches indefinitely near to unity, and the chance approaches near to certainty of the specific not beoming less than the absolute maximum of real roots.

And the total number of arrangements, which is the number of ways in which μ things can be distributed over $(\mu+\nu)$ places, is $\frac{\pi(\mu+\nu)}{\pi\mu\pi\nu}$. Hence we obtain

$$[\mu, \nu, g]=\frac{\pi\mu\pi\nu}{\pi(\mu+\nu-1)}\left\{\frac{(\mu-1)(\mu-2)..(\mu-g+1)\times(\nu-1)(\nu-2)..(\nu-g+1)}{1.2.(g-1)(1.2..g)}\right\}$$

$$=\frac{\pi\mu\pi(\mu-1)\pi\nu\pi(\nu-1)}{\pi g\pi(g-1)\pi(\mu-g)\pi(\nu-g)\pi(\mu+\nu-1)}.$$

[Throughout these investigations $\pi(x)$ is used in the same sense as Πx, to signify the factorial $1.2.3\ldots x$.]

If there should appear any obscurity in the statement of the method by which has been obtained the number of distinct distributions of the μ, ν elements into g groups of each, the reader is referred to the equation in differences obtained further on in this Note, by which all doubt of the correctness of the result will be removed.

(22) For a *trans-rotatory* pencil of rays, to ascertain the probability of the variation-index being $2g+1$.

Imagine a circular arrangement of μ positive elements and ν negative elements containing 2γ variations.

Let this circle be supposed opened out at any point and the variations of the open pencil so formed to be reckoned according to the trans-rotatory law, which is that in passing from one extremity to the other a change is to be seen as a variation, and a variation as a change. If the break is made between two negative or between two positive elements, the number of variations obviously becomes *increased* by one unit; but if between a positive and a negative element, that number becomes decreased by one unit. The number of these latter intervals is 2γ, and of the former $\mu+\nu-2\gamma$.

Hence the probability of the index becoming $2\gamma+1$ is $\frac{\mu+\nu-2\gamma}{\mu+\nu}$, and of its becoming $2\gamma-1$ is $\frac{2\gamma}{\mu+\nu}$.

If, then, we denote the probability to be calculated by $[\mu, \nu, g+\frac{1}{2}]$, it is obvious that we shall have

$$[\mu, \nu, g+\tfrac{1}{2}]=\frac{\mu+\nu-2g}{\mu+\nu}[\mu, \nu, g]+\frac{2(g+1)}{\mu+\nu}[\mu, \nu, g+1].$$

But by the formula previously obtained it will easily be seen that

$$[\mu, \nu, g+1]=\frac{(\mu-g)(\nu-g)}{g(g+1)}[\mu, \nu, g].$$

Hence

$$[\mu, \nu, g+\tfrac{1}{2}]=\frac{[\mu, \nu, g]}{\mu+\nu}\left\{(\mu+\nu-2g)+\frac{2(\mu-g)(\nu-g)}{g}\right\}$$

$$=\left(\frac{2\mu\nu}{g(\mu+\nu)}-1\right)[\mu, \nu, g] \quad \ldots\ldots\ldots\ldots\ldots \quad (*)$$

$$=2\frac{(\pi\mu)^2(\pi\nu)^2}{\pi(g-1)\pi(g+1)\pi(\mu+\nu)\pi(\mu-g)\pi(\nu-g)}-\frac{\pi\mu\pi(\mu-1)\pi\nu\pi(\nu-1)}{\pi(g-1)\pi g\pi(\mu+\nu-1)\pi(\mu-g)\pi(\nu-g)}.$$

When $g=0$ the above expression fails; but reverting to the equation from which it is derived, we obtain

$$(\mu, \nu, \tfrac{1}{2})=\frac{2}{\mu+\nu}[\mu, \nu, 1]=\frac{2\pi\mu\pi\nu}{\pi(\mu+\nu)}.$$

(23) These combined results admit of an easy corroboration, for

$$\Sigma_{\infty}^{0}(\mu, \nu, g+\tfrac{1}{2})=1, \text{ and } \Sigma_{\infty}^{0}(\mu, \nu, g)=1.$$

Hence the equation marked * gives

$$1=[\mu, \nu, \tfrac{1}{2}]+\frac{2\mu\nu}{\mu+\nu}\Sigma\frac{[\mu, \nu, g]}{g}-1.$$

Hence we ought to have

$$\frac{\pi\mu\pi\nu}{\pi(\mu+\nu)}+\frac{\mu\nu}{\mu+\nu}\Sigma\frac{[\mu, \nu, g]}{g}=1, \text{ i. e. } 1+\Sigma\frac{\pi\mu\pi\nu}{\pi(\mu-g)\pi g\pi(\nu-g)\pi g}=\frac{\pi(\mu+\nu)}{\pi\mu\pi\nu};$$

which is true, since the left-hand side of the equation is $1+\mu\nu+\mu\frac{\mu-1}{2}.\nu\frac{\nu-1}{2}+\ldots$, which is obviously the coefficient of x^{ν} in $(1+x)^{\mu}(x+1)^{\nu}$, i. e. in $(1+x)^{\mu+\nu}$.

(24) If we wish to find the chance of the specific superior limit becoming equal to the absolute superior limit, we must write g in the above formulæ equal to ν, that one of the two quantities μ, ν which is not greater than the other, and we shall obtain

$$[\mu, \nu, \nu] \quad =\frac{\pi\mu\pi(\mu-1)}{\pi(\mu+\nu-1)\pi(\mu-\nu)},$$

$$[\mu, \nu, \nu+\tfrac{1}{2}]=\frac{\pi\mu\pi(\mu-1)}{\pi(\mu+\nu)\pi(\mu-\nu-1)};$$

so that, in fact, $[\mu, \nu, \nu+\frac{1}{2}]=[\mu, \nu+1, \nu+1]$, which relation may also be obtained by *à priori* considerations.

(25) With reference to the remark made concerning the mode of obtaining the value of $[\mu, \nu, g]$, I proceed to show how it may be obtained directly by the integration of an equation in differences, and by a method analogous in idea to that by which $[\mu, \nu, g+\frac{1}{2}]$ was made to depend on $[\mu, \nu, g]$. For as in that case we conceived an open pencil to be closed and then reopened, so we may imagine one of the rays to be withdrawn and then reinserted. In this way, observing that the effect of introducing a negative sign into a circle of μ positive and n negative signs consisting of ν distinct groups of each is to produce no change in the number of the groups if inserted between two negative signs, but to increase that number by unity if inserted between two positive signs, we may infer that the probability of ν becoming $\nu+1$, in consequence of such insertion, is $\frac{\mu-\nu}{\mu+\nu}$, and of ν remaining unaltered, is $\frac{n+\nu}{\mu+n}$.

Hence we obtain the equation in differences,

$$[\mu, \nu, g]=\frac{\nu-1+g}{\mu+\nu-1}[\mu, \nu-1, g]+\frac{\mu-g+1}{\mu+\nu-1}[\mu, \nu-1, g-1],$$

in which μ may be considered constant, and ν and g to vary.

The integral must satisfy the further condition that $[\mu, 1, g]$ shall be unity when g is 1, and zero for all values of g greater than 1.

Assume the value of $[\mu, 1, g]$ obtained by the method given in art. (21). This obviously satisfies the initial conditions corresponding to $g=1$. Moreover we may easily deduce from it the equalities

$$[\mu, \nu-1, g-1]=\frac{(g-1)g}{(\mu-g+1)(\nu-g)}[\mu, \nu-1, g], \text{ and } [\mu, \nu, g]=\frac{(\nu-1)\nu}{(\mu+\nu-1)(\nu-g)}[\mu, \nu-1, g].$$

Hence the equation in differences will be satisfied if it be true that

$$\frac{(\nu-1)\nu}{\nu-g}=(\nu-1+g)+\frac{(g-1)g}{\nu-g},$$

which is obviously the case, since $\nu^2-\nu-g^2-g=(\nu-g)(\nu+g-1)$.

Since, then, the assumed value of $[\mu, \nu, g]$ is correctly determined when $\nu=1$, it is obvious, from the form of the equation, that it holds good for all other values of ν, as was to be shown.

(26) From the equation

$$\frac{[\mu, \nu, g+1]}{[\mu, \nu, g]}=\frac{(\mu-g)(\nu-g)}{g(g+1)}$$

making $(\mu-g)(\nu-g)=g(g+1)$ or $g=\frac{\mu\nu}{\mu+\nu+1}$, we may readily infer that the value of g for which the probability $[\mu, \nu, g]$ is greatest is the integer part of $\frac{\mu\nu}{\mu+\nu+1}$, if that quantity is non-integer, or the quantity itself and the number next below it (indifferently) if it is an integer.

(27) If we apply a similar method to $[\mu, \nu, g+\frac{1}{2}]$, we obtain by aid of the formula above given,

$$\frac{[\mu, \nu, g+\frac{1}{2}]}{[\mu, \nu, g-\frac{1}{2}]}=\frac{2\mu\nu-(\mu+\nu)\gamma}{2\mu\nu+\mu+\nu-(\mu+\nu)\gamma}\cdot\frac{(\mu+1)-\nu(\nu+1-\gamma)}{\gamma^2};$$

and equating this ratio to unity, we obtain

$$\frac{2\mu\nu-(\mu+\nu)\gamma}{2\mu\nu+\mu+\nu-(\mu+\nu)\gamma}=\frac{\gamma^2}{(\mu+1)(\nu+1)-(\mu+\nu+2)\gamma};$$

or writing $\mu+\nu=p$, $\mu\nu=q$,

$$(p^2+p)\gamma^2-(3pq+4q+p^2+p)\gamma+2q(q+p+1)=0.$$

The roots of this equation will be both of them real, for its *determinant* is

$$p^2q^2+16pq^2+16q^2+(p^2+p^3)(\mu^2+\nu^2),$$

which is necessarily positive. Hence it follows that there are two positive roots of the equation. Whether there will exist values of g which give actual maxima or minima values, or one and the other to $[\mu, \nu, g+\frac{1}{2}]$, depends on the further condition being satisfied that the values of g in the above equation shall come out, one or both of them, not greater than either of the two numbers μ, ν. The inquiry connected with the satisfaction of this condition may be conducted by means of repeated applications of the

processes of STURM's theorem; but I shall not enter upon it, as it appears to lead to calculations of complexity disproportionate to the interest of the result.

(28) It may be noticed that the *average* value of $[\mu, \nu, g]$ can be calculated without any difficulty. This will be $\Sigma(g[\mu, \nu, g])$, or

$$\frac{\pi\mu\pi\nu}{\pi(\mu+\nu-1)}\Big[1+\frac{(\mu-1)(\nu-1)}{1}+\frac{(\mu-1)(\mu-2)(\nu-1)(\nu-2)}{1.2^2}+\ldots\Big]$$

$$=\frac{\pi\mu\pi\nu}{\pi(\mu+\nu-1)}\cdot\frac{\pi(\mu+\nu-2)}{\pi(\mu-1)\pi(\nu-1)}=\frac{\mu\nu}{(\mu+\nu-1)};$$

so that the average number of variations of sign in a per-rotatory pencil with μ positive and ν negative signs is $\frac{2\mu\nu}{\mu+\nu-1}$, or a little more than the harmonic mean between μ, ν.

In like manner, for a trans-rotatory pencil this number will be

$$\Sigma(2g+1)[\mu, \nu, g+\tfrac{1}{2}]=[\mu, \nu, \tfrac{1}{2}]+\Sigma\Big((2g+1)\Big(\frac{2\mu\nu}{g(\mu+\nu)}-1\Big)[\mu, \nu, g]\Big),$$

which, observing that $\Sigma[\mu, \nu, g]=1$, and $(\mu, \nu, \frac{1}{2})+\frac{2\mu\nu}{\mu+\nu}\ \Sigma\frac{\mu, \nu, g}{g}=2$, gives as the average number of variations of sign $\frac{4\mu\nu}{\mu+\nu}-\frac{2\mu\nu}{\mu+\nu-1}+1$.

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(29) The simplest mode of calculating the value of $[\mu, \nu, g]$ is the following:

Let $[\mu, \nu, g)$, $\big[\mu, \nu, (g-\frac{1}{2})\big)$ denote the probability that an arrangement in open line (in which, as is the case in applying DES CARTES's rule of signs, no account is taken of the relation of the extreme signs to each other) shall contain respectively $2g$ and $2g-1$ variations. Conceive a circular arrangement of γ groups of positive and γ groups of negative signs. If this circle be opened out into a line at an interval between a positive and negative sign (of which there are 2γ), one variation will be lost; but if at any of the remaining $\mu+\nu-\gamma$ intervals, the number of variations remains unaltered. Hence we derive immediately

$$[\mu, \nu, g)=\frac{\mu+\nu-2g}{\mu+\nu}[\mu, \nu, g] \text{ and } [\mu, \nu, g-\tfrac{1}{2})=\frac{2g}{\mu+\nu}[\mu, \nu, g].$$

But we may find $\big[\mu, \nu, g-\frac{1}{2}\big)$ by counting the arrangements which give μ, ν, $2g-1$ variations of sign. These may be all obtained, and without repetition, by intercalating every distribution of μ into g groups with every distribution of ν into the same; and the intercalation may be performed in *two* ways, according as the parcels of the μ signs, or those of the ν signs, are taken first in order. Hence we have

$$[\mu, \nu, g-\tfrac{1}{2})=\frac{2(\mu-1)(\mu-2)\ldots(\mu-g+1)}{1.2\quad\ldots\quad(g-1)}\ \frac{(\nu-1)(\nu-2)\ldots(\nu-g+1)}{1.2\quad\ldots\quad(g-1)}\ \frac{\pi\mu\pi\nu}{\pi(\mu+\nu)}$$

$$=\frac{2\pi\mu\pi(\mu-1)\pi\nu\pi(\nu-1)}{\pi(\mu+\nu)\pi(g-1)\pi(g-1)\pi(\mu-g)\pi(\nu-g)};$$

and thus

$$[\mu, \nu, g]=\frac{\mu+\nu}{2g}[\mu, \nu, g-\tfrac{1}{2}]=\frac{\pi\mu\pi(\mu-1)\pi\nu\pi(\nu-1)}{\pi(\mu+\nu+1)\pi g\pi(g-1)\pi(\mu-g)\pi(\nu-g)},$$

as previously found; also

$$[\mu, \nu, g)=\frac{(\mu+\nu-2g)\pi\mu\pi(\mu-1)\pi\nu\pi(\nu-1)}{\pi(\mu+\nu)\pi g\pi(g-1)\pi(\mu-g)\pi(\nu-g)}.$$

(30) Moreover, we thus see that the average number of variations in an open line with μ positive and ν negative signs, which is

$$\Sigma(2g-1)[\mu, \nu, g-\tfrac{1}{2})+\Sigma 2g[\mu, \nu, g),$$

or

$$\Sigma 2g\big([\mu, \nu, g-\tfrac{1}{2})+[\mu, \nu, g)\big)-\Sigma[\mu, \nu. g-\tfrac{1}{2})\big)$$

will be equal to

$$\Sigma 2g[\mu, \nu, g]-\Sigma\frac{2g}{\mu+\nu}[\mu, \nu, g]=\frac{\mu+\nu-1}{\mu+\nu}\Sigma 2g[\mu, \nu, g]=\frac{\mu+\nu-1}{\mu+\nu}\cdot\frac{2\mu\nu}{\mu+\nu-1}=\frac{2\mu\nu}{\mu+\nu}.$$

The total number of variations and continuations together is $\mu+\nu-1$. Hence the difference between the two is $\frac{4\mu\nu}{\mu+\nu}-(\mu+\nu-1)$, or $\frac{(\mu+\nu)-(\mu-\nu)^2}{\mu+\nu}$; so that the average number of variations is greater than, equal to, or less than that of the continuations, according as the difference between the numbers of the two sets is less than, equal to, or greater than the square root of the entire number of signs. Obviously the average should be the same for the variations as for the continuations if the number of signs, say $n+1$, is given, and each is supposed equally likely to be positive or negative. This is easily verified; for multiplying the probable value of each distribution of signs by the probable value of the number of variations corresponding thereto, we obtain the series

$$\frac{1}{(n+1)2^n}\left\{1.n.(n+1)+2(n-1)(n+1)\frac{n}{2}+3(n-2)\frac{(n+1)n.(n-1)}{1.2.3}+\ldots\right\}=\frac{n(n+1)2^{n-1}}{(n+1)2^n}=\frac{n}{2}.$$

This is the final average of the number of variations of sign, and will be equal to that of the continuations, since the entire number of the two together is n.

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Part III.—ON THE NATURE OF THE ROOTS OF THE GENERAL EQUATION OF THE FIFTH DEGREE.

(31) In a foot-note, Part II. of this memoir, I have shown that when the discriminant of the canonizant (constituting an invariant of the twelfth order) of an equation of the fifth degree bears a particular sign, the character of the roots becomes completely determined by the sign of the discriminant of that equation.

This has naturally led me to investigate *de novo* the whole question of the character of the roots of an equation of that degree; and I have succeeded in obtaining under a form of striking and unexpected simplicity the invariantive criteria which serve to ascertain in all cases the nature of the equation as regards the number of real and imaginary roots which it contains; then passing to the expression for these criteria in terms of the roots themselves, I obtain expressions which exhibit the intimate connexion between this subject and a former theory of my own relative to the construction of the conditions for the existence of a given number and grouping of equal roots, which can hardly fail to lead eventually to the extension of the results herein obtained to equations of any odd degree whatever. It is the more needful that these results in a question of so high moment to the advancement of algebraical science should be made public, inasmuch as they do not seem to accord with those obtained by my eminent friend M. Hermite, who has preceded me in this inquiry in a classic memoir, published in the year 1854 in

the ninth volume of the Cambridge and Dublin Mathematical Journal, since which time I am not aware that the subject has been resumed by any other writer. The discrepancy between our conclusions may be only apparent; but there can be no doubt of the superiority of the form in which they are herein presented, inasmuch as only three functions of the coefficients are required by my method, and five by M. HERMITE's. The solution offered by M. HERMITE is confessedly incomplete, but to this great analyst none the less will always belong the honour, not only of having initiated the inquiry, but of having emitted the fundamental conceptions through which it would seem best to admit of successful treatment. The arrow from my hand may have been the first to hit the mark, but it was his hand which had previously shaped, bent, and strung the bow.

Our methods of procedure, however, are widely dissimilar, and by employing my well-known canonical form for odd-degreed binary quantics, long since given to the world, I have succeeded in evading all necessity for the colossal labours of computation required in M. HERMITE's method, and am able to impart to my conclusions the clearness and certainty of any elementary proposition in geometry, not scrupling to avail myself for such purpose of that copious and inexhaustible well-spring of notions of continuity which is contained in our conception of space, and which renders it so valuable an auxiliary to Mathematic, whose sole proper business seems to me to be the development of the three germinal ideas—of which continuity is one and order and number the other two*.

SECTION I.—*Preparation of the General Binary Quantic of the Fifth Degree.*

(32) Let $$(a, b, c, d, e, i\between x, y)^5 = \mathrm{F}(x, y),$$
a cubic covariant of F is the canonizant C, where C represents the determinant

$$\begin{matrix} a & b & c & d \\ b & c & d & e \\ c & d & e & i \\ y^3 & -y^2x & yx^2 & -x^3. \end{matrix}$$

Let us first suppose that this form does not vanish identically, and has at least two distinct factors ξ, η linear functions of x, y, where of course ξ, η are each of them determinate to a constant factor *près*; giving any value to the constant factor for either of them, we may write $\mathrm{F}(x, y) = \Phi(\xi, \eta) = (\alpha, \beta, \gamma, \delta, \varepsilon, \iota\between\xi, \eta)^5$, and the canonizant of Φ with respect to ξ, η becomes the determinant T, where T represents

$$\begin{matrix} \alpha & \beta & \gamma & \delta \\ \beta & \gamma & \delta & \varepsilon \\ \gamma & \delta & \varepsilon & \\ \eta^3 & -\eta^2\xi & \eta\xi^2 & -\xi^3. \end{matrix}$$

* Herein I think one clearly discerns the internal grounds of the coincidence or parallelism, which observation has long made familiar, between the mathematical and musical ἔθος. May not Music be described as the Mathematic of sense, Mathematic as Music of the reason? the soul of each the same! Thus the musician *feels* Mathematic, the mathematician *thinks* Music,—Music the dream, Mathematic the working life—each to receive its consummation from the other when the human intelligence, elevated to its perfect type, shall shine forth glorified in some future MOZART-DIRICHLET or BEETHOVEN-GAUSS—a union already not indistinctly foreshadowed in the genius and labours of a HELMHOLTZ!

Hence since T to a constant factor *près* is identical with C, the coefficients of $\eta^3$and ξ^3 in the above determinant must vanish in order that $\xi\eta$ may be contained in T.

Hence the two determinants

$$\begin{matrix} \alpha & \beta & \gamma \\ \beta & \gamma & \delta \\ \gamma & \delta & \varepsilon \end{matrix} \quad \text{and} \quad \begin{matrix} \beta & \gamma & \delta \\ \gamma & \delta & \varepsilon \\ \delta & \varepsilon & \iota \end{matrix}$$

both vanish.

Hence either α, β, γ, or otherwise γ, δ, ε, or else the first minors of

$$\begin{vmatrix} \beta & \gamma \\ \gamma & \delta \\ \delta & \varepsilon \end{vmatrix}$$

are each zero.

The first two suppositions must be excluded, since either of them would lead to the conclusion of T, and therefore C, being a perfect cube, contrary to hypothesis. The last supposition implies either that β, γ, δ, or otherwise that γ, δ, ε, or else that $\beta\delta-\gamma^2$ and $\gamma\varepsilon-\delta^2$ are each zero.

If β, γ, δ are each zero, T becomes a multiple of $\eta^2\xi$; if γ, δ, ε are each zero, T becomes a multiple of $\eta\xi^2$; that is to say, T, and consequently C, contains a square factor; and obviously the converse is true, so that when C contains a square factor F is reducible to the form $au^5+5euv^4+fv^5$. When this is not the case $\delta=\frac{\gamma^2}{\beta}$, $\varepsilon=\frac{\delta^2}{\gamma}=\frac{\gamma^3}{\beta^2}$. Hence

$$F=\left(\alpha-\frac{\beta^2}{\gamma}\right)\xi^5+\frac{\beta}{\gamma}\left(\xi+\frac{\gamma}{\beta}\eta\right)^5+\left(\iota-\frac{\varepsilon^2}{\delta}\right)\eta^5,$$

which is of the form $\omega^5+\phi^5+\psi^5$, ω, ϕ, ψ being linear functions of x, y.

(33) We have supposed C not to be a perfect cube. When it is a perfect cube, say ξ^3, we may assume η any second linear function of x, y; and expressing F in the same manner as before in terms of ξ, η, it is clear that all the first minors of

$$\begin{matrix} \alpha & \beta & \gamma & \delta \\ \beta & \gamma & \delta & \varepsilon \\ \gamma & \delta & \varepsilon & \iota, \end{matrix}$$

except the one obtained by cancelling the last column in the above matrix, must vanish, consequently δ, ε, i must all vanish, so that Φ, and consequently F, must contain a cube factor identical with the canonizant itself.

Lastly, if the canonizant vanish entirely, every first minor in the above matrix, when we write again a, b, c, d, e, i in lieu of α, β, γ, δ, ε, ι, will be zero. Hence either a, b, c, d, or b, c, d, e, or c, d, e, i must each vanish, or else that must be the case with the first minors of

$$\begin{matrix} a & b & c & d \\ b & c & d & e, \end{matrix}$$

or of

$$\begin{matrix} b & c & d & e \\ c & d & e & i, \end{matrix}$$

or of

$$\begin{matrix} a & b & c & d \\ c & d & e & i. \end{matrix}$$

Under the first or third supposition F must contain four equal factors; under the second Φ becomes $a\xi^5+i\eta^5$; under the fourth or fifth it is readily seen that the form becomes

$$a\left(\xi+\frac{b}{a}\eta\right)^5+\left(i-\frac{e^2}{d}\right)\eta^5, \text{ or } \left(a-\frac{b^2}{c}\right)\xi^5+i\left(\eta+\frac{e}{i}\xi\right)^5$$

respectively, so that the second, fourth, and fifth suppositions conduct alike to the form $\omega^5+\phi^5$, a particular case of the preceding one.

It remains only to consider the sixth supposition, viz. that the first minors of

$$\begin{matrix} a & b & c & d \\ c & d & e & i \end{matrix}$$

are all zero.

In this case if we write

$$\sqrt{a}x+\sqrt{c}y=u,$$
$$\sqrt{a}x-\sqrt{c}y=v,$$
$$A+B=\frac{1}{a^{\frac{3}{2}}},$$
$$A-B=\frac{b}{a^2c^{\frac{1}{2}}},$$

and if neither a nor c is zero, it will readily be seen that $F(x, y)$ becomes Au^5+Bv^5 by virtue of the relations

$$d=\frac{c}{a}b, \quad e=\left(\frac{c}{a}\right)^2 a, \quad i=\left(\frac{c}{a}\right)^2 b\,(^{29}).$$

If $a=0$ or $c=0$, the preceding transformation fails.

But unless also $i=0$ or $e=0$ at the same time as $a=0$ or $c=0$, a legitimate transformation similar to the above may be performed by interchanging a, c, x, y with i, a, y, x.

If now

$a=0$, it will easily be seen that a, b, c, d or else a, c, e are each zero.

Similarly, if

$i=0$, it will easily be seen that i, e, d, c or else i, d, b are each zero.

Again, if

$c=0$, it will easily be seen that a, b, c, d or else c, e are each zero;

and if

$d=0$, it will easily be seen that c, d, e, i or else d, b are each zero.

(29) Thus we see that the equation $ax^5+5bx^4+10acx^2+10bcx^2+5ac^2x+bc^2=0$ belongs to the class of soluble forms.

Thus, then, if $a=0$ and $i=0$, all the coefficients, or else all except one, viz. b or e, are zero;

if $a=0$ and $d=0$, all the coefficients, or else only not e and i or only not b or only not i are zero;

so if $i=0$ and $c=0$, all must be zero except b and a or e or a;

if $c=0$ and $d=0$, only e and i or else a and b or else a and i will differ from zero.

Hence, then, in any case there will be at least four equal roots, or else F is of the form ax^5+iy^5.

Thus, then, for the first time has been here rigorously demonstrated, free from all doubt and subject to no exceptions, the following important proposition:

Every binary quantic function *not containing three or more equal roots* is reducible to one or the other of the two following forms,

$$u^5+v^5+w^5, \text{ or } au^5+5euv^4+fv^5.$$

The former is the case when the discriminant of the canonizant is different from zero, the latter when it is equal to zero; for it will be observed that, whether the canonizant has equal roots or totally disappears, its discriminant in both cases alike is zero.

(34) It has been seen that when the quintic has three equal roots the canonizant becomes a perfect cube; and it may not be out of place here to point out what the conditions (necessary and sufficient) are to ensure the quintic having four equal roots. These are all comprised in that of the quadratic covariant vanishing. To prove this, let η be a factor of $F(x, y)$, so that

$$F(x, y)=\Phi(x, \eta)=(\alpha, \beta, \gamma, \delta, \varepsilon, 0 \between x, \eta)^5.$$

Then, since the similar covariant *quoad* x, y must also vanish, we have

$$\alpha\varepsilon-4\beta\delta+\gamma^2=0, \quad -3\beta\varepsilon+2\gamma\delta=0, \quad -4\gamma\varepsilon+3\delta^2=0.$$

If $\varepsilon=0$, then $\delta=0$, $\gamma=0$ by virtue of the two extreme equations, and Φ, and therefore F, contains four equal factors. If ε is not zero,

$$\gamma=\frac{3\delta^2}{4\varepsilon},\ \beta=\frac{\delta^3}{2\varepsilon^2},\ \alpha=\frac{5\delta^4}{16\varepsilon^3}, \text{ and } \Phi \text{ becomes } \frac{5\varepsilon}{16}x\left(\frac{\delta}{\varepsilon}x+2\eta\right)^4;$$

so that, as before, there are four equal factors. Conversely, it is obvious that if there are four equal factors u, so that $\Phi=au^5+5bu^4v$, the quadratic covariant of Φ disappears.

(35) The quadratic covariant also it was which led me to perceive the transformation applied in the antecedent article. For when the first minors of

$$\begin{vmatrix} a & b & c & d \\ c & d & e & f \end{vmatrix}$$

are all zeros, the quadratic covariant becomes

$$4(c^2-bd)x^2+4(d^2-ce)y^2.$$

Supposing neither of those coefficients to vanish, and calling its two factors u and v, and making

$$F(x, y)\Phi(u, v)=(\alpha, \beta, \gamma, \delta, \varepsilon, i \between u, v),$$

it is clear that the minors of

$$\begin{vmatrix} \alpha & \beta & \gamma & \delta \\ \gamma & \delta & \varepsilon & i \end{vmatrix}$$

can no longer all be zero, since in that case we should have

$$4(\gamma^2-\beta\delta)u^2+4(\delta^2-\gamma\varepsilon)v^2$$

containing u, v as factors. Consequently the canonizant of Φ must vanish under one or the other of those remaining suppositions which had been previously shown to conduct to the form au^5+bv^5, or else to the case of three or more equal roots. When the quadratic covariant vanishes, we know that there must be four equal roots; and when it becomes a perfect square but does not vanish, it will be found on examination that the equation has three equal roots.

(36) Returning to the general case, where $\Phi=u^5+v^5+w^5$, and making $\frac{u}{r^{\frac{1}{5}}}+\frac{v}{s^{\frac{1}{5}}}+\frac{w}{t^{\frac{1}{5}}}$ identically zero, and writing u', v', w' for $\frac{u}{r^{\frac{1}{5}}}, \frac{v}{s^{\frac{1}{5}}}, \frac{w}{t^{\frac{1}{5}}}$ respectively, Φ becomes $ru'^5+sv'^5+tw'^5$, or, if we please, $ru^5+sv^5+tw^5$, with the condition $u+v+w=0$.

Moreover u, v, w will all three be factors of the canonizant of F. For taking the canonizant of F with respect to u, v, it becomes

$$\begin{matrix} r-t & -t & -t & -t \\ -t & -t & -t & -t \\ -t & -t & -t & s-t \\ v^3 & -v^2u & vu^2 & -u^3, \end{matrix} \quad \text{or } r\times\left\{\begin{matrix} 1 & 0 & 0 & 0 \\ -1 & -1 & -1 & -1 \\ -1 & -1 & -1 & 1 \\ v^3 & -v^2u & vu^2 & -u^3 \end{matrix}\right\}$$

or $rst(uv^2+vu^2)$, i. e. $-rst(uvw)$.

Hence if $x+ey$, $x+fy$, $x+gy$ are three distinct factors of the canonizant of F with respect to x, y, if we choose the ratios $\lambda : \mu : \nu$ so that $\lambda+\mu+\nu=0$, $e\lambda+f\mu+g\nu=0$, we may make $u=\lambda(x+ey)$; $v=\mu(x+fy)$; $w=\nu(x+gy)$; and shall then have

$$F(x, y)=ru^5+sv^5+tw^5, \text{ with the condition } u+v+w=0,$$

where r, s, t may be found from three equations obtained by identifying any three of the six terms in F with the corresponding terms $ru^5+sv^5+tw^5$ expressed as a function of x, y. These equations being linear, it follows that ru^5, sv^5, tw^5 form a *single and unique* system of functions of x, y.

So when the canonizant has two equal roots and is of the form $C(x+py)(x+qy)^2$; in which case the reduced form is $au^5+5euv^4+fv^5$. The canonizant in respect to u, v becomes

$$\begin{matrix} a & 0 & 0 & 0 \\ 0 & 0 & 0 & e \\ 0 & 0 & e & f \\ v^3 & -v^2u & vu^2 & -u^3, \end{matrix}$$

i. e. ae^2uv^2. Hence, writing

$$u=x+py, \quad v=x+qy, \quad F=au^5+5euv^4+fv^5,$$

a, e, f may be obtained, as before, by means of three linear equations, and the terms au^5, $5euv^4$, fv^5 form a single and unique system.

Finally, when the canonizant vanishes entirely, so that the form becomes au^5+fv^5, the quadratic covariant will take the form $C(x+ey)(x+fy)$; and making $u=x+py$, $v=x+qy$, a, f become determined by means of two linear equations, so that au^5, fv^5 form a single and unique system, as in the preceding cases.

(37) When the canonizant has three distinct roots, they may be all real, or one real and the other two imaginary. In the former case, in the expression $ru^5+sv^5+tw^5$, u, v, w may be considered as all real functions of x, y, and r, s, t will then also all of them be real. In the latter case w may be taken as a real function of x, y, u, v as conjugate imaginary functions; and consequently it is easy to see that, except when r, s are equal to each other, they will constitute a pair of conjugate imaginary quantities: in this case we may take for our canonizant form

$$r\left(\frac{-u+iv}{2}\right)^5+s\left(\frac{-u-iv}{2}\right)^5+tu^5;$$

or, if we please,

$$ru_{\prime}^5+sv_{\prime}^5+tw^5$$

understanding by $u_{\prime}$, $v_{\prime}$ $\frac{-u+iv}{2}$, $\frac{-u-iv}{2}$ respectively. And it should be noticed that the determinant of $u_{\prime}$, $v_{\prime}$ in respect to u, v will be

$$\begin{vmatrix} -\frac{1}{2} & \frac{i}{2} \\ -\frac{1}{2} & \frac{-i}{2} \end{vmatrix}$$

which is i.

(38) Let us proceed briefly to express the invariants of $ru^5+sv^5+tw^5$, which call Φ, with respect to u, v; the corresponding ones of $ru_{\prime}^5+sv_{\prime}^5+tw^5$, which call $\Phi_{\prime}$, in respect to the same variables u, v will be found by attaching to these suitable powers of i.

$$\Phi=(r-t,\ -t,\ -t,\ -t,\ -t,\ s-t\between u, v)^5.$$

Hence its quadratic covariant is the quadratic invariant of

$$\left((r-t)u-tv,\ -tu-tv,\ -tu-tv,\ -tu-tv,\ -tu+(s-t)v \between u',\ v'\right)^4,$$

which is obviously

$$-rtu^2-stv^2+(rs-rt-st)uv.$$

Of this the quadratic invariant is

$$rt\,.\,st-\tfrac{1}{4}(rs-rt-st)^2;$$

or writing $\varrho=st$, $\sigma=tr$, $\tau=rs$, and calling this invariant (I),

$$(\mathrm{I})=-\tfrac{1}{4}(\varrho^2+\sigma^2+\tau^2-2\varrho\sigma-2\sigma\tau-2\tau\varrho).$$

Again, the cubic covariant or canonizant has been already shown to be $rst(u^2v+uv^2)$. Calling the discriminant of this (L), we have

$$(\mathrm{L})=-\tfrac{1}{27}r^4s^4t^4\,(^{30})=-\tfrac{1}{27}\varrho^2\sigma^2\tau^2.$$

Again, to find the discriminant (D) in respect to u, v.

When $ru^5+sv^5+tw^5=0$ has two equal roots, and $u+v+w=0$, it is easy to see that we have $ru^4+\lambda=0$, $sv^4+\lambda=0$, $tw^4+\lambda=0$.

Hence to a constant factor *près* (D) will be the *Norm* of

$$(st)^{\frac{1}{4}}+(tr)^{\frac{1}{4}}+(rs)^{\frac{1}{4}},\ \text{i. e. of}\ \varrho^{\frac{1}{4}}+\sigma^{\frac{1}{4}}+\tau^{\frac{1}{4}}\,(^{31}).$$

To find the value of this norm, suppose $\varrho^{\frac{1}{4}}+\sigma^{\frac{1}{4}}+\tau^{\frac{1}{4}}=0$, then

$$\varrho+\sigma+\tau=2(\varrho^{\frac{1}{2}}\sigma^{\frac{1}{2}}+\sigma^{\frac{1}{2}}\tau+\tau^{\frac{1}{2}}\varrho^{\frac{1}{2}}),$$

and

$$\varrho^2+\sigma^2+\tau^2-2\varrho\sigma-2\varrho\tau-2\sigma\tau=8\varrho^{\frac{1}{2}}\sigma^{\frac{1}{2}}\tau^{\frac{1}{2}}(\varrho^{\frac{1}{2}}+\sigma^{\frac{1}{2}}+\tau^{\frac{1}{2}}).$$

Hence

$$(\varrho^2+\sigma^2+\tau^2-2\varrho\sigma-2\varrho\tau-2\sigma\tau)^2=64\varrho\sigma\tau\{(\varrho+\sigma+\tau)+2(\varrho^{\frac{1}{2}}\sigma^{\frac{1}{2}}+\sigma^{\frac{1}{2}}\tau^{\frac{1}{2}}+\tau^{\frac{1}{2}}\varrho^{\frac{1}{2}})\}=128\varrho\sigma\tau(\varrho+\sigma+\tau).$$

Hence (D) must contain $(\mathrm{J})^2-128\varrho\sigma\tau(\varrho+\sigma+\tau)$ as a factor; and since when $t=0$, $\varrho=0$, $\sigma=0$, and $(\mathrm{D})=\tau^4=(\mathrm{J})^2$, it is clear that $(\mathrm{D})=(\mathrm{J})^2-128(\mathrm{K})$, where

$$(\mathrm{K})=\varrho\sigma\tau(\varrho+\sigma+\tau).$$

(39) Although in the investigation in view (K) will only figure as an abbreviation of $\frac{(\mathrm{D})-(\mathrm{J})^2}{128}$, it may not be amiss to indicate a direct process for finding it. Let us for this purpose act upon the Hessian of Φ, treated as a function of u, v twice with the canonizant of Φ converted into an operator by substituting $\frac{d}{dv}$, $-\frac{d}{du}$ in place of u and v.

(30) For this is $\left(0, \frac{rst}{3}, \frac{rst}{3}, 0 \between u, v\right)^3$, and the discriminant of $\left(a, b, c, d \between u, v\right)^3$ is

$$a^2d^2+4ac^3+4db^3-3b^2c^2-6abcd.$$

(31) It is worthy of observation that (J) is also a Norm, viz. of $\varrho^{\frac{1}{2}}+\sigma^{\frac{1}{2}}+\tau^{\frac{1}{2}}$, so that (J) is the discriminant of $ru^3+sv^3+tw^3$. I have not been able to perceive the morphological significancy of this relation.

The Hessian of Φ may be obtained without difficulty under the form

$$rsu^3v^3+stv^3w^3+trw^3u^3 \text{ or } \tau u^3v^3+\varrho v^3w^3+\sigma w^3u^3\ (^{32}).$$

Operating upon this with

$$r^2s^2t^2\left(\frac{d}{dv}\cdot\frac{d}{du}\ \left(\frac{d}{du}-\frac{d}{dv}\right)\right)^2,$$

we obtain $\varrho\sigma\tau(A\tau+B\varrho+C\sigma)$, where

$$A=-2\left(\frac{d}{du}\right)^3\left(\frac{d}{dv}\right)^3u^3v^3=-72;$$

and as we know that this quantity must be of the form $\lambda(K)+\mu(J)^2$, we have $\mu=0$, $\lambda=-72$; so that, denoting the *operator* corresponding to the canonizant by T, and the Hessian by H, we have $(K)=-\frac{1}{72}T^2H\Phi$ (33). This gives a ready practical method for finding the discriminant of a general quintic F by means of the identity $D=J^2+\frac{16}{9}T^2H$, where D is the discriminant, H the Hessian, T the canonizantive operator, and J the quadratic invariant of F in respect to its own variables.

(40) If now we suppose the determinant of u, v in respect to x, y to be μ, where μ is by hypothesis a real quantity, and if we call the

Quadratic invariant in respect to x, y	. .	$-\frac{1}{4}J$,
Discriminant of primitive ,, ,,	. .	D,
Discriminant of the canonizant ,,	. .	$-\frac{1}{27}L$,

we have obviously

$$\left.\begin{array}{l} J=\mu^{10}(\varrho^2+\sigma^2+\tau^2-2\varrho\sigma-2\varrho\tau-2\sigma\tau), \\ K=\mu^{20}\varrho\sigma\tau(\varrho+\sigma+\tau),\quad D=J^2-128K, \\ L=\mu^{30}\varrho^2\sigma^2\tau^2, \end{array}\right\}\text{invariants of }\Phi.$$

This applies to the case where the reduced form is Φ, *i. e.* where the roots of the canonizant are all real, and consequently where $-L$ is negative, *i. e.* L positive.

When L is negative and the reduced form is $\Phi_{\prime}$, then, since the determinant of $u_{\prime}$, v, in respect to $u_{\prime}$ v is $\imath$, we have

$$\left.\begin{array}{l} J=-\mu^{10}(\varrho^2+\sigma^2+\tau^2-2\varrho\sigma-2\varrho\tau-2\sigma\tau), \\ K=\ \ \mu^{20}\varrho\sigma\tau(\varrho+\sigma+\tau),\quad D=J^2-128K, \\ L=-\mu^{30}\varrho^2\sigma^2\tau^2, \end{array}\right\}\text{invariants of }\Phi_{\prime}.$$

By means of the ratios $\frac{L}{J^3}$, $\frac{K}{J^2}$, it is obvious that in either case alike the ratios of ϱ, σ, τ

(32) It will be the quadratic invariant of $ru^3\xi^2+sv^3\eta^2+tw^3\zeta^2$ with respect to ξ, η, $\zeta+\eta+\zeta$ being zero; just as the quadratic covariant of Φ is the quadratic invariant of $ru\xi^4+sv\eta^4+tw\zeta^4$ with regard to the same variables. This latter is in fact $rsuv+stvw+trwu$.

(33) The intervening covariantic form of degree 3 in the variables and 5 in the coefficients, viz. $TH\Phi$, will easily be seen to be

$$rst^2(u^2v-uv^2)+str^2(v^2w-vw^2)+trs^2(w^2u-wu^2).$$

become determinable by means of the same cubic equations, viz.

$$\theta^3-K\theta^2+\frac{K^2-JL}{4}\theta-L^2=0;$$

ϱ, σ, τ will be to each other as the roots of this equation([34]).

(41) Since $ru^5+sv^5+tu^5$ represents a function in x, y with real coefficients, it follows that when L is positive, u, v as well as w being real, $\alpha:\beta:\gamma$ are ratios of real quantities, and the roots of the preceding cubic will be real; when L is negative, u, v becoming conjugate imaginary functions of x, y, whilst w remains real, r, s, unless they are equal, must become conjugate imaginary constants. When r, s, t are all real, ϱ, σ, τ will be so too; and when r, s are imaginary and t real, ϱ, σ will be imaginary and τ real. Thus according as L is positive or negative the roots of θ are or are not all real. Hence understanding by Δ the discriminant of the preceding equation with respect to θ and 1, $\frac{\Delta}{L}$ must be always either zero or negative. We see *à priori* that $\frac{\Delta}{L}$ must be integer, because when $L=0$ the cubic has two equal roots, $\frac{L}{2}$. To compute its value more conveniently, write $K=6k$, $J=12j$. Then the equation becomes

$$(1,\ 2k,\ 3k^2-jL,\ L^2\between\theta,\ -1)^3,$$

of which the discriminant is

$$L^4+4(3k^2-jL)^3+32k^3L^2-12k^2(3k^2-jL)^2-12kL^2(3k^2-jL).$$

Hence

$$\begin{aligned}\frac{\Delta}{L}&=L^3-108k^4j+36k^2j^2L-4j^3L^2+32k^3L\\&\qquad+\ 72k^4j-12k^2j^2L\qquad\quad -36k^3L+12jkL^2\\&=L^3-\ 36k^4j+24k^2j^2L-4j^3L^2-\ 4k^3L+12jkL^2.\end{aligned}$$

Accordingly, multiplying the above equation by $-3\cdot12^2$ in order to avoid fractions, replacing k, j by their values in terms of K, J, and naming G the quantity $-432\,\frac{\Delta}{L}$,

([34]) For since the absolute values of ϱ, σ, τ are not in question, we may consider ϱ, σ, τ as the roots of $\theta^3-K\theta^2+q\theta-r$, so that $\varrho+\sigma+\tau=K$. We have then

$$\frac{\varrho^4\sigma^4\tau^4}{(\varrho\sigma\tau)^3(\varrho+\sigma+\tau)^3}=\frac{L^2}{K^3},\quad\text{or}\quad\frac{r}{K^3}=\frac{L^2}{K^3},$$

which gives $r=L^2$. Again,

$$\frac{\varrho\sigma\tau K^2}{(K^2-4q)^2}=\frac{K^2}{J^2},\quad\text{or}\quad\frac{(K^2-4q)^2}{r}=J^2,\quad\text{or}\quad(K^2-4q)^2=L^2J^2,\quad\text{or}\quad q=\frac{K^2\mp IL}{4}.$$

As regards the sign to be given to JL in q, since

$$\frac{J^3}{L}=\frac{(K^2-4q)^3}{r^2}=\frac{(K^2-4q)^3}{L^4},$$

we have $(K^2-4q)^3=J^3L^3$. Hence

$$q=\frac{K^2-1\tfrac{1}{3}JL}{4}.$$

Consequently

$$q=\frac{K^2-JL}{4},\ \text{and } not\ \frac{K^2+JL}{4}.$$

positive, or to speak more strictly non-negative, we have

$$G=JK^4+8LK^3-2J^2LK^2-72JL^2K-432L^3+J^3L^2\,(^{35}).$$

It is evident that G must be identical to a positive numerical factor *près* with the function which M. HERMITE denotes by I^2 (36).

(35) It will be observed that when $J=0$ and $L=0$, G vanishes. This is easily verifiable *à priori*; for when $J=0$ and $L=0$, the reduced form has been seen to be ax^5+5exy^4, of which the canonizant is

$$\begin{vmatrix} a & 0 & 0 & 0 \\ 0 & 0 & 0 & e \\ 0 & 0 & e & 0 \\ y^3 & -y^2x & yx^2 & -x^3 \end{vmatrix}$$

which equals $aexy^2$.

Hence the form and its canonizant have a common factor x, and consequently their resultant vanishes; hence $I=0$ and $G=I^2=0$. G also vanishes when $K=0$ and $L=0$, which is also easily verifiable; for then the reduced form becomes u^5+v^5, of which the canonizant vanishes, and consequently the resultant of the form and its canonizant becomes intensely zero; which accounts for the high power of K in (JK^4), the sole term of G in which L does not appear.

(36) (a) Compare expression for $16I^2$, Cambridge and Dublin Journal, p. 203. This will be found to contain nine terms, and to rise as high as the fifth power in Δ (which to a constant factor *près* is identical with my J); whereas in $\frac{-\Delta}{L}$ there are only six terms, and no power of J beyond the third. This seems to indicate that the K and L are more fortunately chosen than M. HERMITE's J_2, J_3, which are invariants of the like degrees 8 and 12. It is of course evident that the following relations exist between M. HERMITE's Δ_1, J_2, J_3 and the J, K, L of this paper,

$$\Delta=lJ,$$
$$J_2=mJ^2+nK,$$
$$J_3=pJ^3+qJK+rL,$$

where l, m, n, p, q, r are certain numerical quantities. Until these are ascertained, it is impossible to confront M. HERMITE's results with my own, to ascertain whether or not they are identical in substance, and, if not, wherein the difference consists. I therefore subjoin the necessary calculations for effecting this important object.

Let us first take the form $x^5+5exy^4+y^5$. The quadratic covariant of this is $x(ex+y)$.

Accordingly, to obtain M. HERMITE's A, B, C, C′, B′, A′ (Cambridge and Dublin Journal, vol. ix. p. 179), we must make

$$x=X; \quad ex+y=Y,$$

which gives (vide C. and D. J. p. 180)

$$F=X^5+5eX(Y-eX)^4+(Y-eX)^5$$
$$=(A, B, C, C', B', A' \between X, Y)^5,$$

where

$$A=1+4e^5,\ B=-3e^4,\ C=2e^3,\ C'=-e^2,\ B'=0,\ A'=1.$$

Accordingly (vide C. and D. J. p. 184),

$$AA'-3BB'+2CC'=1+4e^5-4e^5=1 \qquad =\sqrt{\Delta},$$
$$AA'+BB'-2CC'=1+4e^5+4e^5=1+8e^5=\frac{I_1}{2\sqrt{\Delta^3}},$$
$$AA'+5BB'+10CC'=1+4e^5-20e^5=1-16e^5=\frac{I_2}{2\sqrt{\Delta^5}}.$$

Hence

$$\Delta=1,\quad I_1=2+16e^5,\quad I_2=2-32e^5.$$

Again (vide C. and D. J. p. 186. § vii.),

$$8J_1=I_1-\Delta^2=1+16e^5,\quad 24J_2=I_2-2I_1\Delta+\Delta^3=-1-64e^5;$$

(42) In fact M. HERMITE's octodecimal invariant is most simply obtained as the resultant of the primitive quartic and its canonizant. Using the reduced forms for these two

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but J_1, J_2 are subsequently *without warning* (compare expressions for AA′, BB′, CC′, pp. 186, 192) renamed J_2, J_3; so that

$$8J_2=1+16e^5,\quad 24J_3=-1-64e^5.$$

The corresponding values of J, K, L have been already calculated, and we have found

$$J=1,\quad K=-2e^5,\quad L=0.$$

Hence

$$A=1,\quad \frac{1}{8}+2e^5=B-2Ce^5,\quad \frac{-1}{24}-\frac{64}{24}e^5=D-2Ee^5.$$

Thus

$$A=1,\quad B=\frac{1}{8},\quad C=-1,\quad D=-\frac{1}{24},\quad E=\frac{4}{3}.$$

To find F, take another form convenient for the purpose, as $x^5+10dx^2y^3+y^5$.

Taking the emanant of this $(x,\ 0,\ dy,\ dx,\ y \between x',\ y')^4$, the quadratic covariant is obviously $xy+3d^2y^2$, so that $J=1$.

Also its discriminant is

$$\begin{vmatrix} 1 & 0 & 0 & d \\ 0 & 0 & d & 0 \\ 0 & d & 0 & 1 \\ y^3 & -y^2x & yx^2 & -x^3 \end{vmatrix}$$

$$\text{viz. } d^3y^3-d(-dx^3+y^2x)=d^3y^3-dy^2x+d^2x^3,$$

of which the discriminant is

$$d^{10}+4d^2\left(\frac{-d}{3}\right)^3=d^{10}-\tfrac{4}{27}d^5.$$

Hence by definition $\qquad L=e-\frac{27}{4}d^{10}+d^5.$

Again, to find A, B, C, C′, B′, A′, we must write

$$x+3d^2y=X,$$
$$y=Y,$$

and we have then

$$(X-3d^2Y)^5+10d(X-3d^2Y)^2Y^3+Y^5=(A,\ B,\ C,\ C',\ B',\ A' \between X,\ Y)^5.$$

Since $J=1$ and K is of the eighth order only in the coefficients, it is obvious that neither J^3 nor JK can contain a term involving d^{10}. In order therefore to find F, it will be sufficient to compare the coefficient of d^{10} in J_3 and in L.

Now $\quad A=1,\quad B=-3d^2,\quad C=9d^4,\quad C'=27d^5+d,\quad B'=81d^8-12d^3,\quad A'=243d^{10}+90d^5+1.$

Also $\Delta=J=1$. Hence neglecting all but the terms which bring in d^{10}, $24J_3$ (p. 186, Memoir) is tantamount to I_2, and I_2 (p. 186) is tantamount to

$$2(243d^{10}-5\,.\,3\,.\,81d^{10}+10\,.\,9\,.\,27d^{10}),$$

which is

$$12\times 243d^{10}.$$

Hence in J_3 the term containing d^{10} is $\frac{243}{2}$.
Hence $-\frac{27}{4}F=\frac{243}{2}$, or $F=-18$.
Hence we have, finally,

$$\Delta=J,$$
$$J_2=-K+\tfrac{1}{8}J^2,$$
$$J_3=-18L+\tfrac{4}{3}JK-\tfrac{1}{24}J^3;$$

functions,

$$ru^5+sv^5-t(u+v)^5;\quad rstuv(u+v),$$

and conversely,

$$J=\Delta,$$
$$K=-\frac{J}{2}+\frac{1}{8}\Delta^2$$
$$L=-\frac{J_3}{18}-\frac{2}{27}\Delta J_2+\frac{1}{8}\Delta^3.$$

Unhappily a further step is wanting to bring M. HERMITE's results to the final test of comparison; for the value of AA′ (p. 192) does not agree with that given for AA′ (p. 186) by simply changing J_1, J_2 into J_2, J_3 respectively; a further change of Δ into 2Δ becomes necessary to make the ratios of AA′, BB′, CC′ (p. 192) accord with the ratios of the same quantities at p. 186. Finally, even after making this change the expression for $16I^2$ (p. 203) does not accord (even to a constant coefficient *près*) with that with which it is meant to be identical, viz. $16I_3^2$ (p. 187); so that after great labour I am still baffled in my attempt to ascertain the agreement or discrepancy of my conclusions with those of my precursor in the inquiry. As will appear hereafter, the two sets of conclusions are undoubtedly discrepant in form; but whether they are so in substance or not, or rather whether they are or not in contradiction to each other, requires a close examination to discover, the more especially because, as will hereafter be shown, there is a certain necessary element of indeterminateness in the scheme of invariantive conditions which serve to fix the character of the roots. It is greatly to be lamented that so valuable a paper as M. HERMITE's should be to some extent marred, in respect of the important end it would serve as a term of comparison, by the existence of these numerical and notational inaccuracies. I have spent hours upon hours in endeavouring to reconcile these several texts of the same memoir, and, after all my labour, the work is left unperformed without which the truth as between the two methods cannot be elicited. I feel, however, as confident of the correctness of my own conclusions as of the truth of any proposition in Euclid.

([b]) It is worthy of notice that there is a failing case in M. HERMITE's process for finding I^2 in terms of Δ, J_2, J_3, just as there is one in mine for finding G in terms of J, K, L,—the failure of the process, however, in neither case entailing any corresponding defect in the results obtained. The process employed in this memoir fails when $L=0$: for then the general form $ru^5+sv^5+tw^5$ is superseded by the supplementary one, $au^5+5euv^4+fv^5$. M. HERMITE's fails when J (the J of *this* memoir) $=0$; for then the quadratic invariant becomes a perfect square, and the substitution of its factors in place of the original variables becomes inadmissible, since the two former coincide.

([c]) It may be as well here to notice the form which M. HERMITE's two linear covariants assume when referred to the canonical form above written. The quadratic covariant being $rsuv+stvw+trwu$, if we operate with the correlative of this obtained by writing in it $\frac{d}{dv}$, $-\frac{d}{du}$, $\frac{d}{du}-\frac{d}{dv}$ in lieu of u, v, w, viz.

$$-rs\,\frac{d}{du}\frac{d}{dv}-st\frac{d}{du}\left(\frac{d}{du}-\frac{d}{dv}\right)+tr\frac{d}{dv}\left(\frac{d}{du}-\frac{d}{dv}\right)$$

upon the primitive, we obtain to a factor *près* the canonizant $rstuvw$, which has been already obtained; repeating the process, it is easy to see that the first linear covariant of the fifth degree in the coefficient assumes the simple form $rst(stu+trv+rsw)$, or $rst(\varrho u+\sigma v+\tau w)$. Taking again the correlative of this, viz.

$$rst\left(\varrho\frac{d}{dv}-\sigma\frac{d}{du}+\tau\left(\frac{d}{du}-\frac{d}{dv}\right)\right),$$

and operating with it upon $rsuv+stvw+trwu$, it will be found without difficulty that the second linear covariant of the seventh degree in the coefficients becomes

$$rst\{(\sigma-\tau)(\sigma+\tau-\varrho)u+(\tau-\varrho)(\tau+\varrho-\sigma)v+(\varrho-\sigma)(\varrho+\sigma-\tau)w\},$$

which is distinguishable in species from the former one by its symmetry being only of the hemihedral kind.

([d]) It may not be out of place to notice here that the Hessian of the canonical form will be found to be

$$\varrho v^3w^3+\sigma w^3u^3+\tau u^3v^3.$$

their resultant in respect to u, v is obviously

$$(rst)^5(r-s)(s-t)(t-r)\,(^{37}),$$

(c) Again, if we write

$$\begin{aligned} &rst(\rho u+\sigma v+\tau w)=\xi \\ &rst(w-\tau)(\sigma+\tau-\rho)u+(\tau-\rho)(\tau+\rho-\sigma)v+(\rho-\sigma)(\rho+\sigma-\tau)w)=\eta, \\ &u+v+w \qquad\qquad =0, \end{aligned}$$

and from these equations deduce the values of u, v, w, and substitute them in $ru^5+sv^5+tw^5$, we shall obtain M. HERMITE's "forme-type" expressed in terms of the parameters of the reduced form, and every coefficient therein will be invariantive.

The resultant of the equations above written (on making $\xi=0$, $\zeta=0$) will appear in the denominator of each such coefficient. Hence it appears, from M. HERMITE's expressions (Camb. and Dubl. Math. Journal, vol. ix. p. 193), where J_3 will be seen to enter into the denominator of A, B, C, C′, B′, A′, that this resultant to a factor *près* is his J_3. Its value may easily be calculated, and will be found to be

$$\rho\sigma\tau(\rho+\sigma+\tau)^3-4(\rho+\sigma+\tau)(\rho\sigma+\rho\tau+\sigma\tau)+9\rho\sigma\tau=JK+9L.$$

Accordingly as L (to use Dr. SALMON's convenient elliptical expression) is the condition of the failure of my *general* reduced form, so is $9L+JK$ the condition of the failure of M. HERMITE's "forme-type." As particular cases of this last failure, we may suppose $J=0$, $L=0$, or $K=0$, $L=0$. In the former case the reduced form is ax^5+5ex^4y, of which the simplest quadratic and cubic covariants are respectively aex^2; ae^2y^2x. Thus to find L, the first linear covariant, we have to operate upon ae^2y^2x with $ae\left(\frac{d}{dy}\right)^2$, which gives a^2e^3x; and to find L_2, we have to operate on $(aex^2)^2$ with $ae^2\left(\frac{d}{dx}\right)^2\frac{d}{dy}$, or, if we please (according to M. HERMITE's method), with $\left(a^2e^3\frac{d}{dy}\right)$ on aex^2, showing that L_2 vanishes, but L_1 continues to subsist. When, secondly, $K=0$, $L=0$, the reduced form is ax^5+ey^5, and the canonizant disappears entirely, so that the first, and consequently also the second, linear covariants, each of them becomes a *null*.

(37) By aid of the reduced forms of the invariants J, K, L, I given in the text, it is easy to prove that every other invariant, say Ω of a quintic, is a rational integral function of these four. In what follows, let a parenthesis enclosing the symbol of any invariant signify its value when any two of the quantities u, v, w in the reduced form $ru^5+sv^5+tw^5$; $[u+v+w=0]$ are taken as the independent variables. We have then

$$(J)=\rho^2+\sigma^2+\tau^2-2\rho\sigma-2\rho\tau-2\sigma\tau,\quad (K)=\rho\sigma\tau(\rho+\sigma+\tau),\quad (L)=\rho^2\sigma^2\tau^2,\quad (I)=\rho^2\sigma^2\tau^2(\rho-\sigma)(\sigma-\tau)(\tau-\rho),$$

ρ, σ, τ meaning st, tr, st.

The degree of Ω must be of the degree $4m$ or $4m+2$. 1. Let it be of the form $4m$. Then, since the interchange of any two of the variables u, v, w must leave (Ω) unaltered, (Ω) will be unaltered by the interchange of any two of the letters r, s, t, and is consequently a symmetric function of ρ, σ, τ, the roots of the equation

$$\theta^3-\frac{(K)}{(L)^{\frac{1}{2}}}\theta^2+\frac{(K)^2-(J)(L)}{(L)}-(L^{\frac{1}{2}})=0.$$

Hence

$$(\Omega)=\frac{F\big((J),(K),(L)\big)}{(L)^{2m}},$$

F denoting a rational integral function-form of the quantities it affects. Consequently

$$\Omega=\frac{F(J, K, L)}{L^{2m}}.$$

Hence since Ω cannot become infinite when $L=0$, which merely implies that the general form reduces to

$$(a, 0, 0, 0, e, i\between x, y)^5,$$

$\Omega=\Phi(J, K, L)$, a rational integral function of J, K, L.

2. If the degree Ω is of the form $4m+2$, (Ω) will be a function of r, s, t, which changes its sign when u and v

and consequently, if we call I the resultant in respect to x, y, we have

$$\pm\mathrm{I}=\mu^{45}\varrho^2\sigma^2\tau^2(\sigma-\varrho)(\tau-\sigma)(\varrho-\tau^2),$$

and

$$\mathrm{I}^2=\mu^{90}\varrho^4\sigma^4\tau^4(\sigma-\varrho)^2(\tau-\sigma)^2(\varrho-\tau)^2$$
$$=\mu^{30}(\sigma-\varrho)^2(\tau-\sigma)^2(\varrho-\tau)^2\mathrm{L}^2.$$

(43) Thus we see that the two quantities G, I^2, which are both rational integral functions of the degree 36 in the coefficients of $\mathrm{F}(x, y)$, cannot one vanish without the other, at all events when L is not equal to zero. This is sufficient to show that they are identical to a numerical factor *près*, whatever L may be, zero or not zero ([38]), and consequently that the quantity called G, proved to be positive upon the supposition of L not being zero, must also remain positive when L is zero, because it is in fact the square of a rational function of the coefficients. But we may also prove this independently by virtue of the supplementary reduced form $au^5+5euv^4+fv^5$ applicable to the case of L zero.

For when L=0, G becomes JK^4; so that the condition "G not negative" implies simply that J is positive unless K vanishes.

Now the canonizant, when it does not vanish, i. e. when e is not zero, contains v^2u as a factor, and, its coefficients being real, u, v are both of them necessarily real functions of x, y. Consequently J, which by definition is $-4\times$discriminant of quadratic covariant, becomes $-4\mu^{10}\times$discriminant of $au(eu+fv)$ in respect to u, v, which $=\mu^{10}a^2f^2$, μ being real. Consequently J is positive, since the reality of u, v implies that of a, e, f, when e is not zero. When e *is* zero u, v may be either real or imaginary; for u^5+v^5 may be real whether u, v be real or conjugate imaginary functions of x, y; but in that case K, which is found by operating twice upon the Hessian with a canonizant turned into an operator, vanishes, since then all the coefficients of the canonizant vanish ([39]). Hence the rule that G cannot be negative is seen to be true, whatever L may be.

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or any two of its quantities u, v, w, are interchanged, such interchange having the effect of introducing as a multiplier the $5(2m+1)$th power of the determinant of substitution (-1). Hence (Ω) is of the form

$$(\varrho-\sigma)(\sigma-\tau)(\tau-\varrho)\mathrm{F}(\varrho, \sigma, \tau),\ \text{i. e.}\ \frac{(\mathrm{I})\cdot\mathrm{F}(\varrho, \sigma, \tau)}{(\mathrm{L})^{\frac{5}{2}}},$$

which again is of the form

$$\frac{(\mathrm{I})\cdot\mathrm{F}\big((\mathrm{J}), (\mathrm{K}), (\mathrm{L})\big)}{(\mathrm{L})^{2m-8}},$$

so that Ω is of the form

$$\frac{\mathrm{I}.\mathrm{F}(\mathrm{J}, \mathrm{K}, \mathrm{L})}{\mathrm{L}^{2m-8}}.$$

Hence since, as before, Ω cannot become infinite when L=0, and since, furthermore, I does not vanish (for if so then G, which is I^2, would vanish) when L=0, Ω must be of the form $\mathrm{I}\Phi(\mathrm{J}, \mathrm{K}, \mathrm{L})$. Q. E. D.

([38]) For if $\mathrm{Q}^2=\mathrm{KI}^2$ for an indefinite number of systems of values of a, b, c, d, e, f, of which Q, I are rational integral functions, Q^2 and KI^2 must be *absolutely* identical; this of course is the case when Q^2 and KI^2, as proved in the text, are known to be identical for all values of a, b, c, d, e, f which do not make L zero.

([39]) ([a]) In the more general form $au^5+5euv^4+fv^5$, taking $\mu=1$. The canonizant is ae^2uv^2; this squared and

It may be said that the case of three or more equal roots existing in F (x, y) has been

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turned into an operator becomes $a^2e^4\left(\frac{d}{dv}\right)^2\left(\frac{d}{du}\right)^4$, which, applied to the Hessian, viz. $3aeu^4v^2+afu^3v^3-e^2v^6$, after multiplying by $-\frac{1}{72}$, gives $K=-2a^3e^5$, so that $D=J^2-128K=a^4f^4+256a^3e^5$, which is capable of easy verification. In fact D becomes the resultant of au^4+ev^4 and $v^3(4eu+fv)$; v^3 introduces the factor a^3 into D; and further, making $u:v::-f:4e$ and substituting in au^4+ev^4, we obtain the other factor af^4+256e^5.

If we adopt $u^5+5euv^5+v^4$ as the reduced form for the failing case (a form analogous to the well-known one, $u^4+6cu^2v^2+v^4$, for the general quartic), to find e we have $J=\mu^{10}$, $K=-2\mu^{20}e^5$. Hence $e^5=-\frac{K}{2J^2}$; thus when $K=0$, $e=0$.

([b]) By a linear transformation we may always take away any two (except the two first or last) coefficients of a given quintic, but the vanishing of more than two coefficients always corresponds to some invariantive condition. Thus, *ex. gr.*, in the form

$ax^5+5exy^4+fy^5$	$L=0$		
ax^6+fy^5	$L=0$	$K=0$	
ax^5+5exy^4	$L=0$	$J=0$	
$ax^5+10dx^2y^3$	$J=0$	$K=0$	
$ax^5+5bx^4y+10cx^3y^2$	$L=0$	$J=0$	$K=0$.

([c]) The condition for the existence of four equal roots in a quintic is the vanishing of the quadratic covariant; that is to say, we must have

$$ae-4bd+3c^2=0,\quad af-3be+2cd=0,\quad bf-4ce+3d^2=0.$$

The three quantities equated to zero are not separately invariants, but constitute in their *ensemble* an invariantive plexus.

([d]) [It may here be noticed incidentally that the conditions for equal roots in the biquadratic form are as follows. For two equal roots, of course, the discriminant is zero, for three equal roots the two lowest invariants are each zero, and for two pairs of equal roots the Hessian $(A, B, C, D, E)(x, y)^4$ becomes to a factor *près* identical with the primitive $(a, b, c, d, e)(x, y)^4$, so that all the first minors of the matrix

$$\begin{vmatrix} a, & b, & c, & d, & e, & f \\ A, & B, & C, & D, & E, & F \end{vmatrix}$$

vanish. *Quære*, whether the character of the five-rayed pencil (centre at origin), in which a, A; b, B; c, C; d, D; e, E mark points, may not serve to distinguish between the case of four real and four imaginary roots.]

([e]) When $J=0$ and $K=0$, but *not* $L=0$, it is obvious that $\varrho:\sigma:\tau::1:\iota:\iota^2$, ι being any imaginary cube root of unity, and the reduced form is $u^5+\iota v^5+\iota^2w^5$, with the relation $u+v+w=0$.

J and K being zero, D will be so too, and accordingly the equation $u^5+\iota v^5+\iota^2w^5=0$ will have two equal roots. It will easily be found that these equal roots correspond to the system of ratios $u=1$, $v=\iota^2$, $w=\iota$. In fact, if we write $u=1+\varrho$, $v=\iota^2+\iota\varrho$, $w=\iota+\iota^2\varrho$, the equation becomes $u^5+\iota v^5+\iota^2w^5=\varrho^2(30\varrho+3\varrho^3)=0$.

Hence, understanding by ε either of the two prime sixth roots of unity, the complete system of ratios of u, v, w may be expressed as follows:—

$u=1$	$v=\iota^2$	$w=\iota$
$u=1$	$v=\iota^2$	$w=\iota$
$u=1-\sqrt[3]{10}$	$v=\iota^2-\sqrt[3]{10}$	$w=\iota-\iota^2\sqrt[3]{10}$
$u=1+\sqrt[3]{10}\varepsilon$	$v=\varepsilon^4-\sqrt[3]{10}$	$w=\varepsilon^2+\sqrt[3]{10}\varepsilon^5$
$w=1+\sqrt[3]{10}\varepsilon^5$	$v=\varepsilon^4+\sqrt[3]{10}\varepsilon$	$w=\varepsilon^2-\sqrt[3]{10}$.

Thus, when $J=0$ and $K=0$, u, v, w (with the relation $u+v+w=0$) may first be found, in terms of x, y, by

lost sight of; but we know, and it is capable of immediate verification by taking as the

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solving the cubic equation, obtained by equating to zero the canonizant of $(a, b, c, d, e, f \between x, y)$, and then x, y will be known from the above system of values for any two of the quantities u, v, w.

(f) It is obvious that the form $ax^5+dx^2y^3$ gives $J=0$ and $K=0$; but it seems desirable to prove the converse, viz. that when $J=0$ and $K=0$, but not $L=0$, the form is always reducible to $\alpha u^5+10\delta u^2v^3$, which may be done as follows. Since $J=0$ and $K=0$ the discriminant is zero, and we may assume

$$F=ax^5+5bx^4y+10cx^3y^2+10dx^2y^3,$$

and we have $J=$ discriminant of

$$(-4bd+3c^2)\xi^2+2cd\xi\eta+3d^2\eta^2.$$

Hence

$$3d^2(3c^2-4bd)-c^2d^2=0;$$

d cannot be zero, for then we should have $J=0$, $K=0$, $L=0$, contrary to hypothesis. Hence $8c^2-12bd=0$.

If $b=0$ and $c=0$, F is already reduced to the desired form; but if not, $d=\frac{2c^2}{3b}$, and F becomes

$$ax^5+\frac{5b}{6}x^2\left(6x^2y+\frac{12c}{b}xy^2+\frac{8c^2}{b^2}y^3\right);$$

or, making

$$a-\frac{5b}{6}=\alpha, \quad \frac{b}{6}=2\delta, \quad x+\frac{2cy}{b}=v,$$

$F=x^5+10\ x^3v^3$, as was to be shown.

The corresponding converses for the case of $J=0$, $L=0$, and of $K=0$, $L=0$ have been already established.

(g) It will be observed that under a certain point of view L for binary quintics is the analogue of Δ the *discriminant* for binary quartics, the condition of failure in the *general* reduced form in the two cases being $L=0$ and $\Delta=0$ respectively. The mere vanishing of the discriminant in the case of the quintic function, unattended by any other condition, does not affect the nature of the reduced form.

(h) It has been shown previously in the text that when $L=0$ the primitive is reducible to the form

$$(a, 0, 0, 0, e, f \between x, y)^5.$$

Hence if I_{12} is any duodecimal invariant which vanishes when $b=0$, $c=0$, $d=0$, I_{12} must vanish whenever L vanishes, and consequently, since L is of as high a degree as I_{12}, I_{12} must be a numerical multiple of L. In Mr. CAYLEY's Third Memoir on Quintics, "No. 29" represents a duodecimal invariant calculated by M. FAÀ DE BRUNO, and characterized morphologically by Mr. CAYLEY as being that duodecimal invariant in which "the leading coefficient a does not rise above the fourth degree." On examining No. 29 it will be found to contain no term in which b, c, d are all simultaneously absent. Hence it is, by virtue of the above observation, a multiple of my L: to determine the numerical factor, let all the coefficients in the primitive except a, d be supposed zero; then the canonizant becomes

$$\begin{vmatrix} a & 0 & 0 & d \\ 0 & 0 & d & 0 \\ 0 & d & 0 & 0 \\ y^3 & -y^2x & yx^2 & -x^3 \end{vmatrix} = d^3y^3+ad^2x^3.$$

Hence L becomes $-27a^2d^{10}$, but "No. 29" becomes $27a^2d^{10}$. Hence we have the important relation "No. 29" $=-$L, so that No. 29 is a discriminant, an *intrinsic* property of the calculated invariant, which, I believe, was not suspected.

(i) It will at once be recognized that "No. 19" given in Mr. CAYLEY's Second Memoir upon Quantics is identical with the J of this memoir, whence it follows from Mr. CAYLEY's equation (No. 26)=(No. 19)$^2-1152$ No. 26, that K=9 (No. 25). Thus abstraction made of a mere numerical factor, Mr. CAYLEY and myself agree upon perfectly distinct grounds in recognizing K and L as the true simplest invariants of their respective degrees, an accordance as satisfactory as it was unexpected, and which must be considered as setting at rest the question of what should be deemed the, so to say, *staple* invariants of the Binary Quintic.

reduced form $au^5+5bu^4v+10cu^3v^2$, that on such hypothesis all the invariants J, K, L must vanish, so that JK^4 is still non-negative [40].

(44) It is most important to notice that G can only become zero by virtue of two of the quantities ϱ, σ, τ, and therefore of r, s, t becoming equal. When u, v are imaginary, it is the coefficients r, s which must become equal, as otherwise the reduced form would not be a real function of x, y. By equating r to s, and using as an auxiliary variable the ratio $\frac{r}{t}$ or $\frac{s}{t}$, we shall be able to study the composition and inward nature of G with the utmost clearness and facility.

SECTION II.—*On the Criteria which decide the Number of Real and Imaginary Roots.*

(45) Since in the preceding section we have supposed that u, v are always real linear functions of x, y, it is obvious that the character of the roots of the given quintic in x, y is completely identical with that of the roots in the reduced form, and it has been shown that only one reduced form corresponds to a given system of values of J, D, L [41].

Let us suppose J, D, L to be taken as coordinates of a point in space; when J, D, L are so related that the condition G non-negative is satisfied, the point will correspond to an equation with real coefficients, and may be termed a *facultative* point. But when G is negative it will correspond to an equation of the kind alluded to in the recent section of this paper, and there called conjugate: such a point may be termed non-facultative. Thus the whole of space will be divided into two parts, separated by the surface $G=0$, which may be termed respectively facultative and non-facultative (as being made up of facultative or non-facultative points [42])). It is clear that these two portions will be exactly equal, similar, and symmetrical with regard to the axis of D; by which I mean that, if two points be taken in any line perpendicular to the axis of D at equal distances from that axis, one will be facultative and the other non-facultative, as is evident from the fact that when J, L become $-$J, $-$L (K, and therefore D or J^2-128K, remaining unaltered), G is converted into $-$G. Thus by a semirevolution

[40] When the form is $au^5+5euv^4+fv^5$ so that $L=0$, the canonizant, as has been seen before, is ae^2v^2u; the resultant of these two is $a^5e^{10}a^2f=a^7e^{10}f$. Again, $J=a^2f^2$, $K=-2a^3e^5$; thus the square of the resultant $=\frac{1}{16}JK^4$; so that if we call this resultant, which we may take as the definition of the Octodecimal Invariant I, we have $G=16I^2$.

[41] It should be well noticed that the mere ratios $\frac{D}{J^2}$, $\frac{L}{J^3}$ do not suffice to determine the character of the roots. When these ratios are given, it is true that the ratios r, s, t in the reduced form are given, but according as L is positive or negative, the arguments u, v in $ru^5+sv^5+tw^5$ (supposing w to be the real linear function of x, y) will be real or imaginary. When J, L, D are all given *absolutely*, then the character of the roots is completely determined. The *indelible* marks of a quintic function are three in number, viz. the ratios $\frac{K}{J^2}$, $\frac{L}{J^3}$, and the sign of L or J, as for a quartic function they are two in number, viz. $\frac{s^3}{t^2}$ and the sign of s.

[42] It will also be convenient to call the coordinates J, D, L corresponding to any facultative point a facultative system of invariants, and $\frac{D}{J^2}$, $\frac{L}{J^3}$ corresponding to the same (for a *given sign* of J) a facultative system of invariantive ratios.

round the axis of D the facultative and non-facultative portions may be made to exchange places.

(46) The axis of D itself lies on the surface of G, and like every other portion of this surface is facultative, for there is no reason for disallowing G to become zero. Conversely, if, instead of a real equation, we take one of the conjugate class (described in the second section), the whole of the facultative portion of space (except the separating surface G) becomes non-facultative, and the non-facultative part becomes facultative, but G itself remains facultative. When the invariants, or any of them, become imaginary, we are put out of space altogether, and the system can belong neither to a real nor to a conjugate family, but to one with coefficients at the same time imaginary and non-conjugate. $G=0$ (43), it may be remarked, will in all cases be the condition of an equation capable of linear transformation into one of recurrent (44) form; for the reduced form then in general becomes $ru^5+rv^5-t(u+v)^5$. The case when G becomes zero by virtue of $J=0$ and $L=0$, that is to say when the function is reducible by real or imaginary linear substitutions (see footnote (39) (f)) to the form $u(u^4\pm v^4)$, is the one which might for a moment be supposed to offer an exception to the rule; but only the exception is only apparent, since $u(u^4-v^4)$, on writing $u=p+q$, $v=p-q$, becomes $16(p+q)pq(p^2+q^2)$.

(47) To every point in space, it has been remarked, will correspond one particular family of equations all of the same character as regards the number they contain of real or imaginary roots, because capable of being derived from one another by real linear substitutions, such family consisting of an infinite number of ordinary or conjugate equations according as the point is facultative or non-facultative; but it may be well to notice that, conversely, every point does not correspond to a distinct family. In fact every point in the curves $D=pJ^2$, $L=qJ^3$ (p, q being constants) will denote a curve divided into two branches by the origin of coordinates, one of which will be facultative and the other non-facultative; but in each separate branch every point will represent the very same family. Any such separate branch may be termed an isomorphic line; and we see that the whole of space may be conceived as permeated by and made up of such lines radiating out from the origin in all directions.

(48) The origin at which $J=0$, $D=0$, $L=0$, as already noticed, corresponds to the case of three equal roots. The theorem that, when more than half as many roots are equal to each other as there are units in the degree of any binary form, all the invariants vanish, was remarked by myself originally in the very infancy of the subject, before Mr. CAYLEY'S paper, alluded to by M. HERMITE, appeared in Crelle. The method of proof which then occurred to me is the simplest that can be given. For instance, in

(43) I shall hereafter allude to the surface denoted by $G=0$ under the name of the Amphigenous Surface, as being the locus of the points which give birth to real and conjugate forms indifferently.

(44) The roots of recurring equations, geometrically represented, in general go in quadruplets, A, A′; B, B′, where A and B, as also A′, B′, are mutual optical images of each other in respect to a fixed line, and A, A′, as also B, B′, are electrical images of each other in respect to a circle of which the fixed line is a diameter—with liberty, of course, for the images taken in either mode of combination to coalesce so as to reduce the quadruplet to a simple pair.

the case before us, if the quintic have three equal roots, we may reduce it to the form

$$ax^5+5bx^4y+10cx^3y^2.$$

Suppose now, if possible, an invariant of the degree m; the *weight* of each term therein, say $a^rb^sc^t$, in respect to x or y would be the same $\left(\text{viz. } \frac{5m}{2}\right)$, so that we should have

$$5r+4s+3t=\frac{5m}{2}=s+2t, \text{ or } 5s+3s+t=0,$$

and therefore $r=0$, $s=0$, $t=0$, $m=0$. So for a sextic with three equal roots reduced to the form $(a, b, c, 0, 0, 0\between x, y)^6$. Supposing any term in one of its invariants to be $a^rb^sc^t$, we should have

$$6r+5s+4t=\frac{6m}{2}=s+2t, \text{ or } 6r+4s+2t=0,$$

which is absurd, unless $r=0$, $s=0$, $t=0$, $m=0$, and so in general for a binary form of any degree. If in the above example for the degree m only three roots were equal *inter se* (the form assumed being $(a, b, c, d, 0, 0, 0\between x, y)^6$, any term in a supposed invariant being $a^rb^sc^td^u$, where $r+s+t+u=m$, we should have

$$6r+5s+4t+3u=3m=s+2t+3u,$$

and, as before,

$$6r+4s+2t=0, \quad r=0, \quad s=0, \quad t=0;$$

no longer, however, $m=0$, but $m=u$, which is left undetermined.

(49) Before proceeding further it will be proper to consider under what circumstances a variation (in the coefficients of any equation) arbitrary, except that the coefficients are to remain real, can affect the character of the roots.

Let $F(x)=0$ be any algebraical equation with real coefficients, and let $\delta(Fx)$ be the variation of F due to the variation of the coefficients, $dF(x)$ the variation due to the change of x into $x+dx$. If, now, r be a root of $Fx=0$, and $r+dr$ the corresponding root of $F(x)+\delta F(x)=0$, we have

$$Fr=0, \quad F(r+dr)+\delta F(r)=0, \quad \text{or } \delta F(r)+\frac{d}{dr}F(r)dr+\frac{1}{1.2}\left(\frac{d}{dr}\right)^2 Fr(dr)^2+\&c.=0.$$

Hence, unless $\frac{dF}{dr}=0$, i. e. unless there are two equal roots r, we shall have $dr=-\frac{\delta F(r)}{\frac{d}{dr}F(r)}=$ a real quantity; so that the character of the root $r+dr$ will be the same as that of r.

But if

$$\frac{dF}{dr}=0, \quad \frac{d^2F}{dr^2}=0, \quad \ldots \quad \left(\frac{d}{dr}\right)^{i-1}F=0,$$

so that there are i roots r, i being any integer greater than zero, then to find dr we have the equation

$$(dr)^i+\frac{\Pi(i)\delta Fr}{\left(\frac{d}{dr}\right)^i F(r)}=0.$$

Thus dr will have i distinct values; of these, if i is odd, all but one will be imaginary, but if i is even they will be all imaginary, or only all but two imaginary and the remain-

ing two real, according as the sign of $\delta F(r)$ is the same as or the contrary to that of $\left(\frac{d}{dr}\right)^i F(r)$. Accordingly, if r is real [45] and i even, the nature of the *ensemble* of the i roots $r+dr$ will not be the same when $\delta F(r)$ is positive as when $\delta F(r)$ is negative.

(50) So, further, if $Fx=0$ have $2m$ equal roots r, $2n$ equal roots s, and so on, the deduced corresponding groups of roots in $F(x)+\delta F(x)=0$ will, or may at least each of them, undergo a change of character to the extent of one pair of the r group changing their nature with the sign of $\delta F(r)$, one pair of the s group changing their nature with the sign of $\delta F(s)$, and so on; but in no case, except $F(x)$ possess some equal roots (*i. e.* unless its discriminant be zero), can an infinitesimal variation in the constants affect the character of the roots [46].

(51) To every facultative point corresponds a certain set of values of J, D, L; and when these are given, it has been shown that the equation $(a, b, c, d, e, f \between x, y)^5$ is reducible to the form $ru^5+sv^5+tw^5$, where $u+v+w=0$, or to the form $ru_{\prime}^5+sv_{\prime}^5+tw^5$, where

$$u_{\prime}+v_{\prime}+w=0, \text{ and } u_{\prime}=\frac{-w+iv}{2}, \; v_{\prime}=\frac{-w-iv}{2},$$

or to the form $au^5+5euv^4+fv^5$, u, v, w being always real linear functions of x, y, with the sole exception that when $J=0$, $K=0$, $L=0$, the reduced form is

$$au^5+5bu^4v+10cu^3v^2.$$

When these three invariants are not all zero, the coefficients in the reduced form r, s, t or a, e, f are known functions of J, D, L, and the character of the roots is perfectly determinate; so that to every facultative point corresponds an infinite family of equations with real linear coefficients all deducible from each other by real linear substitutions. Thus then, with the sole exception of the origin, every facultative point corresponds to a determinate character of equation, viz. to an equation with four, or two, or no imaginary roots; so that by a bold figure of speech we may be permitted to speak of every point but one in facultative space having a determinate quality, as masculine, feminine, or neuter. The origin alone is exempt from this law, and may be considered to be of epicene gender, since the factor $au^2+5buv+10v^2$ may have its roots real or imaginary. As we travel continuously from point to point in the facultative portion of space we pass from family to family, or, if we please, from an individual of one family to an individual of another family, differing from the former individual by an infinitesimal variation of the constants.

(45) r, although supposed to be one of a group of equal roots, is not *necessarily* real, for it may belong to a factor $(x^2+2e\cos\theta+e^2)^2$.

(46) Compare this statement with the corresponding one given by M. HERMITE, Camb. and Dub. Journal, vol. ix. p. 204, where only one parameter is supposed to undergo a change. I think that greater breadth and at the same time greater precision and clearness are gained by the mode of exposition employed in the text above. It will be observed that for a change of character to be possible when the function passes through a phase of equal roots, it is not enough that there shall exist a group of equal roots r, but there must be an even number of such roots in the group, and, furthermore, the equal roots must be *real*; when this last supposition is not satisfied, no change in the character of dr will affect the character of $r+dr$: an instructive exemplification of this remark will occur in the sequel.

(52) If, then, we insulate any portion of facultative space, and in the block so insulated it is possible to pass from one point to any other—that is to say, if we can draw a *continuous* curve of any sort from one point to another without passing out of the block, and without cutting or touching the plane $D=0$, then by virtue of the principle just laid down, we see that all the points in such block have the same character, and the nature of the roots will be the same in the infinite number of families, each containing an infinite number of individuals which the points in that block severally represent. Now imagine a block taken so extensive as to admit of no further augmentation, except accompanied with a violation of the condition of the capability of free communication between point and point without cutting or touching the surface D; such a block may be termed a *region*, and the whole of facultative space will be capable of subdivision into a certain number of these regions. This being supposed effected, the character of each region will be known when we know the character of a single point in it; that is to say, every region will have a determinate character of positive, negative, or neuter. It will presently be shown that the number of such regions is only three([47]) (the least number it could be to meet the three cases of four, two, or no imaginary roots), one masculine, one feminine, one neuter; and consequently there will be but three cases to consider when the invariantive coordinates J, D, L are given; according as J, D, L belong to one or the other of these three regions, the equation to which they belong will have all its roots real, or only one real, or three real and two imaginary. The origin, it need hardly be added, constitutes a region *per se*, in which, so to say, the characters of masculine and feminine are blended.

(53) Let it be observed that we can see *à priori* that, were it not for the distinction between facultative and non-facultative portions of space, it would be impossible for each point corresponding to a given system of invariants to possess an unequivocal character; for in such case there would necessarily be free continuous communication possible between all the points on each side of D *inter se*, and consequently we should be landed in the absurdity of conceiving the general equation of the fifth degree not to admit of division into cases of four, two, or no imaginary roots; D being negative, we know, would imply two roots, and not more than two, being imaginary; and accordingly D positive would imply either that four roots are imaginary or none—not sometimes one and sometimes the other, but in all cases alike four imaginary, to the exclusion of the supposition of the roots being all real, or else of all the roots being real and never four imaginary. Thus we see that the mere fact of a given system of invariants communicating a definite character to the roots, implies the necessity of the invariants exercising a restraining action over each other's limits, and that where this restraint does not exist it is impossible that the character of the roots can be determined by the values of the invariants.

([47]) It is clear from the definition, that a *region* can only be bounded by G the amphigenous surface, and D the plane of the discriminant: and granted (as will be shown hereafter) that G and D *touch* each other in only one continuous line, it becomes obvious *à priori* that there can be but two regions on one side of D and a single region on the other.

(54) This is precisely what happens in biquadratic equations. In such we know the fundamental invariants t, s, or, if we please, t, Δ (where $\Delta=s^3+27t^2$), are perfectly independent and subject to no equation of condition; so that if we consider t, Δ as the coordinates of points in a plane, the whole of the plane will be made up of facultative points. When Δ is negative, *i. e.* for representative points lying on one side of the line Δ, it is true we know that there is just one pair of imaginary roots constituting what may be termed the neuter case; but when the representative points lie on the other side of this plane, they cannot be said to be either masculine or feminine, but will every one of them possess that epicene character which is peculiar to the origin alone in the case of quintic forms. A single example will make this clear.

Take the two reduced forms

$$u^4+6(1+\varepsilon)u^2v^2+v^4,$$
$$\omega^4+6(1-\varepsilon)\omega^2\theta^2+\theta^4,$$

where u, v are real linear functions of x, y, and ω, θ conjugate imaginary ones of the same; and suppose s, the quadrinvariant in respect to x, y, to be the same for both forms. For greater convenience of computation consider ε to be infinitesimal.

Then in the one case the t is of the same sign as

$$(1+\varepsilon)\left(1-(1+\varepsilon)^2\right), \text{ i. e. } -2\varepsilon,$$

and in the other the t is of the contrary sign to

$$(1-\varepsilon)\left(1-(1-\varepsilon)^2\right), \text{ i. e. } \quad 2\varepsilon,$$

so that t is of the same *sign* (viz. negative) in each case.

Again, in the two cases respectively

$$\frac{t^2}{s^3}=\frac{4\varepsilon^2}{1+3(1\pm\varepsilon)^2}=4\varepsilon^2.$$

Hence t as well as s, and consequently t and Δ are alike for both forms.

But in the one first written the roots are of the same nature as those of $u^4+6u^2v^2+v^4$, i. e. are all impossible, and in the other of the same nature as in

$$\left(\frac{u+\iota v}{2}\right)^4+6\left(\frac{u+\iota v}{2}\right)^2\left(\frac{u-\iota v}{2}\right)^2+\left(\frac{u-\iota v}{2}\right)^4=0,$$

where u, v are real linear functions of x, y and $i=\sqrt{-1}$, in which case the roots are all possible. Thus we see that the very same values of t, Δ may correspond either to the case of four real or four imaginary roots, showing that the point t, Δ is what we have termed *epicene*. If we choose to take s, t as the coordinates, the same remarks would apply, except that Δ instead of a straight line would become a semicubical parabola. All the points on one side of this curve would have a definite neuter character, but those on the opposite side would be neither masculine nor feminine, but epicene.

(55) With a view to its subsequent distribution into regions, I now proceed to ascertain the form of that moiety of space which I have termed facultative.

Let $J^2=qK$, $J^3=\nu L$. Then

$$\frac{G}{J^9}=\frac{1}{q^4}+\frac{8}{\nu q^3}-\frac{2}{\nu q^2}-\frac{72}{\nu^2 q}-\frac{432}{\nu^3}+\frac{1}{\nu^2}, \text{ and } \frac{D}{J^2}=1-\frac{128}{q}.$$

We may for the moment make abstraction of the section of G made by the plane of L; that being done, and J, K, L being referred to the form $ru^5+sv^5+tw^5$ or $ru_i^5+sv_i^5+tw^5$, calling μ^{10}, M, and, as before, using ϱ, σ, τ to denote st, tr, rs, we have

$$\begin{aligned}\pm J &=M(\varrho^2+\sigma^2+\tau^2-2\varrho\sigma-2\varrho\tau-2\sigma\tau),\\ K&=M^2\varrho\sigma\tau(\varrho+\sigma+\tau),\\ \pm L&=M^3\varrho^2\sigma^2\tau^2.\end{aligned}$$

Now when $G=0$, we may suppose $\varrho=\sigma$, $\frac{\tau}{\varrho}=\frac{\tau}{\sigma}=\theta+4$, θ being a new auxiliary variable. We have then

$$\begin{aligned}\pm J &=M(\tau^2-4\varrho\tau) &&=M\varrho\tau\theta,\\ K&=M^2\varrho^2\tau(2\varrho+\tau)&&=M^2\varrho^2\tau^2\left(1+\frac{2}{\theta+4}\right),\\ \pm L&=M^3\varrho^4\tau^2 &&=M^3\varrho^3\tau^3\frac{1}{\theta+4},\end{aligned}$$

and consequently

$$\nu=\frac{J^3}{L}=\theta^4+4\theta^3,$$

$$q=\frac{J^2}{K}=\frac{\theta^2(\theta+4)}{\theta+6}.$$

(56) In general we have $\theta^4+4\theta^3-\nu=0$.

By a well-known corollary to DESCARTES's rule this equation can never have more than two real roots; when ν is positive there will always be two real roots of opposite signs; but when ν is negative and inferior to a certain negative limit, *all the roots become imaginary.* When ν lies between zero and that limit, two roots of θ will be real and both negative. To find that limit we may make $4\theta^3+12\theta^2=0$, or $\theta=-3$, which gives $\nu=81-108=-27$.

(57) When $D=0$, $q=\frac{J^2}{K}=128$, i. e. $\theta^3+4\theta^2-128\theta-768=0$, or $(\theta+8)^2(\theta-12)=0$; so that the roots of θ, when $D=0$, are -8, -8, 12, and the corresponding values of ν are 2^{11}, 2^{11}, $2^{10}27$.

If now we make $\theta^4+4\theta^3=2^{11}$, one of the real values of θ we know is -8, and the other will be the real root of the cubic equation $\theta^3-4\theta^2+32\theta-256=0$.

When $\theta=5$, the left-hand side of the equation $=125+160-100-256=-71$.

When $\theta=6$, the left-hand side of the equation $=216+192-144-256=\quad 8$.

Hence the real root lies between 5 and 6, and q lies between $\frac{225}{11}$ and $\frac{360}{12}$. Thus $q<30$ and $\frac{D}{J^2}=1-\frac{128}{q}$ is negative.

Again, if we take $\theta^4+4\theta^3=27\cdot2^{10}$, and take out the root $\theta=12$, the resulting cubic becomes

$$\theta^3+16\theta^2+192\theta+2304=0,$$

where it will easily be seen the real root lies between -12 and -16.

When $\theta=-12$,

$$q=\theta^2\frac{\theta+4}{\theta+6}=144\times\frac{8}{6}=192\,;$$

and when $\theta=-16$,

$$q=256\times\frac{12}{10}=307\tfrac{1}{5}.$$

Moreover, when q is a maximum or minimum, it will readily be found that $\theta^3+11\theta+24=0$; so that $\theta=-3$, or $\theta=-8$. Hence for the value of θ found from the above cubic $q<192$ and $\frac{D}{J^2}=1-\frac{128}{q}$ is *positive.*

(58) When $J=0$, $\nu=0$; and when $L=0$, $\nu=\infty$.
For these two cases it will be more simple to dispense with the auxiliary variable θ, and to revert to the original equation between J, K, L.

Accordingly, when $J=0$, we find $8LK^3-432L^3=0$. Hence

$$L=0,\ \text{ or } K^3=54L^2,\ \text{ i. e. } \left(\frac{-D}{128}\right)^3=54L^2;$$

so that the complete section of G made by the coordinate plane J becomes a straight line, viz. the axis of D, and a semicubical parabola whose axis is the negative part of D. When J is very nearly zero, ν becomes a positive or negative infinitesimal in the equation $\theta^4+4\theta^3=\nu$.

One real root of this equation is $\theta=\left(\frac{\nu}{4}\right)^{\frac{1}{3}}$.

The other is $-4+\delta$, where $\left(4(-4)^3+12(-4)^2\right)\delta=\nu$,

or
$$\delta=-\frac{\nu}{64}.$$

Now
$$\frac{K^3}{L^2}=\left(\frac{\theta+6}{\theta+4}\right)^3(\theta+4)^2=\frac{(\theta+6)^3}{(\theta+4)}.$$

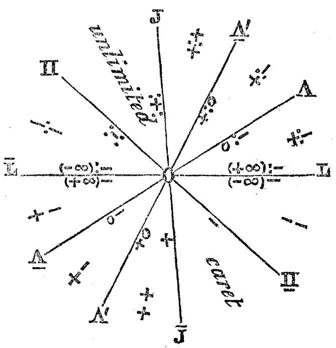
The first value of θ gives $K^3=54L^2$ to an infinitesimal *près*; the other value gives

$$K^3=-\frac{512}{\nu}L^2,$$

or, to an infinitesimal *près*,

$$\left(\frac{D}{128}\right)^3=\frac{512}{\nu}L^2;$$

so that D passes from $+\infty$ to $-\infty$, *i. e.* $\frac{J^3}{L}$ passes through zero.



(59) In the annexed figure(48), the plane of the paper represents the plane of D, *i. e.* the plane for which $D=0$; $JO\bar{J}$ is the axis of J, OJ being the positive and $O\bar{J}$ the negative direction; $LO\bar{L}$ is the axis of L, OL being the positive and $O\bar{L}$ the negative direction. In order to avoid any appearance of an attempt at a practicably impossible accuracy of drawing, I use straight lines to

(48) I shall refer, when I have occasion to do so, to this figure, which contains a synopsis of the whole theory, under the name of the Dial figure.

denote cubical parabolas, and pay no attention whatever to relative magnitudes, but only to the order or progression of magnitudes, using the lines which are drawn in the figure not as *copies* but as *symbols* of the actual curves which are to be mentally imagined.

Thus the line $\mathrm{JO\bar{J}}$ is used to represent the straight line $L=0$; $\Lambda'\mathrm{O}\underline{\Lambda}'$ the cubical parabola $J^3=27\cdot2^{10}L$; $\Lambda\mathrm{O}\underline{\Lambda}$ the cubical parabola $J^3=2^{11}L$; $\Pi\mathrm{O}\underline{\Pi}$ the cubical parabola $J^3=-27L$([49]).

It will be observed that certain combinations of *plus*, *zero*, *minus*, positive and negative *infinity* are placed along the lines and inside the sectorial spaces. The meaning of these will be sufficiently obvious from what has preceded. They refer to the signs of the two values of D in the surface G for each point in the line or sector along or within which they are placed. At every point along the line $\mathrm{O\bar{J}}$, $\frac{D}{J^2}$ has only one value, and that positive; along $\Lambda'\mathrm{O}\underline{\Lambda}'$, $\frac{D}{J^2}$ has two values, one positive and the other zero. Along $\Lambda\mathrm{O}\Lambda$, $\frac{D}{J^2}$ has two values, one positive the other negative. Immediately below $\mathrm{\bar{L}OL}$ two values, one $+\infty$, the other finite and negative. Immediately above $\mathrm{\bar{L}OL}$ two values, one $-\infty$, the other finite and negative. Along $\Pi\mathrm{O}\underline{\Pi}$ one value, finite and negative.

Moreover D has been shown to be never zero, except along $\Lambda'\mathrm{O}\underline{\Lambda}'$, $\Lambda\mathrm{O}\underline{\Lambda}$. Hence it is obvious that *inside* $\Lambda'\mathrm{O\bar{J}}$ and the opposite sector D has two values, both *plus*; inside the next pairs of opposite sectors two values, one *plus*, the other *minus*; inside the next pair of sectors also two values, one plus, the other minus; inside the next pair of sectors two values both *minus*, and in the pair of sectors left vacant, for which $\nu<-27$, it has been shown that D becomes impossible.

([49]) It has been shown in the preceding articles that corresponding to the line $\mathrm{\bar{J}OJ}$ and to the line $\Pi\mathrm{O}\underline{\Pi}$, the vertical ordinate D of the amphigenous surface $(G=0)$ has only one value positive for the former, negative for the latter; along the line $\Lambda'\mathrm{O}\underline{\Lambda}'$ two values, one positive the other negative; for the space between $\Lambda\mathrm{O}\Lambda'$, $\mathrm{\bar{L}OL}$ indefinitely near to the latter two values, one positively infinite, the other negative; and for the space indefinitely near to the same on the opposite of it, two values, one negatively infinite, the other negative. These results are collected and represented symbolically in the Table annexed.

$$\begin{array}{cccccc}
\bar{J} & \Lambda' & \Lambda & \multicolumn{2}{c}{\bar{L}} & \Pi \\
 & + & 0 & \overbrace{(+\infty)\quad -} & & \\
+ & & & & & - \\
 & 0 & - & - & (-\infty) &
\end{array}$$

Thus, corresponding to the upper sheet of G, we have the succession

$$\begin{array}{cccccc}
+ & + & 0 & (+\infty) & - & -
\end{array}$$

and to the lower sheet

$$\begin{array}{cccccc}
+ & 0 & - & - & (-\infty) & -
\end{array}$$

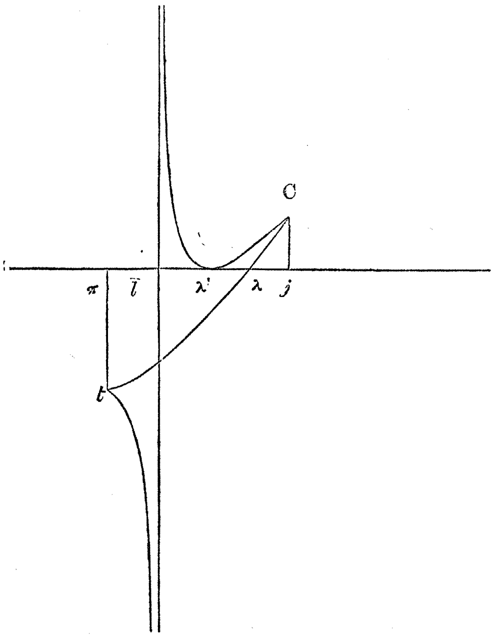
the two sheets coming together at a cuspidal edge above $\mathrm{\bar{J}OJ}$ and below $\Pi\mathrm{O}\bar{\Pi}$.

Moreover these are the only positions of the line revolving in the plane of D corresponding to which a change in the nature of D can take place, and thus we can without further examination fill up the Table, giving the nature of D for the intervening spaces, and may thus obtain the Table embodied in the *dial-figure* above, viz.,

$$\begin{array}{cccccccccc}
\bar{J} & & \Lambda' & & \Lambda & & \multicolumn{2}{c}{\bar{L}} & & \Pi \\
 & + & + & + & 0 & + & \overbrace{(+\infty)\quad -} & & - & \\
+ & & & & & & & & & - \\
 & + & 0 & - & - & - & - & (-\infty) & - &
\end{array}$$

(60) Thus it will be seen that the surface G consists of two opposite portions precisely similar and symmetrical in respect to the axis of D.

Let us trace that one of these whose ground-plan is comprised within the sector $\Pi O\bar{J}$. It will consist of two sheets coming to a cuspidal edge (a common parabola) in the superior part of the plane of L. The upper sheet will touch the plane of D in $O\Lambda$ [50], and, remaining above the plane of D, approach continually to the plane of J as an asymptotic plane. The lower sheet will cut the plane of D in $O\Lambda'$, pass under the plane of D, cut the plane of J, progress to a maximum distance from it, and then approach indefinitely to J as its asymptotic plane. This will become apparent by taking a vertical section of this portion, cutting the lines $O\bar{L}$, $O\bar{J}$; for the nature of the flow of the two branches of the section will evidently be as figured below, where j, λ, λ', l, π represent the points in which the lines $O\bar{J}$, $O\Lambda'$, $O\Lambda$, $O\bar{L}$, $O\Pi$ are cut by the secant plane. [It should be particularly noticed that this figure is only intended to exhibit, under its most general aspect, the nature of the flow of the two branches of the curve; it is drawn in other respects almost at random, and makes no pretension whatever to giving a representation of the actual form of the curve.]



No part of the surface G lies under or above the sector ΠOJ, except the axis of D. The cusp C, where the two branches meet, is the intersection of the cutting plane with the parabola $J=D^2$ lying in the plane of L, and there will be another cusp at t, the point of maximum recession from the plane of J.

(61) I now proceed to discriminate, by aid of this surface, the facultative from the non-facultative portion of space.

If in the expression for G as a function of J, K, L we substitute for K its value $-\frac{D}{128}+\frac{J^2}{128}$, we obtain $G=\frac{J}{(128)^4}D^4+$ terms involving only lower powers of D; so that, calling D_1, D_2 the two real values of D in the upper and lower sheets of G respectively corresponding to any point J, L,

$$G=J(D-D_1)(D-D_2)Q,$$

Q being a quantity essentially positive.

Hence when J is negative the *facultative* points in any line parallel to D will be those for which D lies between D_1, D_2, but when J is positive, the facultative points must be exterior to the segment D_1D_2; I denote this difference in the figure by placing a colon between the signs in each sector for which J is positive, indicating thereby that the facultative points lie between $+\infty$ and D_1, and between D_2 and $-\infty$; but where no

[50] For the value of D for this sheet is zero all along $O\Lambda$, and positive on either side of it.

colon is interposed, then it is to be understood that the facultative points lie between D_1 and D_2. Thus, if we turn back for a moment to the section of G last drawn, the whole of the space included between the two branches and the asymptote is facultative, because up to the asymptote J is negative, and beyond the asymptote the whole of the space not included between the asymptote and the lower branch is facultative, because beyond the asymptote J becomes positive. Thus, then, we see that the whole of that portion of the plane which lies on the left-hand side of the entire curve is facultative, and the portion on the right-hand side of the same non-facultative; the curve separating facultative from non-facultative space as a coast-line, indefinitely extended, separates land from water; so that there is, as of course we might have anticipated, no break of continuity in passing through the plane J.

If we take a corresponding section of the opposite portion of space corresponding to the ground-plan $\mathrm{JL}\underline{\Pi}$, it is obvious that precisely the contrary takes place, because the sign of J is opposite in the opposite sectors; so that what was facultative becomes non-facultative, and *vice versâ*.

(62) It is now clear that the whole of the facultative part of space is divided into three, and only three of the *regions* previously defined. One region will consist of that portion of it which is entirely under the plane of D: the second region will be so much of the upper portion as stands upon the acute sector $\bar{\mathrm{J}}\mathrm{O}\Lambda$; and the third of so much of the remainder of this portion as stands on the sector $\Lambda\mathrm{OJJO}\underline{\Pi}$ (51). Again, as regards the second region, the line $\mathrm{O}\Lambda'$ is quite inoperative against its unity, because we have vertical ordinates above $\mathrm{O}\Lambda'$ through which free communication can take place between the blocks over $\mathrm{JO}\Lambda'$ and $\Lambda'\mathrm{O}\Lambda$; but when we come to $\mathrm{O}\Lambda$, where G touches the plane of D, there we have an effective line of demarcation between the adjoining blocks *above* the plane of D; for it is impossible to pass from one into the other without going under D and coming up again through that plane, or else descending to the line $\mathrm{O}\Lambda$ and so meeting the plane of D (52).

(51) It will be borne in mind that the whole of the infinite prism, both above and below, standing on $\Pi\mathrm{OJ}$ belongs to *facultative* space: the prism standing on the opposite section $\bar{\mathrm{J}}\mathrm{O}\bar{\Pi}$, or, to speak more strictly, on the *inside* of this last-named sector, is wholly unfacultative. The facultative line D which passes through O is completely isolated from the facultative portion which stands over $\Lambda\mathrm{O}\bar{\mathrm{J}}$, except at the point O (which we are forbidden to pass through if we would remain in the same region), and is of course a rectilinear edge to the facultative prism above referred to.

(52) Two superior regions we know *à priori* must exist to correspond respectively to the two cases of five and of one real root. Moreover we know *à priori* that two regions can only meet on the plane of D, and an inspection of the *dial-figure* shows that only $\mathrm{O}\Lambda$ can be such line. Thus without completely making out the geometry of the question as regards the remarkable line ($J=0$, $L=0$) (the axis of D) which lies on the surface G, we may feel assured that the upper part of this line (which is easily found to belong to the 1-real-root region) cannot have any point except the origin in common with the 5-real-roots region, since otherwise these two regions would communicate along this line and merge into one. When it is considered that G is a surface of the ninth order in J, D, L, it will not appear surprising that some difficulty arises in forming a mental conception of certain of its local properties; on the contrary, the subject of wonder rather is that enough can be ascertained about it in a very brief compass to shed all the needful light upon the analytical problem which it illustrates.

(63) It remains only to fix the characters of the several regions; but this requires no calculation to effect, for we know that when D is negative there is one and only one pair of imaginary roots. This disposes of the first of the regions above enumerated. Again, we know that when L is positive so that the reduced form is the superlinear equation $ru^5+sv^5+tw^5=0$, u, v, w being *real* functions, D being also positive, there must be four imaginary roots, as follows from the theory of the second section. Hence the third region has for its character two pairs of imaginary roots; and consequently the only remaining region, the second described, must correspond to the case of no imaginary roots, since otherwise we should be absurdly assuming the impossibility in any case of a quintic equation having all its roots real.

(64) It may, however, be an additional satisfaction to see how the change of character comes to pass at the critical line OΛ from one to five real roots.

Along the line OΛ we have found that, calling the reduced form $ru_i^5+sv_i^5+tw^5$,

$$r=s \qquad \frac{\tau}{\varrho}=\frac{rs}{st}=\frac{r}{t}=\theta+4=-4.$$

Hence the equation becomes

$$4u_i^5+4v_i^5+(u_i+v_i)^5=0,$$

u_i, v_i being of the form $\frac{-u+iv}{2}$, $\frac{-u-iv}{2}$, because L is negative.

Hence $u_i+v_i=0$, or

$$4(u_i^4-u_i^3v_i+u_i^2v_i^2-u_iv_i^3+v_i^4)+(u_i+v_i)^4=0,$$

$$\text{i. e. } 5u_i^4+10u_i^2v_i^2+v_i^4=0,$$

$$\text{i. e. } (u_i^2+v_i^2)=0;$$

so that there are two pairs of equal roots of $\frac{u_i}{v_i}$, viz. $\pm\iota$; to these values of $\frac{u_i}{v_i}$ correspond

$$\frac{u-iv}{u+\iota v}=\iota, \quad \frac{u-\iota v}{u+\iota v}=-\iota.$$

Hence

$$(1-\iota)u=(\iota-1)v, \text{ or } (1+\iota)u=(\iota+1)v;$$

so that the two pairs of equal roots of $\frac{u}{v}$ are ±1, the outstanding root corresponding to $u_i+v_i=0$ being $\frac{u}{v}=0$.

Now, *still keeping upon the surface* G, which we know is facultative, let θ become $-8+4\varepsilon$, where ε is an infinitesimal, then

$$\delta\left(\frac{J^3}{L}\right)=\delta\nu=(4\theta^3+12\theta^2)\delta\theta=-5120\varepsilon;$$

also the supposed equation becomes

$$(4-4\varepsilon)(u_i^5+v_i^5)+(u_i+v_i)^5=0,$$

or

$$(\iota v-u)^5-(\iota v+u)^5+8(1+\varepsilon)u^5=0;$$

or, calling $\frac{v}{u}=\varrho$,

$$(\iota\varrho-1)^5-(\iota\varrho+1)^5+8(1+\varepsilon)=0.$$

Let $\varrho=\pm 1+\sigma$, where σ is an infinitesimal. Hence

$$\left(-10(\pm\iota-1)^5+10(\pm\iota+1)^5\right)\sigma^2-8\varepsilon=0,$$

or

$$20(1-10+5)\sigma^2-8\varepsilon=0,$$

or

$$\sigma^2=\frac{-\varepsilon}{10}=+\frac{1}{51200}\delta\left(\frac{J^3}{L}\right).$$

Hence calling σ_1, σ_2 the two values of σ, the four roots that at OΛ were 1, 1, -1, -1 become $1+\sigma_1$, $1+\sigma_2$, $-1+\sigma_1$, $-1+\sigma_2$, when $\frac{J^3}{L}$ becomes varied by $\delta\left(\frac{J^3}{L}\right)$, and consequently become all real if $\frac{J^3}{L}$ is increased, and all imaginary if $\frac{J^3}{L}$ is decreased, *i. e.* become real or imaginary according as the line OΛ sways towards or away from $O\bar{J}$, conformably with what has been shown on other grounds.

It will be noticed that in the line OΛ produced in the opposite direction, *i. e.* along the line $O\underline{\Lambda}$, L being positive, the reduced form is

$$4(u^5+v^5)+(u+v)^5=0,$$

and the roots of $\frac{u}{v}$ become $\frac{u}{v}=-1$, $\frac{u}{v}=\pm\iota$, $\frac{u}{v}=\pm\iota$; so that, according to the canon laid down at the commencement of this discussion (see foot-note ([46])), no change in the character of the roots can possibly take place along OΛ, and accordingly we have seen that this curved line does not correspond to any demarcation of regions.

(65) It is easy to express the conditions to be satisfied by the coordinates of a point according as it lies in one or another of the three regions which have now been mapped out, and it is clear that we have the following rule:

When D is negative the equation has two imaginary roots.

When D is positive the equation has *no* imaginary roots, provided the two criteria J and $2^{11}L-J^3$ are both negative([53]); but if either of these is zero or positive, there are two pairs of imaginary roots([54]).

The duodecimal criterion-invariant, $2^{11}L-J^3$, and the invariants of the like order, $27\cdot 2^{16}L-J^3$, $-27L-J^3$, I shall henceforth call Λ, Λ′, Π respectively. It has been just above shown that the three invariants J, D, Λ of the 4th, 8th, and 12th orders respectively are sufficient for ascertaining the character of the roots of the quintic to which they appertain.

([53]) Observe that this implies L also being negative; so that $2^{11}-\frac{J^3}{L}$ is positive and $\frac{J^3}{L}<2^{11}$.

([54]) ([a]) Observe that in general when $2^{11}L-J$ is zero there are no facultative points above the plane of D, but when J and $2^{11}L-J$, and consequently L and J are both simultaneously zero, a facultative right line springs into existence, viz. the axis of D extending both above and below the plane of D. The reduced form of equation (as previously demonstrated) corresponding to this singular line is $u^5\pm uv^4=0$.

([b]) It may further be noticed that on each side of the line $O\underline{\Lambda}$ the limits of D are between positive infinity and a positive quantity, and between negative infinity and a negative quantity; so that as we pass from $O\underline{\Lambda}$ to either side of it no facultative point can be found lying in the plane of D, showing that we cannot pass by a real infinitesimal variation of coefficients from an equation with two pairs of equal imaginary roots to an equation with a single pair of equal roots, as is apparent also on purely analytical grounds.

(66) The assertion that the *whole* of facultative space is divisible into three regions, in strictness requires a slight modification. It is obvious that the plane of D itself cannot be said to belong to any of the regions; and in order to make our theory quite complete, so as to furnish criteria applicable to equations having equal roots, and to enable us to distinguish between the case of the unequal roots being all three real, or two imaginary and one real, we must examine what takes place in this plane, and under what circumstances a passage from one point of it to another will or may be accompanied with a change of character in the roots.

If the roots of $f(x)=0$ are supposed to be a, a, c, d, e, where c, d, e are unequal, on varying the constants of fx in such a manner that the variation of the discriminant D is zero, the two equal roots a, a will remain equal. Now *in general* we have $\delta f(a)+f''(a)\frac{(da)^2}{2}=0$; if this, under the particular supposition made, continued to obtain, da would have two distinct values, and the two equal roots would cease to continue to be equal, contrary to hypothesis. Hence we see that $D=0$, $\delta D=0$ necessarily implies $\delta f(a)=0$ [55], and consequently $\delta f(a+da)$ is no longer δfa, but $\delta f'ada$; so that we obtain $da=0$, or $da=-\frac{2\delta f'a}{f''a}$, and no change of character in the five roots results. If, however, the original roots are a, a, c, c, e, then, as shown in the general case, δc will have two distinct values, which will be both real or both imaginary. Accordingly we see that in

[55] (a) This is a somewhat curious theorem (whether new or otherwise I know not) thus incidentally established in the text, viz. that if $D(f)$ represent the discriminant of f, and if $D(f)=0$ and $\delta D(f)=0$, then when $f=0$ we must have $\delta(f)=0$. The very simplest example that can be chosen will serve to illustrate this proposition. Let

$$f=ax^2+2bxy+cy^2.$$

Suppose

$$D(f)=ac-b^2=0,$$

and also

$$\delta D(f)=a\delta c+c\delta a-2b\delta b=0,$$

we have

$$\delta(f)=x^2\delta a+2xy\delta b+y^2\delta c.$$

Now if $f=0$ we may write $x=b$, $y=-a$, and δf becomes

$$\begin{aligned}&b^2\delta a-2ab\delta b+a^2\delta c\\ &=b^2\delta a-2ab\delta b+2ab\delta b-ac\delta a\\ &=(b^2-ac)\delta a=0,\end{aligned}$$

according to the theorem.

If we make $f=(x,1)^n$, D we know becomes a syzygetic function of f and $f'\left(\text{meaning by the latter } \frac{df}{dx}\right)$. Hence since δD vanishes when $fx=0$, $D=0$, and $\delta f(x)=0$, we learn that $\delta(D)$ is a syzygetic function of $(f, f', \delta f)$.

The theorem thus stated easily admits of extension to the higher variations of D, and so extended takes I believe the following form:

$$\delta^i(D)= \text{ a syzygetic function of } (f, f', f'', \ldots f^i, \delta f).$$

(b) Professor CAYLEY has since informed me that the theorem in (55) (a), about whose originality I was in doubt, will be found in SCHLÄFLI's 'De Eliminatione.' This is not the first unconscious plagiarism I have been guilty of towards this eminent man, whose friendship I am proud to claim. A much more glaring case occurs in a note by me in the 'Comptes Rendus,' on the twenty-seven straight lines of cubic surfaces, where I believe I have followed (like one walking in his sleep), down to the very nomenclature and notation, the substance of a portion of a paper inserted by SCHLÄFLI in the 'Mathematical Journal,' which bears my name as one of the editors upon its face!

the plane of D no change can possibly take place except in crossing the line which corresponds to a family of *two pairs* of equal roots.

(67) It has already been pointed out, in a foot-note, that we cannot pass facultatively from $O\underline{\Lambda}$ to either side of this curve line. Hence the separation of the plane of D into subregions can only take place along the line $O\Lambda$, and it remains but to ascertain the character of the points on either side of this line, which we know, therefore, *à priori*, must possess opposite characters, since otherwise we should be admitting the absurd proposition of its being impossible to construct an equation of the fifth degree having two equal roots without the remaining three being always of *one character*, either all real or all not real. Let us, then, ascertain the character of the points in OJ for which $D=0$, $L=0$, and J is positive([56]).

Since $L=0$, the reduced form is $u^5+5euv^4+v^5$.

This equation, by DESCARTES'S rule, must contain imaginary roots. Hence in the sector $\Lambda O\bar{J}$ the roots are all real, and in the remainder of the facultative portion of the plane (from which it may be noticed the sector $\underline{\Lambda}O\bar{J}$ is excluded) two of the roots are imaginary.

Along $O\Lambda$ itself there are, as already observed, two pairs of real equal roots, and along $O\underline{\Lambda}$ two pairs of imaginary equal roots. Thus, finally, we have the *complete rule.*

If D is negative,	2 roots imaginary.	
If D is positive.		
When J, Λ are both negative,	0 roots imaginary.	
„ J, Λ are *not* both negative,	4 roots imaginary.	
If D is zero.		
When J, Λ are both negative,	0 roots imaginary	1 pair of equal roots.
„ J, Λ are *not* both negative,	2 roots imaginary	
„ J is negative, Λ zero,	0 roots imaginary	2 pairs of equal roots.
„ J is positive, Λ zero,	4 roots imaginary	
„ J is zero, Λ zero,		3 equal roots([56 bis]).

Thus we see that our space referred to an arbitrary origin, and with the invariants J, D, Λ for the coordinates, has been first divided into facultative and non-facultative space. The former has then been resolved prismatically into two regions above and one below the plane of D. The plane of D itself, or the facultative part of it, into two

([56]) We could not take J negative, for the facultative points of D in $\bar{J}$ are two positive quantities. See dial figure.

([56 bis]) When $D=0$, $\Lambda=0$, there are two pairs of equal roots. If J is negative these pairs are both real. If J is positive they are both imaginary. When J is zero there are no longer two pairs, but a single triad of equal roots. This perfectly explains what at first sight has the air of a paradox, viz. that the discrimination between the two kinds of double equality of an apparently equal order of generality that may subsist between the roots of an equation, depends on the fulfilment or failure of an algebraical equality. The fact is, as shown above, that there are not, as commonly supposed, two, but three kinds of double equality, according as there are two pairs of real, two pairs of imaginary, or one triad of equal roots; and the last is a sort of transition case between the other two.

planar regions on opposite sides of the line $\Lambda O \underline{\Lambda}$; and again this line into two linear regions on either side of the origin O, which last corresponds to the case of three equal roots, and constitutes a region or microcosm in itself.

(68) It may as well be noticed here that the ambiguity of character in the points representing the different families of biquadratic forms when t and D are taken as the coordinates (and the same would be true if s and D were employed), which prevails when these points lie above the line $D=0$, equally obtains along this line itself. For the reduced form, when $D=0$, is $ax^4+4bx^3y+6cx^2y^2$. In that case, calling the determinant of transformation μ, we have $s=3\mu^{12}c^2$, $D=-\mu^{24}c^3$; and thus, whatever s and D may be, the character of the unequal roots is left undecided.

It may also be noticed that the blending of characters at the *origin* for the quintic form is not precisely of the same nature as that for the points above the line D in the biquadratic form; for at these points it is the cases of 4 and 0 imaginaries which become undistinguishable invariantively; whereas at the origin for quintics the reduced form becomes $ax^5+5bx^4y+10x^3y^2$, and the characters left undistinguished are those of 4 and of 2 imaginary roots—unless, indeed, we consider equal real roots as belonging indifferently to the class of real and imaginary; on which supposition all the three genders (so to say), masculine, feminine, and neuter, become blended together at that point. But if we consider equal real roots as exclusively of the real class, then the *origin* for quartics ceases to be epicene; for when there are three equal roots all of them must be real. Thus the origin in quintics is the only epicene point, and in quartics the only non-epicene point—understanding by epicene the blending of the masculine (4 *imaginary roots*) and feminine (*no imaginary roots*) characters.

(69) We may draw some further important inferences from an inspection of the "dial figure," or the section of facultative space which follows it.

Within the prism $JO\Lambda'$ ([57]) it will be observed D is always positive ([58]). Hence, when J is negative and Λ' is negative, all the roots *must* be real, and the necessity for using the criterion D is done away with.

Again, when J and L are both negative, D is always negative, so that just two of the roots must be imaginary; and in this case also it becomes unnecessary to apply the criterion D.

Again, since there is no facultative prism corresponding to $\underline{\Pi}OJ$, the combination of L and D, both negative, can never occur unless $\underline{\Pi}$ is negative.

When L is negative, but J not negative, there may be two or four imaginary roots, according to the sign of D; but all the roots cannot be real.

(70) M. HERMITE's rule is as follows. For remarks on the relation between his Δ, J_2, J_3 and the J, K, L of this paper, see foot-note ([36]). D is still the discriminant.

If D is negative (of course) two roots are imaginary.

If D is positive.

([57]) By which I mean within the facultative prism of which $JO\Lambda'$ is the section made by the plane of D.

([58]) The vertical section of facultative space in this supposition (see figure) is the area $\lambda C\lambda'$, which lies wholly *above* the plane of D.

When Δ is negative, $25\Delta^3-3.2^{10}J_3$ negative and J_3 positive, no roots are imaginary.

Δ is negative, $25\Delta^3-3.2^{10}J_3$ positive, $25\Delta^3-2^{11}J_3$ negative, no roots are imaginary.

Δ is positive, ... four roots are imaginary.

Δ is negative, $25\Delta^3-3.2^{10}J_3$ positive, $25\Delta^3-2^{11}J_3$ positive, four roots are imaginary([59]).

(71) What is the effect of the condition "Λ *positive* or *negative*," as the case may be? or rather, how does this condition arise? The ground of it is simply this, that $\Lambda=0$ represents a cylindrical surface passing through the curve $O\Lambda$ (see dial figure), which curve is the *edge* of separation between two regions of opposite characters above the plane of D; the cylinder in question cuts the facultative position of space below the plane of D, but above this plane (except along the vertical line $J=0$, $L=0$, *i. e.* the axis of D) it passes exclusively through non-facultative space, never again cutting or meeting the surface G (the facultative boundary). Now it is clear that any surface whatever which passes through $O\Lambda$ and never meets the surface G above the plane $D=0$, except along the axis of D (*i. e.* the line $J=0$, $L=0$), may be substituted for Λ([60]) and will serve equally well with Λ to distinguish between the masculine and feminine regions of space. $\Lambda-\varrho JD$ will fulfil the condition of passing through the line $O\Lambda$,

([59]) ([a]) The last four conditions ought to tally (and be in effect coextensive) with the two given by me for the case of D positive. The third of them, viz. the case of D positive Δ positive, I have already noticed, as inferences from the dial figure; for M. HERMITE's Δ, if not identical with my J, is at all events a positive multiple of it. I do not see how the case of Δ negative, $25\Delta^3-3.2^{10}J_3$ negative with D positive, is met by this system of criteria, since J_3, as well as Δ, may be negative consistently with the second condition. I have not been able to ascertain whether in the memoir such a combination is shown to be impossible. M. HERMITE admits, and indeed has been always aware of, the existence of a *lacuna* in the conditions above stated, which, I understand from him, it is his intention at some future time to fill up, and thus to complete his original solution. In the meanwhile he has been led to study the question from a different point of view, and has succeeded in obtaining a new set of criteria adequate to a complete solution of the question without calling in the aid of the principle of continuity. In this new system my Λ criterion is replaced by an invariant of the twenty-fourth degree, which is of course an objection as far as it goes, but in no wise diminishes the extraordinary interest that attaches to this altered mode of approaching the question, which bears to his original method and my own the same relation as the proof of STURM's theorem by the law of inertia for quadratic forms bears to that given by STURM himself.

([b]) It is apparent from the fact that when $D=0$, G (M. HERMITE's I^2) becomes $(25\Delta^3-3.2^{10}J_3)(25\Delta^3-2^{11}J_3)^2$ (Camb. and Dub. Journal, vol. ix. p. 206), that the factors of this product are respectively of the form $a\Lambda'+bJD$, $c\Lambda+eJD$, a, b, c, e being certain numerical quantities. This gives rise to a singular reflection, *to wit*, that my own criteria for the case of D positive may be varied by the addition of a term λDJ to Λ (λ being a numerical coefficient), provided λ lies within certain limits, the form of the criteria in all other respects remaining unchanged. This proposition, fraught with the most important consequences, and not unlikely to lead to an entire revolution in the mode of attacking the general problem of criteria, I proceed to establish in the text.

([60]) The surface to be employed will be $\Lambda-\varrho JD$, which call M. Λ and M (or at least their upper portions above the plane of D) may then be regarded as the two sides of a sack, of infinite dimensions, open at the top, and seamed together at the bottom, along the curved line $D=0$, $\Lambda=0$, and in the vertical direction along the straight line $J=0$, $L=0$. The surface Λ serving as a screen of separation between the two upper regions, it is clear that M will serve equally well as such screen, provided no superior facultative points lie in the interior of the sack.

whose equation is $\Lambda=0$, $D=0$, and obviously is the only invariant not exceeding the twelfth order capable of so doing; it only remains to ascertain within what limits the numerical coefficient ϱ must be taken so as to fulfil the condition that the combined equations $\Lambda-\varrho JD=0$, $G=0$ shall be incapable of being satisfied by any positive value of D.

(72) Substituting for Λ and D their values, the equation to be combined with $G=0$ becomes

$$J^3-2^{11}L+\varrho J(J^2-128K)=0.$$

Returning to the notation of art. (55), and dividing by JK, this equation, when $G=0$, becomes

$$q-2^{11}\frac{q}{\nu}+\varrho(q-128)=0,$$

or

$$(1+\varrho)q\nu-2^{11}q=128\varrho\nu,$$

which, substituting for q, ν in terms of θ, gives

$$\frac{(1+\varrho)\theta^5(\theta+4)^2}{\theta+6}=2^{11}\frac{\theta^3+4\theta^2}{\theta+6}-128\varrho\theta^3(\theta+4),$$

or

$$(\theta+4)\theta^2(\theta+8)\big((\theta^3-4\theta^2+32\theta-256)+(\theta^3-4\theta^2-96\theta)\varrho\big)=0.$$

When $\theta+8=0$, $D=0$, see art. (57); neglecting, then, this factor, the condition to be satisfied is that when from the equation

$$(\theta+4)\theta^2\big((\theta^3-4\theta^2+32\theta-256)\varrho+(\theta^3-4\theta^2-96\theta)\big)=0$$

a value of θ has been deduced, the values of D corresponding thereto shall not be a positive finite quantity.

(73) Now

$$\frac{D}{J^2}=1-\frac{128(\theta+6)}{\theta^2(\theta+4)}=\frac{\theta^3+4\theta^2-128(\theta+6)}{\theta^2(\theta+4)}=\frac{(\theta+8)^2(\theta-12)}{\theta^2(\theta+4)}.$$

If $\theta=0$, or $\theta+4=0$, since D cannot be infinite, we have $J=0$, so that $\Lambda-\varrho JD$ becomes identical with the original criterion Λ. Hence the factor $(\theta+4)\theta^2$ in the quantity just above equated to zero may be neglected, and the condition to be fulfilled by ϱ is that if θ be any root of the equation

$$\frac{-\theta^3+4\theta^2-32\theta+256}{\theta^3-4\theta^2-96\theta}=\varrho,$$

θ shall be between -4 and 12; this equation on making $\theta=-4\phi$, so that $1>\phi>-3$, becomes

$$-\varrho=\frac{\phi^3+\phi^2+2\phi+4}{\phi^3+\phi^2-6\phi},$$

or, writing $\sigma=\frac{-1-\varrho}{4}$,

$$\sigma=\frac{2\phi+1}{\phi^3+\phi^2-6\phi}=\frac{2\phi+1}{(\phi-2)\phi(\phi+3)}.$$

(74) We wish to ascertain what values of σ will be incompatible with the violation of the limits just assigned to ϕ, and accordingly we must inquire what is the range of values assumed by σ when $\phi>1$ or $\phi<-3$; any values of σ *not* included within this range will be admissible for the purpose in view.

When $\phi < -3$, σ is always positive, and proceeds continuously from ∞ to 0 as ϕ passes from $-3-\varepsilon$ (ε being infinitesimal) to $-\infty$. Consequently σ must not be allowed to have any positive value. When $\phi=\infty$, $\sigma=0$, and when $\phi=1$, $\sigma=-\frac{3}{4}$.

Hence, if no minimum value of σ (i. e. no maximum value of $-\sigma$) occurs between $\phi=1$, $\phi=\infty$, σ may have any value between 0 and $-\frac{3}{4}$; but if such a minimum value, $-$M, where M $>\frac{3}{4}$, should exist, the admissible values of σ would become more enlarged, and might be taken between 0 and $-$M.

Making then $\delta\sigma=0$, we have

$$\frac{2}{2\phi+1}=\frac{3\phi^2+2\phi-6}{\phi^3+\phi^2-6\phi},$$

or

$$4\phi^3+5\phi^2+2\phi-6=0;$$

which, substituting $1+\psi$ for ϕ, becomes

$$4\psi^3+17\psi^2+24\psi+5=0;$$

so that there can be no real root of the equation in ϕ greater than unity.

Hence the admissible values of σ are defined by the inequalities $0>\sigma>-\frac{3}{4}$,

$$\text{i. e. } 0>-\frac{1+\varrho}{4}>-\frac{3}{4}, \qquad \text{or } 0>-(1+\varrho)>-3, \qquad \text{or } 2>\varrho>-1.$$

(75) We have thus obtained the complete solution of the problem of assigning invariantive criteria, such that their signs (positive, negative, or zero) shall serve to fix the nature of the roots. These criteria we now see are

$$\text{J, D, } \Lambda+\mu\text{JD},$$

where μ (the negative, it must be noticed, of ϱ) is any numerical quantity intermediate between 1 and -2 [61].

(76) This important modification of the original criteria J, D, Λ I proceed to apply to the problem of obtaining the *simplest* and *most symmetrical* expression for the criteria in terms of the roots of the equation. Let a, b, c, d, e be the roots, and write

$$Z=\Sigma\{(a-b)^2(a-c)^2(b-c)^2(a-d)^4(a-e)^4(b-d)^4(b-e)^4(c-d)^4(c-e)^4\},$$

or say

$$Z=\Sigma\left\{\zeta(a, b, c)\begin{pmatrix} a\ b\ c \\ de \end{pmatrix}\right\}\text{[62]}.$$

[61] Strictly it has only been proved that the surface $\Lambda+\mu$JD, which passes through the line Λ, D, contains no superior facultative points except those comprised in the line L$=0$, J$=0$. It is, I think, not difficult to see from this, that, if in the "*sack*" formed between Λ and $\Lambda+\mu$JD any such points were contained, L$=0$, J$=0$, *i. e.* the axis of D would be a double or multiple line on the surface G, which is easily disproved by examining the algebraical form of G in art. 41, where K represents $\frac{-\text{D}+\text{J}^2}{128}$; any obscurity, however, which may be supposed to cling to this view is immaterial, as a demonstration capable of being followed *in plano* and leaving nothing to be desired in point of perspicuity, will be found in the Note appended to this Part.

[62] Agreeable to the meaning assigned to ζ and to a couple of rows of letters in my memoir on Syzygetic Relations, in the Philosophical Transactions.

Then, since each letter occurs the same number of times (12) in each term, Z will be an invariant.

(77) Again, suppose any two roots to become equal, say that e becomes d, then Z reduces to the single term $\zeta(a, b, c)\begin{pmatrix} a & b & c \\ d & d & \end{pmatrix}$; for any such factor as $\zeta(a, b, d)$ will be accompanied with the factor $\begin{pmatrix} a & b & d \\ c & d & \end{pmatrix}$ which vanishes.

If, further, we suppose any two of the letters a, b, c to become equal, then Z disappears entirely, since on that supposition $\zeta(a, b, c)$ vanishes. Hence Z is an invariant of the twelfth order, possessing the property of vanishing when the equation to which it belongs has two pairs of equal roots. Hence Z is of the form $p\Lambda+q\mathrm{JD}$, and it becomes of importance to ascertain the value of the ratio $\frac{q}{p}$.

To do this let us suppose $e=0$, $a=-b$, $c=-d$.

The ten terms in Z correspond to the following ten partitions:—

(1)	(2)	(3)	(4)
abc	abd	acd	bcd
de	ce	be	ae
	(5)	(6)	
	abe	cde	
	cd	ab	
(7)	(8)	(9)	(10)
ace	bde	ade	bce
bd	ac	bc	ad

(78) The corresponding values of the terms will be

$$4a^2(a^2-c^2)^2.16(a^2c^2)8^2(a^2-c^2)^4;\ 4a^2(a^2-c^2)^2 16a^2c^2(a^2-c^2)^4;\ 4c^2(a^2-c^2)^2.16a^2c^2(a^2-c^2)^4;$$
$$4c^2(a^2-c^2)^2 16a^2c^2(a^2-c^2)^4;\ 4a^6c^8(a^2-c^2)^2;\ 4c^6a^8(a^2-c^2)^2;\ (a-c)^2 256a^4c^4(a+c)^8;$$
$$a^2c^2(a-c)^2 256a^4c^4.a^4c^4(a+c)^8;\ (a+c)^2 256a^4c^4(a-c)^8;\ (a+c)^2 256a^4c^4(a-c)^8.$$

Collecting and simplifying these terms, and observing that

$$(a-c)^2(a+c)^8+(a+c)^2(a-c)^8=(a^2-c^2)\big((a+c)^6+(a-c)^6\big)=4(a^4-c^4)(a^4+14a^2c^2+c^4),$$

we find

$$\mathrm{Z}=128(a^2+c^2)a^8c^8(a^2-c^2)^6+4(a^2+c^2)a^6c^6(a^2-c^2)^8$$
$$+1024(a^2+c^2)(a^4+14a^2c^2+c^4)(a^2-c^2)^2a^{10}c^{10}.$$

Let $(a^2-c^2)^2=p$, $a^2c^2=q$, and let $\mathrm{Z}_{\prime}=\frac{\mathrm{Z}}{(a^2+c^2)q^3}$. Then

$$\mathrm{Z}_{\prime}=16384pq^3+1024p^2q^2+128p^3q+4p^4$$
$$=2^{14}pq^3+2^{10}p^2q^2+2^7p^3q+2^2p^4.$$

(79) We must now calculate J, D, L:

$$\mathrm{D}=\frac{1}{5^5}\zeta(a,\ -a,\ c,\ -c,\ 0)$$

$$=\frac{1}{5^5}4a^6c^6(a^2-c^2)^4\,;$$

or writing

$$\mathrm{D}=\frac{\mathrm{D}}{q^3},$$

$$\mathrm{D}_{\prime}=\frac{4}{5^5}p^2.$$

Again, for J. The form to which it belongs is

$$x^5-(a^2+c^2)x^3y^2+a^2c^2xy^4,$$

or

$$\left(1,\ 0,-\frac{a^2+c^2}{10},\ 0,\ \frac{a^2c^2}{5},\ 0\between x,\ y\right)^5;$$

so that the coefficients of the biquadratic Emanant are

$$x\,;\quad -\frac{a^2+c^2}{10}y\,;\quad -\frac{a^2+c^2}{10}x\,;\quad \frac{a^2c^2}{5}y\,;\quad \frac{a^2c^2}{5}x.$$

Hence the quadratic covariant becomes

$$\frac{a^2c^2}{5}x^2+\frac{2}{25}(a^2+c^2)a^2c^2y^2+\frac{3}{100}(a^2+c^2)^2x^2$$

$$=\frac{20a^2c^2+3(a^2+c^2)^2}{100}x^2+\frac{2}{25}(a^2+c^2)(a^2c^2)y^2.$$

Hence, by definition, J (which $=-4\times$ Discriminant of the Quadratic Covariant)

$$=-\frac{4}{1250}(a^2c^2)(a^2+c^2)\left(3(a^2-c^2)^2+32a^2c^2\right);$$

and making

$$\mathrm{J}_{\prime}=\frac{\mathrm{J}}{(a^2+c^2)q},$$

$$\mathrm{J}_{\prime}=-\frac{6}{625}p-\frac{64}{625}q=-\frac{6}{5^4}p-\frac{2^8}{5^5}q.$$

Finally, to calculate L. The canonizant of the form

$$\begin{array}{cccc} 1 & 0 & \mathrm{A} & 0 \\ 0 & \mathrm{A} & 0 & \mathrm{B} \\ \mathrm{A} & 0 & \mathrm{B} & 0 \\ y^3; & -xy^2; & x^2y; & -x^3 \end{array}$$

is

$$(\mathrm{A}^3-\mathrm{AB})x^3+(\mathrm{B}^2-\mathrm{A}^2\mathrm{B})xy^2,$$

of which the discriminant is

$$-4\frac{\mathrm{AB}^3}{27}(\mathrm{A}^2-\mathrm{B})^4,$$

where

$$A=-\frac{a^2+c^2}{10}, \qquad B=\frac{a^2c^2}{5}.$$

Hence, by definition,

$$L=AB^3(A^2-B)^4=-\frac{1}{125.10^9}(a^2+b^2)(a^6b^6)\big((a^2-b^2)^2-16a^2b^2\big);$$

and making

$$L_{\prime}=-\frac{L}{(a^2+c^2)q^3},$$

$$L_{\prime}=\frac{1}{125.10^9}(p-16q)^4=-\frac{1}{5^{12}.2^7}(p^2-16q)^4.$$

(80) Now let us write

$$\frac{1}{5^{12}}Z=\eta L+eJD\,(^{63})+\varepsilon J^3.$$

This gives

$$\frac{1}{5^{12}}Z_{\prime}=eqJ_{\prime}D_{\prime}+\varepsilon(p+4q)J_{\prime}^3+\eta L,$$

or

$$4p^4+128q^3p^3+1024q^2p^2+16384pq^3$$
$$=125(256p^2q^2+24p^3q)e+(p+4q)(6p+64q)^3\varepsilon+\frac{1}{2^7}(p-16q)^4\eta),$$

by means of which identity we can obtain linear equations for finding the values of e, ε, η.

Thus, equating the coefficients of p^4, q^4, p^3q respectively, we obtain

$$4=216\varepsilon+\frac{1}{2^7}\eta,$$

$$4.64^3\varepsilon+\frac{16^4}{2^7}\eta=0,$$

which gives $\eta=-2^{11}\varepsilon$ (as it ought to do),

$$128=(24\times 125)e+(4\times 216+108\times 64)\varepsilon+64\ 2^{11}\varepsilon$$
$$=3000e+8800\varepsilon.$$

Hence

$$200\varepsilon=4, \quad \varepsilon=\frac{1}{50}, \quad \eta=-\frac{2^{10}}{25},$$

$$3000e=128-176=-48, \quad e=-\frac{2}{125} \text{ and } \frac{e}{\varepsilon}=-\frac{4}{5}.$$

In order to verify the value of e, let $p=-4$, $q=1$; then, assuming the correctness of the above determinations, we ought to find

$$4^5-128.4^3+1024.16+16384=125(256.16-24.64).\frac{-2}{125}+\frac{1}{128}.160000.-2^{11}.\frac{1}{50},$$

or

$$2^{10}(1-8+16-64)=(-32.256+48.64)-\frac{8}{25}\times 160000,$$

or

$$2^{10}(-55)=-5120-25.2048=2^{10}(-5-50),$$

which is right.

(63) Since Z has been proved to be of the form $p\Lambda+qJD$, we know *à priori* the value of $\frac{\varepsilon}{\eta}$; but I have thought it safer to determine ε, η independently, as an additional check upon the accuracy of the computations.

(81) Thus

$$-Z=\frac{5^{10}}{2}\left(2^{11}L-J^3+\frac{4}{5}JD\right)$$

$$=\frac{5^{10}}{2}\left(\Lambda+\frac{4}{5}JD\right);$$

and accordingly we have proved that $-Z$ is of the form $(\Lambda+\frac{4}{5}JD)$; and consequently, since $\frac{4}{5}$ lies within the allowed limits 1 and -2, $-Z$ may be used to replace Λ in the system of criteria.

(82) On examining the composition of Z, it will be found to have a remarkable relation to the lower criterion J.

J we know is, to a numerical factor *près*, of the form

$$\Sigma\{(d-e)^4\zeta(a, b, c)\},$$

ζ denoting, as usual, the squared product of the differences of the quantities which it affects; and Z, it will readily be seen, is of the form

$$\left(\zeta(a, b, c, d, e)\right)^2\Sigma\frac{1}{\zeta(a, b, c)(d-e)^4};$$

and the squared factor is always positive whatever the roots may be, for ζ is always real.

Hence the essential part of our rule thus transformed comes to this, that if

$$\Sigma\left(\zeta(a, b, c)\times(d-e)^4\right) \text{ and } \Sigma\left\{\left(\zeta(a, b, c)\right)^{-1}(d-e)^{-4}\right\}$$

are both of them positive, then when the discriminant is positive, so that the case of two of the five quantities a, b, c, d, e being conjugate and the other three real is excluded, and the choice lies between supposing all or only one of them real, we are able to affirm that they will all be real. Nothing could be easier than to multiply tests expressed by simple symmetric functions of differences of the roots, any infringement of which would contradict the hypothesis of all the five letters denoting real quantities; the difficulty consists in discovering a system of the least number that will suffice of decisive tests, such that not only their infringement shall contradict the hypothesis of imaginary roots, but whose fulfilment shall ensure the roots being all real. This is what has been proved to be effected by means of the invariants J, D, $\Lambda+\frac{4}{5}JD$.

In the case before us it is clear that when the roots are all real, each of the sums above written must be positive and greater than zero. That their being both positive and greater than zero is inconsistent with four of the letters a, b, c, d, e being imaginary would probably not admit of an easy direct demonstration.

Z we have seen is only a particular value of the general invariant $\Lambda+\mu JD$, which may be called M, where μ is an arbitrary constant limited to lie between 1 and -2.

(83) It may be well to notice the effect of using *as a criterion*, in conjunction with J and D, the value of M corresponding to either extreme value of μ. In such case, supposing M to become zero, it might for a moment appear doubtful to which region

that point representing the family of forms is to be referred. But since the doubt can only arise when J is negative and D positive, and since by hypothesis we have $\Lambda = -\mu JD$, we see that Λ takes the sign of μ; and consequently the sign of M, when it becomes zero, is to be understood as following the sign of μ, i. e. as positive when μ is 1 and negative when μ is -2.

(84) The method above given for ascertaining the nature of the roots of a quintic involves the use of only three criteria. It may be inquired how many would become needful in applying STURM'S method. In the case of a cubic equation only the last of the two Sturmian criteria comes into use; and it seems therefore desirable to ascertain whether all four of the Sturmian criteria applicable to that case are required, or whether a smaller number are sufficient. I speak of four criteria, inasmuch as the leading terms fx and $f'x$ cannot be considered as such, their signs being fixed; so that we are at liberty to consider them both positive. Suppose all six Sturmian functions to be written down, including fx (a function of x of the fifth degree) and $f'x$, and let us characterize by the index (r, s) any succession of signs of the leading coefficients which contain r continuations and s variations, and which therefore will correspond to the case of $(r-s)$ roots.

The total number of cases to be considered are the sixteen following:

(5, 0)	+	+	+	+	+	+
(4, 1)	+	+	+	+	+	−
	+	+	+	+	−	−
	+	+	+	−	−	−
	+	+	−	−	−	−
(3, 2)	+	+	+	+	−	+
	+	+	+	−	+	+
	+	+	+	−	−	+
	+	+	−	+	+	+
	+	+	−	−	+	+
	+	+	−	−	−	+
(2, 3)	+	+	+	−	+	−
	+	+	−	+	+	−
	+	+	−	+	−	−
	+	+	−	−	+	−
(1, 4)	+	+	−	+	−	+

the successions corresponding to the indices (2, 3), (1, 4) will become impossible, as corresponding to a *negative* number of real roots. An inspection of the eleven cases corresponding to the indices (5, 0), (4, 1), (3, 2) will show that no *ternary* combination of signs in the third, fourth, and sixth columns belongs to any of the three characters (5, 0), (4, 1), (3, 2) exclusively, and consequently all four signs must be used; and therefore, if the method of STURM is employed, four criteria are indispensable for determining

effectually the character of the roots in an equation of the fifth degree(64); whereas in the symmetrical and invariantive method which I have employed three have been seen to suffice.

In an equation of the seventh degree the case of 0 or 4 will be distinguishable from that of 2 or 6 imaginary roots by the sign of the discriminant, and then again the case of 0 from that of 4, and of 2 from that of 6, by other invariantive criterion-systems. So for an equation of the ninth degree, the first separation will be that of the 0, 4, or 8 case from that of 2 or 6; then it may be conjectured the 2 case will be invariantively separated from the 6, and the 0 or 8 from that of 4, and, finally, 0 and 8 from each other—the reduction of cases apparently depending upon the relation of the index of the equation to the powers of the number 2. This much we know (from art. 49) as matter of certainty, that no single criterion other than the discriminant can ever serve to distinguish one form of roots from another so that all other criteria must accompany each other in groups; and accordingly the scheme of criteria established in the foregoing investigation is in kind the very simplest *à priori* conceivable.

(64) (a) For an equation of the nth degree there are $n-1$ variable criteria, each capable of being $+$ or $-$, and thus giving rise to 2^{n-1} conceivable diversities of combination. The actual number possible, however, is considerably less than this; and I find by an easy method that this number, when n is odd, is $2^{n-2}+\dfrac{\Pi(n-1)}{2\left(\Pi\frac{n-1}{2}\right)^2}$, and when n is even, is $2^{n-2}+\dfrac{\Pi(n-1)}{\Pi\frac{n}{2}\,\Pi\left(\frac{n}{2}-1\right)}$.

(b) Not quite foreign to this subject is the inquiry as to the comparative probability of each different succession or each different family of successions possessing equivalent characters; and, as connected therewith, the comparative probability of a certain specified number of the roots of an equation of a given degree being real and the remainder imaginary. In the simplest case of a quadratic equation of which the coefficients are real but otherwise arbitrary, I find that upon the particular hypothesis of the squares of the three coefficients being limited by one and the same quantity, the probability of the roots being imaginary is $\frac{31}{72}-\frac{\log 2}{12}$, or $\cdot 3727932$, a little less than $\frac{3}{8}$, this being the value of the integral $\int_0^{\frac{1}{2}} db \int_{b^2}^{1} da\left(1-\frac{b^2}{a}\right)$; but we are not at liberty to infer from this the value of the probability in question when the coefficients are left absolutely unlimited. A case in point, as illustrating the effect of imposing a limit in questions of this kind, occurs in the problem (which I raised in my lectures on Partitions) of finding the probability that four points placed at hazard in a plane will form the angles of a reentrant quadrilateral, which Professor CAYLEY has shown is exactly $\frac{1}{4}$ in the absence of any limit. For if ABCD be the four points, and ABC the greatest of the four triangles of which they may be regarded as the angular points, and if through A, B, C be drawn lines parallel to BC, CA, AB respectively, the triangle $\alpha\beta\gamma$ so formed will be four times as great as ABC, and the point D must be somewhere within $\alpha\beta\gamma$, otherwise ABC would not be less than each of the three other triangles ABD, BCD, CAD; and consequently, since D must lie within ABC when the quadrilateral is reentrant, the probability in question is $\frac{ABC}{\alpha\beta\gamma}$, or $\frac{1}{4}$. Now it is easy to see, by using the very same construction, that if any contour whatever be imposed as a limit upon the positions of the four points, the probability referred to will exceed $\frac{1}{4}$ by a finite quantity—a result somewhat paradoxical, since *à priori* one would have supposed that the value of it for the case of *no limit* would be the *mean* of the values corresponding to the respective suppositions of every possible form of limit.

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Note on the arbitrary constant which appears in one of the criteria for distinguishing the case of four from that of no imaginary roots, and on the curve whose coordinates express the limiting relations of all the octodecimal invariants of a binary quintic, &c.

(85) The appearance of an arbitrary constant in a criterion is a circumstance so unexampled and remarkable that I have thought it desirable to give a more complete, or at least a more palpable proof of the validity of the substitution of $\Lambda+\mu\mathrm{JD}$ for Λ than that furnished in the foregoing text, where some indistinctness arises from the difficulty of raising up in the mind a clear conception of the form of the amphigenous surface, and the two portions of space which it separates. That difficulty is entirely obviated, and the theory rendered palpable to the senses by the following investigation, where the problem is so handled as to involve the contemplation of two dimensions only of space. We have in general

$$\mathrm{D}=\mathrm{J}^2-128\mathrm{K}, \quad \Lambda=2048\mathrm{L}-\mathrm{J}^3,$$

and at the amphigenous surface (see art. 57)

$$\frac{\mathrm{K}}{\mathrm{J}^2}=\frac{\theta+6}{(\theta+4)\theta^2}, \quad \frac{\mathrm{L}}{\mathrm{J}^3}=\frac{1}{(\theta+4)\theta^3}.$$

Let

$$\theta=4\varphi, \quad y=\frac{\mathrm{D}}{\mathrm{J}^2}, \quad x=\frac{\Lambda}{\mathrm{J}^3}.$$

Then

$$y=1-128\frac{\theta+6}{(\theta+4)\theta^2}=1-\frac{8\varphi+12}{\varphi^2(\varphi+1)}=\frac{(\varphi+2)^2(\varphi-3)}{\varphi^2(\varphi+1)},$$

$$x=-1+\frac{2048}{(\theta+4)\theta^3}=-1+\frac{8}{\varphi^4+\varphi^3}=\frac{-(\varphi+2)(\varphi^3-\varphi^2+2\varphi-4)}{\varphi^3(\varphi+1)};$$

and consequently

$$\delta y=\frac{4(\varphi+2)(4\varphi+3)}{\varphi^3(\varphi+1)^2}\delta\varphi, \quad \delta x=-\frac{8(4\varphi+3)}{\varphi^4(\varphi+1)^2}\delta\varphi, \quad \frac{\delta y}{\delta x}=-\frac{\varphi^2+2\varphi}{2}:$$

x, y may be considered as the coordinates (inclined to each other at any angle) of a curve of the fourth order, whose form, so far as is essential to the object in view, I proceed to determine. It is obvious, furthermore, that this curve will be a section of the amphigenous surface made by the plane $\mathrm{J}=1$.

(86) This curve will be seen to consist of four branches, coming together in pairs or two cusps, so as to form two distinct horns(65). For when $\varphi=\infty$, or $\varphi=-\frac{3}{4}$, δy, δx will

(65) (a) Since $$\varphi^4+\varphi^3-\frac{\mathrm{J}^3}{256\mathrm{L}}=0,$$
we see at once, from DESCARTES's rule, that φ can never have more than two real values to one of $\frac{\mathrm{L}}{\mathrm{J}^3}$, or consequently of x, and consequently there can only be two values of y to each of x.

(b) When $\mathrm{J}=0$, the cusp of the left-hand horn and the two points of intersection of the dexter horn with the axis of L coincide at the origin; the upper branch of the latter and the linear of the former become the lower and upper parts of the axis of D, whilst the lower and upper branches of the same respectively become the left and right-hand branches of the semicubical parabola $27.2^{22}\mathrm{L}^2=-\mathrm{D}^3$.

each of them be zero. Hence there is a cusp at the point where $x=-1$, $y=1$ (66), and again at the point where

$$x=-1+\frac{8\times 256}{81-108}=-76\tfrac{23}{27}, \quad y=\frac{(\frac{5}{4})^2(-\frac{9}{4})}{(\frac{3}{4})^2\frac{1}{4}}=-25.$$

(87) When $\phi=0$, and also when $\phi=-1$, x and y each become infinite; when $\phi=\pm\infty$, x and y each become unity.

As ϕ passes from $+\infty$ to 0, δy is always negative, and x always positive; so that there will be one branch of the curve (CMP in Plate XXV.) extending from $x=-1$ to $x=+\infty$, for which y commences at $y=1$, which cuts the axis of x when $\phi=3$, i. e. $x=-\frac{25}{27}$ (67), and which, for the remaining part of its course, lies completely under the axis of x, becoming infinite when x becomes indefinitely great.

Again, as ϕ passes from $-\infty$ to -1, δx remains always positive, but δy is negative so long as $\phi<-2$ vanishes when $\phi=2$, and ever afterwards continues positive. Thus there is a second branch, COQ, which starts from the cusp C, touches the axis of x at the origin, ever afterwards remaining positive, and increasing up to positive infinity.

Since when $\phi=\infty$, $\frac{\delta y}{\delta x}=\infty$, the tangent at C is parallel to the axis of y, and consequently the two branches which start from C lie on the same side of the tangent, so that the cusp at this point is of the second or ramphoidal kind; in Professor CAYLEY'S nomenclature a cusp-node, and equivalent to the union of a double point and a cusp of the first kind.

There remains to account for the values of ϕ in the interval between 0 and -1. Throughout this interval y and x remain both of them negative, and $\frac{\delta y}{\delta x}=-\frac{\phi(\phi+2)}{2}$ (68, 69) is always positive.

There will thus be two branches, in each of which x and y increase simultaneously in the negative direction, coming to a cusp necessarily of the first kind at the point $x=-76\frac{23}{27}$, $y=-25$, one branch corresponding to the values of ϕ from $-\frac{3}{4}$ to 0, the other to the values of ϕ from $-\frac{3}{4}$ to -1, both of them lying completely under the axis of x, and becoming respectively infinite at the extreme values of ϕ (0 and -1).

(66) Where this branch cuts the axis of y we have $\phi^3-\phi^2+2\phi-4=0$, of which the real root will be a trifle less than $\frac{3}{2}$.

(67) From this it is easily seen that, whatever may be supposed to be the inclination of the axes x, y, the curve in question is rectifiable by means of elliptic functions; for $\frac{ds}{d\phi}$ will be expressible as a rational function of ϕ and the square root of a quartic function of ϕ. The same conclusion will hold for the curve obtained by making J constant when J, together with any invariant of the eighth and any of the twelfth order, are taken as the coordinates of the amphigenous surface.

(68) To ascertain which range of ϕ gives the superior and which the inferior outline of the sinister horn, let $\phi=\varepsilon$, an infinitesimal; then $\phi^4+\phi^3=\varepsilon^3$, and the other value of ϕ is $-1-\eta$, where $\eta=\varepsilon^3$. Hence the two values of y corresponding to ϕ nearly zero and ϕ nearly -1 respectively will be

$$y_1=-\frac{12\varepsilon}{\varepsilon^3}=-\frac{12}{\varepsilon^2} \quad\text{and}\quad y_2=\frac{-4(-1-\eta)}{\varepsilon^3}=\frac{4}{\varepsilon^3}.$$

Thus y_1 is negative for ε positive or negative, but y_2 is positive in the one case and negative in the other, as

Again,

$$2y-x+5=\frac{\phi+2}{\phi^4+\phi^3}\Big((2\phi^3-2\phi^2+2\phi)+(\phi^3-\phi^2+2\phi-4)\Big)+5$$

$$=\frac{\phi+2}{\phi^3}(3\phi^2-6\phi-4)+5=\frac{8\phi^3-16\phi-8}{\phi^3}.$$

Hence when $\phi=-1$, for which value of ϕ x and y both become infinite, $2y-x+5=0$; hence the straight line $2y-x+5=0$, represented by AN in the diagram, will be an asymptote to the curve (70).

If now we draw the straight line $2y-x=0$, represented by OB in the figure and join OC, the curvilinear triangle OCM will be completely under OC, and the curvilinear infinite sector XOP completely under OB.

(88) What we have to prove is, that so long as μ lies between 2 and 1, so long may $\Lambda+\mu$JD be substituted as a criterion in lieu of Λ, it being remembered that Λ only plays the part of a criterion when D is positive and J is not positive. Hence, since when J$=0$ $\Lambda+\mu$JD and Λ coincide, we have only to show that, so long as D is positive and J is negative, $\Lambda+\mu$JD and Λ will bear the same sign for all such values of J, D, L as constitute a facultative system, *i. e.* coordinates to a facultative point in space.

Now at any facultative point G (the function of the amphigenous surface), or say rather $G(J, K, L)>0$, or $\frac{1}{J^3}G\left(1, \frac{D}{J^2}, \frac{L}{J^3}\right)>0$, and consequently considering $\frac{D}{J^2}$, $\frac{L}{J^2}$ as the coordinates of a plane curve, the line $G\left(1, \frac{D}{J^2}, \frac{L}{J^3}\right)=0$ (the sign of J being fixed) will separate those points for which J, K, L constitute a facultative system from those

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already seen for the dexter horn. We see also that y_2 becomes indefinitely greater than y_1, so that it is the value of ϕ near to -1 which gives the inferior branch; and consequently the superior branch of the sinister horn belongs to the range from $-\frac{3}{4}$ to 0, and the inferior to the range from $-\frac{3}{4}$ to -1.

(69) It may further be noticed that each horn so called is a true horn, being destitute of any point of contrary flexure, except at infinity; for otherwise we should have

$$\frac{d^2y}{dx^2}=\frac{d\phi}{dx}\cdot\frac{d\frac{dy}{dx}}{d\phi}=-\frac{d\phi}{dx}(\phi+1)=\frac{(\phi+1)^3\phi^4}{8(4\phi+3)}=0,$$

which implies $\phi=0$ or $\phi=-1$, for each of which values of ϕ x and y become infinite. It will be seen hereafter that it is only for the value corresponding to $\phi=0$ that there does exist at infinity a point of inflexion.

(70) The two points where the asymptote cuts the curve will be found by writing

$$\frac{\phi^3-2\phi-1}{\phi+1}=\phi^2-\phi-1=0,$$

which gives

$$\phi=\frac{1\pm\sqrt{5}}{2}.$$

The superior sign corresponds to a point x, y in the inferior branch of the dexter horn, and the lower sign, for which $\phi>-\frac{3}{4}$, to the superior branch of the sinister horn. It is easy to see that there can be no other asymptote; for x, y only become infinite when $\phi=-1$, or $\phi=0$; so that if $\lambda x+\mu y+\nu$ is an asymptote, it must contain $(\phi+1)^2$, or ϕ^2 as a factor. The first condition is only satisfied when $\lambda:\mu:\nu::-1:2:5$; and the latter cannot be satisfied at all.

in which J, K, L constitute a non-facultative one. But the curve above traced is obviously a homographic derivative of that line $\left(\text{for G is the resultant of } \frac{K}{J^2}=\frac{\theta+6}{(\theta+4)\theta^2}, \frac{L}{J^3}=\frac{I}{(\theta+4)\theta^3}\right)$.

Hence this latter curve will also separate systems of values of J, D, Λ corresponding to facultative from those corresponding to non-facultative points. Moreover when J is negative and D positive, it has been shown (see dial figure) that the values of D (in facultative systems) corresponding to finite values of J are *limited* in magnitude; hence, upon the same suppositions, facultative systems of J, D, Λ will correspond to the interior and contour of the curve we have been considering.

(89) Accordingly, since D is supposed positive, our sole concern will be with the curvilinear triangle CMO and the infinite sector QOX, and we have to show that for all points not exterior to those areas Λ and $\Lambda+\mu JD$ have the same sign; that is to say, $1+\mu\frac{JD}{\Lambda}$, or $1+\mu\frac{y}{x}$ is *positive.*

When y and x have opposite signs (as is the case in the triangle CMO), all negative values of μ, and when y and x have the same signs (as is the case in the sector XOQ), all positive values of μ obviously make $1+\mu\frac{y}{x}$ positive. But furthermore $\frac{y}{x}$, which is -1 for the line OC, is greater than -1 for all points in the triangle just named; and again, $\frac{y}{x}$, which is $\frac{1}{2}$ for OB (the parallel to the asymptote through O), will be less than $\frac{1}{2}$ for all points in the sector QOX. Thus, then, as regards points either in the triangle or in the sector, $\frac{y}{x}$ is always intermediate between -1 and $\frac{1}{2}$; so that when μ lies between 1 and -2, $1+\mu\frac{y}{x}$ will be always positive, and Λ and $\Lambda+\mu JD$ will bear the same sign O, so that $\Lambda+\mu JD$ may be used to replace Λ as a criterion. Q.E.D.

(90). It is apparent from the nature of the preceding demonstration that Λ may be replaced by an invariant containing not one merely, but an infinite number of arbitrary constants (limited), provided we are indifferent to the degree which the substitute for Λ may assume. To this end we have only to draw any algebraical curve $F(x, y)=0$ passing through the origin, and with its parameter subject to such conditions of inequality as will ensure the mixtilinear triangle and sector COM, XOQ lying on opposite sides of the curve. If its degree be n, the number of parameters in F left arbitrary within limits will be $\frac{n^2+3n-2}{2}$, and $\varepsilon F(\Lambda, JD)$, where ε means one of the two quantities $+1$ or -1, may be used as a criterion in lieu of Λ. For instance, a common parabola with its axis coincident with that of x and passing through O will obviously serve as a screen between these figures; its equation will be $y^2-x=0$, and the invariant $D^2-J\Lambda$, which is of the sixteenth degree in the coefficients, will serve together with J and D to fix the nature of the roots; so in general we may obtain invariants of any degree of the form $4i$ from twelve

upwards. Thus M. HERMITE, by a method not introducing the notion of continuity, has found one of the twenty-fourth degree, which he has been so obliging as to communicate to me, viz. $(D_1-5\Delta J_2)^2+(9D-25\Delta^2)J_2^2$, where $D_1=16J_3+25\Delta J_2$; and D is his discriminant, which I cannot safely attempt to express in terms of x, y for want of a certain knowledge of the arithmetical relations between his Δ, J_2, J_3 D, and my own J, K, L; but were this transformation effected, the curve so represented must, *ex necessitate*, pass clear between the triangle and sector above referred to, or else the invariant in question could not be a criterion. I have ascertained without difficulty that it passes through the origin and represents one of the principal species of NEWTON's diverging parabolas.

(91) The curve which we have been discussing will, on reference to PLÜCKER's 'Algebraischen Curven,' p. 193, be seen to belong to his sixteenth species of curves of the fourth order having two double points; but as in reality one of these is tantamount to the union of two, it may be considered as having three, the maximum possible number of such points, and consequently comes under the operation of CLEBSCH's rule, given in the last Number of CRELLE's Journal, and accordingly its coordinates have been seen to be rational functions of a single variable. The equation connecting x, y may of course be obtained by means of a simple and obvious substitution operated upon the G of art. 41, or it may be found directly by writing

$$\frac{x+1}{8}=\xi=\frac{1}{\varphi^4+\varphi^3}\quad\frac{y-1}{4}=\eta=-\frac{2\varphi+3}{\varphi^3+\varphi^2},$$

whence we obtain

$$\varphi^4+\varphi^3-\frac{1}{\xi}=0, \quad \ldots \ldots \ldots \quad (1)$$

$$2\varphi^2+3\varphi+\frac{\eta}{\xi}=0. \quad \ldots \ldots \ldots \quad (2)$$

Calling $\varphi_1\varphi_2$ the two roots of equation (2), making

$$\left(\varphi_1^4+\varphi_1^3-\frac{1}{\xi}\right)\left(\varphi_2^4+\varphi_2^3-\frac{1}{\xi}\right)=0,$$

and substituting the values of the symmetric functions of φ_1, φ_2 found from the same equation, we obtain without difficulty

$$\eta^4-\xi\eta^3-8\xi\eta^2+36\xi^2\eta+16\xi^2-27\xi^3=0$$

for the equation in question. The curve thus denoted I propose to call the Bicorn. Its figure is given in Plate XXV., in which ξ, η are taken at right angles, but they may of course be supposed to be inclined at any angle whatever. If we now assume at pleasure any two new axes U, V in the place of the Bicorn, the coordinates u, v will be always respectively proportional to two invariants of the twelfth order of the given quintic, whose particular forms will depend upon the positions of the two new axes so taken. If one of these axes, say that of u, be made coincident with the axis of ξ, v will be proportional to JD, and u to some other invariant of the twelfth degree. When

this is the case, then in general v, as u travels from one end of infinity to the other, will sometimes have four, and sometimes two, or else sometimes two and sometimes no real values, as will be obvious by inspection of the figure. There is, however, one direction of the axis of v which will cause v in all cases to have two, and only two real values. This direction is that of the line joining the two cusps. At the node-cusp, for which $\phi=\infty$, $\xi=0$, $\eta=0$; at the other cusp, for which $\phi=-\frac{3}{4}$, $\xi=-\frac{256}{27}$, $\eta=-\frac{32}{3}$. Hence the equation of the joining line is $9\xi-8\eta=0$. Now $\frac{K}{J^2}=-\frac{\eta}{32}$, $\frac{L}{J^3}=\frac{\xi}{256}$. Hence along this line $9L+JK=0$; and consequently, if the axis of v be taken parallel to this line and passing through the origin, whilst u is proportional to $9L+JK$, v will be proportional to JD; and thus we see that for every value of $9L+JK$, which is HERMITE'S J_3 (see foot-note (34)(e)), D at the amphigenous surface (*i. e.* when $G=0$, and therefore when HERMITE'S $I=0$) will always have two, and only two real values. This perfectly agrees with M. HERMITE'S conclusion (71), and in an unexpected manner confirms the correctness of the concordance established, in the foot-note cited, between his J_3 and my J, K, L. Had M. HERMITE employed any duodecimal invariant whatever other than J_3, a mere inspection of the Bicorn shows that a similar conclusion could not have obtained.

(92) The intersections of the curve whose equation is written in the preceding article with infinity evidently lie in the lines $\eta^3=0$, $\eta-\xi=0$. This latter is the equation to a line parallel to the asymptote which touches the highest and lowest of the four branches of the curve, and corresponds to the value -1 of ϕ. Thus we see that there is a point of inflexion corresponding to the point at infinity at which the second and third branches of the Bicorn may be conceived to unite. It is easy to show that the Bicorn has no double tangent; for we have seen that

$$\frac{dy}{dx}=-\frac{\phi^2+2\phi}{2},$$

and consequently the values of ϕ corresponding to the two supposed points of contact may be regarded as the two roots ϕ_1, ϕ_2 of the equation $\phi^2+2\phi+2\lambda=0$, and we shall have

$$-\frac{2\phi_1+3}{\phi_1^3+\phi_1^2}+\frac{2\phi_2+3}{\phi_2^3+\phi_2^2}=\lambda\left(\frac{2}{\phi_1^4+\phi_1^3}-\frac{2}{\phi_2^4+\phi_2^3}\right),$$

$$i.\ e.\ -(2\phi_1+3)(\phi_2^3+\phi_2^2)+(2\phi_2+3)(\phi_1^3+\phi_1^2)=(\phi_2^4+\phi_2^3)-(\phi_1^4+\phi_1)^3,$$

or

$$4\lambda\,.\,(-2)+4\lambda+3(4-2\lambda)+6\big(-2(4-4\lambda)+(4-2\lambda)\big)=0,$$

or

$$(-8+4-6+8-2)\lambda+12-6-8+4=0,$$

$$i.\ e.\ -4\lambda+2=0,\ \lambda=\tfrac{1}{2},\ \phi^2+2\phi+1=0,$$

and the two values of ϕ coincide, contrary to hypothesis.

It is also easy to find its class; for when $\frac{d\eta}{d\xi}$ corresponds to any point in which the

(71) Lemma 3, p. 202, Cambridge and Dublin Journal, vol. ix.

curve is met by a tangent drawn from the point whose ξ, η coordinates are a, b, we have

$$\left(\frac{2\phi+3}{\phi^3+\phi^2}+b\right)+\frac{d\eta}{d\xi}\left(\frac{1}{\phi^4+\phi^3}-a\right)=0;$$

but

$$\frac{d\eta}{d\xi}=2\frac{dy}{dx}=-(\phi^2+2\phi);$$

hence

$$\frac{(2\phi+3)-(\phi+2)}{\phi^3+\phi^2}+(\phi^2+2\phi)a+b=0;$$

hence

$$a\phi^4+2a\phi^3+b\phi^2+1=0;$$

and ϕ having four values, four tangents (real or imaginary) can be drawn to the Bicorn from every point in its plane. It is thus of the fourth order, fourth class, possesses a common cusp and a cusp-node, no double tangent, and one point of inflexion at infinity. These results accord with those given by PLÜCKER (Algebraischen Curven, p. 222).

(93) The canonical form of the equation to the Bicorn is $(pr+q^2)^2+pq^3=0$, as seen in PLÜCKER, p. 193, where $p=0$, $r=0$, $q=0$ will obviously be the equations to the tangent at the node-cusp, to the tangent at the common cusp, and to the line of junction of the two cusps. This leads to a remarkable transformation of the invariant G of art. (41). Thus we may write $p=\xi$, $q=\mu(9\xi-8\eta)$; and to find r, we must draw the tangent to the lower cusp, for which $\phi=-\frac{3}{4}$, which gives

$$\xi=-\frac{256}{27},\quad \eta=-\frac{32}{3},\quad \frac{d\eta}{d\xi}=-\frac{15}{16}\text{(72)};$$

consequently we may write $r=\lambda(144\eta-135\xi+256)$, and then proceed to satisfy, by assigning suitable values to λ, μ, ν, the identity

$$\left(\lambda(144\eta\xi-135\xi^2+256\xi)+\mu^2(8\eta-9\xi)^2\right)^2+\mu^3\xi(8\eta-9\xi)^3$$
$$=\nu(\eta^4-\eta^3\xi-8\eta^2\xi+36\eta\xi^2+16\xi^2-27\xi^3)=\nu\,.\,2^{20}\text{G}.$$

On performing the necessary calculations it will be found that

$$\lambda=-\frac{1}{2^{12}},\quad \mu=\frac{1}{2^6},\quad \nu=\frac{1}{2^{12}}.$$

Hence we see that J^3G may be expressed under the form $(\text{LL}_1+c\text{J}_3^2)^2+e\text{LJ}_3^3$, where L_1 is a new duodecimal invariant, and c, e are two known numbers; in fact

$$\text{J}^3\text{G}=\left(\text{L}(18\text{JK}+135\text{L}^2-\text{J}^3\text{L})+(\text{JK}+9\text{L})^2\right)^2+64\text{L}(\text{JK}+9\text{L})^3.$$

I am indebted to my friend Dr. HIRST for these references to the immortal work of PLÜCKER.

(94) The existence has been demonstrated of a linear asymptote which is a tangent

(72) I find, by a calculation which offers no difficulty, that the value of ϕ at the point where this tangent cuts the curve will be given by the equation

$$-256\phi^4-256\phi^3+288\phi^2+432\phi+135=0;$$

and taking away the factor $(4\phi+3)^3$ which belongs to the cusp, there remains $\phi=\frac{5}{4}$, which corresponds to a point in the lower branch of the superior horn.

at infinity to the first and fourth branch. A cubic asymptote touches the intermediate branches in the point at infinity corresponding to $\varphi=0$. For we have

$$\xi=\frac{1}{\varphi^3(1+\varphi)}=\frac{1}{\varphi^2}(1-\varphi+\varphi^2-\varphi^3\ldots.);$$

and writing v for $-\eta$,

$$v=\frac{3+2\varphi}{\varphi^2(1+\varphi)}=\frac{1}{\varphi^2}(3-\varphi+\varphi^2-\varphi^3\ldots.),$$

$$v^{\frac{1}{2}}=\frac{3^{\frac{1}{2}}}{\varphi^3}\left(\varphi^2-\frac{1}{6}\varphi^3+\ldots.\right),\quad v^{\frac{3}{2}}=\frac{3^{\frac{1}{2}}}{\varphi^3}\left(3-\frac{3}{2}\varphi+\frac{13}{8}\varphi^2-\frac{27}{16}\varphi^3\ldots.\right).$$

Hence we may determine

A, B, C, D so that $Av^{\frac{3}{2}}+Bv+Cv^{\frac{1}{2}}+D-\xi$ shall $=\lambda\omega^n+\mu\omega^{n'}+\ldots.$,

and I find

$$A=\frac{1}{3^{\frac{1}{2}}},\qquad B=-\frac{1}{6},\qquad C=\frac{7}{72},\qquad D=-\frac{2}{9}.$$

Thus the cubic asymptote will have for its equation

$$\left(\xi+\frac{1}{6}v+\frac{2}{9}\right)^2=3v\left(\frac{v}{9}+\frac{7}{72}\right)^2,$$

which is a divergent cubic parabola with a conjugate point, viz. the point for which

$$v=-\frac{7}{8},\qquad \xi+\frac{1}{6}v+\frac{2}{9}=0,\qquad \text{or } \eta=\frac{7}{8},\qquad \xi=-\frac{9}{128}.$$

(95) It is obvious from the preceding article, that we may expand ξ in terms of v by the descending series

$$\xi=Av^{\frac{3}{2}}+Bv+Cv^{\frac{1}{2}}+D+\frac{E}{v^{\frac{1}{2}}}+\ldots.$$

But we may also obtain an ascending series for ξ in terms of v which will exhibit the nature of the curve of the cusp-node at which point $\varphi=\infty$. Let $\varphi=\frac{1}{\omega}$, then

$$\xi=\frac{1}{\varphi^3(\varphi+1)}=\frac{\omega^4}{1+\omega}=\omega^4(1-\omega+\omega^2-\omega^3\ldots.),$$

$$v=\frac{2\varphi+3}{\varphi^2(\varphi+1)}=\omega^2\left(\frac{2+3\omega}{1+\omega}\right)=\omega^2(2+\omega-\omega^2+\omega^3\ldots).$$

Hence

$$\begin{aligned}
v^2 &=\omega^4(4+4\omega-3\omega^2+2\omega^3\ldots),\\
v^{\frac{5}{2}} &=\omega^4(4\sqrt{2}\,\omega+5\sqrt{2}\,\omega^2-\tfrac{25}{8}\sqrt{2}\,\omega^3\ldots),\\
v^3 &=\omega^4(8\omega^2+12\omega^3\ldots),\\
v^{\frac{7}{2}} &=\omega^4(\sqrt{2}\,\omega^3\ldots),\\
\&c. &=\&c.
\end{aligned}$$

from which we may easily deduce

$$\xi=2\left(\frac{v}{2}\right)^2-\left(\frac{v}{2}\right)^{\frac{5}{2}}+\frac{7}{4}\left(\frac{v}{2}\right)^3-\frac{109}{32}\left(\frac{v}{2}\right)^{\frac{7}{2}}, \text{ \&c.,}$$

in which it will be observed that the indices of the powers of v are precisely complementary to those in the preceding expansion, the two series of indices together comprising all multiples of $\frac{1}{2}$ from positive to negative infinity.

(96) We now see how, supposing the curve to be given with ξ and η at any angle, the axes corresponding to $\frac{\mathrm{K}}{\mathrm{J}^2}$, $\frac{\mathrm{L}}{\mathrm{J}^3}$ may be defined: viz., the origin of coordinates will be at the cusp-node; η, along which $\frac{\mathrm{K}}{\mathrm{J}^2}$ is reckoned, will be in the direction of the tangent at that point; and ξ, along which $\frac{\mathrm{L}}{\mathrm{J}^3}$ is reckoned, will be the axis of that common parabola which at the same point has the closest contact with the given curve.

It seems desirable, with a view to a more complete comprehension of the form of the amphigenous surface, i. e. the *limiting surface* of invariantive parameters, to ascertain the nature of the systems of plane sections of it, parallel to each of the three coordinate planes. The sections parallel to J, which are curves of the fourth order, have been already satisfactorily elucidated. It remains to consider briefly the sections parallel to J and D, which will be curves of the ninth order.

(97) When L is constant, writing $\mathrm{J}=z$, $\mathrm{D}=y$, where for facility of reference we may conceive y horizontal and z vertical, and making $\mathrm{L}=\frac{k^3}{256}$, we have

$$z^3=k^3\phi^3(\phi+1),\quad y=z^2\frac{(\phi+2)^2(\phi-3)}{\phi^2(1+\phi)}=k^2\frac{(\phi-3)(\phi+2)^2}{(1+\phi)^{\frac{1}{3}}},$$

$$\frac{\delta y}{y}=\frac{2}{3}\,\frac{(\phi-1)(4\phi+3)}{(\phi+2)(\phi-3)(\phi+1)}\delta\phi,\quad \frac{\delta z}{z}=\frac{1}{3}\,\frac{4\phi+3}{\phi(\phi+1)}\delta\phi,\quad \frac{\delta z}{\delta y}=\frac{1}{2k}\,\frac{(\phi+1)^{\frac{2}{3}}}{(\phi-1)(\phi+2)}\delta\phi,$$

when	$\phi=-1$,	$z=0$,	$y=\infty$,
,,	$\phi=-\frac{3}{4}$,	$\delta y=0$,	$\delta z=0$,
,,	$\phi=0$,	$z=0$,	$y=-12k^2$,
,,	$\phi=1$,	$\frac{\delta y}{\delta z}=0$,	
,,	$\phi=+\infty$,	$z=+\infty$,	$y=+\infty$,
,,	$\phi=-2$,	$y=0$,	$\frac{\delta z}{\delta y}=\infty$,
,,	$\phi=-\infty$,	$z=+\infty$,	$y=+\infty$.

Hence it appears that the curve consists of three branches, two coming together at an ordinary cusp at the point corresponding to $\phi=-\frac{3}{4}$, and the third completely separate. The nature of the sign of k does not affect the nature of the curve. If, for greater clearness, k be supposed positive, the first branch, having the negative part of

the axis of y for its asymptote, lies entirely in the $-y$, $-z$ quadrant, and is always convex to the axis of y; the second branch, joining the first at a cusp corresponding to $\varphi=-\frac{3}{4}$, is concave to the origin, cuts the axis of y negatively and of z positively, and goes off to infinity; the third branch, having the positive part of the axis of y for its asymptote, lies in the $+y$, $+z$ quadrant, is always convex to the axis of z, which it touches at a point below that where it is cut by the second branch, and also goes off to infinity, lying entirely under the second branch. A straight line, according to the direction in which it is drawn, may cut the curve in one, three, or five real points.

(98) When D is constant, writing $\mathrm{J}=z$, $\mathrm{L}=x$, we have

$$z^2=\mathrm{D}\frac{\varphi^2(\varphi+1)}{(\varphi+2)^2(\varphi-3)}, \quad x=\frac{\mathrm{D}z}{(\varphi-3)\varphi(\varphi+2)^2}.$$

The form of the curve changes with the sign of D. For sections parallel to and above the plane of D, we may make

$$\mathrm{D}=c^2, \quad \tau^2=\frac{\varphi+1}{\varphi-3}, \quad \text{or} \quad \varphi=\frac{3\tau^2+1}{\tau^2-1};$$

then the complete equation-system to the curve will be

$$z=c\tau\frac{3\tau^2+1}{5\tau^2-1}, \quad x=c^3\tau\frac{(\tau^2-1)^4}{4(5\tau^2-1)^3},$$

it being unnecessary to affect c with a double sign, since z and x change their signs with that of τ.

Also

$$\frac{\delta x}{x}=\frac{(\tau^2+1)(15\tau^2+1)\delta\tau}{\tau(\tau^2-1)(5\tau^2-1)}, \qquad \frac{\delta z}{z}=\frac{(\tau^2-1)(15\tau^2+1)\delta\tau}{\tau(3\tau^2+1)(5\tau^2-1)},$$

$$\delta x=\frac{c^3}{4}\frac{(\tau^2+1)(15\tau^2+1)(\tau^2-1)^3}{(5\tau^2-1)^4}\delta\tau, \quad \delta z=c\frac{(15\tau^2+1)(\tau^2-1)}{(5\tau^2-1)^2}\delta\tau,$$

$$\frac{dx}{\delta z}=\frac{c^2}{4}\frac{(\tau^2+1)(\tau^2-1)^2}{(5\tau^2-1)^2}.$$

To the values of τ included between $+\sqrt{\frac{1}{5}}$ and $-\sqrt{\frac{1}{5}}$ will correspond one branch of the curve passing through the origin, where it has a point of contrary flexure, and extending to infinity in both directions.

When $(5\tau^2-1)$ is positive $\frac{x}{\tau}$ is always positive; and when $\tau^2=1$,

$$\delta x=0, \quad \delta z=0, \quad \frac{\delta x}{\delta z}=0.$$

Hence there will be a cusp of the second kind when $x=0$, $z=\pm c$, the axis of z being a tangent to the curve at each cusp. One pair of branches has its cusp at the point $x=0$, $z=c$, and the values of x, z increase indefinitely in the respective branches as τ passes from 1 to $+\infty$ and from 1 to $\sqrt{\frac{1}{5}}$. This pair lies in the $+x$, $+z$ quadrant, and there will be a precisely similar and similarly situated pair in the $-x$, $-z$ quadrant. Thus there will be in all one infinite ∫-formed branch passing through the origin, and

two detached pairs of infinite branches lying in opposite quadrants([73]). The value $\frac{1}{5}$ for τ^2, it will of course be seen, corresponds to -2 for ϕ, and gives, as it ought to do, the position of the cusp.

(99) Finally, for sections parallel to the plane of the discriminant and lying below it, making $D=-k^2$, $t^2=\frac{1+\phi}{3-\phi}$, we obtain in like manner

$$z=kt\frac{3t^2-1}{5t^2+1},\quad x=k^3t\frac{(t^2+1)^4}{4(5t^2+1)^3},\quad \frac{\delta x}{x}=\frac{(t^2-1)(15t^2-1)}{t(t^2+1)(5t^2+1)}\delta t,\quad \frac{\delta z}{z}=\frac{(t^2+1)(15t^2-1)}{t(3t^2-1)(5t^2+1)},$$

$$\delta x=\frac{k^3}{4}\frac{(t^2-1)(15t^2-1)(t^2+1)^3}{(5t^2+1)^4},\quad \delta z=k\frac{(15t^2-1)(t^2+1)}{(5t^2+1)^2},\quad \frac{\delta x}{\delta z}=\frac{k^2}{4}\frac{(t^2-1)(t^2+1)^2}{(5t^2+1)^2}.$$

When $t^2=\frac{1}{15}$ there will be an ordinary cusp, and when $t^2=1$, $\frac{\delta x}{\delta z}=0$.

There will therefore be three branches,—one corresponding to the values of t between $-\sqrt{\frac{1}{15}}$ and $+\sqrt{\frac{1}{15}}$, the other two to values of t between these limits and $-$ and $+$ infinity respectively. The middle branch passes through the origin, where it undergoes an inflexion, and comes to a cusp at a finite distance from the origin in two opposite quadrants. The connected branch at each cusp crosses the axis of x, sweeps convexly towards the axis of z, arrives at a minimum distance from it, and then goes off to infinity.

The value $\frac{1}{15}$ for t^2 corresponds to $-\frac{3}{4}$ for ϕ, and gives, as it ought to do, the cusp-node. In fact the values $\phi=-\frac{3}{4}$, $\phi=-2$ correspond respectively to a cuspidal and to a cusp-nodal line in the limiting surface whose sections we have been considering.

When the cutting plane is that of D itself, the section becomes a double cubic parabola and a single cubical parabola crossing each other at the origin.

([73]) Let ε be an infinitesimal, and $\theta^2=1+\varepsilon$; then

$$\delta z=\frac{4}{c^2}\frac{(4+5\varepsilon)^2}{(2+\varepsilon)\varepsilon^2}\delta x=\frac{32}{c^2}(1+2\varepsilon)\frac{\delta x}{\varepsilon^2}.$$

Hence at either cusp the branch the further removed from the axis of x corresponds to the values of θ^2 between 1 and ∞, and the inferior branch to its values between 1 and $\frac{1}{5}$; so that the order of continuity of the five branches of the curve may be read as follows:—from the infinite point in the higher branch of the upper pair to its cusp; thence to the infinite point in the connected branch, which is contiguous to the infinite point in the opposite extremity of the middle branch; thence along this branch to its contrary infinite extremity; thence to the infinite point in the upper branch of the inferior pair; along that branch to its cusp; and thence, finally, along the lower branch to infinity.

Description of the Plates.

PLATE XXIV.

The (ε, η) equation is $(1, \varepsilon, \varepsilon^2, \eta^2, \eta, 1 \between x, y)^5=0$, of which two roots are always imaginary; its extreme criteria are 0, 0; its middle criteria $\varepsilon^4-\varepsilon\eta^2$, $\eta^4-\eta\varepsilon^2$,

$$p=\varepsilon\eta-1, \quad \sigma=(\varepsilon^3-\eta^2)(\varepsilon^2-\eta^3).$$

Points (p, σ) above the discriminatrix indicate 2 pairs of associated roots in the (ε, η) equation.

Points (p, σ) on the discriminatrix indicate 2 equal roots in the (ε, η) equation.

Points (p, σ) under the discriminatrix indicate 3 solitary roots in the (ε, η) equation.

Points (p, σ) above the equatrix indicate ε, η real and unequal.

Points (p, σ) on the equatrix indicate ε, η equal.

Points (p, σ) under the equatrix indicate ε, η imaginary and conjugate.

Points (p, σ) above the loop of the indicatrix indicate middle criteria not *both* positive.

Points (p, σ) on the loop of the indicatrix indicate middle criteria of opposite signs.

Points (p, σ) under the loop of the indicatrix indicate middle criteria not *both* negative.

The discriminatrix is a closed curve, the *whole* of which is figured on the Plate, and is shaped somewhat like a harp: it has a cusp of the fourth order at the origin.

The equatrix consists of two branches coming together at a cusp at the distance 1 from the origin; the upper branch touches the horizontal axis at the origin; the lower branch, after touching the discriminant at a single point, sweeps out from and round it, cutting the vertical axis at the distance 4 below the origin. Both branches go off to infinity to the right, and lie completely under the horizontal axis. Where the lower branch touches the discriminatrix, the discriminant of the (ε, η) equation passes through zero without changing its sign.

The indicatrix consists of a single branch extending indefinitely in both directions. It passes from infinity below and to the left until, at the distance 1 from the origin, it touches the axis, which at the origin it crosses at an angle of 45°, after which it goes off to infinity in the positive direction. Its *loop* extends from $p=0$ to $p=-1$. The two portions of it figured in the Plate join on together, coming to a maximum at a great distance below the horizontal axis. The narrow tract marked "Region of Real parameters" is that portion of the harp-shaped space for which alone, ε, η being *real*, the (ε, η) equation can have more than one real root. The areas of each of the three regions into which the discriminatrix is divided by the equatrix and indicatrix may readily be expressed numerically in terms of algebraic and inverse circular functions only.

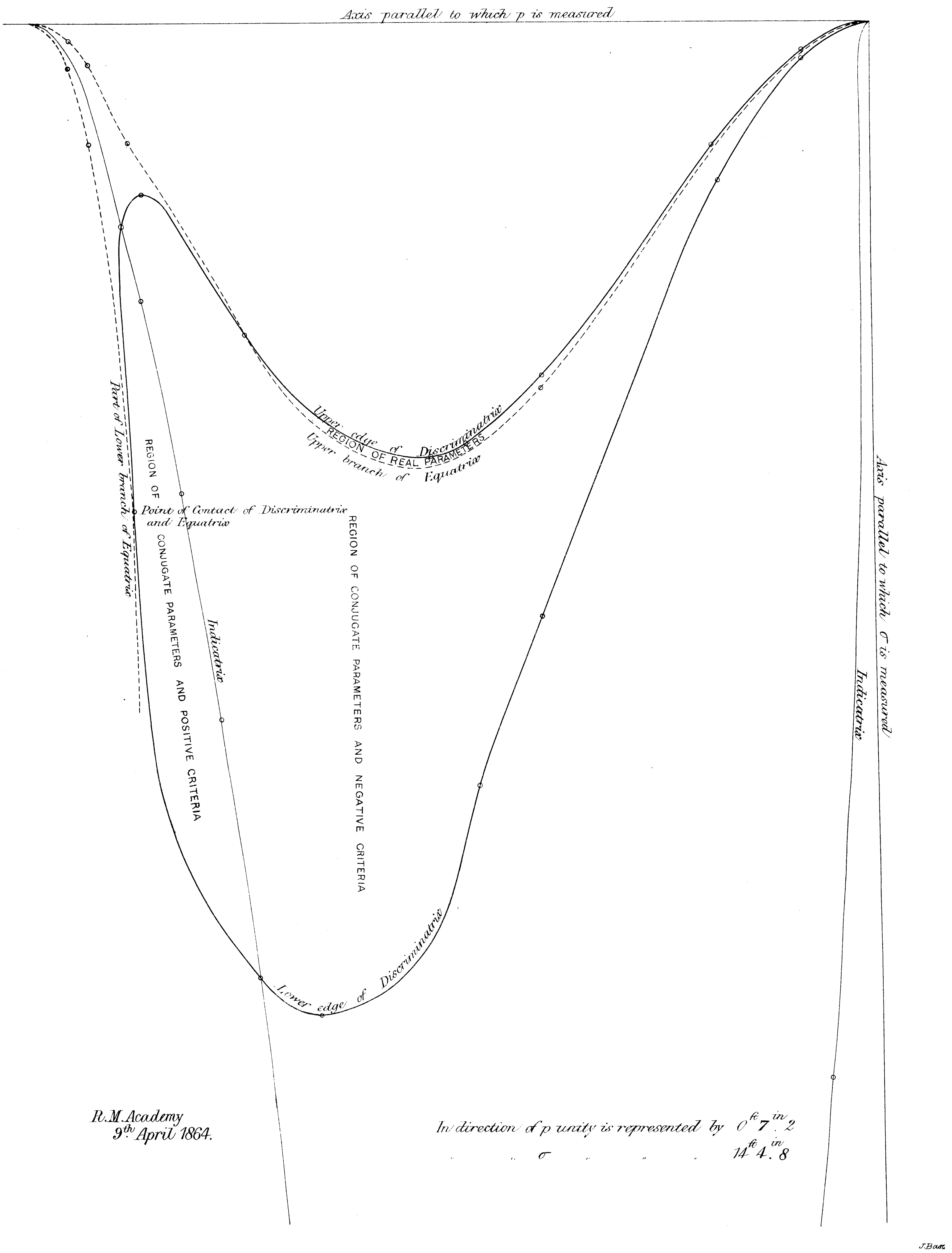
I am indebted to Gentleman Cadet S. L. JACOB, of the Royal Military Academy, for the tracing of the curves of which the above Plate is a somewhat imperfect reproduction.

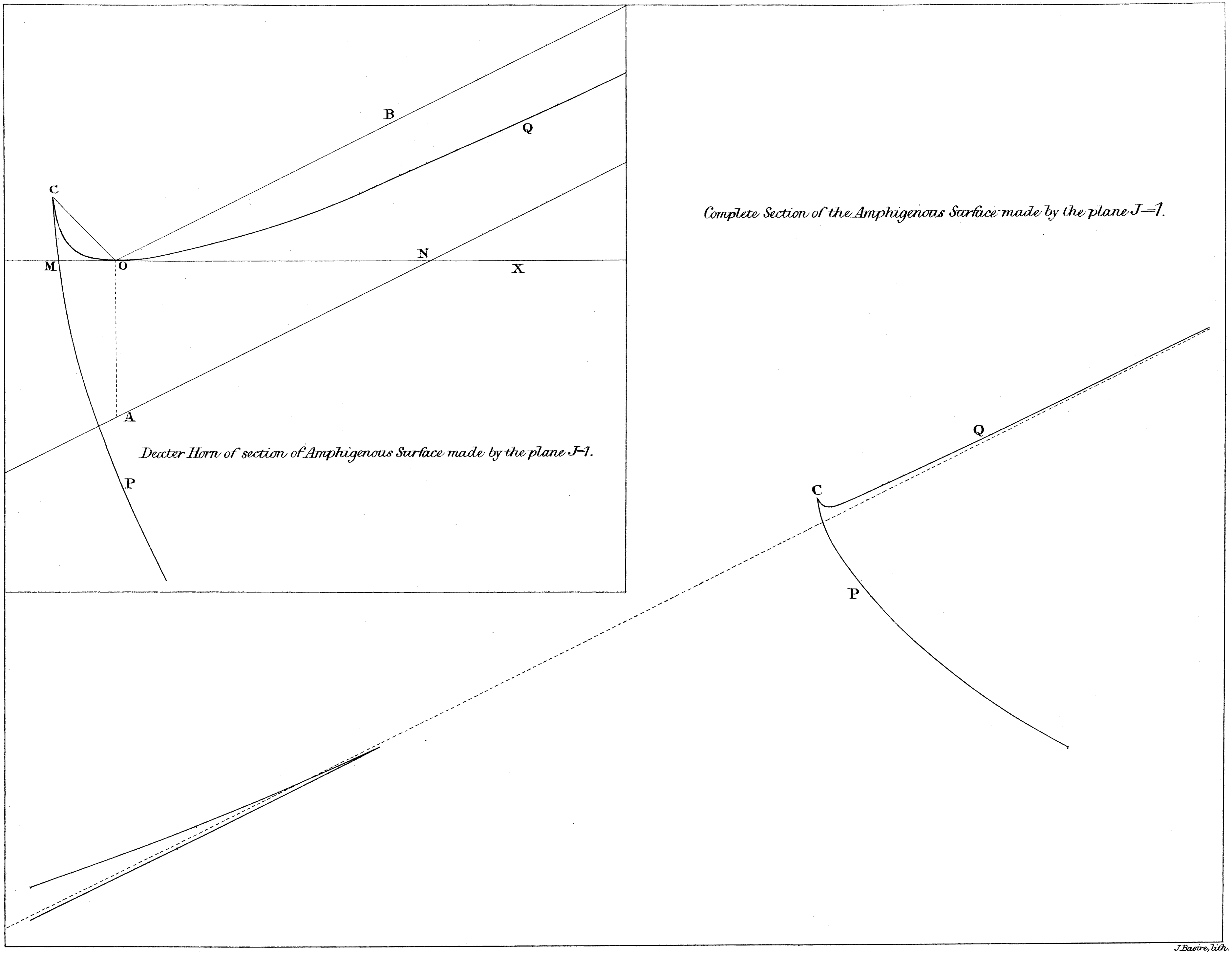
PLATE XXV.

Described in text, p. 658.

CONTENTS.

SUPPLEMENTAL REFERENCES.


Axis parallel to which p is measured
Upper edge of Discriminatrix
REGION OF REAL PARAMETERS
Upper branch of Equatrix
Part of Lower branch of Equatrix
REGION OF CONJUGATE PARAMETERS AND POSITIVE CRITERIA
Point of Contact of Discriminatrix and Equatrix
REGION OF CONJUGATE PARAMETERS AND NEGATIVE CRITERIA
Indicatrix
Axis parallel to which σ is measured
Indicatrix
Lower edge of Discriminatrix
R. M. Academy
9th April 1864.
In direction of p unity is represented by 0ft 7in.2
" " σ " " " " 14ft 4in.8
J. Basire



G. S. Pritchard del.[t]

J. Basire, lith.