XIV. *On the Sextactic Points of a Plane Curve.*

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Received June 15,—Read June 15, 1865.

THE beautiful equation given by Professor CAYLEY (Proceedings of the Royal Society, vol. xiii. p. 553) for determining the sextactic points of a plane curve, and deduced, as I understand, by the method of his memoir "On the Conic of Five-pointic Contact" (Philosophical Transactions, vol. cxlix. p. 371), led me to inquire how far the formulæ of my own memoir "On the Contact of Curves" (Philosophical Transactions, vol. clvii. p. 41) were applicable to the present problem.

The formulæ in question are briefly as follows: If $U=0$ be the equation of the curve, $H=0$ that of its Hessian, and $V=(a, b, c, f, g, h)(x, y, z)^2=0$ that of the conic of five-pointic contact; and if, moreover, α, β, γ being arbitrary constants,

$$\left.\begin{array}{l}\delta=\alpha x+\beta y+\gamma z,\\ \square=(\gamma\partial_y U-\beta\partial_z U)\partial_x+(\alpha\partial_z U-\gamma\partial_x U)\partial_y+(\beta\partial_x U-\alpha\partial_y U)\partial_z,\end{array}\right\}\quad . \quad . \quad . \quad (1)$$

then, writing as usual

$$\left.\begin{array}{c}\partial_x U=u,\ \partial_y U=v,\ \partial_z U=w;\ \partial_x H=p,\ \partial_y H=q,\ \partial_z H=r,\\ \partial_x^2 U=u_1,\ ..\ \partial_y\partial_z U=u',\ ..\quad \partial_x^2 H=p_1,\ ..\ \partial_y\partial_z H=p',\ ..\\ \mathfrak{A}=v_1w_1-u'^2,\ ..\quad \mathfrak{F}=v'w'-u_1u',\ ..\\ v\gamma-w\beta=\lambda,\ w\alpha-u\gamma=\mu,\ u\beta-v\alpha=\nu,\end{array}\right\}\quad . \quad . \quad . \quad . \quad (2)$$

the values of the ratios $a:b:c:f:g:h$ are determined by the equations

$$V=0,\ \square V=0,\ \square^2 V=0,\ \square^3 V=0,\ \square^4 V=0.\quad . \quad . \quad . \quad . \quad . \quad (3)$$

Now, if at the point in question the curvature of U be such that a sixth consecutive point lies on the conic V, the point is called a sextactic point; and the condition for this will be (in terms of the above formulæ) $\square^5 V=0$. From the six equations $V=0$, $\square V=0$, .. $\square^5 V=0$, the quantities a, b, c, f, g, h can be linearly eliminated; and the result will be an equation which, when combined with $U=0$, will determine the ratios $x:y:z$, the coordinates of the sextactic points of U. But the equation so derived contains (beside other extraneous factors) the indeterminate quantities α, β, γ, to the degree 15, which consequently remain to be eliminated. Instead therefore of proceeding as above, I eliminate α, β, γ beforehand, in such a way that ($W=0$ representing any one of the series $V=0$, $\square V=0$, .. from which α, β, γ have been already

eliminated) the equations $W=0$, $\square W=0$, $\square^2 W=0$ are replaced by

$$\frac{\partial_x W}{u}=\frac{\partial_y W}{v}=\frac{\partial_z W}{w}=\frac{\Delta W}{\varpi H} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

where ϖ is a numerical factor, and

$$\Delta=(\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{F}, \mathfrak{G}, \mathfrak{H})(\partial_x, \partial_y, \partial_z)^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

To this preliminary transformation the first section of the paper is devoted*. The second section contains the actual elimination of the constants of the conic, and the reduction of the resultant to six forms, $\mathfrak{L}=0$, $\mathfrak{M}=0$, $\mathfrak{N}=0$, $\mathfrak{L}'=0$, $\mathfrak{M}'=0$, $\mathfrak{N}'=0$; of which $\mathfrak{L}$ and $\mathfrak{L}'$, $\mathfrak{M}$ and $\mathfrak{M}'$, $\mathfrak{N}$ and $\mathfrak{N}'$ differ respectively only by one and the same numerical factor, viz. $(n-2)^3$. All these forms, however, contain extraneous factors, the determination of which is the object of the remainder of the paper. The third section is devoted to the establishment of some formulæ of reduction, the demonstrations of which are rather too long to be conveniently inserted in what would otherwise be their more natural place (§ 4). Besides these I have established many others of a like nature; but the specimens here given will doubtless suffice to suggest the mode of proof of the rest to any one desirous of pursuing the subject further. In the fourth section it is shown that all six forms $\mathfrak{L}$, .. $\mathfrak{L}'$, .. are divisible by the Hessian of U, and that $\mathfrak{L}$, $\mathfrak{L}'$ are also divisible by u^3, $\mathfrak{M}$, $\mathfrak{M}'$ by v^3, and $\mathfrak{N}$, $\mathfrak{N}'$ by w^3, and that the result of these divisions is a single expression of the degree $12n-27$.

§ 1. *Preliminary Transformation.*

The first two equations of the system (3) are, as is well known, equivalent to the following, viz.

$$\frac{\partial_x V}{u}=\frac{\partial_y V}{v}=\frac{\partial_z V}{w}=\theta, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

where θ is indeterminate. The third equation, viz. $\square^2 V=0$, when written in full, is

$$0=\square\lambda\partial_x V+\square\mu\partial_y V+\square\nu\partial_z V+\lambda^2\partial_x^2 V+\mu^2\partial_y^2 V+\nu^2\partial_z^2 V+2(\mu\nu\partial_y\partial_z V+\nu\lambda\partial_z\partial_x V+\lambda\mu\partial_x\partial_y V). \quad (7)$$

Now n being the degree of U, we may without difficulty establish the following formulæ given by CAYLEY (*l. c.* p. 381):

$$\left.\begin{aligned}
(n-1)u^2&=-\mathfrak{B}z^2+2\mathfrak{F}zy-\mathfrak{C}y^2,\\
(n-1)v^2&=-\mathfrak{C}x^2+2\mathfrak{G}xz-\mathfrak{A}z^2,\\
(n-1)w^2&=-\mathfrak{A}y^2+2\mathfrak{H}yx-\mathfrak{B}x^2,\\
(n-1)vw&=-\mathfrak{F}x^2-\mathfrak{G}xy-\mathfrak{H}xz+\mathfrak{A}yz,\\
(n-1)wu&=-\mathfrak{F}yx+\mathfrak{G}y^2-\mathfrak{H}yz+\mathfrak{B}zx,\\
(n-1)vw&=-\mathfrak{F}zx-\mathfrak{G}zy+\mathfrak{H}z^2+\mathfrak{C}xy,
\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

* In a paper recently published in the 'Quarterly Journal of Mathematics,' vol. vii. p. 114, I have given a transformation having the same object in view; but its form is partial and in some sense incomplete, and the mode of proof less direct and obvious than that given in the text.

whence writing

$$\Phi=(a, b, c, f, g, h)(\alpha, \beta, \gamma)^2, \quad \ldots \ldots \ldots \quad (9)$$

we may derive

$$\left.\begin{aligned}
(n-1)\lambda^2 &=-\delta^2\mathfrak{A}+2\delta x(\mathfrak{A}\alpha+\mathfrak{H}\beta+\mathfrak{G}\gamma)-x^2\Phi,\\
(n-1)\mu^2 &=-\delta^2\mathfrak{B}+2\delta y(\mathfrak{H}\alpha+\mathfrak{B}\beta+\mathfrak{F}\gamma)-y^2\Phi,\\
(n-1)\nu^2 &=-\delta^2\mathfrak{C}+2\delta z(\mathfrak{G}\alpha+\mathfrak{F}\beta+\mathfrak{C}\gamma)-z^2\Phi,\\
(n-1)\mu\nu &=-\delta^2\mathfrak{F}+\delta z(\mathfrak{H}\alpha+\mathfrak{B}\beta+\mathfrak{F}\gamma)+\delta y(\mathfrak{G}\alpha+\mathfrak{F}\beta+\mathfrak{C}\gamma)-yz\Phi,\\
(n-1)\nu\lambda &=-\delta^2\mathfrak{G}+\delta x(\mathfrak{G}\alpha+\mathfrak{F}\beta+\mathfrak{C}\gamma)+\delta z(\mathfrak{A}\alpha+\mathfrak{H}\beta+\mathfrak{G}\gamma)-zx\Phi,\\
(n-1)\lambda\mu &=-\delta^2\mathfrak{H}+\delta y(\mathfrak{A}\alpha+\mathfrak{H}\beta+\mathfrak{G}\gamma)+\delta x(\mathfrak{H}\alpha+\mathfrak{B}\beta+\mathfrak{F}\gamma)-xy\Phi.
\end{aligned}\right\} \quad \ldots \quad (10)$$

But, as will be found on calculating the expressions,

$$\left.\begin{aligned}
(n-1)\square\lambda &=\delta(\mathfrak{A}\alpha+\mathfrak{H}\beta+\mathfrak{G}\gamma)-x\Phi,\\
(n-1)\square\mu &=\delta(\mathfrak{H}\alpha+\mathfrak{B}\beta+\mathfrak{F}\gamma)-y\Phi,\\
(n-1)\square\nu &=\delta(\mathfrak{G}\alpha+\mathfrak{F}\beta+\mathfrak{C}\gamma)-z\Phi,
\end{aligned}\right\} \quad \ldots \ldots \ldots \quad (11)$$

so that

$$\left.\begin{aligned}
(n-1)^2\lambda^2 &=-\delta^2\mathfrak{A}+2(n-1)x\square\lambda+x^2\Phi,\\
(n-1)^2\mu^2 &=-\delta^2\mathfrak{B}+2(n-1)y\square\mu+y^2\Phi,\\
(n-1)^2\nu^2 &=-\delta^2\mathfrak{C}+2(n-1)z\square\nu+z^2\Phi,\\
(n-1)^2\mu\nu &=-\delta^2\mathfrak{F}+(n-1)(y\square\nu+z\square\mu)+yz\Phi,\\
(n-1)^2\nu\lambda &=-\delta^2\mathfrak{G}+(n-1)(z\square\lambda+x\square\nu)+zx\Phi,\\
(n-1)^2\lambda\mu &=-\delta^2\mathfrak{H}+(n-1)(x\square\mu+y\square\lambda)+xy\Phi.
\end{aligned}\right\} \quad \ldots \ldots \quad (12)$$

Hence, if m be the degree of V,

$$\begin{aligned}
&(n-1)^2\{\lambda^2\partial_x^2\mathrm{V}+\mu^2\partial_y^2\mathrm{V}+\nu^2\partial_z^2\mathrm{V}+2(\mu\nu\partial_y\partial_z\mathrm{V}+\nu\lambda\partial_z\partial_x\mathrm{V}+\lambda\mu\partial_x\partial_y\mathrm{V})\}\\
&\quad=-\delta^2(\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{F}, \mathfrak{G}, \mathfrak{H})(\partial_x, \partial_y, \partial_z)^2\mathrm{V}+2(n-1)(m-1)(\square\lambda\partial_x\mathrm{V}+\square\mu\partial_y\mathrm{V}+\square\nu\partial_z\mathrm{V}),
\end{aligned}$$

whence, substituting in (7), and bearing in mind that

$$\left.\begin{aligned}
(n-1)(\mathfrak{A}u+\mathfrak{H}v+\mathfrak{G}w)&=\mathrm{H}x,\\
(n-1)(\mathfrak{H}u+\mathfrak{B}v+\mathfrak{F}w)&=\mathrm{H}y,\\
(n-1)(\mathfrak{G}u+\mathfrak{F}v+\mathfrak{C}w)&=\mathrm{H}z,
\end{aligned}\right\} \quad \ldots \ldots \ldots \quad (13)$$

we have

$$(n-1)^2\left(1+\frac{2(m-1)}{n-1}\right)(\square\lambda\partial_x\mathrm{V}+\square\mu\partial_y\mathrm{V}+\square\nu\partial_z\mathrm{V})-\delta^2(\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{F}, \mathfrak{G}, \mathfrak{H})(\partial_x, \partial_y, \partial_z)^2\mathrm{V}=0.$$

But

$$\begin{aligned}
\square\lambda\partial_x\mathrm{V}+\square\mu\partial_y\mathrm{V}+\square\nu\partial_z\mathrm{V}&=\theta(u\square\lambda+v\square\mu+w\square\nu)\\
&=\frac{\theta\delta}{n-1}\{(\mathfrak{A}u+\mathfrak{H}v+\mathfrak{G}w)\alpha+(\mathfrak{H}u+\mathfrak{B}v+\mathfrak{F}w)\beta+(\mathfrak{G}u+\mathfrak{F}v+\mathfrak{C}w)\gamma\}\\
&=\frac{\theta\delta^2}{(n-1)^2}\mathrm{H},
\end{aligned}$$

so that (7) finally takes the forms

$$(\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{F}, \mathfrak{G}, \mathfrak{H})(\partial_x, \partial_y, \partial_z)^2 V - \left(1+\frac{2(m-1)}{n-1}\right)\theta H = 0\,, \quad . \quad . \quad . \quad (14)$$

or, in the case where V is a conic, and consequently $m=2$,

$$\mathfrak{A}a+\mathfrak{B}b+\mathfrak{C}c+2(\mathfrak{F}f+\mathfrak{G}g+\mathfrak{H}h)-\tfrac{1}{2}\frac{n+1}{n-1}\theta H=0\,; \quad . \quad . \quad . \quad . \quad . \quad (15)$$

and in general making $\varpi=1+\frac{2(m-1)}{n-1}$, (14) takes the form indicated above, viz.

or

$$\left.\begin{aligned} &\Delta V-\varpi\theta H=0,\\ &\frac{\partial_x V}{u}=\frac{\partial_y V}{v}=\frac{\partial_z V}{w}=\frac{\Delta V}{\varpi H}. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

§ 2. *Elimination of the Constants of the Conic of Five-pointic Contact.*

Before proceeding to the application of the formulæ (16) to the investigation of the sextactic points, it will be convenient to premise that if s, t be any two homogeneous functions of x, y, z, the nature of the operation Δ is such that

$$\Delta st=s\Delta t+t\Delta s+2(\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{F}, \mathfrak{G}, \mathfrak{H})(\partial_x s, \partial_y s, \partial_z s)(\partial_x t, \partial_y t, \partial_z t), \quad . \quad . \quad (17)$$

and also that

$$\Delta V=3H, \quad \Delta u=p, \quad \Delta v=q, \quad \Delta w=r. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

This being premised, our first object is to establish an equivalent for $\Box^3 V=0$, divested of the extraneous quantities α, β, γ. Now, since

$$\begin{aligned} &\delta(v\partial_z V-w\partial_y V)=x\Box V,\\ &\delta(w\partial_x V-u\partial_z V)=y\Box V,\\ &\delta(u\partial_y V-v\partial_x V)=z\Box V, \end{aligned}$$

and $\Box\delta=0$, it follows that

$$\begin{aligned} &\delta\Box(v\partial_z V-w\partial_y V)=\lambda\Box V+x\Box^2 V,\\ &\delta\Box(w\partial_x V-u\partial_z V)=\mu\Box V+y\Box^2 V,\\ &\delta\Box(u\partial_y V-v\partial_x V)=\nu\Box V+z\Box^2 V\,; \end{aligned}$$

and consequently not only do $v\partial_z V-w\partial_y V$, $w\partial_x v-u\partial_z V$, $u\partial_y V-v\partial_x V$ vanish with $\Box V$, but, when this is the case, $\Box(v\partial_z V-w\partial_y V)$, .. vanish with $\Box^2 V$. The same will obviously be the case if the operation $\Box$ be continued; so that, in general terms, we may, by operating upon $v\partial_z V-w\partial_y V$, .. with the symbol $\Box$, 0, 1, 2, .. times, form a system of equations equivalent to that formed by operating on V with the same symbol 1, 2, 3, .. times. And if we represent any of the three quantities $v\partial_z V-w\partial_y V$, .. by W, the equations $W=0$, $\Box W=0$, $\Box^2 W=0$ will be equivalent to the system

$$\frac{\partial_x W}{u}=\frac{\partial_y W}{v}=\frac{\partial_z W}{w}=\frac{\Delta W}{\varpi_1 H}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

analogous to (16). More generally, if

$$\Delta_1 = u\Delta - \varpi H\partial_x,$$
$$\Delta_2 = v\Delta - \varpi H\partial_y,$$
$$\Delta_3 = w\Delta - \varpi H\partial_z,$$

and if Δ' stands for any of the three symbols Δ_1, Δ_2, Δ_3, then the equations $V=0$, $\square V=0$ are equivalent to

$$\frac{1}{u}\partial_x V = \frac{1}{v}\partial_y V = \frac{1}{w}\partial_z V;$$

the equations $\square^2 V=0$, $\square^3 V=0$ are equivalent to

$$\frac{1}{u}\partial_x \Delta' V = \frac{1}{v}\partial_y \Delta' V = \frac{1}{w}\partial_z \Delta' V.$$

Similarly, if Δ'' stands for any one of the symbols Δ_1, Δ_2, Δ_3, either the same as Δ' or not, then $\square^4 V=0$, $\square^5 V=0$ are equivalent to

$$\frac{1}{u}\partial_x \Delta''\Delta' V = \frac{1}{v}\partial_y \Delta''\Delta' V = \frac{1}{w}\partial_z \Delta''\Delta' V,$$

and so on indefinitely, for $\square^{2i} V=0$, $\square^{2i+1} V=0$. If the series should terminate with $\square^{2i} V=0$, *e. g.* $\square^6 V=0$, then the last equivalent would be $\Delta'''\Delta''\Delta' V=0$, where Δ''' stands, like Δ'', for any one of the symbols Δ_1, Δ_2, Δ_3 indifferently. The form W, however, presents peculiar advantages for the application of the operations Δ, as will be more fully seen in the sequel. And it follows from what has been said above that, if W retain the same signification as before, we may replace the equations $W=0$, $\square W=0$ (and consequently the equations $\square V=0$, $\square^2 V=0$) by

$$\frac{1}{u}\partial_x W = \frac{1}{v}\partial_y W = \frac{1}{w}\partial_z W,$$

and in the same way the equations $\square^2 W=0$, $\square^3 W=0$ (and consequently $\square^3 V=0$, $\square^4 V=0$) by

$$\frac{1}{u}\Delta'\partial_x W = \frac{1}{v}\Delta'\partial_y W = \frac{1}{w}\Delta'\partial_z W,$$

and so on. I do not, however, propose on the present occasion to pursue the general theory further.

Returning to the problem of the sextactic points, and forming the equations in W (19), we have

$$\left.\begin{aligned}
\frac{1}{u}\partial_x(v\,\partial_z V - w\partial_y V) &= \frac{1}{v}\partial_y(v\,\partial_z V - w\partial_y V) = \frac{1}{w}\partial_z(v\,\partial_z V - w\partial_y V) = \frac{1}{\varpi_1 H}\Delta(v\,\partial_z V - w\partial_y V)\\
\frac{1}{u}\partial_x(w\partial_x V - u\,\partial_z V) &= \frac{1}{v}\partial_y(w\partial_x V - u\,\partial_z V) = \frac{1}{w}\partial_z(w\partial_x V - u\,\partial_z V) = \frac{1}{\varpi_1 H}\Delta(w\partial_x V - u\,\partial_z V)\\
\frac{1}{u}\partial_x(u\,\partial_y V - v\,\partial_x V) &= \frac{1}{v}\partial_y(u\,\partial_y V - v\,\partial_x V) = \frac{1}{w}\partial_z(u\,\partial_y V - v\,\partial_x V) = \frac{1}{\varpi_1 H}\Delta(u\,\partial_y V - v\,\partial_x V).
\end{aligned}\right\} \quad (20)$$

But since W is of the degree n, $\varpi_1=1+\frac{2(n-1)}{n-1}=3$. Also since $\partial_x\mathrm{V}$, $\partial_y\mathrm{V}$, $\partial_z\mathrm{V}$ are linear in x, y, z, it follows that $\Delta\partial_x\mathrm{V}=0$, $\Delta\partial_y\mathrm{V}=0$, $\Delta\partial_z\mathrm{V}=0$; hence, applying the formulæ (17), (18),

$$\Delta v\partial_z\mathrm{V}=q\partial_z\mathrm{V}+2(\mathfrak{A}\,..\,\mathfrak{F}\,..\,)(w',\ v_1,\ u')(\partial_x\partial_z\mathrm{V},\ \partial_y\partial_z\mathrm{V},\ \partial_z^2\mathrm{V}).$$

But since

$$\mathfrak{A}w'+\mathfrak{H}v_1+\mathfrak{G}u'=0,\quad \mathfrak{H}w'+\mathfrak{B}v_1+\mathfrak{G}u'=\mathrm{H},\quad \mathfrak{G}w'+\mathfrak{F}v_1+\mathfrak{C}u'=0,$$

it follows that

$$\Delta v\partial_z\mathrm{V}=q\partial_z\mathrm{V}+2\mathrm{H}\partial_y\partial_z\mathrm{V}.$$

Similarly,

$$\Delta w\partial_y\mathrm{V}=r\partial_y\mathrm{V}+2\mathrm{H}\partial_z\partial_y\mathrm{V},$$

so that (20) become

$$\left.\begin{aligned} q\partial_z\mathrm{V}-r\partial_y\mathrm{V}&=\frac{3\mathrm{H}}{u}(w'\partial_z\mathrm{V}-v'\,\partial_y\mathrm{V}+2vg-2wh)=\,..\\ r\partial_x\mathrm{V}-p\partial_z\mathrm{V}&=\frac{3\mathrm{H}}{u}(v'\,\partial_x\mathrm{V}-u_1\partial_z\mathrm{V}+2wa-2ug)=\,..\\ p\partial_y\mathrm{V}-q\partial_x\mathrm{V}&=\frac{3\mathrm{H}}{u}(u_1\partial_y\mathrm{V}-w'\partial_x\mathrm{V}+2uh-2va)=\,..\,, \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (21)$$

whence, multiplying by p, q, r respectively, and adding, we have

$$\left.0=\begin{vmatrix} p & u_1 & \partial_x\mathrm{V} \\ q & w' & \partial_y\mathrm{V} \\ r & v' & \partial_z\mathrm{V} \end{vmatrix}+2\begin{vmatrix} p & u & a \\ q & v & h \\ r & w & g \end{vmatrix}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (22)$$

Substituting for $\partial_x\mathrm{V}=\theta u$, $\partial_y\mathrm{V}=\theta v$, $\partial_z\mathrm{V}=\theta w$, (22) becomes

$$\left.\begin{vmatrix} p & u & 2a-\theta u_1 \\ q & v & 2h-\theta w' \\ r & w & 2g-\theta v' \end{vmatrix}=0;\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (23)$$

and writing

$$vr-wq=\mathrm{X},\quad wp-ur=\mathrm{Y},\quad uq-vp=\mathrm{Z},$$

$$\left.\begin{aligned} u_1\mathrm{X}+w'\mathrm{Y}+v'\,\mathrm{Z}&=\mathrm{P} \\ w'\mathrm{X}+v_1\,\mathrm{Y}+u'\,\mathrm{Z}&=\mathrm{Q} \\ v'\,\mathrm{X}+u'\,\mathrm{Y}+w_1\mathrm{Z}&=\mathrm{R}, \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (24)$$

(23) takes the form

$$2(a\mathrm{X}+h\mathrm{Y}+g\mathrm{Z})-\theta\mathrm{P}=0;$$

or finally substituting $2(ax+hy+gz)=\theta u$, and forming similar equations in Q and R, we have the system

$$\left.\begin{aligned} a(u\mathrm{X}-x\mathrm{P})+h(u\mathrm{Y}-y\mathrm{P})+g(u\mathrm{Z}-z\mathrm{P})&=0 \\ h(v\,\mathrm{X}-x\mathrm{Q})+b(v\,\mathrm{Y}-y\mathrm{Q})+g(v\,\mathrm{Z}-z\mathrm{Q})&=0 \\ g(w\mathrm{X}-x\mathrm{R})+f(w\mathrm{Y}-y\mathrm{R})+c(w\mathrm{Z}-z\mathrm{R})&=0, \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (25)$$

which may be regarded as the three forms by any one of which $\square^3\mathrm{V}=0$ may be replaced. Before proceeding further, it will be convenient to notice that the quantities $u\mathrm{X}-x\mathrm{P}, \ldots$ are capable of being transformed in a manner which will be useful hereafter, as follows:—

$$
\begin{aligned}
\mathrm{P}x&=\mathrm{X}u_1x+(ww'-vv')px+(v'q-w'r)ux\\
&=\mathrm{X}u_1x+(ww'-vv')(3\overline{n-2}\mathrm{H}-qy-rz)-(v'q-w'r)(vy-wz)\\
&=\mathrm{X}(u_1x+w'y+v'z)+3(n-2)\mathrm{H}(ww'-vv')\\
&=(n-1)u\mathrm{X}+3(n-2)\mathrm{H}(ww'-vv'),
\end{aligned}
$$

i. e.

$$
\left.\begin{aligned}
-u\mathrm{X}+x\mathrm{P}&=(n-2)\{u\mathrm{X}-3\mathrm{H}(vv'-ww')\}\\
-u\mathrm{Y}+y\mathrm{P}&=(n-2)\{u\mathrm{Y}-3\mathrm{H}(wu_1-uv')\}\\
-u\mathrm{Z}+z\mathrm{P}&=(n-2)\{u\mathrm{Z}-3\mathrm{H}(uv'-vu_1)\}.
\end{aligned}\right\} \quad \ldots \ldots \ldots \quad (26)
$$

Returning to (25), and taking any one of the three as W, we shall have for $\square^3\mathrm{V}=0$, $\square^4\mathrm{V}=0$, $\square^5\mathrm{V}=0$,

$$
\left.\begin{aligned}
a\partial_x(u\mathrm{X}-x\mathrm{P})+h\partial_x(u\mathrm{Y}-y\mathrm{P})+g\partial_x(u\mathrm{Z}-z\mathrm{P})-\quad \theta_2u&=0\\
a\partial_y(u\mathrm{X}-x\mathrm{P})+h\partial_y(u\mathrm{Y}-y\mathrm{P})+g\partial_y(u\mathrm{Z}-z\mathrm{P})-\quad \theta_2v&=0\\
a\partial_z(u\mathrm{X}-x\mathrm{P})+h\partial_z(u\mathrm{Y}-y\mathrm{P})+g\partial_z(u\mathrm{Z}-z\mathrm{P})-\quad \theta_2w&=0\\
a\Delta(u\mathrm{X}-x\mathrm{P})+h\Delta(u\mathrm{Y}-y\mathrm{P})+g\Delta(u\mathrm{Z}-z\mathrm{P})-\varpi_2\theta_2\mathrm{H}&=0\,;
\end{aligned}\right\} \quad \ldots \quad (27)
$$

and similar groups may be formed from the other two equations of (25). Now as (27) contain only three out of the six constants $a, \ldots f_1 \ldots$, and the single indeterminate θ_2, they are sufficient for the elimination in view, and give for the equation whereby the sextactic points are to be determined,

$$
\left.\begin{vmatrix}
\partial_x(u\mathrm{X}-x\mathrm{P}) & \partial_x(u\mathrm{Y}-y\mathrm{P}) & \partial_x(u\mathrm{Z}-z\mathrm{P}) & u\\
\partial_y(u\mathrm{X}-x\mathrm{P}) & \partial_y(u\mathrm{Y}-y\mathrm{P}) & \partial_y(u\mathrm{Z}-z\mathrm{P}) & u\\
\partial_z(u\mathrm{X}-x\mathrm{P}) & \partial_z(u\mathrm{Y}-y\mathrm{P}) & \partial_z(u\mathrm{Z}-z\mathrm{P}) & w\\
\Delta(u\mathrm{X}-x\mathrm{P}) & \Delta(u\mathrm{Y}-y\mathrm{P}) & \Delta(u\mathrm{Z}-z\mathrm{P}) & \varpi_2\mathrm{H}
\end{vmatrix}=0,\right\} \quad \ldots \quad (28)
$$

which, in virtue of (26), may also be written in the form

$$
\left.\begin{vmatrix}
\partial_x\{u\mathrm{X}-3\mathrm{H})vv'-ww')\} & \partial_x\{u\mathrm{Y}-3\mathrm{H}(wu_1-uv')\} & \partial_x\{u\mathrm{Z}-3\mathrm{H}(uv'-vu_1)\} & u\\
\partial_y\{u\mathrm{X}-3\mathrm{H})vv'-ww')\} & \partial_y\{u\mathrm{Y}-3\mathrm{H}(wu_1-uv')\} & \partial_y\{u\mathrm{Z}-3\mathrm{H}(uv'-vu_1)\} & v\\
\partial_z\{u\mathrm{X}-3\mathrm{H})vv'-ww')\} & \partial_z\{u\mathrm{Y}-3\mathrm{H}(wu_1-uv')\} & \partial_z\{u\mathrm{Z}-3\mathrm{H}(uv'-vu_1)\} & w\\
\Delta\{u\mathrm{X}-3\mathrm{H})vv'-ww')\} & \Delta\{u\mathrm{Y}-3\mathrm{H}(wu_1-uv')\} & \Delta\{u\mathrm{Z}-3\mathrm{H}(uv'-vu_1)\} & \varpi_2\mathrm{H}
\end{vmatrix}=0,\right\} \quad (29)
$$

with similar expressions in v, Q; w, R. Calling (28) and (29) $\mathfrak{L}$, $\mathfrak{L}'$ respectively, we may designate the entire group of six forms, three of the form (28), and three of the form (29) by

$$
\mathfrak{L}=0,\ \mathfrak{M}=0,\ \mathfrak{N}=0,\ \mathfrak{L}'=0,\ \mathfrak{M}'=0,\ \mathfrak{N}=0. \quad \ldots \quad (30)
$$

And as $\mathfrak{L}$, $\mathfrak{L}'$ differ only in respect of a numerical factor, any other factor that can be predicated of $\mathfrak{L}$ may be affirmed of $\mathfrak{L}'$, and *vice versâ*; and similarly for the other pairs $\mathfrak{M}$, $\mathfrak{M}'$; $\mathfrak{N}$, $\mathfrak{N}'$.

§ 3. *Formulæ of Reduction.*

The degree of the expressions (28) or (29) is $18n-36$; it remains to show that existence of certain extraneous factors, which when divided out will reduce the degree to $12n-27$, and at the same time render the three forms identical. But before entering upon this, it will be convenient to premise the following formulæ, the first group of which are easily verified.

$$\left.\begin{aligned}
yZ-zY&=3(n-2)Hu\\
zX-xZ&=3(n-2)Hv\\
xY-yX&=3(n-2)Hw\\[1ex]
y\partial_xZ-z\partial_xY&=(3n-7)up-(n-1)up+3(n-2)Hu_1\\
y\partial_yZ-z\partial_yY&=(3n-7)uq-(n-1)vp+3(n-2)Hw'\\
y\partial_zZ-z\partial_zY&=(3n-7)ur-(n-1)wp+3(n-2)Hv'\\[1ex]
z\partial_xX-x\partial_xZ&=(3n-7)vp-(n-1)uq+3(n-2)Hw'\\
z\partial_yX-x\partial_yZ&=(3n-7)vq-(n-1)vq+3(n-2)Hv_1\\
z\partial_zX-x\partial_zZ&=(3n-7)vr-(n-1)wq+3(n-2)Hu'\\[1ex]
x\partial_xY-y\partial_xX&=(3n-7)wp-(n-1)ur+3(n-2)Hv'\\
x\partial_yY-y\partial_yX&=(3n-7)wq-(n-1)vr+3(n-2)Hu'\\
x\partial_zY-y\partial_zX&=(3n-7)wr-(n-1)wr+3(n-2)Hw_1.
\end{aligned}\right\}\quad\ldots\ldots\quad(31)$$

And writing

$$\left.\begin{aligned}
-P_1&=Xp_1+Yr'+Zq'\\
-Q_1&=Xr'+Yq_1+Zp'\\
-R_1&=Xq'+Yp'+Zr_1,
\end{aligned}\right\}\quad\ldots\ldots\ldots\ldots\quad(32)$$

then also

$$\left.\begin{array}{lll}
Y\partial_xZ-Z\partial_xY=-(pP+uP_1) & Z\partial_xX-X\partial_xZ=-(qP+vP_1) & X\partial_xY-Y\partial_xX=-(rP+wP_1)\\
Y\partial_yZ-Z\partial_yY=-(pQ+uQ_1) & Z\partial_yX-X\partial_yZ=-(qQ+vQ_1) & X\partial_yY-Y\partial_yX=-(rQ+wQ_1)\\
Y\partial_zZ-Z\partial_zY=-(pR+uR_1) & Z\partial_zX-X\partial_zZ=-(qR+vR_1) & X\partial_zY-Y\partial_zX=-(rR+wR_1),
\end{array}\right\}(33)$$

Moreover, writing with Professor CAYLEY,

$$\left.\begin{array}{l}
(\mathfrak{A},\mathfrak{B},\mathfrak{C},\mathfrak{F},\mathfrak{G},\mathfrak{H})(\partial_x,\partial_y,\partial_z)^2H=\Omega\\
\partial_x\Omega_{\bar{U}}=(\mathfrak{A},\mathfrak{B},\mathfrak{C},\mathfrak{F},\mathfrak{G},\mathfrak{H})(\partial_x,\partial_y,\partial_z)^2p,\ \partial_y\Omega_{\bar{U}}=\ldots,\ \partial_z\Omega_{\bar{U}}=\ldots\\
\partial_x\Omega_{\bar{H}}=(\partial_x\mathfrak{A},\partial_x\mathfrak{B},\partial_x\mathfrak{C},\partial_x\mathfrak{F},\partial_x\mathfrak{G},\partial_x\mathfrak{H})(\partial_x,\partial_y,\partial_z)^2H,\ \partial_y\Omega_{\bar{H}}=\ldots,\partial_z\Omega_{\bar{H}}=\ldots,
\end{array}\right\}(34)$$

and noticing that

$$X\partial_x\Omega_U+Y\partial_y\Omega_U+Z\partial_z\Omega_U=\text{Jac.}\,(U,\ H,\ \Omega_U), \quad . \quad . \quad . \quad . \quad . \quad . \quad (35)$$

and that

$$\left.\begin{aligned}\Delta X&=v\partial_z\Omega_U-w\partial_y\Omega_U\\ \Delta Y&=w\partial_x\Omega_U-u\partial_z\Omega_U\\ \Delta Z&=u\partial_y\Omega_U-v\partial_x\Omega_U,\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (36)$$

then we have

$$\left.\begin{aligned}Y\Delta Z-Z\,\Delta Y&=u\ \text{Jac.}\,(U,\ H,\ \Omega_U) & y\Delta Z-z\,\Delta Y&=(5n-12)\Omega u\\ Z\,\Delta X-X\Delta Z&=v\ \text{Jac.}\,(U,\ H,\ \Omega_U) & z\Delta X-x\Delta Z&=(5n-12)\Omega v\\ X\Delta Y-Y\Delta X&=w\ \text{Jac.}\,(U,\ H,\ \Omega_U) & x\Delta Y-y\Delta X&=(5n-12)\Omega w.\end{aligned}\right\} \quad . \quad . \quad (37)$$

Again, if $\mathfrak{A}'$, $\mathfrak{B}'$, $\mathfrak{C}'$, $\mathfrak{F}'$, $\mathfrak{G}'$, $\mathfrak{H}'$ be the same quantities with respect to H that $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{F}$, $\mathfrak{G}$, $\mathfrak{H}$ are with respect to U, *i. e.* if $\mathfrak{A}'=q_1r_1-p'^2$, .. $\mathfrak{F}'=q'r'-p_1p'$, . . , and if

$$\left.\begin{aligned}\Theta&=(\mathfrak{A}',\ \mathfrak{B}',\ \mathfrak{C}',\ \mathfrak{F}',\ \mathfrak{G}',\ \mathfrak{H}')(u,\ v,\ w)^2\\ \Psi&=(\mathfrak{A},\ \mathfrak{B},\ \mathfrak{C},\ \mathfrak{F},\ \mathfrak{G},\ \mathfrak{H})(p,\ q,\ r)^2,\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (38)$$

then

$$\begin{vmatrix}u_1 & \partial_x Y & \partial_x Z\\ w' & \partial_y Y & \partial_y Z\\ v' & \partial_z Y & \partial_z Z\end{vmatrix}=\text{Jac.}\,(u,\ Y,\ Z,)=\begin{vmatrix}u_1 & v'p-u_1r+wp_1-uq' & u_1q-w'p+ur'-vp_1\\ w' & u'p-w'r+wr'-up' & w'q-v_1p+uq_1-vr'\\ v' & w_1p-v'r+wq'-ur_1 & v'q-u'p+up'-vq'\end{vmatrix}$$

$$\begin{aligned}=Hp^2&-(\mathfrak{H}r'+\mathfrak{B}q_1+\mathfrak{F}p')pu\\ &+(\mathfrak{H}p_1+\mathfrak{B}r'+\mathfrak{F}q')pv\\ &+(\mathfrak{G}p_1+\mathfrak{F}r'+\mathfrak{C}q')wp\\ &+(\mathfrak{G}'u_1+\mathfrak{F}'w'+\mathfrak{C}'v')wu\\ &-(\mathfrak{G}q'+\mathfrak{F}p'+\mathfrak{C}r_1)pu\\ &+(\mathfrak{A}'u_1+\mathfrak{H}'w'+\mathfrak{G}'v')u^2\\ &+(\mathfrak{H}'u_1+\mathfrak{B}'w'+\mathfrak{F}'v')uv\end{aligned}$$

$$\begin{aligned}=Hp^2-\Omega pu&+(\mathfrak{A}p_1+\mathfrak{H}r'+\mathfrak{G}q')pu\\ &+(\mathfrak{H}p_1+\mathfrak{B}r'+\mathfrak{F}q')pv\\ &+(\mathfrak{G}p_1+\mathfrak{F}r'+\mathfrak{C}q')pw\\ &+(\mathfrak{A}'u_1+\mathfrak{H}'w'+\mathfrak{G}'v')u^2\\ &+(\mathfrak{H}'u_1+\mathfrak{B}'w'+\mathfrak{F}'v')uv\\ &+(\mathfrak{G}'u_1+\mathfrak{F}'w'+\mathfrak{C}v')uw\end{aligned}$$

$$=Hp^2-\Omega pu+\frac{3n-7}{n-1}Hp^2+\tfrac{1}{2}u\partial_x\Theta_U$$

Similarly,

$$\text{Jac.}\,(u,\ Z,\ X)=\begin{vmatrix}u_1 & u_1q-w'p+ur'-vp_1 & w'r-v'q+vq'-wr'\\ w' & w'q-v_1p+uq_1-vr' & v_1r-u'q+vp'-wq_1\\ v' & v'q-u'p+up'-vq' & u'r-w_1q+vr_1-wp'\end{vmatrix}$$

$$
\begin{array}{lll}
=\mathrm{H}pq & =\mathrm{H}pq-\Omega pv-(\mathfrak{A}r'+\mathfrak{H}q_1+\mathfrak{G}p')up & =\mathrm{H}pq-\Omega pv+\tfrac{1}{2}v\partial_x\Theta_{\mathrm{U}}+\frac{3n-7}{n-1}\mathrm{H}pq \\
-(\mathfrak{G}q'+\mathfrak{F}p'+\mathfrak{C}r_1)vp & -(\mathfrak{H}r'+\mathfrak{B}q_1+\mathfrak{F}p')uq & +\tfrac{1}{2}v\partial_x\Psi_{\mathrm{U}}-\tfrac{1}{2}u\partial_y\Psi_{\mathrm{U}} \\
+(\mathfrak{G}r'+\mathfrak{F}q_1+\mathfrak{C}p')wp & -(\mathfrak{G}r'+\mathfrak{F}q_1+\mathfrak{C}p')ur & \\
-(\mathfrak{G}r'+\mathfrak{F}q_1+\mathfrak{C}p')ur & +(\mathfrak{A}r'+\mathfrak{H}q_1+\mathfrak{G}p')up & \\
-(\mathfrak{H}r'+\mathfrak{B}q_1+\mathfrak{F}p')uq & +(\mathfrak{H}r'+\mathfrak{B}q_1+\mathfrak{F}p')vp & \\
+(\mathfrak{A}'u_1+\mathfrak{H}'w'+\mathfrak{G}'v')uv & +(\mathfrak{G}r'+\mathfrak{F}q_1+\mathfrak{C}p')wp & \\
+(\mathfrak{G}p_1+\mathfrak{F}r'+\mathfrak{C}q')vr & +(\mathfrak{A}p_1+\mathfrak{H}r'+\mathfrak{G}q')pv & \\
+(\mathfrak{H}p_1+\mathfrak{B}r'+\mathfrak{F}q')vq & +(\mathfrak{H}p_1+\mathfrak{B}r'+\mathfrak{F}q')qv & \\
+(\mathfrak{H}'u_1+\mathfrak{B}'w'+\mathfrak{F}'v')v^2 & +(\mathfrak{G}p_1+\mathfrak{F}r'+\mathfrak{C}q')rv & \\
+(\mathfrak{G}'u_1+\mathfrak{F}'w'+\mathfrak{C}'v')vw & +\tfrac{1}{2}v\partial_x\Theta_{\mathrm{U}} &
\end{array}
$$

Hence

$$
\left.\begin{array}{l}
\text{Jac.}\,(u,\ \mathrm{Y},\ \mathrm{Z})=\frac{4(n-2)}{n-1}\mathrm{H}p^2-(\Omega p-\tfrac{1}{2}\partial_x\Theta_{\mathrm{U}})u \\
\text{Jac.}\,(u,\ \mathrm{Z},\ \mathrm{X})=\frac{4(n-2)}{n-1}\mathrm{H}pq-(\Omega p-\tfrac{1}{2}\partial_x\Theta_{\mathrm{U}})v+\tfrac{1}{2}u\partial_x\Psi_{\mathrm{U}}-\tfrac{1}{2}u\partial_y\Psi_{\mathrm{U}} \\
\text{Jac.}\,(u,\ \mathrm{X},\ \mathrm{Y})=\frac{4(n-2)}{n-1}\mathrm{H}pr-(\Omega p-\tfrac{1}{2}\partial_x\Theta_{\mathrm{U}})w+\tfrac{1}{2}w\partial_x\Psi_{\mathrm{U}}-\tfrac{1}{2}u\partial_z\Psi_{\mathrm{U}} \\
\text{Jac.}\,(v,\ \mathrm{Y},\ \mathrm{Z})=\frac{4(n-2)}{n-1}\mathrm{H}qp-(\Omega q-\tfrac{1}{2}\partial_y\Theta_{\mathrm{U}})u+\tfrac{1}{2}u\partial_y\Psi_{\mathrm{U}}-v\partial_x\Psi_{\mathrm{U}} \\
\text{Jac.}\,(v,\ \mathrm{Z},\ \mathrm{X})=\frac{4(n-2)}{n-1}\mathrm{H}q^2-(\Omega q-\tfrac{1}{2}\partial_y\Theta_{\mathrm{U}})v \\
\text{Jac.}\,(v,\ \mathrm{X},\ \mathrm{Y})=\frac{4(n-2)}{n-1}\mathrm{H}qr-(\Omega q-\tfrac{1}{2}\partial_y\Theta_{\mathrm{U}})w+\tfrac{1}{2}w\partial_y\Psi_{\mathrm{U}}-v\partial_z\Psi_{\mathrm{U}} \\
\text{Jac.}\,(w,\ \mathrm{Y},\ \mathrm{Z})=\frac{4(n-2)}{n-1}\mathrm{H}rp-(\Omega r-\tfrac{1}{2}\partial_z\Theta_{\mathrm{U}})u+\tfrac{1}{2}u\partial_z\Psi_{\mathrm{U}}-\tfrac{1}{2}w\partial_x\Psi_{\mathrm{U}} \\
\text{Jac.}\,(w,\ \mathrm{Z},\ \mathrm{X})=\frac{4(n-2)}{n-1}\mathrm{H}rq-(\Omega r-\tfrac{1}{2}\partial_z\Theta_{\mathrm{U}})v+\tfrac{1}{2}v\partial_z\Psi_{\mathrm{U}}-\tfrac{1}{2}w\partial_y\Psi_{\mathrm{U}} \\
\text{Jac.}\,(w,\ \mathrm{X},\ \mathrm{Y})=\frac{4(n-2)}{n-1}\mathrm{H}r^2-(\Omega r-\tfrac{1}{2}\partial_z\Theta_{\mathrm{U}})w.
\end{array}\right\} \qquad (39)
$$

Again,

$$x\,\text{Jac.}\,(u,\ \mathrm{Y},\ \mathrm{Z})+y\,\text{Jac.}\,(v,\ \mathrm{Y},\ \mathrm{Z})+z\,\text{Jac.}\,(w,\ \mathrm{Y},\ \mathrm{Z})=(n-1)\,\text{Jac.}\,(\mathrm{U},\ \mathrm{Y},\ \mathrm{Z})\,;$$

whence, bearing in mind that

$$x\partial_x\Psi_{\mathrm{U}}+y\partial_y\Psi_{\mathrm{U}}+z\partial_z\Psi_{\mathrm{U}}=2(3n-7)\Psi_{\mathrm{U}},$$
$$x\partial_x\Theta_{\mathrm{U}}+y\partial_y\Theta_{\mathrm{U}}+z\partial_z\Theta_{\mathrm{U}}=2(n-1)\Theta_{\mathrm{U}},$$

because in the differentiations $\mathfrak{A}, \ldots \mathfrak{A}', \ldots$ are supposed constant, it follows that

$$\left.\begin{aligned}
\text{Jac.}(\text{U}, \text{Y}, \text{Z}) &= \frac{12(n-2)^2}{(n-1)^2}\text{H}^2p + \left\{\frac{3n-7}{n-1}\Psi - \frac{3(n-2)}{n-1}\text{H}\Omega + \Theta\right\}u \\
\text{Jac.}(\text{U}, \text{Z}, \text{X}) &= \frac{12(n-2)^2}{(n-1)^2}\text{H}^2q + \left\{\frac{3n-7}{n-1}\Psi - \frac{3(n-2)}{n-1}\text{H}\Omega + \Theta\right\}v \\
\text{Jac.}(\text{U}, \text{X}, \text{Y}) &= \frac{12(n-2)^2}{(n-1)^2}\text{H}^2r + \left\{\frac{3n-7}{n-1}\Psi - \frac{3(n-2)}{n-1}\text{H}\Omega + \Theta\right\}w.
\end{aligned}\right\} \quad . \quad . \quad (40)$$

Again,

$$\begin{vmatrix} u_1 & \partial_x\text{Y} & \partial_x\text{Z} \\ w' & \partial_y\text{Y} & \partial_y\text{Z} \\ v' & \partial_z\text{Y} & \partial_z\text{Z} \end{vmatrix} = \begin{vmatrix} u_1 & v'p-u_1r+wp_1-uq' & u_1q-w'p+ur'-vp_1 \\ w' & u'p-w'r+wr'-up' & w'q-v_1p+uq_1-vr' \\ v' & w_1p-v'r+wq'-ur_1 & v'q-u'p+up'-vq' \end{vmatrix}$$

$$\begin{aligned}
&= p^2\text{H} - (\mathfrak{H}r'+\mathfrak{B}q_1+\mathfrak{F}p')up + (\mathfrak{G}'u_1+\mathfrak{F}'w'+\mathfrak{C}'v')wu \\
&\quad + (\mathfrak{G}p_1+\mathfrak{F}r'+\mathfrak{C}q')wp + (\mathfrak{A}'u_1+\mathfrak{H}'w'+\mathfrak{G}'v')u^2 \\
&\quad - (\mathfrak{G}q'+\mathfrak{F}p'+\mathfrak{C}r_1)up + (\mathfrak{H}'u_1+\mathfrak{B}'w'+\mathfrak{F}'v')uv \\
&\quad + (\mathfrak{H}p_1+\mathfrak{B}r'+\mathfrak{F}q')vp \\
&= p^2\text{H} + \frac{3n-7}{n-1}p^2\text{H} - \Omega up + (\mathfrak{A}', \ldots \mathfrak{F}', \ldots)(u, v, w)(u_1, w', v')u.
\end{aligned}$$

Whence

$$\left.\begin{aligned}
\text{Jac.}(u, \text{Y}, \text{Z}) &= \frac{4(n-2)}{n-1}\text{H}p^2 - \Omega up + (\mathfrak{A}', \ldots \mathfrak{F}', \ldots)(u, v, w)(u_1, w', v')u \\
\text{Jac.}(u, \text{Z}, \text{X}) &= \frac{4(n-2)}{n-1}\text{H}pq - \Omega uq + (\mathfrak{A}', \ldots \mathfrak{F}', \ldots)(u, v, w)(w', v_1, u')u \\
\text{Jac.}(u, \text{X}, \text{Y}) &= \frac{4(n-2)}{n-1}\text{H}pr - \Omega ur + (\mathfrak{A}', \ldots \mathfrak{F}', \ldots)(u, v, w)(v', u', w_1)u.
\end{aligned}\right\}. \quad (41)$$

A similar process of reduction conducts to the relation

$$\begin{aligned}
\text{Jac.}(\text{X}, \text{Y}, \text{Z}) &= -(\mathfrak{A}, \ldots \mathfrak{F}, \ldots)(p, q, r)(p_1, r', q')\text{X} - (\mathfrak{A}', \ldots \mathfrak{F}', \ldots)(u, v, w)(u_1, w', v')\text{X} \\
&\quad - (\mathfrak{A}, \ldots \mathfrak{F}, \ldots)(p, q, r)(r', q_1, p')\text{Y} - (\mathfrak{A}', \ldots \mathfrak{F}', \ldots)(u, v, w)(w', v_1, u')\text{Y} \\
&\quad - (\mathfrak{A}, \ldots \mathfrak{F}, \ldots)(p, q, r)(q', p', r_1)\text{Z} - (\mathfrak{A}', \ldots \mathfrak{F}', \ldots)(u, v, w)(v', u', w_1)\text{Z} \\
&= -\text{Jac.}(\text{U}, \text{H}, \Psi_\text{U}) - \text{Jac.}(\text{U}, \text{H}, \Theta_\text{U}).
\end{aligned}$$

Whence also

$$\left.\begin{aligned}
\text{Jac.}(u\text{X}, u\text{Y}, u\text{Z}) &= u^3\text{Jac.}(\text{X}, \text{Y}, \text{Z}) + u^2\{\text{X Jac.}(u, \text{Y}, \text{Z}) + \text{Y Jac.}(\text{X}, u, \text{Z}) + \text{Z Jac.}(\text{X}, \text{Y}, u)\} \\
&= -u^3\text{Jac.}(\text{U}, \text{H}, \Psi_\text{U}).
\end{aligned}\right\}(42)$$

§ 4.

The resultant equation which, when combined with that of the original curve, will determine the sextactic points, was exhibited in § 2 under six different forms, there designated by

$$\mathfrak{L}=0,\quad \mathfrak{M}=0,\quad \mathfrak{N}=0,\quad \mathfrak{L}'=0,\quad \mathfrak{M}'=0,\quad \mathfrak{N}'=0.$$

Now since $\mathfrak{L}$ and $\mathfrak{L}'$, $\mathfrak{M}$ and $\mathfrak{M}'$, $\mathfrak{N}$ and $\mathfrak{N}'$ respectively differ only by the numerical factor $(n-2)^3$, we shall, in seeking to discover the extraneous factors, employ either $\mathfrak{L}, \ldots$ or $\mathfrak{L}', \ldots$ as most convenient for the purpose. And in the first place it will be shown that H is a factor of all these expressions. Putting $H=0$, $\mathfrak{L}'$ becomes

$$\left.\begin{array}{llll} \partial_x uX & \partial_x uY & \partial_x uZ & u \quad =0\,; \\ \partial_y uX & \partial_y uY & \partial_y uZ & v \\ \partial_z uX & \partial_z uY & \partial_z uZ & w \\ \Delta uX & \Delta uY & \Delta uZ & \varpi_2 H \end{array}\right\} \quad \ldots\ldots\ldots \quad (43)$$

also

$$\left.\begin{array}{l} \Delta uX=pX+u\Delta X+2H\partial_x X \\ \Delta uY=pY+u\Delta Y+2H\partial_x Y \\ \Delta uZ=pZ+u\Delta Z+2H\partial_x Z\,; \end{array}\right\} \quad \ldots\ldots\ldots \quad (44)$$

so that the above equation, written in full, is

$$\begin{array}{llll} u_1 X+u\partial_x X & u_1 Y+u\partial_x Y & u_1 Z+u\partial_x Z & u \\ w'X+u\partial_y X & w'Y+u\partial_y Y & w'Z+u\partial_y Z & v \\ v'X+u\partial_z X & v'Y+u\partial_z Y & v'Z+u\partial_z Z & w \\ pX+u\Delta X+2H\partial_x X & pY+u\Delta Y+2H\partial_x Y & pZ+u\Delta Z+2H\partial_x Z & \varpi_2 H. \end{array}$$

Although this expression contains terms explicitly multiplied by H, which might on the present supposition be omitted, it will still perhaps be worth while to develope it completely. Expanding in the usual way, it becomes

$$u^2X \begin{array}{llll} u_1 & \partial_x Y & \partial_x Z & u \\ w' & \partial_y Y & \partial_y Z & v \\ v' & \partial_z Y & \partial_z Z & w \\ p & \Delta Y & \Delta Z & \varpi_2 H \end{array} + u^2Y \begin{array}{llll} u_1 & \partial_x Z & \partial_x X & u \\ w' & \partial_y Z & \partial_y X & v \\ v' & \partial_z Z & \partial_z X & w \\ p & \Delta Z & \Delta X & \varpi_2 H \end{array} + u^2Z \begin{array}{llll} u_1 & \partial_x X & \partial_x Y & u \\ w' & \partial_y X & \partial_y Y & v \\ v' & \partial_z X & \partial_z Y & w \\ p & \Delta X & \Delta Y & \varpi_2 H \end{array} + u^3 \begin{array}{llll} \partial_x X & \partial_x Y & \partial_x Z & u \\ \partial_y X & \partial_y Y & \partial_y Z & v \\ \partial_z X & \partial_z Y & \partial_z Z & w \\ \Delta X & \Delta Y & \Delta Z & \varpi_2 H \end{array}$$

$$+H \begin{array}{llll} u_1 X+u\partial_x X & u_1 Y+u\partial_x Y & u_1 Z+u\partial_x Z & u \\ w'X+u\partial_y X & w'Y+u\partial_y Y & w'Z+u\partial_y Z & v \\ v'X+u\partial_z X & v'Y+u\partial_z Y & v'Z+u\partial_z Z & w \\ 2\partial_x X & 2\partial_x Y & 2\partial_x Z & \varpi_2. \end{array}$$

In this the coefficient of $-p$

$$=\tfrac{1}{2}\left\{\begin{array}{lll}\partial_x\mathrm{X} & \mathrm{Z}\partial_x\mathrm{Y}-\mathrm{Y}\partial_x\mathrm{Z} & u\\ \partial_y\mathrm{X} & \mathrm{Z}\partial_y\mathrm{Y}-\mathrm{Y}\partial_y\mathrm{Z} & v\\ \partial_z\mathrm{X} & \mathrm{Z}\partial_z\mathrm{Y}-\mathrm{Y}\partial_z\mathrm{Z} & w\end{array}+\begin{array}{lll}\partial_x\mathrm{Y} & \mathrm{X}\partial_x\mathrm{Z}-\mathrm{Z}\partial_x\mathrm{X} & u\\ \partial_y\mathrm{Y} & \mathrm{X}\partial_y\mathrm{Z}-\mathrm{Z}\partial_y\mathrm{X} & v\\ \partial_z\mathrm{Y} & \mathrm{X}\partial_z\mathrm{Z}-\mathrm{Z}\partial_z\mathrm{X} & w\end{array}+\begin{array}{lll}\partial_x\mathrm{Z} & \mathrm{Y}\partial_x\mathrm{X}-\mathrm{X}\partial_x\mathrm{Y} & u\\ \partial_y\mathrm{Z} & \mathrm{Y}\partial_y\mathrm{X}-\mathrm{X}\partial_y\mathrm{Y} & v\\ \partial_z\mathrm{Z} & \mathrm{Y}\partial_z\mathrm{X}-\mathrm{X}\partial_z\mathrm{Y} & w\end{array}\right\}$$

$$=\tfrac{1}{2}\left\{\begin{array}{lll}p\partial_x\mathrm{X}+q\partial_x\mathrm{Y}+r\partial_x\mathrm{Z} & \mathrm{P} & u\\ p\partial_y\mathrm{X}+q\partial_y\mathrm{Y}+r\partial_y\mathrm{Z} & \mathrm{Q} & v\\ p\partial_z\mathrm{X}+q\partial_z\mathrm{Y}+r\partial_z\mathrm{Z} & \mathrm{R} & w\end{array}+\begin{array}{lll}u\partial_x\mathrm{X}+v\partial_x\mathrm{Y}+w\partial_x\mathrm{Z} & \mathrm{P}_1 & u\\ u\partial_y\mathrm{X}+v\partial_y\mathrm{Y}+w\partial_y\mathrm{Z} & \mathrm{Q}_1 & v\\ u\partial_z\mathrm{X}+v\partial_z\mathrm{Y}+w\partial_z\mathrm{Z} & \mathrm{R}_1 & w\end{array}\right\}=\begin{array}{lll}\mathrm{P}_1 & \mathrm{P} & u\\ \mathrm{Q}_1 & \mathrm{Q} & v\\ \mathrm{R}_1 & \mathrm{R} & w.\end{array}$$

Now

$$\begin{array}{lll}u & u_1 & \mathrm{P}_1\\ v & w' & \mathrm{Q}_1\\ w & v' & \mathrm{R}_1\end{array}=\frac{1}{n-1}\{z(\mathfrak{H}\mathrm{P}_1+\mathfrak{B}\mathrm{Q}_1+\mathfrak{F}\mathrm{R}_1)-y(\mathfrak{G}\mathrm{P}_1+\mathfrak{F}\mathrm{Q}_1+\mathfrak{C}\mathrm{R}_1)\}$$

$$\begin{array}{lll}u & w' & \mathrm{P}_1\\ v & v_1 & \mathrm{Q}_1\\ w & u' & \mathrm{R}_1\end{array}=\frac{1}{n-1}\{x(\mathfrak{G}\mathrm{P}_1+\mathfrak{F}\mathrm{Q}_1+\mathfrak{C}\mathrm{R}_1)-z(\mathfrak{A}\mathrm{P}_1+\mathfrak{H}\mathrm{Q}_1+\mathfrak{G}\mathrm{R}_1)\}$$

$$\begin{array}{lll}u & v' & \mathrm{P}_1\\ v & u' & \mathrm{Q}_1\\ w & w_1 & \mathrm{R}_1\end{array}=\frac{1}{n-1}\{y(\mathfrak{A}\mathrm{P}_1+\mathfrak{H}\mathrm{Q}_1+\mathfrak{G}\mathrm{R}_1)-x(\mathfrak{H}\mathrm{P}_1+\mathfrak{B}\mathrm{Q}_1+\mathfrak{F}\mathrm{R}_1)\};$$

so that multiplying these equations by X, Y, Z respectively, and adding,

$$\left.\begin{aligned}\begin{array}{lll}u & \mathrm{P} & \mathrm{P}_1\\ v & \mathrm{Q} & \mathrm{Q}_1\\ w & \mathrm{R} & \mathrm{R}_1\end{array}&=\frac{1}{n-1}\{(\mathfrak{A}\mathrm{P}_1+\mathfrak{H}\mathrm{Q}_1+\mathfrak{G}\mathrm{R}_1)(y\mathrm{Z}-z\mathrm{Y})\\&\qquad+(\mathfrak{H}\mathrm{P}_1+\mathfrak{B}\mathrm{Q}_1+\mathfrak{F}\mathrm{R}_1)(z\mathrm{X}-x\mathrm{Z})\\&\qquad+(\mathfrak{G}\mathrm{P}_1+\mathfrak{F}\mathrm{Q}_1+\mathfrak{C}\mathrm{R}_1)(x\mathrm{Y}-y\mathrm{X})\}\\&=\frac{3(n-2)}{n-1}\mathrm{H}\{(\mathfrak{A}u+\mathfrak{H}v+\mathfrak{G}w)\mathrm{P}_1+(\mathfrak{H}u+\mathfrak{B}v+\mathfrak{F}w)\mathrm{Q}_1+(\mathfrak{G}u+\mathfrak{F}v+\mathfrak{C}w)\mathrm{R}_1\}\\&=\frac{3(n-2)}{(n-1)^2}\mathrm{H}^2(\mathrm{P}_1x+\mathrm{Q}_1y+\mathrm{R}_1z)\\&=\frac{3(n-2)(3n-7)}{(n-1)^2}\mathrm{H}^2(\mathrm{X}p+\mathrm{Y}q+\mathrm{Z}r)\\&=0.\end{aligned}\right\}\quad(45)$$

Hence the whole expression

$$
=u^2\left\{\begin{vmatrix} u_1 & u & Y\partial_x Z-Z\partial_x Y & \partial_x X \\ w' & v & Y\partial_y Z-Z\partial_y Y & \partial_y X \\ v' & w & Y\partial_z Z-Z\partial_z Y & \partial_z X \\ . & \varpi_2 H & Y\Delta Z-Z\Delta Y & \Delta X \end{vmatrix}+\begin{vmatrix} u_1 & u & Z\partial_x X-X\partial_x Z & \partial_x Y \\ w' & v & Z\partial_y X-X\partial_y Z & \partial_y Y \\ v' & w & Z\partial_z X-X\partial_z Z & \partial_z Y \\ . & \varpi_2 H & Z\Delta X-X\Delta Z & \Delta Y \end{vmatrix}+\begin{vmatrix} u_1 & u & X\partial_x Y-Y\partial_x X & \partial_x Z \\ w' & v & X\partial_y Y-Y\partial_y X & \partial_y Z \\ v' & w & X\partial_z Y-Y\partial_z X & \partial_z Z \\ . & \varpi_2 H & X\Delta Y-Y\Delta X & \Delta Z \end{vmatrix}\right\}
$$

$$
+u^3\begin{vmatrix} \partial_x X & \partial_x Y & \partial_x Z & u \\ \partial_y X & \partial_y Y & \partial_y Z & v \\ \partial_z X & \partial_z Y & \partial_z Z & w \\ \Delta X & \Delta Y & \Delta Z & \varpi_2 H \end{vmatrix};
$$

or in virtue of (33),

$$
=u^2\left\{\begin{vmatrix} u_1 & u & -(pP+uP_1) & \partial_x X \\ w' & v & -(pQ+uQ_1) & \partial_y X \\ v' & w & -(pR+uR_1) & \partial_z X \\ . & \varpi_2 H & u\,\mathrm{Jac.}(U,H,\Omega_U) & \Delta X \end{vmatrix}+\begin{vmatrix} u_1 & u & -(qP+vP_1) & \partial_x Y \\ w' & v & -(qQ+vQ_1) & \partial_y Y \\ v' & w & -(qR+vR_1) & \partial_z Y \\ . & \varpi_2 H & v\,\mathrm{Jac.}(U,H,\Omega_U) & \Delta Y \end{vmatrix}+\begin{vmatrix} u_1 & u & -(rP+wP_1) & \partial_x Z \\ w' & v & -(rQ+wQ_1) & \partial_y Z \\ v' & w & -(rR+wR_1) & \partial_z Z \\ . & \varpi_2 H & w\,\mathrm{Jac.}(U,H,\Omega_U) & \Delta Z \end{vmatrix}\right\}
$$

$$
+u^3\begin{vmatrix} \partial_x X & \partial_x Y & \partial_x Z & u \\ \partial_y X & \partial_y Y & \partial_y Z & v \\ \partial_z X & \partial_z Y & \partial_z Z & w \\ \Delta X & \Delta Y & \Delta Z & \varpi_2 H \end{vmatrix}
$$

$$
=2u^2\varpi_2 H\begin{vmatrix} u_1 & P_1 & P \\ w' & Q_1 & Q \\ v' & R_1 & R \end{vmatrix}+u^2\,\mathrm{Jac.}(U,H,\Omega_U)\begin{vmatrix} u_1 & u & P \\ w' & v & Q \\ v' & w & R \end{vmatrix}+u^2\,\mathrm{Jac.}(U,H,\Omega_U)\begin{vmatrix} u_1 & u & P \\ w' & v & Q \\ v' & w & R \end{vmatrix}+u^3\begin{vmatrix} \partial_x X & \partial_x Y & \partial_x Z & u \\ \partial_y X & \partial_y Y & \partial_y Z & v \\ \partial_z X & \partial_z Y & \partial_z Z & w \\ \Delta X & \Delta Y & \Delta Z & \varpi_2 H \end{vmatrix}.
$$

But

$$
\begin{vmatrix} u_1 & P_1 & P \\ w' & Q_1 & Q \\ v' & R_1 & R \end{vmatrix}=Z(\mathfrak{H}P_1+\mathfrak{B}Q_1+\mathfrak{F}R_1)-Y(\mathfrak{G}P_1+\mathfrak{F}Q_1+\mathfrak{C}R_1)
$$

$$
=u(\mathfrak{A}\ \mathfrak{B}\ \mathfrak{C}\ \mathfrak{F}\ \mathfrak{G}\ \mathfrak{H})(p\ q\ r)(P_1Q_1R_1)-p(\mathfrak{A}\ \mathfrak{B}\ \mathfrak{C}\ \mathfrak{F}\ \mathfrak{G}\ \mathfrak{H})(u\ v\ w)(P_1Q_1R_1)
$$

$$
=u(\mathfrak{A}\ \mathfrak{B}\ \mathfrak{C}\ \mathfrak{F}\ \mathfrak{G}\ \mathfrak{H})(p\ q\ r)(P_1Q_1R_1),
$$

since

$$
(\mathfrak{A}\ \mathfrak{B}\ \mathfrak{C}\ \mathfrak{F}\ \mathfrak{G}\ \mathfrak{H})(u,\ v,\ w)P_1Q_1R_1)=\frac{1}{n-1}H(P_1x+Q_1y+R_1z)=0,
$$

also

$$
\begin{aligned}
(\mathfrak{A}\ \mathfrak{B}\ \mathfrak{C}\ \mathfrak{F}\ \mathfrak{G}\ \mathfrak{H})(P_1Q_1R_1) &= (\mathfrak{A}\,..)(p\ q\ r)(p_1\ r'\ q')X \\
&\quad +(\mathfrak{A}\,..)(p\ q\ r)(r'\ q_1\ p')Y \\
&\quad +(\mathfrak{A}\,..)(p\ q\ r)(q'\ p'\ r_1)Z \\
&= \mathrm{Jac.}(U,\ H,\ \Psi_U).
\end{aligned}
$$

Hence the whole expression above written

$$=2u^3\left\{\varpi_2\,\text{Jac.}\,(\text{U, H, }\Psi_{\text{U}})+\frac{3(n-2)}{n-1}\,\text{H Jac.}\,(\text{U, H, }\Omega_{\text{U}})\right\}\text{H}+u^3\begin{vmatrix}\partial_x\text{X}\,..\,u\\ \partial_y\text{X}\,..\,v\\ \cdot\quad\cdot\end{vmatrix}.$$

But

$$\begin{vmatrix}\partial_x\text{X} & \partial_x\text{Y} & \partial_x\text{Z} & u\\ \partial_y\text{X} & \partial_y\text{Y} & \partial_y\text{Z} & v\\ \partial_z\text{X} & \partial_z\text{Y} & \partial_z\text{Z} & w\\ \Delta\text{X} & \Delta\text{Y} & \Delta\text{Z} & \end{vmatrix}=-\frac{12(n-2)^2}{(n-1)^2}\,\text{H}^2\,\text{Jac.}\,(\text{U, H, }\Omega_{\text{U}}),$$

and

$$\begin{vmatrix}\partial_x\text{X} & \partial_x\text{Y} & \partial_x\text{Z}\\ \partial_y\text{X} & \partial_y\text{Y} & \partial_y\text{Z}\\ \partial_z\text{X} & \partial_z\text{Y} & \partial_z\text{Z}\end{vmatrix}=-\,\text{Jac.}\,(\text{U, H, }\Psi_{\text{U}})-\text{Jac.}\,(\text{U, H, }\Theta_{\text{U}}).$$

Hence, finally, the whole expression

$$\left.\begin{aligned}&=u^3\text{H}\left\{\varpi_2\,\text{Jac.}\,(\text{U, H, }\Psi_{\text{U}})+\left(\frac{6(n-2)}{n-1}-\frac{12(n-2)^2}{(n-1)^2}\right)\text{H Jac.}\,(\text{U, H, }\Omega)-\varpi_2\,\text{Jac.}\,(\text{U, H, }\Theta_{\text{U}})\right\}\\ &=u^3\text{H}\left\{\varpi_2\Big(\text{Jac.}\,(\text{U, H, }\Psi_{\text{U}})-\text{Jac.}\,(\text{U, H, }\Theta_{\text{U}})\Big)-\frac{6(n-2)(n-3)}{(n-1)^2}\,\text{H Jac.}\,(\text{U, H, }\Omega_{\text{U}})\right\},\end{aligned}\right\}\quad(46)$$

which is therefore divisible by $\text{H}u^3$. Consequently H is a factor of all the expressions $\mathfrak{L}_1\,..\,\mathfrak{L}'\,..$, which was to be proved.

Although not absolutely necessary to our argument, it is perhaps worth while to show, as may readily be done, that $\mathfrak{L}$ is divisible by u. Omitting the terms explicitly multiplied by u in the first three columns, the equation becomes

$$\left.\begin{vmatrix}u_1\text{X}-\partial_x x\text{P} & u_1\text{Y}-\partial_x y\text{P} & u_1\text{Z}-\partial_x z\text{P} & u\\ w'\text{X}-\partial_y x\text{P} & w'\text{Y}-\partial_y y\text{P} & w'\text{Z}-\partial_y z\text{P} & v\\ v'\,\text{X}-\partial_z x\text{P} & v'\,\text{Y}-\partial_z y\text{P} & v'\,\text{Z}-\partial_z z\text{P} & w\\ p\,\text{X}-\Delta x\text{P}+2\text{H}\partial_x\text{X} & p\,\text{Y}-\Delta y\text{P}+2\text{H}\partial_x\text{Y} & p\,\text{Z}-\Delta z\text{P}+2\text{H}\partial_x\text{Z} & \varpi_z\text{H}\end{vmatrix}=0.\right\}\quad(47)$$

In this the coefficient of ϖ_2H,

$$\text{P}=(\text{Y}z-\text{Z}y)\begin{vmatrix}w'\partial_y\text{P}\\ v'\partial_z\,\text{P}\end{vmatrix}+\text{P}(\text{Z}x-\text{X}z)\begin{vmatrix}v'\partial_z\text{P}\\ u_1\partial_x\text{P}\end{vmatrix}+\text{P}(\text{X}y-\text{Y}x)\begin{vmatrix}u_1\partial_x\text{P}\\ w'\partial_y\text{P}\end{vmatrix}$$

$$+\text{P}^2(u_1\text{X}+w'\text{Y}+v'\text{Z})-\text{P}^2(x\partial_x\text{P}+y\partial_y\text{P}+z\partial_z\text{P})-\text{P}^3,$$

which, writing

$$\text{K}=\begin{vmatrix}u & u_1 & \partial_x\text{P}\\ v & w' & \partial_y\text{P}\\ w & v' & \partial_z\text{P}\end{vmatrix}$$

$$=-(n-2)(3\text{HK}+5\text{P}^2)\text{P}.$$

Similarly, it will be found that the coefficients of

$$\begin{gathered}(pX-\Delta xP+2H\partial_x X)\\(pY-\Delta yP+2H\partial_x Y)\\(pZ-\Delta z P+2H\partial_x Z)\end{gathered}$$

are

$$\begin{gathered}-(n-2)(3HK+5P^2)u,\\-(n-2)(3HK+5P^2)v,\\-(n-2)(3HK+5P^2)w\end{gathered}$$

respectively; and consequently the whole expression

$$\begin{aligned}&=-(n-2)(3HK+5P^2)\{(pX-\Delta xP+2H\partial_x X)u\\&\qquad\qquad+(pY-\Delta yP+2H\partial_x Y)v\\&\qquad\qquad+(pZ-\Delta z P+2H\partial_x Z)w+\varpi_2 HP\}\\&=-(n-2)(3HK+5P^2)\{-2HP-2(\mathfrak{A}\ldots)(u,v,w)(\partial_x P,\partial_y P,\partial_z P)+\varpi_2 HP\}\\&=-(n-2)(3HK+5P^2)\left\{-2-\frac{10(n-2)}{n-1}+\varpi_2\right\}HP.\end{aligned}$$

But $\varpi_2=1+\frac{10(n-2)}{n-1}$, so that the above expression

$$=(n-2)(3HK+P^2)HP.$$

Now

$$\left.\begin{aligned}-(n-2)(3HK+P^2)&=\begin{array}{llll}u & v & w & nU\\ u_1 & w' & v' & (n-1)(u-u)\\ p & q & r & 3(n-2)H\\ \partial_x P & \partial_y P & \partial_z & 5(n-2)P\end{array}\\&=\begin{array}{llll}u & u & w & xu\ +yv\ +zw\\ u_1 & w' & v' & xu_1+yw'+zv'\quad-(n-1)u\\ p & q & r & xp\ +yq\ +zr\\ \partial_x P & \partial_y P & \partial_z P & x\partial_x P+y\partial_y P+z\partial_z P\end{array}\\&=-(n-1)u\ \begin{array}{lll}u & p & \partial_x P\\ v & q & \partial_y P\\ w & r & \partial_z P;\end{array}\end{aligned}\right\}\quad . \quad (48)$$

so that the whole expression is divisible by u. Similarly, it might be shown that M_1 or M' is divisible by v, and N or N' by w.

It follows from what has gone before that $\mathfrak{L}$, $\mathfrak{M}$, $\mathfrak{N}$, $\mathfrak{L}'$, $\mathfrak{M}'$, $\mathfrak{N}'$ are all divisible by H, that $\mathfrak{L}$, $\mathfrak{L}'$ are divisible by u, $\mathfrak{M}$, $\mathfrak{M}'$ by v, $\mathfrak{N}$, $\mathfrak{N}'$ by w, and consequently dividing

out those factors, the three expressions $\mathfrak{L}$, $\mathfrak{M}$, $\mathfrak{N}$ are of the form

$$\left.\begin{aligned} Au^2+B_1u+C_1=0,\\ Av^2+B_2v+C_2=0,\\ Aw^2+B_3w+C_3=0, \end{aligned}\right\} \qquad (49)$$

in which the coefficients of u^2, v^2, w^2 are the same, viz. the expressions given in (46). From these equations it follows that

$$\frac{B_1u+C_1}{u^2}=\frac{B_2v+C_2}{v^2}=\frac{B_3w+C_3}{w^2}. \qquad (50)$$

But as u, v, w do not in general vanish simultaneously, these relations can hold good only in virtue of B_1 being divisible by u, and C_1 by u^2; B_2 by v, and C_2 by v^2; B_3 by w and C_3 by w^2. Whence, finally, $\mathfrak{L}$ is divisible by Hu^3, $\mathfrak{M}$ by Hv^3, $\mathfrak{N}$ by Hw^3; and the degree of the equation is reduced to

$$(18n-36)-3(n-2)-3(n-1)=12n-27.$$

Also, since the ratios $(B_1u+C_1):u^2$, $(B_2v+C_2):v^2$, $(B_3w+C_3):w^2$ are in virtue of (50) equal (say $=B$), it follows that $\mathfrak{L}$, $\mathfrak{M}$, $\mathfrak{N}$, $\mathfrak{L}'$, $\mathfrak{M}'$, $\mathfrak{N}'$ all lead to the same result, viz. $A+B=0$, which it was our object to prove.