XX. *On the Secular Changes in the Elements of the Orbit of a Satellite revolving about a Tidally distorted Planet.*

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Introduction.

THE following paper treats of the effects of frictional tides in a planet on the orbit of its satellite. It is the sequel to three previous papers on a similar subject.*

The investigation has proved to be one of unexpected complexity, and this must be my apology for the great length of the present paper. This was in part due to the fact that it was found impossible to consider adequately the changes in the orbit of the satellite, without a reconsideration of the parallel changes in the planet. Thus some of the ground covered in the previous paper on "Precession" had to be retraversed; but as the methods here employed are quite different from those used before, this repetition has not been without some advantage.

* "On the Bodily Tides of Viscous and Semi-elastic Spheroids, and on the Ocean Tides upon a Yielding Nucleus," Phil. Trans., Part I., 1879.

"On the Precession of a Viscous Spheroid, and on the remote History of the Earth," Phil. Trans., Part II., 1879.

"On Problems connected with the Tides of a Viscous Spheroid," Phil. Trans., Part II., 1879.

These papers are hereafter referred to as "Tides," "Precession," and "Problems" respectively.

There is also a fourth paper, treating the subject from a different point of view, viz.: "The Determination of the Secular Effects of Tidal Friction by a Graphical Method," Proc. Roy. Soc., No. 197, 1879. And lastly a fifth paper of more recent date, "On the Analytical Expressions which give the History of a Fluid Planet of Small Viscosity, attended by a Single Satellite," Proc. Roy. Soc., No. 202, 1880.

It will probably conduce to the intelligibility of what follows, if an explanatory outline of the contents of the paper is placed before the reader. Such an outline must of course contain references to future procedure, and cannot therefore be made entirely intelligible, yet it appears to me that some sort of preliminary notions of the nature of the subject will be advantageous, because it is sometimes difficult for a reader to retain the thread of the argument amidst the mass of details of a long investigation, which is leading him in some unknown direction.

Part VIII. contains a general review of the subject in its application to the evolution of the planets of the solar system. This is probably the only part of the paper which will have any interest to the general reader.

The mathematical reader, who merely wishes to obtain a general idea of the results, is recommended to glance through the present introduction, and then to turn to Part VII., which contains a summary, with references to such parts of the paper as it was not desirable to reproduce. This summary does not contain any analysis, and deals more especially with the physical aspects of the problem, and with the question of the applicability of the investigation to the history of the earth and moon, but of course it must not be understood to contain references to every point which seems to be worthy of notice. I think also that a study of Part VII. will facilitate the comprehension of the analytical parts of the paper.

Part I. contains an explanation of the peculiarities of the method of the disturbing function as applied to the tidal problem. At the beginning there is a summary of the meaning to be attached to the principal symbols employed. The problem is divided into several heads, and the disturbing function is partially developed in such a way that it may be applicable either to finding the perturbations of the satellite, or of the planet itself.

In Part II. the satellite is supposed to move in a circular orbit, inclined to the fixed plane of reference. It here appears that the problem may be advantageously subdivided into the following cases: 1st, where the permanent oblateness of the planet is small, and where the satellite is directly perturbed by the action of a second large and distant satellite such as the sun; 2nd, where the planet and satellite are the only two bodies in existence; 3rd, where the permanent oblateness is considerable, and the action of the second satellite is not so important as in the first case. The first and second of these cases afford the subject for the rest of this part, and the laws are found which govern the secular changes in the inclination and mean distance of the satellite, and the obliquity and diurnal rotation of the planet.

Part III. is devoted to the third of the above cases. It was found necessary first to investigate the motion of a satellite revolving about a rigid oblate spheroidal planet, and perturbed by a second satellite. Here I had to introduce the conception of a pair of planes, to which the motions of the satellite and planet may be referred. The problem of the third case is then shown to resolve itself into a tracing of the secular changes in the positions of these two "proper" planes, under the influence of tidal

friction. After a long analytical investigation differential equations are found for the rate of these changes.

Part IV. contains the numerical integration of the differential equations of Parts II. and III., in application to the case of the earth, moon, and sun, the earth being supposed to be viscous.

Part V. contains the investigation of the secular changes of the eccentricity of the orbit of a satellite, together with the corresponding changes in the planet's mode of motion.

Part VI. contains a numerical integration of the equations of Part V. in the case of the earth and moon. The objects of Parts VII. and VIII. have been already explained.

In the abstract of this paper in the Proceedings of the Royal Society,* certain general considerations are adduced which throw light on the nature of the results here found. This general reasoning is not reproduced here, because it is incapable of leading us to definite results, and it was only used there as a substitute for analysis.

I.

THE THEORY OF THE DISTURBING FUNCTION.

§ 1. *Preliminary considerations.*

In the theory of disturbed elliptic motion the six elements of the orbit may be divided into two groups of three.

One set of three gives a description of the nature of the orbit which is being described at any epoch, and the second set is required to determine the position of the body at any instant of time. In a speculative inquiry like the present one, where we are only concerned with very small inequalities which would have no interest unless their effects could be cumulative from age to age, so that the orbit might become materially changed, it is obvious that the secular changes in the second set of elements need not be considered.

The three elements whose variations are not here found are the longitudes of the perigee, the node, and the epoch; but the subsequent investigation will afford the materials for finding their variations if it be desirable to do so.

The first set of elements whose secular changes are to be traced are, according to the ordinary system, the mean distance, the eccentricity, and the inclination of the orbit. We shall, however, substitute for the two former elements, viz.: mean distance and eccentricity, two other functions which define the orbit equally well; the first of these is a quantity proportional to the square root of the mean distance, and the second is the ellipticity of the orbit. The inclination will be retained as the third element.

The principal problem to be solved is as follows:—

* No. 200, 1879.

A planet is attended by one or more satellites which raise frictional tides (either bodily or oceanic) in their planet; it is required to find the secular changes in the orbits of the satellites due to tidal reaction.

This problem is however intimately related to a consideration of the parallel changes in the inclination of the planet's axis to a fixed plane, and in its diurnal rotation.

It will therefore be necessary to traverse again, to some extent, the ground covered by my previous paper "On the Precession of a Viscous Spheroid."

In the following investigation the tides are supposed to be a bodily deformation of the planet, but a slight modification of the analytical results would make the whole applicable to the case of oceanic tides on a rigid nucleus.* The analysis will be such that the results may be applied to any theory of tides, but particular application will be made to the case where the planet is a homogeneous viscous spheroid, and the present paper is thus a continuation of my previous ones on the tides and rotation of such a spheroid.

The general problem above stated may be conveniently divided into two:—

First, to find the secular changes in mean distance and inclination of the orbit of a satellite moving in a circular orbit about its planet.

Second, to find the secular change in mean distance, and eccentricity of the orbit of a satellite moving in an elliptic orbit, but always remaining in a fixed plane.

As stated in the introductory remarks, it will also be necessary to investigate the secular changes in the diurnal rotation and in the obliquity of the planet's equator to the plane of reference.

The tidally distorted planet will be spoken of as the earth, and the satellites as the moon and sun.

This not only affords a useful vocabulary, but permits an easy transition from questions of abstract dynamics to speculations concerning the remote history of the earth and moon.

§ 2. *Notation—Equation of variation of elements.*

The present section, and the two which follow it, are of general applicability to the whole investigation.

For reasons which will appear later it will be necessary to conceive the earth to have two satellites, which may conveniently be called Diana and the moon. The following are the definitions of the symbols employed.

The time is t, and the suffix 0 to any symbol indicates the value of the corresponding quantity initially, when $t=0$. The attraction of unit masses at unit distance is μ.

For the earth, let—

$M=$ mass in ordinary units; $a=$ mean radius; $w=$ density, or mass per unit volume, the earth being treated as homogeneous; $g=$ mean gravity; $\mathfrak{g}=\frac{2}{5}g/a$;

* Or, as to Part III., on a nucleus which is sufficiently plastic to adjust itself to a form of equilibrium.

C, A= the greatest and least moments of inertia of the earth; if we neglect the ellipticity they will be equal to $\frac{2}{5}Ma^2$; $n=$ angular velocity of diurnal rotation; $\psi=$ longitude of autumnal equinox measured along the ecliptic from a fixed point in the ecliptic—the ecliptic being here a name for a plane fixed in space; $i=$ obliquity of ecliptic; χ the angle between a point fixed on the equator and the autumnal equinox; ρ the radius vector of any point measured from the earth's centre.

For Diana, let—

$c=$ mean distance; $\xi=(c/c_0)^{\frac{1}{2}}$; $\Omega=$ mean motion; e= eccentricity of orbit; $\eta=$ ellipticity of orbit; $\varpi=$ longitude of perigee; $j=$ inclination of orbit to ecliptic; $N=$ longitude of node; $\epsilon=$ longitude of epoch; $m=$ mass; $\nu=$ ratio of earth's mass to Diana's or M/m; $l=$ true longitude measured from the node; $\theta=$ true longitude measured from the autumnal equinox; $\tau=\frac{3}{2}\mu m/c^3$, so that $\tau=3\Omega^2/2(1+\nu)$, also $\tau=\tau_0/\xi^6$; r the radius vector measured from earth's centre.

Also $\lambda=\Omega/n$; $\mathfrak{m}$ the ratio of the earth's moment of momentum of rotation to that of the orbital motion of Diana (or the moon) and the earth round their common centre of inertia.

For the moon let all the same symbols apply when accents are added to them.

Where occasion arises to refer merely to the elements of a satellite in general, the unaccented symbols will be employed.

Let R be the disturbing function as ordinarily defined in works on physical astronomy.

Other symbols will be defined as the necessity for them arises.

Then the following are the well-known equations for the variation of the mean distance, eccentricity, inclination, and longitude of the node.

$$\frac{dc}{dt}=\frac{2\Omega c^2}{\mu(M+m)}\frac{d\mathrm{R}}{d\epsilon} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$\frac{d\mathrm{e}}{dt}=\frac{\Omega c}{\mu(M+m)}\left[\frac{1-\mathrm{e}^2}{\mathrm{e}}\frac{d\mathrm{R}}{d\epsilon}-\frac{\sqrt{1-\mathrm{e}^2}}{\mathrm{e}}\left(\frac{d\mathrm{R}}{d\epsilon}+\frac{d\mathrm{R}}{d\varpi}\right)\right] \quad . \quad . \quad . \quad . \quad (2)$$

$$-\frac{dj}{dt}=\frac{\Omega c}{\mu(M+m)}\frac{1}{\sqrt{1-\mathrm{e}^2}}\left[\frac{1}{\sin j}\frac{d\mathrm{R}}{dN}+\tan\tfrac{1}{2}j\left(\frac{d\mathrm{R}}{d\epsilon}+\frac{d\mathrm{R}}{d\varpi}\right)\right] \quad . \quad . \quad . \quad . \quad (3)$$

$$\sin j\,\frac{dN}{dt}=\frac{\Omega c}{\mu(M+m)}\frac{1}{\sqrt{1-\mathrm{e}^2}}\frac{d\mathrm{R}}{dj} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

The last of these equations will only be required in Part III.

Now let $\mathrm{R}=\mathrm{WC}(M+m)/Mm$; then if we substitute this value for R in each of the equations (1–4), it is clear that the right hand side of each will involve a factor $\Omega c\mathrm{C}/\mu Mm$.

Then let

$$k=\frac{\mathrm{C}}{\mu Mm}\Omega_0 c_0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \tag{5}$$

(For a homogeneous earth $\frac{\mathrm{C}}{\mu Mm}=\frac{2\nu}{5g}$, and $\Omega_0 c_0=\left[(ga^2)\frac{1+\nu}{\nu}\right]^{\frac{1}{3}}\Omega_0^{\frac{1}{3}}$. Thus if we put

$$s=\tfrac{2}{5}\left[\left(\frac{a\nu}{g}\right)^2(1+\nu)\right]^{\frac{1}{3}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \tag{6}$$

$$k=s\Omega_0^{\frac{1}{3}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \tag{7}$$

$s^{\frac{3}{4}}$ is a time, being about $3^{\text{hrs.}}\ 4\tfrac{1}{2}^{\text{min.}}$ for the homogeneous earth. k is also a time, being about 57 minutes, with the present orbital angular velocity of the moon, and the earth being homogeneous).

Then since $\Omega=\Omega_0\xi^{-3}$, $c=c_0\xi^2$, therefore

$$\frac{\mathrm{C}}{\mu Mm}\Omega c=\frac{k}{\xi} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \tag{8}$$

Again, since $(c/c_0)^{\frac{1}{2}}=\xi$, therefore

$$\frac{1}{c}\frac{dc}{dt}=\frac{2}{\xi}\frac{d\xi}{dt} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \tag{9}$$

and since $\eta=1-\sqrt{1-\mathrm{e}^2}$, therefore

$$\frac{d\eta}{dt}=\frac{\mathrm{e}}{\sqrt{1-\mathrm{e}^2}}\frac{d\mathrm{e}}{dt} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \tag{10}$$

Then substituting for R in terms of W in the four equations (1–4), and using the transformations (8–10), we get,

$$\frac{d\xi}{dt}=k\frac{d\mathrm{W}}{d\epsilon} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \tag{11}$$

$$\frac{d\eta}{dt}=-\frac{k}{\xi}\left(\eta\frac{d\mathrm{W}}{d\epsilon}+\frac{d\mathrm{W}}{d\varpi}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \tag{12}$$

and if the orbit be circular, so that $\mathrm{e}=0$, $d\mathrm{W}/d\varpi=0$,

$$-\frac{dj}{dt}=\frac{k}{\xi}\left(\frac{1}{\sin j}\frac{d\mathrm{W}}{dN}+\tan\tfrac{1}{2}j\frac{d\mathrm{W}}{d\epsilon}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \tag{13}$$

$$\sin j\frac{dN}{dt}=\frac{k}{\xi}\frac{d\mathrm{W}}{dj} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \tag{14}$$

These are the equations of variation of elements which will be used below. The last two (13) and (14) will only be required in the case where the orbit is circular.

The function W only differs from the ordinary disturbing function by a constant factor, and so W will be referred to as the disturbing function.

I will now explain why it has been convenient to depart from ordinary usage, and will show how the same disturbing function W may be used for giving the perturbations of the rotation of the planet.

In the present problem all the perturbations, both of satellites and planet, arise from tides raised in the planet.

The only case treated will be where the tidal wave is expressible as a surface spherical harmonic of the second order.

Suppose then that $\rho=a+\sigma$ is the equation to the wave surface, superposed on the sphere of mean radius a.

Then the potential V of the wave σ, at an external point ρ, must be given by

$$V=\tfrac{4}{5}\pi\mu wa\left(\frac{a}{\rho}\right)^3\sigma \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

Here w is the density of the matter forming the wave; in our case of a homogeneous earth, distorted by bodily tides, w is the mean density of the earth. (If we contemplate oceanic tides, the subsequent results for the disturbing function must be reduced by the factor $\frac{2}{11}$, this being the ratio of the density of water to the mean density of the earth.)

Now suppose the external point ρ to be at a satellite whose mass, radius vector, and mean distance are m, r, c. Then if we put $\tau=\frac{3}{2}\mu m/c^3$, and observe that $C=\frac{8}{15}\pi wa^5$, we have

$$V=\frac{C}{m}\tau\left(\frac{c}{r}\right)^3\frac{\sigma}{a} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

Where σ is the height of tide, where the wave surface is pierced by the satellite's radius vector.

But the ordinary disturbing function R for this satellite is this potential V augmented by the factor $(M+m)/M$, because the planet must be reduced to rest. Hence our disturbing function

$$W=\tau\left(\frac{c}{r}\right)^3\frac{\sigma}{a} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

where σ is the height of tide at the place where the wave surface is pierced by r.

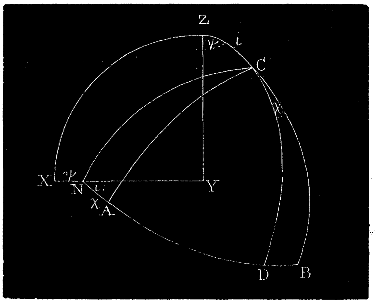
Now let us turn to the case of the planet as perturbed by the attraction of the same satellite on the same wave surface. The whole force function of the action of the satellite on the planet is, by (16), clearly equal to

$$m\left[\frac{M}{c}+\frac{C}{m}\tau\left(\frac{c}{r}\right)^3\frac{\sigma}{a}\right]$$

The latter term of this expression will give the perturbing couples; it is equal to C W.

In the accompanying fig. 1 let X Y Z be axes fixed in space, and (adopting the phraseology for the case of the earth) let X Y be the ecliptic; let A B C be axes fixed in the planet; let χ be the angle A N or B C D; i the obliquity of the ecliptic; ψ the longitude of the autumnal equinox from the fixed point X in the ecliptic.

Fig. 1.



Now suppose W to be expressed in terms of χ, i, ψ.

Then the perturbing couples, which act on the planet, are

$$C\frac{dW}{di} \text{ about N, tending to increase } i.$$

$$C\frac{dW}{d\psi} \text{ about Z, tending to increase } \psi.$$

$$C\frac{dW}{d\chi} \text{ about C, tending to increase } \chi.$$

Now let $\mathfrak{L}$, $\mathfrak{M}$, $\mathfrak{N}$ be the perturbing couples acting about A, B, C respectively. Then must

$$C\frac{dW}{d\psi}=-\mathfrak{L}\sin i\sin\chi-\mathfrak{M}\sin i\cos\chi+\mathfrak{N}\cos i$$

$$C\frac{dW}{di}=-\mathfrak{L}\cos\chi+\mathfrak{M}\sin\chi$$

$$C\frac{dW}{d\chi}=\mathfrak{N}$$

Whence

$$\frac{\mathfrak{L}}{C}=\frac{1}{\sin i}\left(\cos i\frac{dW}{d\chi}-\frac{dW}{d\psi}\right)\sin\chi-\frac{dW}{di}\cos\chi$$

$$\frac{\mathfrak{M}}{C}=\frac{1}{\sin i}\left(\cos i\frac{dW}{d\chi}-\frac{dW}{d\psi}\right)\cos\chi+\frac{dW}{di}\sin\chi$$

$$\frac{\mathfrak{N}}{C}=\frac{dW}{d\chi}$$

But if ω_1, ω_2, ω_3 be the component angular velocities of the planet about A, B, C respectively, and if we may neglect $(C-A)/A$ compared with unity, the equations of motion may be written

$$\frac{d\omega_1}{dt}=\frac{\mathfrak{L}}{C},\ \frac{d\omega_2}{dt}=\frac{\mathfrak{M}}{C},\ \frac{d\omega_3}{dt}=\frac{\mathfrak{N}}{C}$$

as was shown in section (6) of my previous paper on "Precession."

Then since $\chi=nt$, we have by integration,

$$\omega_1=-\frac{1}{n\sin i}\left(\cos i\frac{dW}{d\chi}-\frac{dW}{d\psi}\right)\cos\chi-\frac{1}{n}\frac{dW}{di}\sin\chi$$

$$\omega_2=\frac{1}{n\sin i}\left(\cos i\frac{dW}{d\chi}-\frac{dW}{d\psi}\right)\sin\chi-\frac{1}{n}\frac{dW}{di}\cos\chi$$

Then substituting these values in the geometrical equations,

$$\frac{di}{dt}=-\omega_1\cos\chi+\omega_2\sin\chi$$

$$\sin i\frac{d\psi}{dt}=-\omega_1\sin\chi-\omega_2\cos\chi$$

We have finally,

$$\left.\begin{aligned} n\sin i\frac{di}{dt}&=\cos i\frac{dW}{d\chi}-\frac{dW}{d\psi}\\ n\sin i\frac{d\psi}{dt}&=\frac{dW}{di}\\ \frac{dn}{dt}&=\frac{dW}{d\chi}\end{aligned}\right\}\quad\ldots\ldots\ldots\quad(18)$$

These are the equations which will be used for determining the perturbations of the planet's rotation.

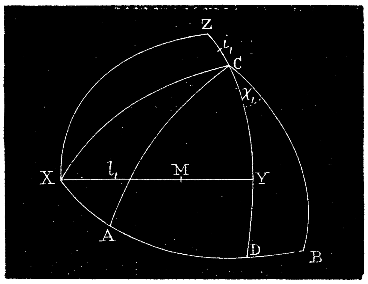
We now see that the same disturbing function W will serve for finding both sets of perturbations.

It is clear that it is not necessary in the above investigation that σ should actually be a tide wave; it may just as well refer to the permanent oblateness of the planet. Thus the ordinary precession and nutations may be determined from these formulas.

§ 3. *To find spherical harmonic functions of Diana's coordinates with reference to axes fixed in the earth.*

Let A, B, C be rectangular axes fixed in the earth, C being the pole and AB the equator.

Fig. 2.



Let X, Y, Z be a second set of rectangular axes, XY being the plane of Diana's orbit.

Let M be the projection of Diana in her orbit.

Let $i_{,}$=ZC, the obliquity of the equator to the plane of Diana's orbit.

$\chi_{,}$=AX=BCY.

$l_{,}$=MX, Diana's longitude from the node X.

Let $\left.\begin{aligned} M_1 &= \cos MA \\ M_2 &= \cos MB \\ M_3 &= \cos MC \end{aligned}\right\}$ Diana's direction-cosines referred to A, B, C.

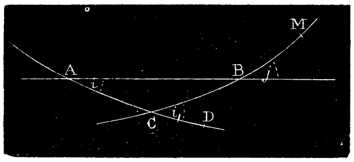
Then

$$\left.\begin{aligned} M_1 &= \cos l_{,} \cos \chi_{,} + \sin l_{,} \sin \chi_{,} \cos i_{,} \\ M_2 &= -\cos l_{,} \sin \chi_{,} + \sin l_{,} \cos \chi_{,} \cos i_{,} \\ M_3 &= \sin l_{,} \sin i_{,} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

We may observe that M_2 is derivable from M_1 by writing $\chi_{,}+\frac{1}{2}\pi$ in place of $\chi_{,}$.

These expressions refer to the plane of Diana's orbit, but we must now refer to the ecliptic.

Fig. 3.



In fig. 3, let A be the autumnal equinox, B the ascending node of the orbit, C the intersection of the orbit with the equator, being the X of fig. 2, and let D be a point fixed in the equator, being the A of fig. 2.

Then if we refer to the sides and angles of the spherical triangle A B C by the letters a, b, c, A, B, C as is usual in works on spherical trigonometry, we have

A=i, the obliquity of the ecliptic.

B=j, the inclination of the orbit.

$\pi-$ C=$i_{,}$=ZC of fig. 2.

c =N, the longitude of the node measured from A, for at present we may suppose $\psi=0$, without loss of generality.

Then let $\chi = \mathrm{DA}$, and we have

$$\chi - \mathrm{b} = \mathrm{DC} = \chi_{,}$$

Again, if M be Diana in her orbit, $\mathrm{MB} = l$, and since $\mathrm{MC} = l_{,}$, therefore

$$l + \mathrm{a} = l_{,}$$

Whence

$$\begin{aligned} \cos\chi_{,} &= \cos\chi \cos\mathrm{b} + \sin\chi \sin\mathrm{b} \\ \sin\chi_{,} &= \sin\chi \cos\mathrm{b} - \cos\chi \sin\mathrm{b} \\ \cos l_{,} &= \cos l \cos\mathrm{a} - \sin l \sin\mathrm{a} \\ \sin l_{,} &= \sin l \cos\mathrm{a} + \cos l \sin\mathrm{a} \end{aligned}$$

Substituting these values in the first of (19) we have

$$\begin{aligned} \mathrm{M}_1 = {} & \cos\chi \cos l\,(\cos\mathrm{a}\cos\mathrm{b} - \sin\mathrm{a}\sin\mathrm{b}\cos i_{,}) + \sin\chi \cos l\,(\cos\mathrm{a}\sin\mathrm{b} + \sin\mathrm{a}\cos\mathrm{b}\cos i_{,}) \\ & - \cos\chi \sin l\,(\sin\mathrm{a}\cos\mathrm{b} + \cos\mathrm{a}\sin\mathrm{b}\cos i_{,}) - \sin\chi \sin l\,(\sin\mathrm{a}\sin\mathrm{b} - \cos\mathrm{a}\cos\mathrm{b}\cos i_{,}) \end{aligned}$$

Now $\cos i_{,} = -\cos\mathrm{C}$, and

$$\begin{aligned} \cos\mathrm{a}\cos\mathrm{b} + \sin\mathrm{a}\sin\mathrm{b}\cos\mathrm{C} &= \cos\mathrm{c} = \cos N \\ \cos\mathrm{a}\sin\mathrm{b} - \sin\mathrm{a}\cos\mathrm{b}\cos\mathrm{C} &= \sin\mathrm{a}\,[\cot\mathrm{a}\sin\mathrm{b} - \cos\mathrm{b}\cos\mathrm{C}] = \sin\mathrm{a}\cot\mathrm{A}\sin\mathrm{C} \\ &= \cos i \sin N \\ \sin\mathrm{a}\cos\mathrm{b} - \cos\mathrm{a}\sin\mathrm{b}\cos\mathrm{C} &= \sin\mathrm{b}\,[\cot\mathrm{b}\sin\mathrm{a} - \cos\mathrm{a}\cos\mathrm{C}] = \sin\mathrm{b}\cot\mathrm{B}\sin\mathrm{C} \\ &= \cos j \sin N \\ \sin\mathrm{a}\sin\mathrm{b} + \cos\mathrm{a}\cos\mathrm{b}\cos\mathrm{C} &= \sin\mathrm{a}\sin\mathrm{b} + \cos\mathrm{c}\cos\mathrm{C} - \sin\mathrm{a}\sin\mathrm{b}\cos^2\mathrm{C} \\ &= \sin\mathrm{a}\sin\mathrm{b}\sin^2\mathrm{C} + \cos\mathrm{c}(-\cos\mathrm{A}\cos\mathrm{B} + \sin\mathrm{A}\sin\mathrm{B}\cos\mathrm{c}) \\ &= \sin\mathrm{A}\sin\mathrm{B}\sin^2\mathrm{c} + \sin\mathrm{A}\sin\mathrm{B}\cos^2\mathrm{c} - \cos\mathrm{A}\cos\mathrm{B}\cos\mathrm{c} \\ &= \sin i \sin j - \cos i \cos j \cos N \end{aligned}$$

Then substituting in the expression for M_1,

$$\begin{aligned} \mathrm{M}_1 = {} & \cos\chi \cos l \cos N + \sin\chi \cos l \sin N \cos i - \cos\chi \sin l \sin N \cos j \\ & - \sin\chi \sin l\,(\sin i \sin j - \cos i \cos j \cos N) \end{aligned}$$

Let $P = \cos\frac{1}{2}i$, $Q = \sin\frac{1}{2}i$, $p = \cos\frac{1}{2}j$, $q = \sin\frac{1}{2}j$

Then

$$\begin{aligned} \mathrm{M}_1 = {} & (P^2+Q^2)(p^2+q^2)\cos\chi\cos l\cos N + (P^2-Q^2)(p^2+q^2)\sin\chi\cos l\sin N \\ & -(P^2+Q^2)(p^2-q^2)\cos\chi\sin l\sin N + (P^2-Q^2)(p^2-q^2)\sin\chi\sin l\cos N \\ & -4PQpq\sin\chi\sin l \\ = {} & P^2p^2\cos(\chi-l-N) + P^2q^2\cos(\chi+l-N) + Q^2p^2\cos(\chi+l+N) \\ & + Q^2q^2\cos(\chi-l+N) + 2PQpq\,[\cos(\chi+l) - \cos(\chi-l)]. \qquad (20) \end{aligned}$$

Since M_2 is derivable from M_1 by writing $\chi_{,}+\frac{1}{2}\pi$ for $\chi_{,}$, therefore it is also derivable by writing $\chi+\frac{1}{2}\pi$ for χ. Hence $-M_2$ is the same as M_1, save that sines replace cosines

Again $M_3=\sin l_{,} \sin i_{,}=\sin l \cos a \sin i_{,}+\cos l \sin a \sin i_{,}$

But $\sin a \sin i_{,}=\sin i \sin N=2PQ \sin N$

And $\cos a \sin i_{,}=\sin i \cot a \sin c=\sin i (\cot A \sin B+\cos c \cos B)$

$$=\cos i \sin j+\sin i \cos j \cos N$$
$$=2pq(P^2-Q^2)+2PQ(p^2-q^2) \cos N$$

Therefore

$$M_3=2PQ[p^2 \sin (l+N)-q^2 \sin (l-N)]+2pq(P^2-Q^2) \sin l \quad . \quad . \quad . \quad (21)$$

For the sake of future developments it will be more convenient to replace the sines and cosines in the expressions for the M's by exponentials, and for brevity the $\sqrt{-1}$ will be omitted in the indices.

Then

$2M_1=e^{\chi-l-N}[Pp-Qqe^{N}]^2+e^{\chi+l+N}[Qp+Pqe^{-N}]^2+$ the same with the signs of the indices of the exponentials changed,

$-2M_2\sqrt{-1}=$ the same with sign of second line changed,

$M_3\sqrt{-1}=e^{l+N}[Pp-Qqe^{-N}][Qp+Pqe^{-N}]-$ same with signs of the indices of the exponentials changed.

Now let

$$\left.\begin{array}{l} \varpi=Pp-Qqe^{N}, \quad \kappa=Qp+Pqe^{N} \\ \underline{\varpi}=Pp-Qqe^{-N}, \quad \underline{\kappa}=Qp+Pqe^{-N} \end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (22)$$

From these definitions it appears that ϖ and κ are two imaginary functions, which oscillate between the real values $\cos \frac{1}{2}(i+j)$ and $\cos \frac{1}{2}(i-j)$, and $\sin \frac{1}{2}(i+j)$ and $\sin \frac{1}{2}(i-j)$ as the node of the orbit moves round.

Also let $\theta=l+N$, the true longitude of Diana measured from the autumnal equinox. Strictly speaking, when longitudes are measured from a fixed point in the ecliptic $\theta=l+N-\psi$, but in the present investigation nothing is lost by regarding ψ as zero; in § (12), and in Part III., we shall have to introduce ψ.

Then

$$\left.\begin{array}{rl} 2M_1= & \varpi^2 e^{\chi-\theta}+\underline{\kappa}^2 e^{\chi+\theta}+\underline{\varpi}^2 e^{-\chi+\theta}+\kappa^2 e^{-\chi-\theta} \\ 2M_2\sqrt{-1}= & -\varpi^2 e^{\chi-\theta}-\underline{\kappa}^2 e^{\chi+\theta}+\underline{\varpi}^2 e^{-\chi+\theta}+\kappa^2 e^{-\chi-\theta} \\ M_3\sqrt{-1}= & \underline{\varpi\kappa} e^{\theta}-\varpi\kappa e^{-\theta} \end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad (23)$$

The object of the present investigation is to find the following spherical harmonic functions of the second degree of M_1, M_2, M_3, viz.:

$$\mathrm{M}_1^2-\mathrm{M}_2^2,\ 2\mathrm{M}_1\mathrm{M}_2,\ 2\mathrm{M}_2\mathrm{M}_3,\ 2\mathrm{M}_1\mathrm{M}_3,\ \tfrac{1}{3}-\mathrm{M}_3^2$$

Then by adding the squares of the first and second of (23), we have

$$\begin{aligned}2(\mathrm{M}_1^2-\mathrm{M}_2^2)= &\ \varpi^4e^{2(\chi-\theta)}+2\varpi^2\underline{\kappa}^2e^{2\chi}+\underline{\kappa}^4e^{2(\chi+\theta)}\\ &+\underline{\varpi}^4e^{-2(\chi-\theta)}+2\underline{\varpi}^2\kappa^2e^{-2\chi}+\kappa^4e^{-2(\chi+\theta)}\quad\ldots\ldots\quad(24)\end{aligned}$$

From (20) we know that M_1 has the form $\Sigma\mathrm{A}\cos(\chi+\mathrm{B})$, and $-\mathrm{M}_2$ the form $\Sigma\mathrm{A}\sin(\chi+\mathrm{B})$; therefore $(\mathrm{M}_1+\mathrm{M}_2)2^{-\frac{1}{2}}$ has the form $\Sigma\mathrm{A}\cos(\chi+\frac{1}{4}\pi+\mathrm{B})$, and $(\mathrm{M}_1-\mathrm{M}_2)2^{-\frac{1}{2}}$ the form $\Sigma\mathrm{A}\sin(\chi+\frac{1}{4}\pi+\mathrm{B})$. Hence if we write $\chi-\frac{1}{4}\pi$ for χ in $\mathrm{M}_1^2-\mathrm{M}_2^2$, we obtain $-2\mathrm{M}_1\mathrm{M}_2$. Therefore from (24) we obtain

$$\begin{aligned}-4\mathrm{M}_1\mathrm{M}_2\sqrt{-1}= &\ \varpi^4e^{2(\chi-\theta)}+2\varpi^2\underline{\kappa}^2e^{2\chi}+\underline{\kappa}^4e^{2(\chi+\theta)}\\ &-\underline{\varpi}^4e^{-2(\chi-\theta)}-2\underline{\varpi}^2\kappa^2e^{-2\chi}-\kappa^4e^{-2(\chi+\theta)}\quad\ldots\ldots\quad(25)\end{aligned}$$

The $\sqrt{-1}$ appears on the left hand side because $e^{\frac{\pi}{2}}=-(-1)^{-\frac{1}{2}}$, $e^{-\frac{\pi}{2}}=(-1)^{-\frac{1}{2}}$.
It is also easy to show that,

$$\begin{aligned}2\mathrm{M}_2\mathrm{M}_3= &-\varpi^3\kappa e^{\chi-2\theta}+\varpi\underline{\kappa}(\varpi\underline{\varpi}-\kappa\underline{\kappa})e^{\chi}+\underline{\varpi}\underline{\kappa}^3e^{\chi+2\theta}\\ &-\underline{\varpi}^3\underline{\kappa}e^{-(\chi-2\theta)}+\underline{\varpi}\kappa(\underline{\varpi}\varpi-\underline{\kappa}\kappa)e^{-\chi}+\varpi\kappa^3e^{-(\chi+2\theta)}\quad\ldots\ldots\quad(26)\end{aligned}$$

$$\begin{aligned}2\mathrm{M}_1\mathrm{M}_3\sqrt{-1}= &-\varpi^3\kappa e^{\chi-2\theta}+\varpi\underline{\kappa}(\varpi\underline{\varpi}-\kappa\underline{\kappa})e^{\chi}+\underline{\varpi}\underline{\kappa}^3e^{\chi+2\theta}\\ &+\underline{\varpi}^3\kappa e^{-(\chi-2\theta)}-\underline{\varpi}\kappa(\underline{\varpi}\varpi-\underline{\kappa}\kappa)e^{-\chi}-\varpi\kappa^3e^{-(\chi+2\theta)}\quad\ldots\ldots\quad(27)\end{aligned}$$

$$\tfrac{1}{3}-\mathrm{M}_3^2=\tfrac{1}{3}-2\varpi\underline{\varpi}\kappa\underline{\kappa}+\underline{\varpi}^2\underline{\kappa}^2e^{2\theta}+\varpi^2\kappa^2e^{-2\theta}\quad\ldots\ldots\quad(28)$$

It may be here noted that $\varpi\underline{\varpi}+\kappa\underline{\kappa}=1$, so that

$$\tfrac{1}{3}-2\varpi\underline{\varpi}\kappa\underline{\kappa}=\tfrac{1}{3}(\varpi^2\underline{\varpi}^2-4\varpi\underline{\varpi}\kappa\underline{\kappa}+\kappa^2\underline{\kappa}^2)$$

These five formulas (24) to (28) are clearly equivalent to the expansion of the harmonic functions as a series of sines and cosines of angles of the form $\alpha\chi+\beta l+\gamma N$. It remains to explain the uses to be made of these expressions.

§ 4. *The disturbing function.*

In the theory of the disturbing function the differentiation with respect to the elements of the orbit of the disturbed body is an artifice to avoid the determination of the three component disturbing forces, by means of differentiation with regard to

the radius vector, longitude and latitude. In the present problem we have to determine the perturbation of a satellite under the influence of the tides raised by itself and by another satellite. Where the tides are raised by the satellite itself, the elements of that satellite's orbit of course enter in the disturbing function in expressing the state of tidal distortion of the planet, but they also enter as expressing the position of the satellite. It is clear that, in effecting the differentiations above referred to, we must only regard the elements of the orbit as entering in the disturbing function in the latter sense. Hence it follows that even although there may be only one satellite, yet in the evaluation of the disturbing function we must suppose that there are two satellites, viz.: one a tide-raising satellite and another a disturbed satellite.

In this place, where the planet is called the earth, the tide-raising satellite may be conveniently called Diana, and the satellite whose motion is disturbed may be called the moon. After the formation of the differential equations Diana may be made identical with the moon or with the sun at will, or the analysis may be made applicable to a planet with any number of satellites.

As above stated, unaccented symbols will be taken to apply to Diana, and accented symbols to the moon.

The first step, then, is to find the tidal distortion due to Diana.

Let M be the projection of Diana on the celestial sphere concentric with the earth, and P the projection of any point in the earth.

Let $\rho\xi$, $\rho\eta$, $\rho\zeta$ be the rectangular coordinates of P and $r\mathrm{M}_1$, $r\mathrm{M}_2$, $r\mathrm{M}_3$ the rectangular coordinates of Diana referred to axes A, B, C fixed in the earth.

Then since ρ, r are radii vectores, ξ, η, ζ and M_1, M_2, M_3 are direction-cosines.

The tide-generating potential V (of the second degree of harmonics, which will be alone considered) at P is given by

$$\mathrm{V}=\tfrac{3}{2}\frac{\mu m}{r^3}\rho^2(\cos^2\mathrm{PM}-\tfrac{1}{3})$$

according to the usual theory.

Now

$$\cos\mathrm{PM}=\xi\mathrm{M}_1+\eta\mathrm{M}_2+\zeta\mathrm{M}_3$$

and

$$\cos^2\mathrm{PM}-\tfrac{1}{3}=2\xi\eta\mathrm{M}_1\mathrm{M}_2+2\frac{\xi^2-\eta^2}{2}\,\frac{\mathrm{M}_1{}^2-\mathrm{M}_2{}^2}{2}+2\eta\zeta\mathrm{M}_2\mathrm{M}_3+2\xi\zeta\mathrm{M}_1\mathrm{M}_3$$
$$+\tfrac{3}{2}\frac{\xi^2+\eta^2-2\zeta^2}{3}\,\frac{\mathrm{M}_1{}^2+\mathrm{M}_2{}^2-2\mathrm{M}_3{}^2}{3}$$

Also by previous definition, $\tau=\frac{3}{2}\mu m/c^3$; so that

$$\tfrac{3}{2}\frac{\mu m}{r^3}=\frac{\tau}{(1-e^2)^3}\left[\frac{c(1-e^2)}{r}\right]^3$$

Now let

$$\mathrm{X}=\left[\frac{c(1-e^2)}{r}\right]^{\frac{3}{2}}\mathrm{M}_1,\ \mathrm{Y}=\left[\frac{c(1-e^2)}{r}\right]^{\frac{3}{2}}\mathrm{M}_2,\ \mathrm{Z}=\left[\frac{c(1-e^2)}{r}\right]^{\frac{3}{2}}\mathrm{M}_3\ .\ \ .\ \ .\ \ .\quad(29)$$

Then clearly

$$V \div \frac{\tau}{(1-e^2)^3}\rho^2 = 2\xi\eta XY + 2\,\frac{\xi^2-\eta^2}{2}\,\frac{X^2-Y^2}{2} + 2\eta\zeta YZ + 2\xi\zeta XZ + \tfrac{3}{2}\,\frac{\xi^2+\eta^2-2\zeta^2}{3}\,\frac{X^2+Y^2-2Z^2}{3}$$

Now assume that the five functions $2XY$, X^2-Y^2, YZ, XZ, $X^2+Y^2-2Z^2$ are each expressed as a series of simple time-harmonics; it will appear below that this may always be done. We now have V expressed as the sum of five solid harmonics $\rho^2\xi\eta$, $\rho^2(\xi^2-\eta^2)$, &c., each multiplied by a simple time-harmonic. According to any tidal theory each such term must raise a tide expressible by a surface harmonic of the same type, and multiplied by a simple time-harmonic of the same speed; moreover, each such tide must have a height which is some fraction of the corresponding equilibrium tide of a perfectly fluid spheroid, but the simple time-harmonic will in general be altered in phase.

Now if $r=a+\sigma$ be the equation to the wave-surface, corresponding to a generating potential $V=[\tau/(1-e^2)^3]\,\rho^2\,2\xi\eta XY$, then when the spheroid is *perfectly fluid*, $\sigma/a=[\tau/\mathfrak{g}(1-e^2)^3]\;2\xi\eta XY$, where $\mathfrak{g}=\frac{2}{5}g/a$, according to the ordinary equilibrium theory of tides. (It will now be assumed that we are dealing with bodily tides of the spheroid; if the tides were oceanic a slight modification would have to be introduced.)

In a frictional fluid, the tide σ will be reduced in height and altered in phase.

Let $\mathfrak{X}\mathfrak{Y}$ represent a function of the same form as XY, save that each simple time-harmonic term of XY is multiplied by some fraction expressive of reduction of height of tide, and that the argument of each such simple harmonic term is altered in phase; the constants so introduced will be functions of the constitution of the spheroid, and of the speed of the harmonic terms. Also extend the same notation to the other functions of X, Y, Z which occur in V.

Then it is clear that, if $r=a+\sigma$ be the equation to the complete wave surface corresponding to the potential V,

$$(1-e^2)^3\frac{\mathfrak{g}}{\tau}\frac{\sigma}{a} = 2\xi\eta\,\mathfrak{X}\mathfrak{Y} + 2\frac{\xi^2-\eta^2}{2}\,\frac{\mathfrak{X}^2-\mathfrak{Y}^2}{2} + 2\eta\zeta\,\mathfrak{Y}\mathfrak{Z} + 2\xi\zeta\,\mathfrak{X}\mathfrak{Z} + \tfrac{3}{2}\,\frac{\xi^2+\eta^2-2\zeta^2}{3}\,\frac{\mathfrak{X}^2+\mathfrak{Y}^2-2\mathfrak{Z}^2}{3} \quad . \quad (30)$$

This expression shows that σ is a surface harmonic of the second order.

Then by (17) we have for the disturbing function for the moon, due to Diana's tides,

$$W=\tau'\left(\frac{c'}{r'}\right)^3\left(\frac{\sigma}{a}\right)$$

where σ is the height of tide, at the point where the moon's radius vector pierces the wave surface.

Hence in the expression (30) for σ, we must put

$$\xi = M_1', \ \eta = M_2', \ \zeta = M_3'$$

Then by analogy with (29), let

$$X' = \left[\frac{c'(1-e'^2)}{r'}\right]^{\frac{3}{2}} M_1', \ Y' = \left[\frac{c'(1-e'^2)}{r'}\right]^{\frac{3}{2}} M_2', \ Z' = \left[\frac{c'(1-e'^2)}{r'}\right]^{\frac{3}{2}} M_3'$$

and we have

$$W = \frac{\tau\tau'}{\mathfrak{g}} \frac{1}{(1-e^2)^3(1-e'^2)^3}\left[2X'Y'\,\mathfrak{X}\mathfrak{Y} + 2\frac{X'^2-Y'^2}{2}\,\frac{\mathfrak{X}^2-\mathfrak{Y}^2}{2} + 2Y'Z'\,\mathfrak{Y}\mathfrak{Z} + 2X'Z'\,\mathfrak{X}\mathfrak{Z} + \tfrac{3}{2}\,\frac{X'^2+Y'^2-2Z'^2}{3}\,\frac{\mathfrak{X}^2+\mathfrak{Y}^2-2\mathfrak{Z}^2}{3}\right] \quad . \quad . \quad . \quad . \quad (31)$$

This is the required expression for the disturbing function on the moon, due to Diana's tides.

So far the investigation is general, but we now have to develop this function so as to make it applicable to the several problems to be considered.

II.

SECULAR CHANGES IN THE INCLINATION OF THE ORBIT OF A SATELLITE.

§ 5. *The perturbed satellite moves in a circular orbit inclined to a fixed plane.—Subdivision of the problem.*

In this case $e=0$, $e'=0$, $r=c$, $r'=c'$, so that the functions X, Y, Z and X′, Y′, Z′ are simply the direction cosines of Diana and the moon, referred to the axes A, B, C fixed in the earth. Hence $X=M_1$, $Y=M_2$, $Z=M_3$, and the five formulas (24–8) give the functions X^2-Y^2, $2XY$, $2YZ$, $2ZX$, $\frac{1}{3}-Z^2$. In order to form the functions in gothic letters we must express these functions as simple time-harmonics.

The formulas (24) to (28) are equivalent to the expression of the five functions as a series of terms of the type A cos $(\alpha\chi+\beta\theta+\gamma N+\delta)$. Now χ is the angle between a point fixed on the equator and the autumnal equinox, and therefore (neglecting alterations in the diurnal rotation and the precessional motion) increases uniformly with the time, being equal to $nt+$a constant, which constant may be treated as zero by a proper choice of axes A, B, C.

θ is the true longitude measured from the autumnal equinox, and is equal to $\Omega t+\epsilon-\psi$, since the orbit is circular; also ψ may for the present be put equal to zero, without any loss of generality.

Then if in forming the expressions for the state of tidal distortion of the earth we neglect the motion of the node, the five functions are expressed as a series of simple time-harmonics of the type A cos $(\alpha nt+\beta\Omega t+\zeta)$.

The corresponding term in the corresponding gothic-letter function will be KA cos $(\alpha nt+\beta\Omega t+\zeta-\mathrm{k})$, where K is the fraction by which the tide is reduced and k is the alteration of phase.

It appears, from the inspection of the five formulas (24–8), that there are tides of seven speeds, viz.: $2(n-\Omega)$, $2n$, $2(n+\Omega)$, $n-2\Omega$, n, $n+2\Omega$, 2Ω.

The following schedule gives the symbols to be introduced for reduction of tide and alteration of phase or lag.

	Semi-diurnal.			Diurnal.			Fortnightly.
	Slow.	Sidereal.	Fast.	Slow.	Sidereal.	Fast.	
Speed	$2(n-\Omega)$,	$2n$,	$2(n+\Omega,)$	$n-2\Omega$,	n,	$n+2\Omega$,	2Ω
Fraction of equilibrium tide .	F_1	F	F_2	G_1	G	G_2	H
Retardation of phase or lag. .	$2\mathrm{f}_1$	2f	$2\mathrm{f}_2$	g_1	g	g_2	2h

The gothic-letter functions may now at once be written down from (24–8).

Thus,

$$
\begin{aligned}
2(\mathfrak{X}^2-\mathfrak{Y}^2)=&\mathrm{F}_1\varpi^4e^{2(\chi-\theta)-2\mathrm{f}_1}+\mathrm{F}2\varpi^2\underline{\kappa}^2e^{2\chi-2\mathrm{f}}+\mathrm{F}_2\underline{\kappa}^4e^{2(\chi+\theta)-2\mathrm{f}_2}\\
&+\mathrm{F}_1\underline{\varpi}^4e^{-2(\chi-\theta)+2\mathrm{f}_1}+\mathrm{F}2\underline{\varpi}^2\kappa^2e^{-2\chi+2\mathrm{f}}+\mathrm{F}_2\kappa^4e^{-2(\chi+\theta)+2\mathrm{f}_2}\quad .\ .\ .\quad (32)
\end{aligned}
$$

$$
-4\mathfrak{X}\mathfrak{Y}\sqrt{-1}=\text{the same, with second line of opposite sign}\quad .\ .\ .\ .\quad (33)
$$

$$
\begin{aligned}
2\mathfrak{Y}\mathfrak{Z}=&-\mathrm{G}_1\varpi^3\kappa e^{\chi-2\theta-\mathrm{g}_1}+\mathrm{G}\varpi\underline{\kappa}(\varpi\underline{\varpi}-\underline{\kappa}\kappa)e^{\chi-\mathrm{g}}+\mathrm{G}_2\underline{\varpi}\underline{\kappa}^3e^{\chi+2\theta-\mathrm{g}_2}\\
&-\mathrm{G}_1\underline{\varpi}^3\underline{\kappa}e^{-(\chi-2\theta)+\mathrm{g}_1}+\mathrm{G}\underline{\varpi}\kappa(\underline{\varpi}\varpi-\underline{\kappa}\kappa)e^{-\chi+\mathrm{g}}+\mathrm{G}_2\varpi\kappa^3e^{-(\chi+2\theta)+\mathrm{g}_2}.\quad .\ .\quad (34)
\end{aligned}
$$

$$
2\mathfrak{X}\mathfrak{Z}\sqrt{-1}=\text{the same, with second line of opposite sign}\quad .\ .\ .\ .\ .\ .\quad (35)
$$

$$
\tfrac{1}{3}-\mathfrak{Z}^2=\tfrac{1}{3}-2\varpi\underline{\varpi}\underline{\kappa}\kappa+\mathrm{H}\underline{\varpi}^2\underline{\kappa}^2e^{2\theta-2\mathrm{h}}+\mathrm{H}\varpi^2\kappa^2e^{-2\theta+2\mathrm{h}}\quad .\ .\ .\ .\ .\quad (36)
$$

The fact that there is no factor of the same kind as H in the first pair of (36) results from the assumption that the tides due to the motion of the nodes of the orbit are the equilibrium tides unaltered in phase.

The formulas for $2(\mathrm{X}'^2-\mathrm{Y}'^2)$, $-4\mathrm{X}'\mathrm{Y}'\sqrt{-1}$, $2\mathrm{Y}'\mathrm{Z}'$, $2\mathrm{X}'\mathrm{Z}'\sqrt{-1}$, $\frac{1}{3}-\mathrm{Z}'^2$ are found by symmetry, by merely accenting all the symbols in the five formulas (24–8) for the M functions. In the use made of these formulas this accentuation will be deemed to be done.

At present we shall not regard χ as being accented, but in § 12 and in Part III. we shall have to regard χ as also accented.

We now have to develop the several products of the X′ functions multiplied by the $\mathfrak{X}$ functions.

Before making these multiplications, it must be considered what are the terms which are required for finding secular changes in the elements, since all others are superfluous for the problem in hand.

Such terms are clearly those in which θ and θ' are wanting, and also those where $\theta-\theta'$ occurs, for these will be wanting in θ when Diana is made identical with the moon. It follows therefore that we need only multiply together terms of the like speeds. In the following developments all superfluous terms are omitted.

Semi-diurnal terms.

These are $2X'Y'\mathfrak{X}\mathfrak{Y}+2\frac{X'^2-Y'^2}{2}\frac{\mathfrak{X}^2-\mathfrak{Y}^2}{2}$.

If we multiply (24) (with accented symbols) by (32), and (25) (with accented symbols) by (33), and subtract the latter from the former, we see that χ disappears from the expression, and that,

$$8X'Y'\mathfrak{X}\mathfrak{Y}+2(X'^2-Y'^2)(\mathfrak{X}^2-\mathfrak{Y}^2)=\text{First line of (24)} \times \text{second of (32)}$$
$$+\text{Second of (25)} \times \text{first of (33)}$$

Then as far as we are concerned

$$2X'Y'\mathfrak{X}\mathfrak{Y}+2\frac{X'^2-Y'^2}{2}\frac{\mathfrak{X}^2-\mathfrak{Y}^2}{2}$$
$$=\tfrac{1}{4}\left[F_1\varpi^4\underline{\varpi}'^4e^{2(\theta'-\theta)-2f_1}+4F\varpi^2\underline{\kappa}^2\underline{\varpi}'^2\kappa'^2e^{-2f}+F_2\underline{\kappa}^4\kappa'^4e^{-2(\theta'-\theta)-2f_2}\right]$$
$$+\tfrac{1}{4}\left[F_1\underline{\varpi}^4\varpi'^4e^{-2(\theta'-\theta)+2f_1}+4F\underline{\varpi}^2\kappa^2\varpi'^2\underline{\kappa}'^2e^{2f}+F_2\kappa^4\underline{\kappa}'^4e^{2(\theta'-\theta)+2f_2}\right] \quad . \tag{37}$$

If χ had been accented in the X' functions, we should have had $2(\chi-\chi')$ in all the indices of exponentials of the first line, and $-2(\chi-\chi')$ in all the indices of the second line. These three pairs of terms will be called W_{I}, W_{II}, W_{III}.

Diurnal terms.

These are $2Y'Z'\mathfrak{Y}\mathfrak{Z}+2X'Z'\mathfrak{X}\mathfrak{Z}$.

If the multiplications be performed as in the previous case, it will be found that χ disappears in the sum of the two products, and, as far as concerns terms in $\theta'-\theta$ and those independent of θ and θ', we have

$$2Y'Z'\mathfrak{Y}\mathfrak{Z}+2X'Z'\mathfrak{X}\mathfrak{Z}$$
$$=G_1\varpi^3\underline{\kappa}\,\underline{\varpi}'^3\kappa'e^{2(\theta'-\theta)-g_1}+G\varpi\underline{\kappa}(\varpi\underline{\varpi}-\underline{\kappa}\kappa)\underline{\varpi}'\kappa'(\underline{\varpi}'\varpi'-\underline{\kappa}'\kappa')e^{-g}+G_2\underline{\varpi}\,\underline{\kappa}^3\varpi'\kappa'^3e^{-2(\theta'-\theta)-g_2}$$
$$+G_1\underline{\varpi}^3\underline{\kappa}\varpi'^3\kappa'e^{-2(\theta'-\theta)+g_1}+G\underline{\varpi}\kappa(\varpi\underline{\varpi}-\underline{\kappa}\kappa)\varpi'\underline{\kappa}'(\underline{\varpi}'\varpi'-\underline{\kappa}'\kappa')e^{g}+G_2\varpi\kappa^3\underline{\varpi}'\underline{\kappa}'^3e^{2(\theta'-\theta)+g_2} \quad . \tag{38}$$

If χ had been accented in the X' functions we should have had $\chi-\chi'$ in all the

indices of the exponentials of the first line, and $-(\chi-\chi')$ in all the indices of the second line. These three pairs of terms will be called W_1, W_2, W_3.

Fortnightly term.

This is $\frac{3}{2}(\frac{1}{3}-Z'^2)(\frac{1}{3}-\mathfrak{Z}^2)$.

Multiplying (36) by (28) when the symbols are accented, and only retaining desired terms,

$$\frac{3}{2}(\tfrac{1}{3}-Z'^2)(\tfrac{1}{3}-\mathfrak{Z}^2)=\frac{3}{2}(\tfrac{1}{3}-2\varpi\underline{\varpi}\kappa\underline{\kappa})(\tfrac{1}{3}-2\varpi'\underline{\varpi}'\kappa'\underline{\kappa}')+\tfrac{3}{2}\mathrm{H}\underline{\varpi}^2\underline{\kappa}^2\varpi'^2\kappa'^2e^{-2(\theta'-\theta)-2h}$$
$$+\tfrac{3}{2}\mathrm{H}\varpi^2\kappa^2\underline{\varpi}'^2\underline{\kappa}'^2e^{2(\theta-\theta)+2h} \quad . \quad (39)$$

Even if χ had been accented in the X' functions, neither χ or χ' would have entered in this expression. These terms will be called W_0.

Then the sum of the three expressions (37), (38), and (39), when multiplied by $\tau\tau'/\mathfrak{g}$, is equal to W, the disturbing function.

If Diana be a different body from the moon the terms in $\theta'-\theta$ are periodic, and the only part of W, from which secular changes in the moon's mean distance and inclination can arise, are the sidereal semi-diurnal and diurnal terms, viz.: those in F and G, and also the term independent of H in (39). These terms being independent of θ' are independent of ϵ', the moon's epoch. Hence it follows that, as far as concerns the influence of Diana's tides upon the moon, $dW/d\epsilon'$ is zero, and we conclude that—*the tides raised by any one satellite can produce directly no secular change in the mean distance of any other satellite.**

But Diana being still distinct from the moon, the F-, G-, and part of the fortnightly term, which are independent of θ, do involve N and N'; for W contains terms of the forms $e^{\pm\alpha N}$, $e^{\pm\alpha N'}$, $e^{\pm(\alpha N+\beta N')}$, also it has terms independent of N, N'. Hence dW/dN' will contain terms of the form $e^{\pm\alpha N'}$, $e^{\pm(\alpha N+\beta N')}$, or their equivalent sines or cosines.

Now by hypothesis there are two disturbing bodies, and we know by lunar theory that the direct influence of Diana on the moon is such as to tend to make the nodes of the moon's orbit revolve on the ecliptic; on the other hand, there is a direct influence of the permanent oblateness of the earth on the nodes of the moon's orbit.

If the oblateness of the earth be large, the result of the joint influence of these two causes may be such as either to make the nodes of the moon's orbit rotate with a very unequal angular velocity, or perform oscillations (possibly large ones) about a mean position. If this be the case the mean value of dW/dN' may differ considerably from zero. This case is considered in detail in Part III. of this paper.

If on the other hand the oblateness be small the nodes of the orbit revolve with a

* If there be a rigorous relationship between the mean motions of a pair of satellites this may not be true. This appears to be (at least very nearly) the case between two pairs of satellites of the planet Saturn.

sensibly uniform angular velocity on the ecliptic. This is the case at present with the earth and moon. Here then dW/dN', as far as concerns the influence of Diana's tides on the moon, is sensibly periodic according to simple harmonic functions of the time. From this we conclude that:—

If the nodes of the satellites' orbits revolve uniformly on the plane of reference, then the tides raised by any one satellite can produce no secular change in the inclination of the orbit of any other satellite.

There are thus two cases in which the problem is simplified by our being permitted to consider only the case of identity between Diana and the moon:

1st. Where there are two or more satellites, but where the nodes of the perturbed satellite's orbit revolve with sensible uniformity on the plane of reference.

2nd. Where the planet and satellite are the only bodies in existence.

In these two cases, after differentiation of the disturbing function with respect to the accented elements, we shall be able to drop the accents.

There is also a third case in which Diana's tides *will* produce a secular effect on the inclination of the moon's orbit, and this is where the nodes of the moon's orbit either revolve irregularly or oscillate. This case is enormously more complicated than the others, and forms the subject of Part III. of this paper; I have only attempted to solve it on the supposition of the smallness both of the inclination of the orbit, and of the obliquity of the ecliptic.

The first of these three cases is that which actually represents the moon and earth, together with solar perturbation of the moon at the present time.

In tracing the configuration of the lunar orbit backwards from the present state, we shall start with the first case; this will graduate into the third, and from this it will pass to a state represented to a very close degree of approximation by the second.

We are not at present concerned to know what are the conditions under which there may be approximate uniformity in the motion of the nodes; this will be investigated below.

We will begin with the first of the three cases, and will find also the rate of change of the diurnal rotation and of the obliquity of the planet.

The second case will then be taken, and afterwards the third case will have to be discussed almost *ab initio* in Part III.

§ 6. *Secular change of inclination of the orbit of a satellite, where there is a second disturbing body, and where the nodes revolve with sensible uniformity on the fixed plane of reference.*

By (13) the equation giving the change of inclination is

$$-\frac{\xi'}{k'}\frac{dj'}{dt}=\frac{1}{\sin j'}\frac{dW}{dN'}+\tan\tfrac{1}{2}j'\frac{dW}{d\epsilon'}$$

As shown above, however, we need here only deal with a single satellite, so that Diana and the moon may be considered as identical and the accents may be dropped to all the symbols, except in the differential coefficients of W. Also we need only maintain the distinction between Diana and the moon as regards N, N' and ϵ, ϵ'; and after the differentiations of W these distinctions must also be dropped. Hence ϖ only differs from ϖ', κ from κ', $\underline{\varpi}$ from $\underline{\varpi}'$, and $\underline{\kappa}$ from $\underline{\kappa}'$ in the accentuation of N.

Also since $\theta=\Omega t+\epsilon$, $\theta'=\Omega' t+\epsilon'$, therefore we may replace $\theta'-\theta$ in the three expressions (37–9) by $\epsilon'-\epsilon$.

If we put $\sin j=2pq$, $\tan\frac{1}{2}j=q/p$, and write $\phi(N,\epsilon)$ for the operation $\frac{1}{2pq}\frac{d}{dN'}+\frac{q}{p}\frac{d}{d\epsilon'}$, putting $N=N'$, $\epsilon=\epsilon'$ after differentiation; then from (13) we have

$$-\frac{\xi}{k}\frac{dj}{dt}=\phi(N,\epsilon)\mathrm{W}$$

Also for brevity, let $\phi(N)=\frac{1}{2pq}\frac{d}{dN'}$, $\phi(\epsilon)=\frac{q}{p}\frac{d}{d\epsilon'}$; so that $\phi(N,\epsilon)=\phi(N)+\phi(\epsilon)$.

The terms corresponding to the tides of the seven speeds will now be taken separately, the coefficients in ϖ, κ will be developed, and the terms involving $N'-N$ selected, the operation $\phi(N,\epsilon)$ performed, and then N' put equal to N, and ϵ to ϵ'. For the sake of brevity the coefficient $\tau^2/\mathfrak{g}$ will be dropped and will be added in the final result. The component parts of W taken from the equations (37–9) will be indicated as W_{I}, W_{II}, $\mathrm{W}_{\mathrm{III}}$ for the slow, sidereal, and fast semi-diurnal parts; as W_1, W_2, W_3 for the slow, sidereal, and fast diurnal parts; and as W_0 for the fortnightly part.

Slow semi-diurnal terms $(2n-2\Omega)$.

$$\mathrm{W}_{\mathrm{I}}=\tfrac{1}{4}\mathrm{F}_1\left[\varpi^4\underline{\varpi}'^4e^{2(\epsilon'-\epsilon)-2\mathrm{f}_1}+\underline{\varpi}^4\varpi'^4e^{-2(\epsilon'-\epsilon)+2\mathrm{f}_1}\right] \quad . \quad . \quad . \quad . \quad . \quad (40)$$

Let

$$\mathrm{w}_{\mathrm{I}}=\tfrac{1}{4}\varpi^4\underline{\varpi}'^4e^{2(\epsilon'-\epsilon)-2\mathrm{f}_1}$$

Since

$$\varpi=Pp-qQe^N$$

Therefore

$$\varpi^4=P^4p^4-4P^3Qp^3qe^N+6P^2Q^2p^2q^2e^{2N}-4PQ^3pq^3e^{3N}+Q^4q^4e^{4N}$$

$$\underline{\varpi}'^4=\text{ the same with } -N' \text{ in place of } N$$

Therefore

$$\mathrm{w}_{\mathrm{I}}=\tfrac{1}{4}\{P^8p^8+16P^6Q^2p^6q^2e^{N-N'}+36P^4Q^4p^4q^4e^{2(N-N')}+16P^2Q^6p^2q^6e^{3(N-N')}$$
$$+Q^8q^8e^{4(N-N')}\}e^{2(\epsilon'-\epsilon)-2\mathrm{f}_1}$$

Therefore

$$\mathrm{w}_{\mathrm{I}}=\Sigma\mathrm{A}_nP^{8-2n}Q^{2n}p^{8-2n}q^{2n}e^{n(N-N')+2(\epsilon'-\epsilon)-2\mathrm{f}_1}$$

where $n=0, 1, 2, 3, 4$.

Then

$$\phi(N)\mathrm{w}_{\mathrm{I}}=\Sigma\frac{n}{2\sqrt{-1}}\mathrm{A}_n P^{8-2n}Q^{2n}p^{7-2n}q^{2n-1}e^{-2\mathrm{f}_1}$$

$$\phi(\epsilon)\mathrm{w}_{\mathrm{I}}=-\Sigma\frac{4}{2\sqrt{-1}}\mathrm{A}_n P^{8-2n}Q^{2n}p^{7-2n}q^{2n+1}e^{-2\mathrm{f}_1}$$

Therefore by addition

$$\phi(N,\epsilon)\mathrm{w}_{\mathrm{I}}=\Sigma[n(p^2+q^2)-4q^2]P^{8-2n}Q^{2n}p^{7-2n}q^{2n-1}\frac{e^{-2\mathrm{f}_1}}{2\sqrt{-1}}$$

Now when

$$\begin{array}{lll} n=0, & \mathrm{A}_n=\tfrac{1}{4}, & n(p^2+q^2)-4q^2=-4q^2 \\ \quad =1, & \quad =4, & \qquad\qquad =p^2-3q^2 \\ \quad =2, & \quad =9, & \qquad\qquad =2(p^2-q^2) \\ \quad =3, & \quad =4, & \qquad\qquad =3p^2-q^2 \\ \quad =4, & \quad =\tfrac{1}{4}, & \qquad\qquad =4p^2 \end{array}$$

If we had taken the second term of W_{I} we should have had the same coefficients but multiplied by $-e^{2\mathrm{f}_1}/2\sqrt{-1}$ instead of by $e^{-2\mathrm{f}_1}/2\sqrt{-1}$. Therefore, since $(e^{2\mathrm{f}_1}-e^{-2\mathrm{f}_1})/2\sqrt{-1}=\sin 2\mathrm{f}_1$

$$\phi(N,\epsilon)\mathrm{W}_{\mathrm{I}}=-\mathrm{F}_1\sin 2\mathrm{f}_1[-P^8p^7q+4P^6Q^2p^5q(p^2-3q^2)+18P^4Q^4p^3q^3(p^2-q^2) \\ +4P^2Q^6pq^5(3p^2-q^2)+Q^8pq^7]$$

Then let

$$\mathfrak{F}_1=\tfrac{1}{4}[P^8p^6-4P^6Q^2p^4(p^2-3q^2)-18P^4Q^4p^2q^2(p^2-q^2)-4P^2Q^6q^4(3p^2-q^2)-Q^8q^6]\,. \quad (41)$$

and remembering that $2pq=\sin j$, we have

$$\phi(N,\epsilon)\mathrm{W}_{\mathrm{I}}=2\mathfrak{F}_1\mathrm{F}_1\sin 2\mathrm{f}_1\sin j \quad . \; . \; . \; . \; . \; . \; . \; . \quad (42)$$

Sidereal semi-diurnal terms ($2n$).

$$\mathrm{W}_{\mathrm{II}}=\mathrm{F}[\varpi^2\underline{\kappa}^2\underline{\varpi}'^2\kappa'^2e^{-2\mathrm{f}}+\underline{\varpi}^2\kappa^2\varpi'^2\underline{\kappa}'^2e^{2\mathrm{f}}] \quad . \; . \; . \; . \; . \; . \; . \quad (43)$$

Here the epoch is wanting, so that $\phi(N,\epsilon)=\phi(N)$.

Let

$$\mathrm{w}_{\mathrm{II}}=(\varpi\underline{\kappa\varpi}'\kappa')^2$$

$$\varpi=Pp-Qqe^{N},\ \underline{\kappa}=Qp+Pqe^{-N}$$

$$\varpi\underline{\kappa}=PQ(p^2-q^2)+pq(P^2e^{-N}-Q^2e^{N})$$

$$\underline{\varpi}'\kappa'=PQ(p^2-q^2)+pq(P^2e^{N'}-Q^2e^{-N'})$$

$$\sqrt{\mathrm{w}_{\mathrm{II}}}=P^2Q^2(p^2-q^2)^2+PQpq(p^2-q^2)[P^2(e^{N'}+e^{-N})-Q^2(e^{-N'}+e^{N})]$$
$$+p^2q^2[P^4e^{-(N-N')}+Q^4e^{(N-N')}-P^2Q^2(e^{N+N'}+e^{-(N+N')})]$$

$$\mathrm{w}_{\mathrm{II}}=P^4Q^4[(p^2-q^2)^4-4p^2q^2(p^2-q^2)^2+4p^4q^4]+4P^6Q^2p^2q^2(p^2-q^2)^2e^{-(N-N')}$$
$$+4P^2Q^6p^2q^2(p^2-q^2)^2e^{N-N'}+p^4q^4[P^8e^{-2(N-N')}+Q^8e^{2(N-N')}]$$

$$\phi(N)\mathrm{w}_{\mathrm{II}}=-\frac{1}{2\sqrt{-1}}\left[4pq(p^2-q^2)^2P^2Q^2(P^4-Q^4)+2p^3q^3(P^8-Q^8)\right]$$

If we had operated on the other term of W_{II} we should have got the same with the opposite sign, and $e^{2\mathrm{f}}$ in place of $e^{-2\mathrm{f}}$.

Then let

$$\mathfrak{F}=\tfrac{1}{2}(P^2-Q^2)\{2(p^2-q^2)^2P^2Q^2+p^2q^2(P^4+Q^4)\} \quad . \quad . \quad . \quad . \quad (44)$$

and we have

$$\phi(N,\epsilon)\mathrm{W}_{\mathrm{II}}=2\mathfrak{F}\mathrm{F}\sin 2\mathrm{f}\sin j \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (45)$$

Fast semi-diurnal terms $(2n+2\Omega)$.

$$\mathrm{W}_{\mathrm{III}}=\tfrac{1}{4}\mathrm{F}_2[\underline{\kappa}^4\kappa'^4e^{-2(\epsilon'-\epsilon)-2\mathrm{f}_2}+\kappa^4\underline{\kappa}'^4e^{2(\epsilon'-\epsilon)+2\mathrm{f}_2}] \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (46)$$

Since κ is obtained from ϖ by writing Q for P, and $-P$ for Q, therefore by writing $-2\mathrm{f}_2$ for $2\mathrm{f}_1$, and interchanging Q's and P's we may write down the result by symmetry with the slow semi-diurnal terms. Then let

$$\mathfrak{F}_2=\tfrac{1}{4}[Q^8p^6-4P^2Q^6p^4(p^2-3q^2)-18P^4Q^4p^2q^2(p^2-q^2)-4P^6Q^2q^4(3p^2-q^2)-P^8q^6] \,. \quad (47)$$

and

$$\phi(N,\epsilon)\mathrm{W}_{\mathrm{III}}=-2\mathfrak{F}_2\mathrm{F}_2\sin 2\mathrm{f}_2\sin j \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (48)$$

Slow diurnal terms.

$$\mathrm{W}_1=\mathrm{G}_1[\varpi^3\underline{\kappa\varpi}'^3\kappa'e^{2(\epsilon'-\epsilon)-\mathrm{g}_1}+\varpi^3\underline{\kappa\varpi}'^3\kappa'e^{-2(\epsilon'-\epsilon)+\mathrm{g}_1}] \quad . \quad . \quad . \quad . \quad . \quad . \quad (49)$$

Let $\mathrm{w}_1=\varpi^3\underline{\kappa\varpi}'^3\kappa'e^{2(\epsilon'-\epsilon)-\mathrm{g}_1}$.

For the moment let $\mathrm{I}=\frac{1}{2}i$, then since $\varpi=Pp-Qqe^N$, and since $P=\cos\mathrm{I}$, $Q=\sin\mathrm{I}$, therefore $d\varpi/d\mathrm{I}=-\kappa$, and therefore $d\varpi^4/d\mathrm{I}=-4\varpi^3\kappa$.

Hence (see slow semi-diurnal terms)

$$\varpi^3\kappa=P^3Qp^4+P^2(P^2-3Q^2)p^3qe^N-3PQ(P^2-Q^2)p^2q^2e^{2N}+Q^2(3P^2-Q^2)pq^3e^{3N}-PQ^3q^4e^{4N}$$

$\underline{\varpi}'^3\underline{\kappa}'=$ same with $-N'$ for N

Hence

$$\mathrm{w}_1=[P^6Q^2p^8+P^4(P^2-3Q^2)^2p^6q^2e^{N-N'}+9P^2Q^2(P^2-Q^2)^2p^4q^4e^{2(N-N')}$$
$$+Q^4(3P^2-Q^2)^2p^2q^6e^{3(N-N')}+P^2Q^6q^8e^{4(N-N')}]\,e^{2(\epsilon'-\epsilon)-\mathrm{g}_1}$$

$$\phi(N)\mathrm{w}_1=\frac{e^{-\mathrm{g}_1}}{2\sqrt{-1}}\Big[\qquad P^4(P^2-3Q^2)^2p^5q+18P^2Q^2(P^2-Q^2)^2p^3q^3$$
$$+3Q^4(3P^2-Q^2)^2pq^5+4P^2Q^6\frac{q^7}{p}\Big]$$

$$\phi(\epsilon)\mathrm{w}_1=-\frac{e^{-\mathrm{g}_1}}{2\sqrt{-1}}\Big[4P^6Q^2p^7q+4P^4(P^2-3Q^2)^2p^5q^3+36P^2Q^2(P^2-Q^2)^2p^3q^5$$
$$+4Q^4(3P^2-Q^2)^2pq^7+4P^2Q^6\frac{q^9}{p}\Big]$$

Adding

$$\phi(N,\epsilon)\mathrm{w}_1=-\frac{e^{-\mathrm{g}_1}}{2\sqrt{-1}}[4P^6Q^2p^7q-P^4(P^2-3Q^2)^2p^5q(p^2-3q^2)$$
$$-18P^2Q^2(P^2-Q^2)^2p^3q^3(p^2-q^2)-Q^4(3P^2-Q^2)^2pq^5(3p^2-q^2)-4P^2Q^6pq^7]$$

Then let

$$\mathfrak{G}_1=\tfrac{1}{4}[4P^6Q^2p^6-P^4(P^2-3Q^2)^2p^4(p^2-3q^2)-18P^2Q^2(P^2-Q^2)^2p^2q^2(p^2-q^2)$$
$$-Q^4(3P^2-Q^2)^2q^4(3p^2-q^2)-4P^2Q^6q^6]\quad . \quad (50)$$

and we have

$$\phi(N,\epsilon)\mathrm{W}_1=2\mathfrak{G}_1\mathrm{G}_1\sin\mathrm{g}_1\sin j\quad .\ .\ .\ .\ .\ .\ .\quad (51)$$

Sidereal diurnal terms (n).

$$\mathrm{W}_2=\mathrm{G}[\varpi\kappa(\varpi\underline{\varpi}-\kappa\underline{\kappa})\underline{\varpi}'\underline{\kappa}'(\varpi'\underline{\varpi}'-\kappa'\underline{\kappa}')e^{-\mathrm{g}}+\underline{\varpi}\underline{\kappa}(\varpi\underline{\varpi}-\kappa\underline{\kappa})\varpi'\kappa'(\varpi'\underline{\varpi}'-\kappa'\underline{\kappa}')e^{\mathrm{g}}]\quad . \quad (52)$$

Here the epoch is wanting, so that $\phi(N,\epsilon)=\phi(N)$.

Let

$$\mathrm{w}_2 = \varpi\underline{\kappa}(\varpi\underline{\varpi}-\kappa\underline{\kappa})\underline{\varpi}'\kappa'(\varpi'\underline{\varpi}'-\kappa'\underline{\kappa}')$$

$$\varpi\underline{\kappa} = PQ(p^2-q^2)+pq(P^2e^{-N}-Q^2e^{N})$$

$$\varpi\underline{\varpi}-\kappa\underline{\kappa} = (P^2-Q^2)(p^2-q^2)-2PQpq(e^{N}+e^{-N})$$

$$\varpi\underline{\kappa}(\varpi\underline{\varpi}-\kappa\underline{\kappa}) = PQ(P^2-Q^2)[(p^2-q^2)^2-2p^2q^2]+P^2(P^2-3Q^2)pq(p^2-q^2)e^{-N}$$
$$-Q^2(3P^2-Q^2)pq(p^2-q^2)e^{N}-2PQp^2q^2(P^2e^{-2N}-Q^2e^{2N})$$

$\underline{\varpi}'\kappa'(\varpi'\underline{\varpi}'-\kappa'\underline{\kappa}')=$ the same with $-N'$ instead of N

$$\mathrm{w}_2 = P^2Q^2(P^2-Q^2)^2[(p^2-q^2)^2-2p^2q^2]^2+P^4(P^2-3Q^2)^2p^2q^2(p^2-q^2)^2e^{-(N-N')}$$
$$+Q^4(3P^2-Q^2)^2p^2q^2(p^2-q^2)^2e^{N-N'}+4P^2Q^2p^4q^4(P^4e^{-2(N-N')}+Q^4e^{2(N-N')})$$

$$\phi(N)\mathrm{w}_2 = -\frac{1}{2\sqrt{-1}}\{pq(p^2-q^2)^2(P^4(P^2-3Q^2)^2-Q^4(3P^2-Q^2)^2)+8P^2Q^2(P^4-Q^4)p^3q^3\}$$

Now

$$P^4(P^2-3Q^2)^2-Q^4(3P^2-Q^2)^2=(P^2-Q^2)(P^4+Q^4-6P^2Q^2)$$

Put therefore

$$\mathfrak{G}=\tfrac{1}{4}(P^2-Q^2)\{(p^2-q^2)^2(P^4+Q^4-6P^2Q^2)+8P^2Q^2p^2q^2\} \quad . \quad . \quad . \quad . \quad (53)$$

and we have

$$\phi(N,\epsilon)\mathrm{W}_2=2\mathfrak{G}\mathrm{G}\sin\mathrm{g}\sin j \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (54)$$

Fast diurnal terms $(n+2\Omega)$.

$$\mathrm{W}_3=\mathrm{G}_2[\varpi\underline{\kappa}^3\underline{\varpi}'\kappa'^3e^{-2(\epsilon'-\epsilon)-\mathrm{g}_2}+\varpi\kappa^3\underline{\varpi}'\underline{\kappa}'^3e^{2(\epsilon'-\epsilon)+\mathrm{g}_2}] \quad . \quad . \quad . \quad . \quad . \quad . \quad (55)$$

By an analogy similar to that by which the fast semi-diurnal was derived from the slow, we have

$$\mathfrak{G}_2=\tfrac{1}{4}[4P^2Q^6p^6-Q^4(3P^2-Q^2)^2p^4(p^2-3q^2)-18P^2Q^2(P^2-Q^2)^2p^2q^2(p^2-q^2)$$
$$-P^4(P^2-3Q^2)^2q^4(3p^2-q^2)-4P^6Q^2q^6] \quad . \quad (56)$$

and

$$\phi(N,\epsilon)\mathrm{W}_3=-2\mathfrak{G}_2\mathrm{G}_2\sin\mathrm{g}_2\sin j \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (57)$$

Fortnightly terms (2Ω).

$$\mathrm{W}_0=\tfrac{3}{2}[(\tfrac{1}{3}-2\varpi\underline{\varpi}\kappa\underline{\kappa})(\tfrac{1}{3}-2\varpi'\underline{\varpi}'\kappa'\underline{\kappa}')+\mathrm{H}\varpi^2\underline{\kappa}^2\underline{\varpi}'^2\kappa'^2e^{-2(\epsilon'-\epsilon)-2\mathrm{h}}+\mathrm{H}\underline{\varpi}^2\kappa^2\varpi'^2\underline{\kappa}'^2e^{2(\epsilon'-\epsilon)+2\mathrm{h}}] \quad . \quad (58)$$

It will be found that $\phi(N)$ performed on the first term is zero, as it ought to be according to the general principles of energy—for the system is a conservative one as far as regards these terms.

Let

$$\mathrm{w}_0=(\varpi\kappa\underline{\varpi'\kappa'})^2e^{2(\epsilon'-\epsilon)+2\mathrm{h}}$$

$$\varpi\kappa=PQp^2+pq(P^2-Q^2)e^N-PQq^2e^{2N}$$

$$\varpi^2\kappa^2=P^2Q^2p^4+2PQ(P^2-Q^2)p^3qe^N+[(P^2-Q^2)^2-2P^2Q^2]p^2q^2e^{2N}$$
$$-2PQ(P^2-Q^2)pq^3e^{3N}+P^2Q^2q^4e^{4N}$$

$\underline{\varpi'^2\kappa'^2}$= the same with $-N'$ for N

$$\mathrm{w}_0=[P^4Q^4p^8+4P^2Q^2(P^2-Q^2)^2p^6q^2e^{N-N'}+[(P^2-Q^2)^2-2P^2Q^2]p^4q^4e^{2(N-N')}$$
$$+4P^2Q^2(P^2-Q^2)^2p^2q^6e^{3(N-N')}+P^4Q^4q^8e^{4(N-N')}]e^{2(\epsilon'-\epsilon)+2\mathrm{h}}$$

$$\phi(N)\mathrm{w}_0=\frac{e^{2\mathrm{h}}}{2\sqrt{-1}}\Big[\quad 4P^2Q^2(P^2-Q^2)^2p^5q+2[(P^2-Q^2)^2-2P^2Q^2]^2p^3q^3$$
$$+12P^2Q^2(P^2-Q^2)^2pq^5+4P^4Q^4\frac{q^7}{p}\Big]$$

$$\phi(\epsilon)\mathrm{w}_0=-\frac{e^{2\mathrm{h}}}{2\sqrt{-1}}\Big[4P^4Q^4p^7q+16P^2Q^2(P^2-Q^2)^2p^5q^3+4[(P^2-Q^2)^2-2P^2Q^2]^2p^3q^5$$
$$+16P^2Q^2(P^2-Q^2)^2pq^7+4P^4Q^4\frac{q^9}{p}\Big]$$

Adding and arranging the terms

$$\phi(N,\epsilon)\mathrm{w}_0=-\frac{e^{2\mathrm{h}}}{2\sqrt{-1}}pq\{4P^4Q^4(p^6-q^6)-4P^2Q^2(P^2-Q^2)^2(p^2-q^2)^3$$
$$-2p^2q^2(p^2-q^2)[(P^2-Q^2)^2-2P^2Q^2]^2\}$$

Then let

$$\mathfrak{H}=\tfrac{3}{4}\{2P^4Q^4(p^6-q^6)-2P^2Q^2(P^2-Q^2)^2(p^2-q^2)^3$$
$$-p^2q^2(p^2-q^2)[(P^2-Q^2)^2-2P^2Q^2]^2\} \quad . \quad . \quad (59)$$

and we have

$$\phi(N,\epsilon)\mathrm{W}_0=-2\mathfrak{H}\mathrm{H}\sin 2\mathrm{h}\sin j \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (60)$$

This is the last of the seven sets of terms.

Then collecting results from (42–5–8, 51–4–7, 60), we have

$$\frac{1}{\sin j}\frac{dj}{dt}=-\frac{\tau^2}{\mathfrak{g}}\frac{k}{\xi}\{2\mathfrak{F}_1\mathrm{F}_1\sin 2\mathrm{f}_1+2\mathfrak{F}\mathrm{F}\sin 2\mathrm{f}-2\mathfrak{F}_2\mathrm{F}_2\sin 2\mathrm{f}_2+2\mathfrak{G}_1\mathrm{G}_1\sin \mathrm{g}_1$$
$$+2\mathfrak{G}\mathrm{G}\sin \mathrm{g}-2\mathfrak{G}_2\mathrm{G}_2\sin \mathrm{g}_2-2\mathfrak{H}\mathrm{H}\sin 2\mathrm{h}\} \quad . \quad (61)$$

The seven gothic-letter functions defined by (41–4–7, 50–3–6–9) are functions of the sines and cosines of half the obliquity and of half the inclination, but they are reducible to forms which may be expressed in the following manner :—

$$\left.\begin{aligned}
\mathfrak{F}_1+\mathfrak{F}_2 &= \tfrac{1}{4}\cos j\left[1-\tfrac{1}{4}\sin^2 j-2\sin^2 i(1-\tfrac{5}{8}\sin^2 j)+\tfrac{5}{8}\sin^4 i(1-\tfrac{7}{4}\sin^2 j)\right]\\
\mathfrak{F}_1-\mathfrak{F}_2 &= \tfrac{1}{4}\cos i\left[1-\tfrac{3}{4}\sin^2 j-\tfrac{3}{2}\sin^2 i(1-\tfrac{5}{4}\sin^2 j)\right]\\
\mathfrak{G}_1+\mathfrak{G}_2 &= -\tfrac{1}{4}\cos j\left[1-\sin^2 j-\tfrac{7}{2}\sin^2 i(1-\tfrac{10}{7}\sin^2 j)+\tfrac{5}{2}\sin^4 i(1-\tfrac{7}{4}\sin^2 j)\right]\\
\mathfrak{G}_1-\mathfrak{G}_2 &= -\tfrac{1}{4}\cos i\left[1-\tfrac{3}{2}\sin^2 j-3\sin^2 i(1-\tfrac{5}{4}\sin^2 j)\right]\\
\mathfrak{F} &= \tfrac{1}{4}\cos i\left[\tfrac{1}{2}\sin^2 j+\sin^2 i-\tfrac{5}{4}\sin^2 i\sin^2 j\right]\\
\mathfrak{G} &= \tfrac{1}{4}\cos i\left[1-\sin^2 j-2\sin^2 i+\tfrac{5}{2}\sin^2 i\sin^2 j\right]\\
\mathfrak{H} &= -\tfrac{1}{4}\cos j\left[\tfrac{3}{4}\sin^2 j+\tfrac{3}{2}\sin^2 i(1-\tfrac{5}{2}\sin^2 j)-\tfrac{15}{8}\sin^4 i(1-\tfrac{7}{4}\sin^2 j)\right]
\end{aligned}\right\} \quad . \quad (62)$$

These coefficients will be applicable whatever theory of tides be used, and no approximation, as regards either the obliquity or inclination, has been used in obtaining them.

§ 7. *Application to the case where the planet is viscous.*

If the planet or earth be viscous with a coefficient of viscosity v, then according to the theory of viscous tides, when inertia is neglected, the tangent of the phase-retardation or lag of any tide is equal to $19v/2gaw$ multiplied by the speed of that tide ; and the height of tide is equal to the equilibrium tide of a perfectly fluid spheroid multiplied by the cosine of the lag. If therefore we put $\dfrac{2gaw}{19v}=\mathfrak{p}$, we have

$$\tan 2\mathrm{f}_1=\frac{2(n-\Omega)}{\mathfrak{p}},\ \tan 2\mathrm{f}=\frac{2n}{\mathfrak{p}},\ \tan 2\mathrm{f}_2=\frac{2(n+\Omega)}{\mathfrak{p}}$$

$$\tan \mathrm{g}_1=\frac{n-2\Omega}{\mathfrak{p}},\ \tan \mathrm{g}=\frac{n}{\mathfrak{p}},\ \tan 2\mathrm{g}_2=\frac{n+2\Omega}{\mathfrak{p}},\ \tan 2\mathrm{h}=\frac{2\Omega}{\mathfrak{p}}$$

$$F_1=\cos 2\mathrm{f}_1,\ F=\cos 2\mathrm{f},\ F_2=\cos 2\mathrm{f}_2,\ G_1=\cos \mathrm{g}_1,\ G=\cos \mathrm{g},\ G_2=\cos \mathrm{g}_2$$

and $H=\cos 2\mathrm{h}$.

Therefore

$$-\frac{\xi}{k\sin j}\frac{dj}{dt}=\frac{\tau^2}{\mathfrak{g}}\{\mathfrak{F}_1\sin 4\mathrm{f}_1+\mathfrak{F}\sin 4\mathrm{f}-\mathfrak{F}_2\sin 4\mathrm{f}_2+\mathfrak{G}_1\sin 2\mathrm{g}_1 +\mathfrak{G}\sin 2\mathrm{g}-\mathfrak{G}_2\sin 2\mathrm{g}_2-\mathfrak{H}\sin 4\mathrm{h}\} \quad . \quad . \quad . \quad (63)$$

This equation involves such complex functions of i and j, that it does not present to the mind any physical meaning. It will accordingly be illustrated graphically.

For this purpose the case is taken when the planet rotates fifteen times as fast as the

satellite revolves. Then the speeds of the seven tides are proportional to the following numbers: 28, 30, 32 (semi-diurnal); 13, 15, 17 (diurnal); and 2 (fortnightly).

It would require a whole series of figures to illustrate the equation for all values of i and j, and for all viscosities. The case is therefore taken where the inclination j of the orbit to the ecliptic is so small that we may neglect squares and higher powers of $\sin j$. Then the formulas (62) become

$$\begin{aligned}
&\mathfrak{F}_1+\mathfrak{F}_2=\tfrac{1}{4}(1-2\sin^2 i+\tfrac{5}{8}\sin^4 i)\\
&\mathfrak{F}_1-\mathfrak{F}_2=\tfrac{1}{4}\cos i(1-\tfrac{3}{2}\sin^2 i)\\
&\mathfrak{G}_1+\mathfrak{G}_2=-\tfrac{1}{4}(1-\tfrac{7}{2}\sin^2 i+\tfrac{5}{2}\sin^4 i)\\
&\mathfrak{G}_1-\mathfrak{G}_2=-\tfrac{1}{4}\cos i(1-3\sin^2 i)\\
&\mathfrak{F}=\tfrac{1}{4}\cos i\sin^2 i,\ \mathfrak{G}=\tfrac{1}{4}\cos i(1-2\sin^2 i)\\
&\mathfrak{H}=-\tfrac{3}{8}\sin^2 i(1-\tfrac{5}{4}\sin^2 i)
\end{aligned}$$

From these we may compute a series of values corresponding to $i=0°$, $15°$, $30°$, $45°$, $60°$, $75°$, $90°$. (I actually did compute them from the P, Q formulas.)

I then took as five several standards of the viscosity of the planet, such viscosities as would make the lag f_1 of the slow semi-diurnal tide (of speed $2n-2\Omega$) equal to 10°, 20°, 30°, 40°, 44°. Then it is easy to compute tables giving the five corresponding values of each of the following, viz.: $\sin 4\mathrm{f}_1$, $\sin 4\mathrm{f}$, $\sin 4\mathrm{f}_2$, $\sin 2\mathrm{g}_1$, $\sin 2\mathrm{g}$, $\sin 2\mathrm{g}_2$, $\sin 4\mathrm{h}$.

Then these numerical values were appropriately multiplied (with CRELLE's three figure table) by the sets of values before found for the $\mathfrak{F}$'s, $\mathfrak{G}$'s, &c.

From the sets of tables formed, the proper sets were selected and added up. The result was to have a series of numbers which were proportional to $dj/\sin j dt$.

Then the series corresponding to each degree of viscosity were set off in a curve, as shown in fig. 4.

The ordinates, which are generally negative, represent $dj/\sin j dt$, and the abscissæ correspond to i, the obliquity of the planet's equator to the ecliptic.

This figure shows that the inclination j of the orbit will diminish, unless the obliquity be very large.

It appears from the results of previous papers, that the satellite's distance will increase as the time increases, unless the obliquity be very large, and if the obliquity be very large the mean distance decreases more rapidly for large than for small viscosity. This statement, taken in conjunction with our present figure, shows that in general the inclination will decrease as long as the mean distance increases, and *vice versâ*. This is not, however, necessarily true for all speeds of rotation of the planet and revolution of the satellite.

The most remarkable feature in these curves is that they show that, for moderate degrees of viscosity (f_1 less than 20°), the inclination j decreases most rapidly when i

the obliquity is zero; whilst for larger viscosities (f_1 between 20° and 45°), there is a very marked maximum rate of decrease for obliquities ranging from 30° to 40°.

Fig. 4.

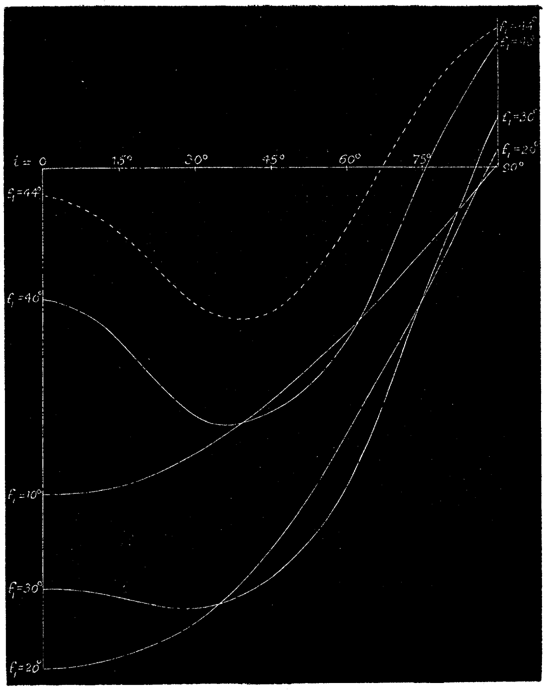


Diagram illustrating the rate of change of the inclination of a satellite's orbit to a fixed plane on which its nodes revolve, for various obliquities and viscosities of the planet $\left(\frac{1}{\sin j}\frac{dj}{dt}\text{ when } j \text{ is small}\right)$.

We now return to the analytical investigation.

If the viscosity be sufficiently small to allow the phase retardations to be small, so that the lag of each tide is proportional to its speed, we may express the lags of all the tides in terms of that of the sidereal semi-diurnal tide, viz.: 2f. Then on this hypothesis we have

$$\frac{\sin 4f_1}{\sin 4f}=1-\lambda,\ \frac{\sin 4f}{\sin 4f}=1,\ \frac{\sin 4f_2}{\sin 4f}=1+\lambda,\ \frac{\sin 2g_1}{\sin 4f}=\tfrac{1}{2}-\lambda,\ \frac{\sin 2g}{\sin 4f}=\tfrac{1}{2}$$

$$\frac{\sin 2g_2}{\sin 4f}=\tfrac{1}{2}+\lambda,\ \frac{\sin 4h}{\sin 4f}=\lambda,\text{ where }\lambda=\frac{\Omega}{n}$$

And

$$-\frac{\xi}{k\sin j}\frac{dj}{dt}=\frac{\tau^2}{\mathfrak{g}}\sin 4f\left[\mathfrak{F}_1+\mathfrak{F}-\mathfrak{F}_2+\tfrac{1}{2}(\mathfrak{G}_1+\mathfrak{G}-\mathfrak{G}_2)-\lambda(\mathfrak{F}_1+\mathfrak{F}_2+\mathfrak{G}_1+\mathfrak{G}_2+\mathfrak{H})\right]$$

But by (62)

$$\mathfrak{F}_1-\mathfrak{F}_2+\tfrac{1}{2}(\mathfrak{G}_1-\mathfrak{G}_2)=\tfrac{1}{8}\cos i\text{ and }\mathfrak{F}+\tfrac{1}{2}\mathfrak{G}=\tfrac{1}{8}\cos i$$

and

$$\mathfrak{F}_1+\mathfrak{F}_2+\mathfrak{G}_1+\mathfrak{G}_2+\mathfrak{H}=0.$$

These results may of course be also obtained when the functions are expressed in terms of P, Q, p, q.

Whence on this hypothesis

$$-\frac{\xi}{k\sin j}\frac{dj}{dt}=\frac{\tau^2}{\mathfrak{g}}\sin 4\mathrm{f}.\tfrac{1}{4}\cos i \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (64)$$

§ 8. *Secular change in the mean distance of a satellite, where there is a second disturbing body, and where the nodes revolve with sensible uniformity on the fixed plane of reference.*

By (11) the equation giving the rate of change of ξ' is

$$\frac{1}{k'}\frac{d\xi'}{dt}=\frac{d\mathrm{W}}{d\epsilon'}$$

As before, we may drop the accents, except as regards ϵ'.

In § 6 we wrote $\phi(\epsilon)$ for the operation $\tan\frac{1}{2}j\,\frac{d}{d\epsilon'}$; hence $\frac{d\mathrm{W}}{d\epsilon'}=\frac{p}{q}\phi(\epsilon)\mathrm{W}$, and by reference to that section the result may be at once written down. We have

$$\frac{1}{k}\frac{d\xi}{dt}=\frac{\tau^2}{\mathfrak{g}}\{2\Phi_1\mathrm{F}_1\sin 2\mathrm{f}_1-2\Phi_2\mathrm{F}_2\sin 2\mathrm{f}_2+2\Gamma_1\mathrm{G}_1\sin \mathrm{g}_1-2\Gamma_2\mathrm{G}_2\sin \mathrm{g}_2-2\Lambda\mathrm{H}\sin 2\mathrm{h}] \quad . \quad (65)$$

Where

$$\left.\begin{aligned}
\Phi_1&=\tfrac{1}{2}[P^8p^8+Q^8q^8+16P^2p^2Q^2q^2(P^4p^4+Q^4q^4)+36P^4Q^4p^4q^4]\\
\Phi_2&=\text{ the same with } Q \text{ and } P \text{ interchanged}\\
\Gamma_1&=2[P^2Q^2(P^4p^8+Q^4q^8)+P^4(P^2-3Q^2)^2p^6q^2+Q^4(3P^2-Q^2)^2p^2q^6\\
&\qquad +9P^2Q^2(P^2-Q^2)^2p^4q^4]\\
\Gamma_2&=\text{ the same with } Q \text{ and } P \text{ interchanged}\\
\Lambda&=3[P^4Q^4(p^8+q^8)+4P^2Q^2(P^2-Q^2)^2p^2q^2(p^4+q^4)\\
&\qquad +p^4q^4[(P^2-Q^2)^2-2P^2Q^2]^2]
\end{aligned}\right\} \quad . \quad (66)$$

These functions are reducible to the following forms

$$\left.\begin{aligned}
2(\Phi_1+\Phi_2)&=1-\sin^2 j+\tfrac{1}{8}\sin^4 j-\sin^2 i(1-2\sin^2 j+\tfrac{5}{8}\sin^4 j)\\
&\qquad +\tfrac{1}{8}\sin^4 i(1-5\sin^2 j+\tfrac{35}{8}\sin^4 j)\\
2(\Phi_1-\Phi_2)&=\cos i\cos j[1-\tfrac{1}{2}\sin^2 j-\tfrac{1}{2}\sin^2 i(1-\tfrac{5}{2}\sin^2 j)]\\
2(\Gamma_1+\Gamma_2)&=\sin^2 j-\tfrac{1}{2}\sin^4 j+\sin^2 i(1-\tfrac{7}{2}\sin^2 j+\tfrac{5}{2}\sin^4 j)\\
&\qquad -\tfrac{1}{2}\sin^4 i(1-5\sin^2 j+\tfrac{35}{8}\sin^4 j)\\
2(\Gamma_1-\Gamma_2)&=\cos i\cos j[\sin^2 j+\sin^2 i(1-\tfrac{5}{2}\sin^2 j)]\\
2\Lambda\quad&=\tfrac{3}{8}\sin^4 j+\sin^2 i(\tfrac{3}{2}\sin^2 j-\tfrac{15}{8}\sin^4 j)\\
&\qquad +\tfrac{3}{8}\sin^4 i(1-5\sin^2 j+\tfrac{35}{8}\sin^4 j)
\end{aligned}\right\} \quad . \quad (67)$$

§ 9. *Application to the case where the planet is viscous.*

As in § 7

$$\frac{1}{k}\frac{d\xi}{dt}=\frac{\tau^2}{\mathfrak{g}}\{\Phi_1 \sin 4\mathrm{f}_1-\Phi_2 \sin 4\mathrm{f}_2+\Gamma_1 \sin 2\mathrm{g}_1-\Gamma_2 \sin 2\mathrm{g}_2-\Lambda \sin 4\mathrm{h}\} \quad . \quad . \quad (68)$$

If j be put equal to zero this equation will be found to be the same as that used as the equation of tidal reaction in the previous paper on "Precession."

If the viscosity be small, with the same notation as before

$$\frac{1}{k}\frac{d\xi}{dt}=\frac{\tau^2}{\mathfrak{g}}\sin 4\mathrm{f}[\Phi_1-\Phi_2+\tfrac{1}{2}(\Gamma_1-\Gamma_2)-\lambda(\Phi_1+\Phi_2+\Gamma_1+\Gamma_2+\Lambda)] \quad . \quad . \quad . \quad (69)$$

Now

$$\Phi_1-\Phi_2+\tfrac{1}{2}(\Gamma_1-\Gamma_2)=\tfrac{1}{2}\cos i \cos j$$

and

$$\Phi_1+\Phi_2+\Gamma_1+\Gamma_2+\Lambda=\tfrac{1}{2}$$

Therefore

$$\frac{1}{k}\frac{d\xi}{dt}=\frac{1}{2}\frac{\tau^2}{\mathfrak{g}}\sin 4\mathrm{f}[\cos i \cos j-\lambda] \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (70)$$

We see that the rate of tidal reaction diminishes as the inclination of the orbit increases.

§ 10. *Secular change in the inclination of the orbit of a single satellite to the invariable plane, where there is no other disturbing body than the planet.*

This is the second of the two cases into which the problem subdivides itself.

If there be only two bodies, then the fixed plane of reference, which was called the ecliptic, may be taken as the invariable plane of the system. It follows from the principle of the composition of moments of momentum that the planet's axis of rotation, the normal to the satellite's orbit and the normal to the invariable plane, necessarily lie in one plane. Whence it follows that the orbit and the equator necessarily intersect in the invariable plane. From this principle it would of course be possible either to determine the motion of the node from the precession of the planet or *vice versâ,* and the change of obliquity of the planet's axis (if any) from the change in the plane of the orbit or *vice versâ;* this principle will be applied later.

We have found it convenient to measure longitudes from a line in the fixed plane, which is instantaneously coincident with the descending node of the equator on the fixed plane. Hence it follows that where there are only two bodies we shall after differentiation have to put $N=N'=0$.

Then since $\varpi' = Pp - Qqe^{N'}$ therefore $\frac{d\varpi'}{dN'} = \frac{1}{\sqrt{-1}} Qq$, and similarly

$$\frac{d\varpi'}{dN'} = -\frac{Qq}{\sqrt{-1}},\ \frac{d\kappa'}{dN'} = -\frac{Pq}{\sqrt{-1}},\ \frac{d\kappa'}{dN'} = \frac{Pq}{\sqrt{-1}},\ \text{when } N'=0.$$

Also after differentiation when $N=0$, $\varpi = \underline{\varpi} = \cos\frac{1}{2}(i+j)$, $\kappa = \underline{\kappa} = \sin\frac{1}{2}(i+j)$

In order to find dj/dt we must, as before, perform $\phi(N, \epsilon)$ on W. Then take the same notation as before for the W's and w's with suffixes.

Slow semi-diurnal term.

$$\frac{d}{dN'}\left(\tfrac{1}{4}\varpi^4\underline{\varpi}'^4\right) = \varpi^7 \frac{d\varpi'}{dN'} = -\frac{\varpi^7 Qq}{\sqrt{-1}}$$

and

$$\phi(N)\left(\tfrac{1}{4}\varpi^4\underline{\varpi}'^4\right) = -\frac{1}{2\sqrt{-1}}\cdot\varpi^7\cdot\frac{Q}{p},\ \text{also } \phi(\epsilon)e^{2(\epsilon'-\epsilon)-2f_1} = -\frac{1}{2\sqrt{-1}}\cdot\frac{4q}{p}e^{-2f_1}$$

Hence

$$\phi(N, \epsilon)\mathrm{w}_{\mathrm{I}} = \frac{e^{-2f_1}}{2\sqrt{-1}}\left[-\varpi^7\frac{Q}{p} - \varpi^8\cdot\frac{q}{p}\right] = -\varpi^7\kappa\frac{e^{-2f_1}}{2\sqrt{-1}}$$

and

$$\phi(N, \epsilon)\mathrm{W}_{\mathrm{I}} = \varpi^7\kappa \mathrm{F}_1 \sin 2\mathrm{f}_1.$$

Sidereal semi-diurnal term.

$$\frac{d\mathrm{w}_{\mathrm{II}}}{dN'} = 2\varpi^3\kappa^3\left(\varpi\frac{d\kappa'}{dN'} + \kappa\frac{d\varpi'}{dN'}\right) = -\frac{2\varpi^3\kappa^3}{\sqrt{-1}}(\varpi Pq + \kappa Qq) = -\frac{2\varpi^3\kappa^3}{\sqrt{-1}}\cdot pq$$

and since $\phi(\epsilon)\mathrm{W}_{\mathrm{II}} = 0$, therefore

$$\phi(N, \epsilon)\ \mathrm{W}_{\mathrm{II}} = 2\varpi^3\kappa^3 \mathrm{F} \sin 2\mathrm{f}$$

Fast semi-diurnal term.

By symmetry

$$\phi(N, \epsilon)\mathrm{W}_{\mathrm{III}} = \varpi\kappa^7 \mathrm{F}_2 \sin 2\mathrm{f}_2$$

Slow diurnal term.

$$\frac{d}{dN'}\underline{\varpi}'^3\underline{\kappa}' = 3\varpi^2\kappa\frac{d\varpi'}{dN'} + \varpi^3\frac{d\kappa'}{dN'} = -\frac{\varpi^2}{\sqrt{-1}}(3\kappa Qq - \varpi Pq)$$

$$\phi(N, \epsilon)\mathrm{w}_1 = -\frac{e^{-\mathrm{g}_1}}{2\sqrt{-1}}\ \frac{\varpi^5\kappa}{p}(3Q\kappa - P\varpi + 4q\varpi\kappa) = -\frac{e^{-\mathrm{g}_1}}{2\sqrt{-1}}\ \varpi^5\kappa(\varpi^2 - 3\kappa^2)$$

and

$$\phi(N, \epsilon)\mathrm{W}_1 = -\varpi^5\kappa(\varpi^2 - 3\kappa^2)\mathrm{G}_1 \sin \mathrm{g}_1$$

Sidereal diurnal term.

$$\frac{d}{dN'}\underline{\varpi'\kappa'}=-\frac{q}{\sqrt{-1}}(Q\kappa+P\varpi)=-\frac{pq}{\sqrt{-1}}\text{ and }\frac{d}{dN'}(\varpi'\underline{\varpi'}-\kappa'\underline{\kappa'})=0$$

Therefore

$$\phi(N,\,\epsilon)\mathrm{w}_2=-\frac{e^{-\mathrm{g}}}{\sqrt{-1}}\varpi\kappa(\varpi^2-\kappa^2)^2$$

and

$$\phi(N,\,\epsilon)\mathrm{W}_2=\varpi\kappa(\varpi^2-\kappa^2)^2\mathrm{G}\sin\mathrm{g}$$

Fast diurnal term.

By symmetry

$$\phi(N,\,\epsilon)\mathrm{W}_3=\varpi\kappa^5(3\varpi^2-\kappa^2)\mathrm{G}_2\sin\mathrm{g}_2$$

Fortnightly term.

$$-\frac{d}{dN'}(\underline{\varpi'}\underline{\kappa'})^2=-\frac{2\varpi\kappa}{\sqrt{-1}}q(\kappa Q-P\varpi)$$

and

$$\phi(N,\,\epsilon)\mathrm{w}_0=-\frac{e^{2\mathrm{h}}}{2\sqrt{-1}}\,\varpi^2\kappa^2\left[2\varpi\kappa(Q\kappa-P\varpi)+4\varpi^2\kappa^2q\right]=\frac{e^{2\mathrm{h}}}{2\sqrt{-1}}2\varpi\kappa(\varpi^2-\kappa^2)$$

Whence

$$\phi(N,\,\epsilon)\mathrm{W}_0=3\varpi^3\kappa^3(\varpi^2-\kappa^2)\mathrm{H}\sin 2\mathrm{h}$$

Then collecting terms we have, on applying the result to the case of viscosity,

$$-\frac{\xi}{k}\frac{dj}{dt}=\frac{\tau^2}{\mathfrak{g}}\left[\tfrac{1}{2}\varpi^7\kappa\sin 4\mathrm{f}_1+\varpi^3\kappa^3\sin 4\mathrm{f}+\tfrac{1}{2}\varpi\kappa^7\sin 4\mathrm{f}_2+\tfrac{3}{2}\varpi^3\kappa^3(\varpi^2-\kappa^2)\sin 4\mathrm{h}\right.$$
$$\left.-\tfrac{1}{2}\varpi^5\kappa(\varpi^2-3\kappa^2)\sin 2\mathrm{g}_1+\tfrac{1}{2}\varpi\kappa(\varpi^2-\kappa^2)^2\sin 2\mathrm{g}+\tfrac{1}{2}\varpi\kappa^5(3\varpi^2-\kappa^2)\sin 2\mathrm{g}_2\right]\,. \quad (71)$$

In the particular case where the viscosity is small, this becomes

$$-\frac{\xi}{k}\frac{dj}{dt}=\frac{1}{2}\frac{\tau^2}{\mathfrak{g}}\sin 4\mathrm{f}\,\varpi\kappa=\frac{1}{4}\frac{\tau^2}{\mathfrak{g}}\sin 4\mathrm{f}\sin(i+j)\quad .\ .\ .\ .\ .\ .\ .\quad (72)$$

The right hand side is necessarily positive, and therefore the inclination of the orbit to the invariable plane will always diminish with the time.

The general equation (71) for any degree of viscosity is so complex as to present no idea to the mind, and it will accordingly be graphically illustrated.

The case taken is where $n/\Omega=15$, which is the same relation as in the previous graphical illustration of § 7.

The general method of illustration is sufficiently explained in that section.

Fig. 5 illustrates the various values which dj/dt (the rate of increase of inclination to the invariable plane) is capable of assuming for various viscosities of the planet, and

for various inclinations of the satellite's orbit to the planet's equator. Each curve corresponds to one degree of viscosity, the viscosity being determined by the lag of the slow semi-diurnal tide of speed $2n-2\Omega$. The ordinates give dj/dt (not as before $dj/\sin j dt$) and the abscissæ give $i+j$, the inclination of the orbit to the equator.

Fig. 5.

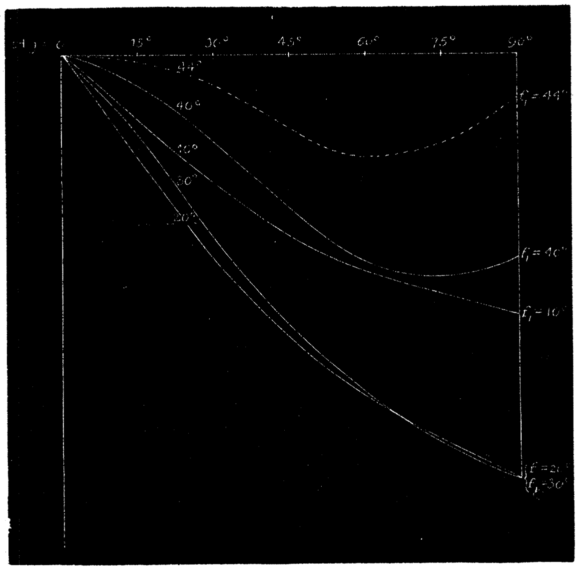


Diagram illustrating the rate of change of the inclination of a single satellite's orbit to the invariable plane, for various viscosities of the planet, and various inclinations of the orbit to the planet's equator $\left(\frac{dj}{dt}\right)$.

We see from this figure that the inclination to the invariable plane will always decrease as the time increases, and the only noticeable point is the maximum rate of decrease for large viscosities, for inclinations of the orbit and equator ranging from 60° to 70°. If n/Ω had been taken considerably smaller than 15, the inclination would have been found to increase with the time for large viscosity of the planet.

§ 11. *Secular change in the mean distance of the satellite, where there is no other disturbing body than the planet.—Comparison with result of previous paper.*

To find the variation of ξ we have to differentiate with respect to ϵ', and the following result may be at once written down

$$\frac{d\xi}{kdt}=\frac{1}{2}\frac{\tau^2}{\mathfrak{g}}[\varpi^8 \sin 4\mathrm{f}_1-\kappa^8 \sin 4\mathrm{f}_2+4\varpi^6\kappa^2 \sin 2\mathrm{g}_1-4\varpi^2\kappa^6 \sin 2\mathrm{g}_2-6\varpi^4\kappa^4 \sin 4\mathrm{h}]. \quad (73)$$

This agrees with the result of a previous paper (viz.: (57) or (79) of "Precession"), obtained by a different method; but in that case the inclination of the orbit was zero, so that ϖ and κ were the cosine and sine of half the obliquity, instead of the cosine and sine of $\frac{1}{2}(i+j)$.

In the case where the viscosity is small this becomes

$$\frac{d\xi}{kdt}=\frac{1}{2}\frac{\tau^2}{\mathfrak{g}}\sin 4\mathrm{f}[\cos(i+j)-\lambda] \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (74)$$

It will now be shown that the preceding result (71) for dj/dt may be obtained by means of the principle of conservation of moment of momentum, and by the use of the results of a previous paper.

It is easily shown that the moment of momentum of orbital motion of the moon and earth round their common centre of inertia is $C\xi/k$, and the moment of momentum of the earth's rotation is clearly Cn. Also j and i are the inclinations of the two axes of moment of momentum to the axis of resultant moment of momentum of the system. Hence

$$\frac{\xi}{k}\sin j=n\sin i$$

By differentiation of which

$$\frac{\xi}{k}\frac{dj}{dt}\cos j=\frac{dn}{dt}\sin i+n\cos i\frac{di}{dt}-\frac{1}{k}\frac{d\xi}{dt}\sin j$$

$$=\left[\frac{dn}{dt}\sin(i+j)+n\cos(i+j)\frac{di}{dt}\right]\cos j-\left[\frac{dn}{dt}\cos(i+j)-n\sin(i+j)\frac{di}{dt}+\frac{1}{k}\frac{d\xi}{dt}\right]\sin j$$

Now from equation (52) of the paper on "Precession," the second term on the right-hand side is zero, and therefore

$$\frac{\xi}{k}\frac{dj}{dt}=\frac{dn}{dt}\sin(i+j)+n\cos(i+j)\frac{di}{dt}$$

But by equations (21) and (16) and (29) of the paper on "Precession" (when ϖ and κ are written for the p, q of that paper)

$$\frac{dn}{dt}=-\frac{\tau^2}{\mathfrak{g}}\left[\tfrac{1}{2}\varpi^8\sin 4\mathrm{f}_1+2\varpi^4\kappa^4\sin 4\mathrm{f}+\tfrac{1}{2}\kappa^8\sin 4\mathrm{f}_2+\varpi^6\kappa^2\sin 2\mathrm{g}_1\right.$$
$$\left.+\varpi^2\kappa^2(\varpi^2-\kappa^2)^2\sin 2\mathrm{g}+\varpi^2\kappa^6\sin 2\mathrm{g}_2\right]$$

$$n\frac{di}{dt}=\frac{\tau^2}{\mathfrak{g}}\left[\tfrac{1}{2}\varpi^7\kappa\sin 4\mathrm{f}_1-\varpi^3\kappa^3(\varpi^2-\kappa^2)\sin 4\mathrm{f}-\tfrac{1}{2}\varpi\kappa^7\sin 4\mathrm{f}_2+\tfrac{1}{2}\varpi^5\kappa(\varpi^2+3\kappa^2)\sin 2\mathrm{g}_1\right.$$
$$\left.-\tfrac{1}{2}\varpi\kappa(\varpi^2-\kappa^2)^3\sin 2\mathrm{g}-\tfrac{1}{2}\varpi\kappa^5(3\varpi^2+\kappa^2)\sin 2\mathrm{g}_2-\tfrac{3}{2}\varpi^3\kappa^3\sin 4\mathrm{h}\right]$$

Then if we multiply the former of these by $\sin(i+j)$ or $2\varpi\kappa$, and the latter by $\cos(i+j)$ or $\varpi^2-\kappa^2$, and add, we get the equation (71), which has already been established by the method of the disturbing function.

It seemed well to give this method, because it confirms the accuracy of the two long analytical investigations in the paper on "Precession" and in the present one.

§ 12. *The method of the disturbing function applied to the motion of the planet.*

In the case where there are only two bodies, viz.: the planet and the satellite, the problem is already solved in the paper on "Precession," and it is only necessary to remember that the p and q of that paper are really $\cos\frac{1}{2}(i+j)$, $\sin\frac{1}{2}(i+j)$, instead of $\cos\frac{1}{2}i$, $\sin\frac{1}{2}i$. This will not be reinvestigated, but we will now consider the case of two satellites, the nodes of whose orbits revolve with uniform angular velocity on the ecliptic. The results may be easily extended to the hypothesis of any number of satellites.

In (18) we have the equations of variation of i, ψ, χ in terms of W. But as the correction to the precession has not much interest, we will only take the two equations

$$\left.\begin{aligned} n\sin i\frac{di}{dt}&=\cos i\frac{d\mathrm{W}}{d\chi'}-\frac{d\mathrm{W}}{d\psi'} \\ \frac{dn}{dt}&=\frac{d\mathrm{W}}{d\chi'} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (75)$$

which give the rate of change of obliquity and the tidal friction.

In the development of W in § 5, it was assumed that ψ, ψ' were zero, and χ, χ' did not appear, because χ was left unaccented in the X′-Y′-Z′ functions.

Longitudes were there measured from the autumnal equinox, but here we must conceive the N, N' of previous developments replaced by $N-\psi$, $N'-\psi'$; also $\Omega t+\epsilon$, $\Omega' t+\epsilon'$ must be replaced by $\Omega t+\epsilon-\psi$, $\Omega' t+\epsilon'-\psi'$.

It will not be necessary to redevelop W for the following reasons.

$\Omega' t+\epsilon'-\psi'$ occurs only in the exponentials, and $N'-\psi'$ does not occur there; and $N'-\psi'$ only occurs in the functions of ϖ and κ, and $\Omega' t+\epsilon'-\psi'$ does not occur there. Hence

$$-\frac{d\mathrm{W}}{d\psi'}=\frac{d\mathrm{W}}{d\epsilon'}+\frac{d\mathrm{W}}{dN'} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (76)$$

Again, it will be seen by referring to the remarks made as to χ, χ' in the development of W in § 5, that we have the following identities :—

$$\left.\begin{array}{l} \text{For semi-diurnal terms,} \\ \qquad \dfrac{d\mathrm{W}_{\mathrm{I}}}{d\chi'}=-\dfrac{d\mathrm{W}_{\mathrm{I}}}{d\epsilon'},\ \dfrac{d\mathrm{W}_{\mathrm{II}}}{d\chi'}=\dfrac{d\mathrm{W}_{\mathrm{II}}}{df},\ \dfrac{d\mathrm{W}_{\mathrm{III}}}{d\chi'}=\dfrac{d\mathrm{W}_{\mathrm{III}}}{d\epsilon'} \\ \text{For diurnal terms,} \\ \qquad \dfrac{d\mathrm{W}_1}{d\chi'}=-\frac{1}{2}\dfrac{d\mathrm{W}_1}{d\epsilon'},\ \dfrac{d\mathrm{W}_2}{d\chi'}=\dfrac{d\mathrm{W}_2}{dg},\ \dfrac{d\mathrm{W}_3}{d\chi'}=\frac{1}{2}\dfrac{d\mathrm{W}_3}{d\epsilon'} \\ \text{For the fortnightly term,} \\ \qquad \dfrac{d\mathrm{W}_0}{d\chi'}=0 \\ \text{Also} \\ \qquad \dfrac{d\mathrm{W}_{\mathrm{II}}}{d\epsilon'}=0,\ \dfrac{d\mathrm{W}_2}{d\epsilon'}=0 \end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (77)$$

Then making use of (76) and (77), and remembering that $\cos i = P^2 - Q^2$, $\sin i = 2PQ$, we may write equations (75), thus

$$(2PQ)n\frac{di}{dt} = \frac{d}{d\epsilon'}[2Q^2\mathrm{W}_{\mathrm{I}} + 2P^2\mathrm{W}_{\mathrm{III}} + \tfrac{1}{2}(P^2+3Q^2)\mathrm{W}_1 + \tfrac{1}{2}(3P^2+Q^2)\mathrm{W}_3 + \mathrm{W}_0] + (P^2-Q^2)\left[\frac{d\mathrm{W}_{\mathrm{II}}}{d\mathrm{f}} + \frac{d\mathrm{W}_2}{d\mathrm{g}}\right] + \frac{d}{dN'}(\Sigma\mathrm{W}) \quad . \quad (78)$$

$$\frac{dn}{dt} = -\frac{d}{d\epsilon'}[\mathrm{W}_{\mathrm{I}} - \mathrm{W}_{\mathrm{III}} + \tfrac{1}{2}\mathrm{W}_1 - \tfrac{1}{2}\mathrm{W}_3] + \frac{d\mathrm{W}_{\mathrm{II}}}{d\mathrm{f}} + \frac{d\mathrm{W}_2}{d\mathrm{g}} \quad . \quad . \quad . \quad . \quad . \quad (79)$$

It is clear that by using these transformations we may put $\psi = \psi' = 0$, $\chi = \chi'$ before differentiation, so that ψ and χ again disappear, and we may use the old development of W.

The case where Diana and the moon are distinct bodies will be taken first, and it will now be convenient to make Diana identical with the sun.

In this case after the differentiations are made we are *not* to put $N = N'$ and $\epsilon = \epsilon'$.

The only terms, out of which secular changes in i and n can arise, are those depending on the sidereal semi-diurnal and diurnal tides, for all others are periodic with the longitudes of the two disturbing bodies. Hence the disturbing function is reduced to W_{II} and W_2. Also $d\mathrm{W}_{\mathrm{II}}/dN'$ and $d\mathrm{W}_2/dN'$ can only contribute periodic terms, because $N - N'$ is not zero, and by hypothesis the nodes revolve uniformly on the ecliptic.

Then if we consider that here p' is not equal p, nor q' to q, we see that, as far as is of present interest,

$$\mathrm{W}_{\mathrm{II}} = 2\mathrm{F}\cos 2\mathrm{f}\; P^4Q^4[(p^2-q^2)^2 - 2p^2q^2][(p'^2-q'^2)^2 - 2p'^2q'^2]$$
$$\mathrm{W}_2 = 2\mathrm{G}\cos \mathrm{g}\; P^2Q^2(P^2-Q^2)^2[(p^2-q^2)^2 - 2p^2q^2][(p'^2-q'^2)^2 - 2p'^2q'^2]$$

Also the equations of variation of i and n are simply

$$(2PQ)n\frac{di}{dt} = (P^2-Q^2)\left[\frac{d\mathrm{W}_{\mathrm{II}}}{d\mathrm{f}} + \frac{d\mathrm{W}_2}{d\mathrm{g}}\right]$$
$$\frac{dn}{dt} = \frac{d\mathrm{W}_{\mathrm{II}}}{d\mathrm{f}} + \frac{d\mathrm{W}_2}{d\mathrm{g}}$$

Then if we put

$$\left.\begin{aligned} \phi &= 2P^4Q^4[(p^2-q^2)^2 - 2p^2q^2][(p'^2-q'^2)^2 - 2p'^2q'^2] \\ &= \tfrac{1}{8}\sin^4 i(1-\tfrac{3}{2}\sin^2 j)(1-\tfrac{3}{2}\sin^2 j') \\ \tfrac{1}{2}\gamma &= P^2Q^2(P^2-Q^2)^2[(p^2-q^2)^2 - 2p^2q^2][(p'^2-q'^2)^2 - 2p'^2q'^2] \\ &= \tfrac{1}{4}\sin^2 i\cos^2 i(1-\tfrac{3}{2}\sin^2 j)(1-\tfrac{3}{2}\sin^2 j') \end{aligned}\right\} \quad . \quad (80)$$

We have

$$\left.\begin{aligned} -\frac{dn}{dt}&=\frac{2\tau\tau'}{\mathfrak{g}}[2\phi\mathrm{F}\sin 2\mathrm{f}+\gamma\mathrm{G}\sin\mathrm{g}] \\ n\frac{di}{dt}&=-\frac{2\tau\tau'}{\mathfrak{g}}[2\phi\mathrm{F}\sin 2\mathrm{f}+\gamma\mathrm{G}\sin\mathrm{g}]\cot i \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (81)$$

It will be noticed that in (81) $2\tau\tau'$ has been introduced in the equations instead of $\tau\tau'$; this is because in the complete solution of the problem these terms are repeated twice, once for the attraction of the moon on the solar tides, and again for that of the sun on the lunar tides.

The case where Diana is identical with the moon must now be considered. This will enable us to find the effects of the moon's attraction on her own tides, and then by symmetry those of the sun's attraction on his tides.

We will begin with the *tidal friction*.

By comparison with (65)

$$\frac{d}{d\epsilon'}[\mathrm{W}_{\mathrm{I}}-\mathrm{W}_{\mathrm{III}}+\tfrac{1}{2}\mathrm{W}_1-\tfrac{1}{2}\mathrm{W}_3]=2\Phi_1\mathrm{F}_1\sin 2\mathrm{f}_1+2\Phi_2\mathrm{F}_2\sin 2\mathrm{f}_2+\Gamma_1\mathrm{G}_1\sin\mathrm{g}_1+\Gamma_2\mathrm{G}_2\sin\mathrm{g}_2 \quad (82)$$

Now when we put $N=N'$ (see (43) and (52))

$$\mathrm{W}_{\mathrm{II}}=2\mathrm{F}\cos 2\mathrm{f}.\mathrm{w}_{\mathrm{II}} \text{ and } \frac{d\mathrm{W}_{\mathrm{II}}}{d\mathrm{f}}=-4\mathrm{F}\sin 2\mathrm{f}.\mathrm{w}_{\mathrm{II}}$$

Also

$$\mathrm{W}_2=2\mathrm{G}\cos\mathrm{g}.\mathrm{w}_2 \text{ and } \frac{d\mathrm{W}_2}{d\mathrm{g}}=-2\mathrm{G}\sin\mathrm{g}.\mathrm{w}_2$$

Then let

$$\begin{aligned}\Phi=2\mathrm{w}_{\mathrm{II}}&=2P^4Q^4[(p^2-q^2)^2-2p^2q^2]^2+8P^2Q^2(P^4+Q^4)p^2q^2(p^2-q^2)^2+2p^4q^4(P^8+Q^8) \\ &=2P^4Q^4(p^2-q^2)^4+8p^2q^2(p^2-q^2)^2P^2Q^2(P^4+Q^4-P^2Q^2)+2p^4q^4(P^8+4P^4Q^4+Q^8) \quad (83)\end{aligned}$$

and let

$$\begin{aligned}\tfrac{1}{2}\Gamma=\mathrm{w}_2&=P^2Q^2(P^2-Q^2)^2[(p^2-q^2)^2-2p^2q^2]^2 \\ &\qquad +[P^4(P^2-3Q^2)^2+Q^4(3P^2-Q^2)^2]p^2q^2(p^2-q^2)^2+4P^2Q^2(P^4+Q^4)p^4q^4 \\ &=P^2Q^2(P^2-Q^2)^2(p^2-q^2)^4+[(P^2-Q^2)^4-6P^2Q^2(P^2-Q^2)^2+8P^4Q^4]p^2q^2(p^2-q^2)^2 \\ &\qquad +8P^2Q^2(P^4+Q^4-P^2Q^2)p^4q^4 \quad (84)\end{aligned}$$

And we have

$$\begin{aligned}-\frac{dn}{dt}=\frac{\tau^2}{\mathfrak{g}}[2\Phi_1\mathrm{F}_1\sin 2\mathrm{f}_1+2\Phi\mathrm{F}\sin 2\mathrm{f}+2\Phi_2\mathrm{F}_2\sin 2\mathrm{f}_2+\Gamma_1\mathrm{G}_1\sin\mathrm{g}_1 \\ +\Gamma\mathrm{G}\sin\mathrm{g}+\Gamma_2\mathrm{G}_2\sin\mathrm{g}_2]. \quad . \quad (85)\end{aligned}$$

This is only a partial solution, since it only refers to the action of the moon on her own tides.

If the second satellite, say the sun, be introduced, the action of the sun on the solar tides may be written down by symmetry, and the elements of the solar (or terrestrial) orbit may be indicated by the same symbols as before, but with accents.

From (85) and (81) the complete solution may be collected.

In the case of viscosity, and where the viscosity is small, it will be found that the solution becomes

$$-\frac{dn}{dt}=\frac{1}{2}\frac{\sin 4\mathrm{f}}{\mathfrak{g}}\Big\{(1-\tfrac{1}{2}\sin^2 i)(\tau^2+\tau'^2)-\tfrac{1}{2}(1-\tfrac{3}{2}\sin^2 i)(\tau^2\sin^2 j+\tau'^2\sin^2 j')$$
$$-\tau^2\frac{\Omega}{n}\cos i\cos j-\tau'^2\frac{\Omega'}{n}\cos i\cos j'+\tfrac{1}{2}\tau\tau'\sin^2 i(1-\tfrac{3}{2}\sin^2 j)(1-\tfrac{3}{2}\sin^2 j')\Big\} \quad . \quad (86)$$

If j and j' be put equal to zero and Ω'/n neglected, this result will be found to agree with that given in the paper on "Precession," § 17, (83).

We will next consider *the change of obliquity.*

The combined effect has already been determined in (81), but the separate effects of the two bodies remain to be found. The terms of different speeds must now be taken one by one.

Slow semi-diurnal term.

$$n\frac{di}{dt}\div\frac{\tau^2}{\mathfrak{g}}=\frac{Q}{P}\frac{dW_{\mathrm{I}}}{d\epsilon'}+\frac{1}{2PQ}\frac{dW_{\mathrm{I}}}{dN'}$$

We had before

$$-\frac{\xi}{k}\frac{dj}{dt}\div\frac{\tau^2}{\mathfrak{g}}=\frac{q}{p}\frac{dW_{\mathrm{I}}}{d\epsilon'}+\frac{1}{2pq}\frac{dW_{\mathrm{I}}}{dN}$$

Now W_{I} is symmetrical with regard to P and p, Q and q, and so are its differentials with regard to ϵ' and N'. The solution may be written down by symmetry with the "slow semi-diurnal" of § 6, by writing P for p and Q for q and *vice versâ.*

Let

$$\mathsf{F}_1=\tfrac{1}{4}\{P^6p^8-4P^4(P^2-3Q^2)p^6q^2-18P^2Q^2(P^2-Q^2)p^4q^4-4Q^4(3P^2-Q^2)p^2q^6-Q^6q^8\} \quad (87)$$

and

$$n\frac{di}{dt}\div\frac{\tau^2}{\mathfrak{g}}=2\mathsf{F}_1\mathrm{F}_1\sin 2\mathrm{f}\sin i \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (88)$$

Sidereal semi-diurnal term.

$$n\frac{di}{dt}\div\frac{\tau^2}{\mathfrak{g}}=\frac{1}{2PQ}\Big[(P^2-Q^2)\frac{dW_{\mathrm{II}}}{d\mathrm{f}}+\frac{dW_{\mathrm{II}}}{dN'}\Big]$$

Now

$$\frac{d\mathrm{W}_{\mathrm{II}}}{d\mathrm{f}} = -2\Phi\mathrm{F}\sin 2\mathrm{f} \quad \text{and} \quad \frac{d\mathrm{W}_{\mathrm{II}}}{dN'} = 4p^2q^2.2\mathfrak{F}\mathrm{F}\sin 2\mathrm{f}$$

Therefore

$$n\frac{di}{dt} \div \frac{\tau^2}{\mathfrak{g}} = 2\mathrm{F}\sin 2\mathrm{f}\left[-\frac{P^2-Q^2}{2PQ}\Phi + \frac{2p^2q^2}{PQ}\mathfrak{F}\right]$$

On substitution from (44) and (83) for Φ and $\mathfrak{F}$ and simplification, we find that if

$$\mathsf{F} = \tfrac{1}{2}\{P^2Q^2(P^2-Q^2)[(p^2-q^2)^2-2p^2q^2]^2+2p^2q^2(p^2-q^2)^2(P^2-Q^2)^3-2p^4q^4(P^6-Q^6)\} \quad (89)$$

then

$$n\frac{di}{dt} \div \frac{\tau^2}{\mathfrak{g}} = -2\mathsf{F}\mathrm{F}\sin 2\mathrm{f}\sin i \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (90)$$

Fast semi-diurnal term.

$$n\frac{di}{dt} \div \frac{\tau^2}{\mathfrak{g}} = \frac{P}{Q}\frac{d\mathrm{W}_{\mathrm{III}}}{d\epsilon'} + \frac{1}{2PQ}\frac{d\mathrm{W}_{\mathrm{III}}}{dN'}$$

Since $\mathrm{W}_{\mathrm{III}}$ is found from W_{I} by writing Q for P, and $-P$ for Q, and $-2\mathrm{f}_2$ for $2\mathrm{f}_1$, therefore in this case ndi/dt is found from its value in the slow semi-diurnal term by the like changes, and if

$$\mathsf{F}_2 = \tfrac{1}{4}\{Q^6p^8+4Q^4(3P^2-Q^2)p^6q^2+18P^2Q^2(P^2-Q^2)p^4q^4+4P^4(P^2-3Q^2)p^2q^6-P^6q^8\} \quad (91)$$

$$n\frac{di}{dt} \div \frac{\tau^2}{\mathfrak{g}} = -2\mathsf{F}_2\mathrm{F}_2\sin 2\mathrm{f}_2\sin i \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (92)$$

Slow diurnal term.

$$n\frac{di}{dt} \div \frac{\tau^2}{\mathfrak{g}} = \frac{1}{2PQ}\left[\frac{P^2+3Q^2}{2}\frac{d\mathrm{W}_1}{d\epsilon'} + \frac{d\mathrm{W}_1}{dN'}\right]$$

$$\frac{d\mathrm{W}_1}{dN'} = -2\mathrm{G}_1\sin \mathrm{g}_1[P^4(P^2-3Q^2)^2p^6q^2+18P^2Q^2(P^2-Q^2)^2p^4q^4 + 3Q^4(3P^2-Q^2)^2p^2q^6+4P^2Q^6q^8]$$

$$\frac{d\mathrm{W}_1}{d\epsilon'} = 2\mathrm{G}_1\sin \mathrm{g}_1.\Gamma_1$$

Substituting these values and simplifying, it will be found that if

$$\mathsf{G}_1 = \tfrac{1}{4}\{P^4(P^2+3Q^2)p^8+2P^2(P^2-3Q^2)^2p^6q^2-9(P^2-Q^2)^3p^4q^4-2Q^2(3P^2-Q^2)^2p^2q^6 - Q^4(3P^2+Q^2)q^8\} \quad (93)$$

Then

$$n\frac{di}{dt} \div \frac{\tau^2}{\mathfrak{g}} = 2\mathsf{G}_1\mathrm{G}_1\sin \mathrm{g}_1\sin i \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (94)$$

Sidereal diurnal term.

$$n\frac{di}{dt}\div\frac{\tau^2}{\mathfrak{g}}=\frac{1}{2PQ}\left[(P^2-Q^2)\frac{d\mathrm{W}_2}{d\mathrm{g}}+\frac{d\mathrm{W}_2}{dN'}\right]$$

$$\frac{d\mathrm{W}_2}{d\mathrm{g}}=-2\mathrm{G}\sin\mathrm{g}(\tfrac{1}{2}\Gamma)\text{ and }\frac{d\mathrm{W}_2}{dN'}=4p^2q^22\mathfrak{G}\mathrm{G}\sin\mathrm{g}$$

Therefore

$$n\frac{di}{dt}\div\frac{\tau^2}{\mathfrak{g}}=2\mathrm{G}\sin\mathrm{g}\left[-\frac{P^2-Q^2}{4PQ}\Gamma+\frac{2p^2q^2}{PQ}\mathfrak{G}\right]$$

On substitution from (53) and (84) for Γ and $\mathfrak{G}$ and simplification, we find that if

$$\mathsf{G}=\tfrac{1}{4}\{(P^2-Q^2)^3[(p^2-q^2)^2-2p^2q^2]^2-2(P^2-Q^2)[(P^2-Q^2)^2-12P^2Q^2]p^2q^2(p^2-q^2)^2$$
$$-4(P^2-Q^2)(P^4+4P^2Q^2+Q^4)p^4q^4\}\quad(95)$$

Then

$$n\frac{di}{dt}\div\frac{\tau^2}{\mathfrak{g}}=-2\mathsf{G}\mathrm{G}\sin\mathrm{g}\sin i\quad\ldots\ldots\quad(96)$$

Fast diurnal term.

$$\frac{di}{dt}\div\frac{\tau^2}{\mathfrak{g}}=\frac{1}{2PQ}\left[\frac{3P^2+Q^2}{2}\frac{d\mathrm{W}_3}{d\epsilon'}+\frac{d\mathrm{W}_3}{dN'}\right]$$

As the fast semi-diurnal is derived from the slow, so here also ; and if

$$\mathsf{G}_2=\tfrac{1}{4}\{Q^4(3P^2+Q^2)p^8+2Q^2(3P^2-Q^2)^2p^6q^2+9(P^2-Q^2)^3p^4q^4-2P^2(P^2-3Q^2)^2p^2q^6$$
$$-P^4(P^2+3Q^2)q^8\}\quad(97)$$

Then

$$n\frac{di}{dt}\div\frac{\tau^2}{\mathfrak{g}}=-2\mathsf{G}_2\mathrm{G}_2\sin\mathrm{g}_2\sin i\quad\ldots\ldots\quad(98)$$

Fortnightly term.

$$\frac{di}{dt}\div\frac{\tau^2}{\mathfrak{g}}=\frac{1}{2PQ}\left(\frac{d\mathrm{W}_0}{d\epsilon'}+\frac{d\mathrm{W}_0}{dN'}\right)$$

If we take the term in W_0 which has 2h positive in the exponential, we have

$$\frac{d\mathrm{W}_0}{dN'}=\frac{3}{2}\frac{e^{2\mathrm{h}}}{2\sqrt{-1}}[8P^2Q^2(P^2-Q^2)^2p^6q^2+4[(P^2-Q^2)^2-2P^2Q^2]^2p^4q^4$$
$$+24P^2Q^2(P^2-Q^2)^2p^2q^6+8P^4Q^4q^8]$$

$$\frac{d\mathrm{W}_0}{d\epsilon'}=-\frac{3}{2}\frac{e^{2\mathrm{h}}}{2\sqrt{-1}}[4P^4Q^4p^8+16P^2Q^2(P^2-Q^2)^2p^6q^2+4[(P^2-Q^2)^2-2P^2Q^2]^2p^4q^4$$
$$+16P^2Q^2(P^2-Q^2)^2p^2q^6+4P^4Q^4q^8]$$

Then if these be added and simplified, it will be found that if

$$\mathsf{H}=\tfrac{3}{4}(p^4-q^4)[(p^4+q^4)P^2Q^2+2p^2q^2(P^2-Q^2)^2] \quad . \quad . \quad . \quad . \quad . \quad (99)$$

Then

$$n\frac{di}{dt}\div\frac{\tau^2}{\mathfrak{g}}=-2\mathsf{H}\mathrm{H}\sin 2\mathrm{h}\sin i \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (100)$$

Then collecting results from the seven equations (88, 90–2–4–6–8, 100),

$$\begin{aligned} n\frac{di}{dt}=\frac{\tau^2}{\mathfrak{g}}\sin i\{&2\mathsf{F}_1\mathrm{F}_1\sin 2\mathrm{f}_1-2\mathsf{F}\mathrm{F}\sin 2\mathrm{f}-2\mathsf{F}_2\mathrm{F}_2\sin 2\mathrm{f}_2+2\mathsf{G}_1\mathrm{G}_1\sin \mathrm{g}_1 \\ &-2\mathsf{G}\mathrm{G}\sin \mathrm{g}-2\mathsf{G}_2\mathrm{G}_2\sin \mathrm{g}_2-2\mathsf{H}\mathrm{H}\sin 2\mathrm{h}\} \quad . \quad (101) \end{aligned}$$

This is only a partial solution, and refers only to the action of the moon on her own tides; the part depending on the sun alone may be written down by symmetry.

The various functions of i and j here introduced admit of reduction to the following forms :—

$$\left.\begin{aligned} \Phi&=\tfrac{1}{4}\{\tfrac{1}{2}\sin^4 i+\tfrac{1}{2}\sin^2 j(4\sin^2 i-5\sin^4 i)+\tfrac{1}{2}\sin^4 j(1-5\sin^2 i+\tfrac{35}{8}\sin^4 i)\} \\ \tfrac{1}{2}\Gamma&=\tfrac{1}{4}\{\sin^2 i-\sin^4 i+\sin^2 j(1-\tfrac{11}{2}\sin^2 i+5\sin^4 i) \\ &\qquad -\sin^4 j(1-5\sin^2 i+\tfrac{35}{8}\sin^4 i)\} \end{aligned}\right\} \quad (102)$$

$$\left.\begin{aligned} \mathsf{F}_1+\mathsf{F}_2&=\tfrac{1}{4}\cos j\{1-\tfrac{3}{4}\sin^2 i-\tfrac{3}{2}\sin^2 j(1-\tfrac{5}{4}\sin^2 i)\} \\ \mathsf{F}_1-\mathsf{F}_2&=\tfrac{1}{4}\cos i\{1-\tfrac{1}{4}\sin^2 i-2\sin^2 j(1-\tfrac{5}{8}\sin^2 i)+\tfrac{5}{8}\sin^4 j(1-\tfrac{7}{4}\sin^2 i)\} \\ \mathsf{G}_1+\mathsf{G}_2&=\tfrac{1}{4}\cos j \\ \mathsf{G}_1-\mathsf{G}_2&=\tfrac{1}{4}\cos i\{1+\tfrac{1}{2}\sin^2 i-\tfrac{1}{2}\sin^2 j(1+5\sin^2 i)-\tfrac{5}{4}\sin^4 j(1-\tfrac{7}{4}\sin^2 i)\} \\ \mathsf{F}&=\tfrac{1}{4}\cos i\{\tfrac{1}{2}\sin^2 i+\sin^2 j(1-\tfrac{5}{2}\sin^2 i)-\tfrac{5}{4}\sin^4 j(1-\tfrac{7}{4}\sin^2 i)\} \\ \mathsf{G}&=\tfrac{1}{4}\cos i\{1-\sin^2 i-\tfrac{7}{2}\sin^2 j(1-\tfrac{10}{7}\sin^2 i)+\tfrac{5}{2}\sin^4 j(1-\tfrac{7}{4}\sin^2 i)\} \\ \mathsf{H}&=\tfrac{1}{4}\cos j\{\tfrac{3}{4}\sin^2 i+\tfrac{3}{2}\sin^2 j(1-\tfrac{5}{4}\sin^2 i)\} \end{aligned}\right\} \quad (103)$$

Φ_1, Φ_2, Γ_1, Γ_2 are given in equations (67), and ϕ and γ in equations (80).

The expressions for F_1 and F_2 are found by symmetry with those for $\mathfrak{F}_1$ and $\mathfrak{F}_2$, by interchanging i and j; the first of equations (62) then corresponds with the second of (103), and *vice versâ*.

From (103) it follows that

$$\mathsf{F}_1-\mathsf{F}_2+\tfrac{1}{2}(\mathsf{G}_1-\mathsf{G}_2)=\tfrac{3}{8}\cos i(1-\tfrac{3}{2}\sin^2 j)$$

and

$$\mathsf{F}+\tfrac{1}{2}\mathsf{G}=\tfrac{1}{8}\cos i(1-\tfrac{3}{2}\sin^2 j)$$

Also

$$\mathsf{F}_1+\mathsf{F}_2+\mathsf{G}_1+\mathsf{G}_2+\mathsf{H}=\tfrac{1}{2}\cos j$$

The complete solution of the problem may be collected from the equations (101) and (81).

In the case of the viscosity of the earth, and when the viscosity is small, we easily find the complete solution to be

$$n\frac{di}{dt}=\frac{\sin 4\mathrm{f}}{\mathfrak{g}}\cdot\tfrac{1}{4}\sin i\cos i\left\{\tau^2(1-\tfrac{3}{2}\sin^2 j)+\tau'^2(1-\tfrac{3}{2}\sin^2 j')-\frac{2\Omega}{n}\tau^2\sec i\cos j\right.$$
$$\left.-\frac{2\Omega'}{n}\tau'^2\sec i\cos j'-\tau\tau'(1-\tfrac{3}{2}\sin^2 j)(1-\tfrac{3}{2}\sin^2 j')\right\} \quad . \quad . \quad (104)$$

This result agrees with that given in (83) of "Precession," when the squares of j and j' are neglected, and when Ω'/n is also neglected.

The preceding method of finding the tidal friction and change of obliquity is no doubt somewhat artificial, but as the principal object of the present paper is to discuss the secular changes in the elements of the satellite's orbit, it did not seem worth while to develop the disturbing function in such a form as would make it applicable both to the satellite and the planet; it seemed preferable to develop it for the satellite and then to adapt it for the case of the perturbation of the planet.

In long analytical investigations it is difficult to avoid mistakes; it may therefore give the reader confidence in the correctness of the results and process if I state that I have worked out the preceding values of di/dt and dn/dt independently, by means of the determination of the disturbing couples $\mathfrak{L}$, $\mathfrak{M}$, $\mathfrak{N}$. That investigation separated itself from the present one at the point where the products of the X′-Y′-Z′ functions and $\mathfrak{X}$-$\mathfrak{Y}$-$\mathfrak{Z}$ functions are formed, for products of the form $\mathrm{Y'Z'}\times\mathfrak{X}\mathfrak{Y}$ had there to be found. From this early stage the two processes are quite independent, and the identity of the results is confirmatory of both. Moreover, the investigation here presented reposes on the values found for dj/dt and $d\xi/dt$, hence the correctness of the result of the first problem here treated was also confirmed.

III.

THE PROPER PLANES OF THE SATELLITE, AND OF THE PLANET, AND THEIR SECULAR CHANGES.

§ 13. *On the motion of a satellite moving about a rigid oblate spheroidal planet, and perturbed by another satellite.*

The present problem is to determine the joint effects of the perturbing influence of the sun, and of the earth's oblateness upon the motion of the moon's nodes, and upon the inclination of the orbit to the ecliptic; and also to determine the effects on the

obliquity of the ecliptic and on the earth's precession. In the present configuration of the three bodies the problem presents but little difficulty, because the influence of oblateness on the moon's motion is very small compared with the perturbation due to the sun; on the other hand, in the case of Jupiter, the influence of oblateness is more important than that of solar perturbation. In each of these special cases there is an appropriate approximation which leads to the result. In the present problem we have, however, to obtain a solution, which shall be applicable to the preponderance of either perturbing cause, because we shall have to trace, in retrospect, the evanescence of the solar influence, and the increase of the influence of oblateness.

The lunar orbit will be taken as circular, and the earth or planet as homogeneous and of ellipticity $\mathfrak{e}$, so that the equation to its surface is

$$\rho=a\{1+\mathfrak{e}(\tfrac{1}{3}-\cos^2\theta)\}$$

The problem will be treated by the method of the disturbing function, and the method will be applied so as to give the perturbations both of the moon and earth.

First consider only the influence of oblateness.

Let ρ, θ be the coordinates of the moon, so that $\rho=c$ and $\cos\theta=\mathrm{M}_3$. Then in the formula (17) § 2, $r=c$ and $\frac{\sigma}{a}=\mathfrak{e}(\frac{1}{3}-\mathrm{M}_3{}^2)$, so that the disturbing function

$$\mathrm{W}=\tau\mathfrak{e}(\tfrac{1}{3}-\mathrm{M}_3{}^2)$$

This function, when suitably developed, will give the perturbation of the moon's motion due to oblateness, and the lunar precession and nutation of the earth.

Then by (21) we have

$$\mathrm{M}_3=\sin i\,[p^2\sin(l+N)-q^2\sin(l-N)]+\sin j\cos i\sin l.$$

Where l is the moon's longitude measured from the node, and N is the longitude of the ascending node of the lunar orbit measured from the descending node of the equator.

Then as we are only going to find secular inequalities, we may, in developing the disturbing function, drop out terms involving l; also we must write $N-\psi$ for N, because we cannot now take the autumnal equinox as fixed.

Then omitting all terms which involve l,

$$\mathrm{M}_3{}^2=\sin^2 i\,[\tfrac{1}{2}(p^4+q^4)-p^2q^2\cos 2(N-\psi)]+\tfrac{1}{2}\sin^2 j\cos^2 i$$
$$+\sin j\sin i\cos i\,[p^2-q^2]\cos(N-\psi)$$

Since $p=\cos\frac{1}{2}j$, $q=\sin\frac{1}{2}j$, we have

$$p^4+q^4=1-\tfrac{1}{2}\sin^2 j,\ p^2q^2=\tfrac{1}{4}\sin^2 j,\ p^2-q^2=\cos j$$

and

$$M_3^2=\tfrac{1}{2}\sin^2 i(1-\tfrac{1}{2}\sin^2 j)+\tfrac{1}{2}\sin^2 j(1-\sin^2 i)$$
$$+\tfrac{1}{4}\sin 2i\sin 2j\cos(N-\psi)-\tfrac{1}{4}\sin^2 i\sin^2 j\cos 2(N-\psi)$$

Now

$$\tfrac{1}{2}(\sin^2 i+\sin^2 j)-\tfrac{3}{4}\sin^2 i\sin^2 j-\tfrac{1}{3}=-\tfrac{1}{3}(1-\tfrac{3}{2}\sin^2 i)(1-\tfrac{3}{2}\sin^2 j)$$

Wherefore

$$W=\tau\mathfrak{C}\{\tfrac{1}{3}(1-\tfrac{3}{2}\sin^2 i)(1-\tfrac{3}{2}\sin^2 j)-\tfrac{1}{4}\sin 2i\sin 2j\cos(N-\psi)$$
$$+\tfrac{1}{4}\sin^2 i\sin^2 j\cos 2(N-\psi)\} \qquad (105)$$

This is the disturbing function.

Before applying it, we will assume that i and j are sufficiently small to permit us to neglect $\sin^2 i\sin^2 j$ compared with unity.

Then

$$\tfrac{1}{3}(1-\tfrac{3}{2}\sin^2 i)(1-\tfrac{3}{2}\sin^2 j)=\tfrac{1}{12}+\tfrac{1}{4}-\tfrac{1}{2}\sin^2 i-\tfrac{1}{2}\sin^2 j+\sin^2 i\sin^2 j-\tfrac{1}{4}\sin^2 i\sin^2 j$$
$$=\tfrac{1}{12}+\tfrac{1}{4}\cos 2i\cos 2j-\tfrac{1}{4}\sin^2 i\sin^2 j$$

Hence, when we neglect the terms in $\sin^2 i\sin^2 j$

$$W=\tfrac{1}{4}\tau\mathfrak{C}\{\tfrac{1}{3}+\cos 2i\cos 2j-\sin 2i\sin 2j\cos(N-\psi)\} \quad . \quad . \quad . \quad . \quad (106)$$

Then since this disturbing function does not involve the epoch or χ, we have by (13), (14), and (18)

$$-\frac{\xi}{k}\sin j\frac{dj}{dt}=\frac{dW}{dN},\qquad \frac{\xi}{k}\sin j\frac{dN}{dt}=\frac{dW}{dj},\qquad -n\sin i\frac{di}{dt}=\frac{dW}{d\psi},\qquad n\sin i\frac{d\psi}{dt}=\frac{dW}{di}$$

Thus as far as concerns the influence of the oblateness on the moon, and the reaction of the moon on the earth,

$$\left.\begin{aligned}
\frac{\xi}{k}\sin j\frac{dj}{dt}&=-\tfrac{1}{4}\tau\mathfrak{C}\sin 2i\sin 2j\sin(N-\psi)\\
\frac{\xi}{k}\sin j\frac{dN}{dt}&=-\tfrac{1}{2}\tau\mathfrak{C}\{\cos 2i\sin 2j+\sin 2i\cos 2j\cos(N-\psi)\}\\
n\sin i\frac{di}{dt}&=\ \ \tfrac{1}{4}\tau\mathfrak{C}\sin 2i\sin 2j\sin(N-\psi)\\
n\sin i\frac{d\psi}{dt}&=-\tfrac{1}{2}\tau\mathfrak{C}\{\sin 2i\cos 2j+\cos 2i\sin 2j\cos(N-\psi)\}
\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (107)$$

If there be no other disturbing body, and if we refer the motion to the invariable plane of the system, we must always have $N=\psi$.

In this case the first and third of (107) become

$$\frac{dj}{dt}=\frac{di}{dt}=0$$

and the second and fourth become

$$\frac{\xi}{k}\sin j\,\frac{dN}{dt}=n\sin i\,\frac{d\psi}{dt}=-\tfrac{1}{2}\tau\mathfrak{k}\sin 2(i+j)$$

But ξ/k is proportional to the moment of momentum of the orbital motion, and n is proportional to the moment of momentum of the earth's rotation, and so by the definition of the invariable plane

$$\frac{\xi}{k}\sin j=n\sin i \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (108)$$

Wherefore $\frac{dN}{dt}=\frac{d\psi}{dt}$, and it follows that the two nodes remain coincident. This result is obviously correct.

In the present case, however, there is another disturbing body, and we must now consider

The perturbing influence of the sun.

Accented symbols will here refer to the elements of the solar orbit.

We might of course form the disturbing function, but it is simpler to accept the known results of lunar theory; these are that the inclination of the lunar orbit to the ecliptic remains constant, whilst the nodes regrede with an angular velocity $\frac{3}{4}\left(\frac{\Omega'}{\Omega}\right)^2\left[1-\frac{3}{8}\frac{\Omega'}{\Omega}\right]\Omega\cos j$.

Now $\frac{3}{4}\left(\frac{\Omega'}{\Omega}\right)^2\Omega=\frac{1}{2}(\frac{3}{2}\Omega'^2)\times\frac{1}{\Omega}=\frac{1}{2}\frac{\tau'}{\Omega}$ in our notation. Hence I shall write $\frac{1}{2}\frac{\tau'}{\Omega}$ for $\frac{3}{4}\left(\frac{\Omega'}{\Omega}\right)^2\left[1-\frac{3}{8}\frac{\Omega'}{\Omega}\right]\Omega$, although if necessary (in Part IV.) I shall use the more accurate formula for numerical calculation.

For the solar precession and nutation we may obtain the results from (107) by putting $j=0$, and τ' for τ.

Thus for the solar effects we have

$$\left.\begin{aligned}\frac{dj}{dt}&=0\\ \frac{dN}{dt}&=-\frac{1}{2}\frac{\tau'}{\Omega}\cos j\\ \frac{di}{dt}&=0\\ n\sin i\,\frac{d\psi}{dt}&=-\tfrac{1}{2}\tau'\mathfrak{k}\sin 2i\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (109)^*$$

* The following seems worthy of remark. By the last of (109) we have $d\psi/dt=-\tau'\mathfrak{k}\cos i/n$.

In this formula $\mathfrak{k}$ is the precessional constant, because the earth is treated as homogeneous.

The full expression for the precessional constant is (2C−A−B)/2C, where A, B, C are the three principal moments of inertia.

Now if we regard the earth and moon as being two particles rotating with an angular velocity Ω about

Then when the system is perturbed both by the oblateness of the earth and by the sun, we have from (107) and (109),

$$\left.\begin{aligned}
\frac{\xi}{k}\sin j\,\frac{dj}{dt}&=-\tfrac{1}{4}\tau\mathfrak{c}\sin 2i\sin 2j\sin(N-\psi)\\
\frac{\xi}{k}\sin j\,\frac{dN}{dt}&=-\tfrac{1}{2}\tau\mathfrak{c}\{\cos 2i\sin 2j+\sin 2i\cos 2j\cos(N-\psi)\}-\tfrac{1}{4}\frac{\tau'}{\Omega}\frac{\xi}{k}\sin 2j\\
n\sin i\,\frac{di}{dt}&=\tfrac{1}{4}\tau\mathfrak{c}\sin 2i\sin 2j\sin(N-\psi)\\
n\sin i\,\frac{d\psi}{dt}&=-\tfrac{1}{2}\tau\mathfrak{c}\{\sin 2i\cos 2j+\cos 2i\sin 2j\cos(N-\psi)\}-\tfrac{1}{2}\tau'\mathfrak{c}\sin 2i
\end{aligned}\right\}\quad(110)$$

The second pair of equations is derivable from the first by writing i for j and j for i; N for ψ and ψ for N; n for ξ/k; n for Ω; and $\frac{1}{2}\mathfrak{c}$ for $\frac{1}{4}$ in the term in τ'.

The first pair of equations may be put into the form

$$\cos 2j\,\frac{d(2j)}{dt}=-\frac{k}{\xi}\tau\mathfrak{c}\sin 2i\cos j\cos 2j\sin(N-\psi)$$

$$\sin 2j\,\frac{dN}{dt}=-\frac{k}{\xi}\tau\mathfrak{c}\{\cos 2i\cos j\sin 2j+\sin 2i\cos i\cos 2i\cos(N-\psi)\}-\tfrac{1}{2}\frac{\tau'}{\Omega}\sin 2j\cos j$$

Now let

$$\left.\begin{aligned}
y&=\tfrac{1}{2}\sin 2j\sin N, & \eta&=\tfrac{1}{2}\sin 2i\sin\psi\\
z&=\tfrac{1}{2}\sin 2j\cos N, & \zeta&=\tfrac{1}{2}\sin 2i\cos\psi
\end{aligned}\right\}\quad\ldots\ldots\quad(111)$$

Therefore

$$\begin{aligned}
2\frac{dz}{dt}&=\cos N\cos 2j\,\frac{d(2j)}{dt}-\sin N\sin 2j\,\frac{dN}{dt}\\
&=\frac{k}{\xi}\tau\mathfrak{c}[\cos j\cos 2j.2\eta+\cos 2i\cos j.2y]+\tfrac{1}{2}\frac{\tau'}{\Omega}\cos j.2y
\end{aligned}$$

or

$$\frac{dz}{dt}=\left(\frac{k\tau\mathfrak{c}}{\xi}\cos 2i\cos j+\tfrac{1}{2}\frac{\tau'}{\Omega}\cos j\right)y+\frac{k\tau\mathfrak{c}}{\xi}\cos j\cos 2j.\eta$$

Again

$$\begin{aligned}
2\frac{dy}{dt}&=\sin N\cos 2j\,\frac{d(2j)}{dt}+\cos N\sin 2j\,\frac{dN}{dt}\\
&=-\frac{k\tau\mathfrak{c}}{\xi}[\cos j\cos 2j.2\zeta+\cos 2i\cos j.2z]-\tfrac{1}{2}\frac{\tau'}{\Omega}\cos j.2z
\end{aligned}$$

their common centre of inertia, then the three principal moments of inertia of the system are $Mmc^2/(M+m)$, $Mmc^2/(M+m)$, 0, and therefore the precessional constant of the system is $\frac{1}{2}$. Thus the formula for dN/dt is precisely analogous to that for $d\psi/dt$, each of them being equal to $\tau'\times$ prec. const. $\times$ cos inclin. $\div$ rotation.

or

$$\frac{dy}{dt} = -\left(\frac{k\tau\mathfrak{e}}{\xi}\cos 2i\cos j + \tfrac{1}{2}\frac{\tau'}{\Omega}\cos j\right)z - \frac{k\tau\mathfrak{e}}{\xi}\cos j\cos 2j.\zeta$$

Now let

$$\left.\begin{aligned} a_1 &= \frac{k\tau\mathfrak{e}}{\xi} \qquad & a_2 &= \tfrac{1}{2}\frac{\tau'}{\Omega} \\ b_1 &= \frac{\tau\mathfrak{e}}{n} \qquad & b_2 &= \frac{\tau'\mathfrak{e}}{n} \end{aligned}\right\} \quad \dots \dots \dots \quad (112)$$

And we have

$$\left.\begin{aligned} \frac{dz}{dt} &= \quad (a_1\cos 2i\cos j + a_2\cos j)y + a_1\cos j\cos 2j.\eta \\ \frac{dy}{dt} &= -(a_1\cos 2i\cos j + a_2\cos j)z - a_1\cos j\cos 2j.\zeta \end{aligned}\right\} \quad \dots \quad (113)$$

and by symmetry from the two latter of (110)

$$\left.\begin{aligned} \frac{d\zeta}{dt} &= \quad (b_1\cos 2j\cos i + b_2\cos i)\eta + b_1\cos i\cos 2i.y \\ \frac{d\eta}{dt} &= -(b_1\cos 2j\cos i + b_2\cos i)\zeta - b_1\cos i\cos 2i.z \end{aligned}\right\} \quad \dots \quad (114)$$

These four simultaneous differential equations have to be solved.

The a's and b's are constant, and if it were not for the cosines on the right the equations would be linear and easily soluble.

It has already been assumed that i and j are not very large, hence it would require large variations of i and j to make considerable variations in the coefficients, I shall therefore substitute for i and j, as they occur explicitly, mean values i_0 and j_0; and this procedure will be justifiable unless it be found subsequently that i and j vary largely.

Then let

$$\left.\begin{aligned} \alpha &= a_1\cos 2i_0\cos j_0 + a_2\cos j_0 \qquad & \beta &= b_1\cos 2j_0\cos i_0 + b_2\cos i_0 \\ \mathrm{a} &= a_1\cos j_0\cos 2j_0 & \mathrm{b} &= b_1\cos i_0\cos 2i_0 \end{aligned}\right\} \quad \dots \quad (115)$$

(Hereafter i and j will be treated as small and the cosines as unity.)

Then

$$\left.\begin{aligned} \frac{dz}{dt} &= \quad \alpha y + \mathrm{a}\eta \\ \frac{dy}{dt} &= -\alpha z - \mathrm{a}\zeta \\ \frac{d\zeta}{dt} &= \quad \beta\eta + \mathrm{b}y \\ \frac{d\eta}{dt} &= -\beta\zeta - \mathrm{b}z \end{aligned}\right\} \quad \dots \dots \dots \quad (116)$$

These equations suggest the solutions

$$z=\Sigma L\cos(\kappa t+\mathrm{m}) \qquad \zeta=\Sigma L'\cos(\kappa t+\mathrm{m})$$
$$y=\Sigma L\sin(\kappa t+\mathrm{m}) \qquad \eta=\Sigma L'\sin(\kappa t+\mathrm{m})$$

Then substituting in (116), we must have

$$-L\kappa=\alpha L+\mathrm{a}L'\,;\quad -L'\kappa=\beta L'+\mathrm{b}L$$

Wherefore

$$\frac{L'}{L}=-\frac{\kappa+\alpha}{\mathrm{a}}=-\frac{\mathrm{b}}{\kappa+\beta}$$

and

$$(\kappa+\alpha)(\kappa+\beta)-\mathrm{ab}=0 \text{ or } \kappa^2+\kappa(\alpha+\beta)+\alpha\beta-\mathrm{ab}=0.$$

This quadratic equation has two real roots (κ_1 and κ_2 suppose), because $(\alpha+\beta)^2-4(\alpha\beta-\mathrm{ab})=(\alpha-\beta)^2+4\mathrm{ab}$ is essentially positive.

Then let

$$\left.\begin{aligned}\kappa_1+\kappa_2&=-(\alpha+\beta)\\ \kappa_1-\kappa_2&=-\{(\alpha-\beta)^2+4\mathrm{ab}\}^{\frac{1}{2}}\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (117)$$

And the solution is

$$\left.\begin{aligned}\tfrac{1}{2}\sin 2j\cos N&=z=L_1\cos(\kappa_1 t+\mathrm{m}_1)+L_2\cos(\kappa_2 t+\mathrm{m}_2)\\ \tfrac{1}{2}\sin 2j\sin N&=y=L_1\sin(\kappa_1 t+\mathrm{m}_1)+L_2\sin(\kappa_2 t+\mathrm{m}_2)\\ \tfrac{1}{2}\sin 2i\cos\psi&=\zeta=L_1'\cos(\kappa_1 t+\mathrm{m}_1)+L_2'\cos(\kappa_2 t+\mathrm{m}_2)\\ \tfrac{1}{2}\sin 2i\sin\psi&=\eta=L_1'\sin(\kappa_1 t+\mathrm{m}_1)+L_2'\sin(\kappa_2 t+\mathrm{m}_2)\\ \text{where}\qquad & \\ \frac{L_1'}{L_1}=-\frac{\kappa_1+\alpha}{\mathrm{a}}&=-\frac{\mathrm{b}}{\kappa_1+\beta}\,;\ \frac{L_2'}{L_2}=-\frac{\kappa_2+\alpha}{\mathrm{a}}=-\frac{\mathrm{b}}{\kappa_2+\beta}\end{aligned}\right\} \quad . \quad . \quad (118)$$

From these equations we have

$$\tfrac{1}{4}\sin^2 2j=L_1^2+L_2^2+2L_1L_2\cos[(\kappa_1-\kappa_2)t+\mathrm{m}_1-\mathrm{m}_2]$$
$$\tfrac{1}{4}\sin^2 2i=L_1'+L_2'+2L_1'L_2'\cos[(\kappa_1-\kappa_2)t+\mathrm{m}_1-\mathrm{m}_2]$$

From this we see that $\sin 2j$ oscillates between $2(L_1+L_2)$ and $2(L_1\backsim L_2)$, and $\sin 2i$ between $2(L_1'+L_2')$ and $2(L_1'\backsim L_2')$.

Let us change the constants introduced by integration, and write $L_1=\frac{1}{2}\sin 2j_0$, $L_2'=\frac{1}{2}\sin 2i_0$.

Then our solution is

$$\left.\begin{aligned}
\sin 2j \cos N &= \sin 2j_0 \cos(\kappa_1 t+\mathrm{m}_1) - \frac{\mathrm{a}}{\kappa_2+\alpha}\sin 2i_0 \cos(\kappa_2 t+\mathrm{m}_2) \\
\sin 2j \sin N &= \sin 2j_0 \sin(\kappa_1 t+\mathrm{m}_1) - \frac{\mathrm{a}}{\kappa_2+\alpha}\sin 2i_0 \sin(\kappa_2 t+\mathrm{m}_2) \\
\sin 2i \cos\psi &= -\frac{\kappa_1+\alpha}{\mathrm{a}}\sin 2j_0 \cos(\kappa_1 t+\mathrm{m}_1) + \sin 2i_0 \cos(\kappa_2 t+\mathrm{m}_2) \\
\sin 2i \sin\psi &= -\frac{\kappa_1+\alpha}{\mathrm{a}}\sin 2j_0 \sin(\kappa_1 t+\mathrm{m}_1) + \sin 2i_0 \sin(\kappa_2 t+\mathrm{m}_2)
\end{aligned}\right\} \quad . \quad . \quad (119)$$

From this it follows that

$$\begin{aligned}
\sin 2i \sin 2j \cos(N-\psi) = {} & -\frac{\kappa_1+\alpha}{\mathrm{a}}\sin^2 2j_0 - \frac{\mathrm{a}}{\kappa_2+\alpha}\sin^2 2i_0 \\
& + \left(1+\frac{\kappa_1+\alpha}{\kappa_2+\alpha}\right)\sin 2i_0 \sin 2j_0 \cos\left[(\kappa_1-\kappa_2)t+\mathrm{m}_1-\mathrm{m}_2\right]
\end{aligned}$$

$$\sin 2i \sin 2j \sin(N-\psi) = \left(1-\frac{\kappa_1+\alpha}{\kappa_2+\alpha}\right)\sin 2i_0 \sin 2j_0 \sin\left[(\kappa_1-\kappa_2)t+\mathrm{m}_1-\mathrm{m}_2\right]$$

Now

$$\begin{gathered}
(\kappa_1+\alpha)(\kappa_2+\alpha) = -(\kappa_1+\alpha)(\kappa_1+\beta) = -\mathrm{ab} \\
\therefore \kappa_1+\kappa_2+2\alpha = \alpha-\beta
\end{gathered}$$

Therefore

$$\left.\begin{aligned}
\sin 2i \sin 2j \cos(N-\psi) = {} & -\frac{1}{\kappa_2+\alpha}\{\mathrm{a}\sin^2 2i_0 - \mathrm{b}\sin^2 2j_0 \\
& -(\alpha-\beta)\sin 2i_0 \sin 2j_0 \cos\left[(\kappa_1-\kappa_2)t+\mathrm{m}_1-\mathrm{m}_2\right]\} \\
\sin 2i \sin 2j \sin(N-\psi) = {} & -\frac{\kappa_1-\kappa_2}{\kappa_2+\alpha}\sin 2i_0 \sin 2j_0 \sin\left[(\kappa_1-\kappa_2)t+\mathrm{m}_1-\mathrm{m}_2\right]
\end{aligned}\right\} \quad (120)$$

From (120) it is clear that the nodes of the lunar orbit will oscillate about the equinoctial line, if

$$\mathrm{a}\sin^2 2i_0 \backsim \mathrm{b}\sin^2 2j_0 \text{ be greater than } (\alpha-\beta)\sin 2i_0 \sin 2j_0,$$

but will rotate (although not uniformly) if the former be less than the latter.

With the present configuration of the earth and moon

$$\mathrm{a}\sin^2 2i_0 \backsim \mathrm{b}\sin^2 2j_0 \text{ is very small compared with } (\alpha-\beta)\sin 2i_0 \sin 2j_0,$$

and the nodes of the lunar orbit revolve very nearly uniformly on the ecliptic; also the inclination of the orbit varies very slightly, as the nodes revolve.

In the investigation in Part II. the secular rate of change in the inclination of the

lunar orbit has been found, on the assumption that the nodes of the lunar orbit rotate uniformly.

It is intended to trace the effects of tidal friction on the earth and moon retrospectively. In the course of the solution the importance of the solar perturbation of the moon, relatively to the influence of the earth's oblateness, will wane; the nodes will cease to revolve uniformly, and the inclination of the lunar orbit and of the equator to the ecliptic will be subject to nutation. The differential equations of Part II. will then cease to be applicable, and new ones will have to be found.

The problem is one of such complication, that I have thought it advisable only to attempt to obtain a solution on the hypothesis of the smallness both of the obliquity and of the inclination of the orbit to the plane of reference or the ecliptic. It seems best however to give the preceding investigation, although it is more accurate than the solution subsequently used.*

The first step towards this further consideration is to obtain a clear idea of the nature of the motions represented by the analytical solutions (118) or (119) of the present problem.

Assuming then i and j to be small, we have from (112) and (115)

$$\alpha=a_1+a_2,\ \mathrm{a}=a_1,\quad \beta=b_1+b_2,\ \mathrm{b}=b_1 \quad . \quad . \quad . \quad . \quad . \quad . \quad (121)$$

$$\left.\begin{aligned} j\cos N &= L_1 \cos(\kappa_1 t+\mathrm{m}_1)+L_2 \cos(\kappa_2 t+\mathrm{m}_2) \\ j\sin N &= L_1 \sin(\kappa_1 t+\mathrm{m}_1)+L_2 \sin(\kappa_2 t+\mathrm{m}_2) \\ i\cos\psi &= L_1' \cos(\kappa_1 t+\mathrm{m}_1)+L_2' \cos(\kappa_2 t+\mathrm{m}_2) \\ i\sin\psi &= L_1' \sin(\kappa_1 t+\mathrm{m}_1)+L_2' \sin(\kappa_2 t+\mathrm{m}_2) \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (122)$$

Take a set of rectangular axes; let the axis of x' pass through the fixed point in the ecliptic from which longitudes are measured, let the axis of z' be drawn perpendicular to the ecliptic northwards, and let the rotation from x' to y' be positive, and therefore consentaneous with the moon's orbital motion.

Then N is the longitude of the ascending node of the lunar orbit, and therefore the direction cosines of the normal to the lunar orbit drawn northwards are,

$$\sin j\cos(N-\tfrac{1}{2}\pi),\ \sin j\sin(N-\tfrac{1}{2}\pi),\ \cos j;\ \text{ or since } j \text{ is small, } j\sin N,\ -j\cos N,\ 1.$$

And ψ is the longitude of the descending node of the equator, and therefore the direction cosines of the earth's axis, drawn northwards are,

$$\sin i\cos(\psi+\tfrac{1}{2}\pi),\ \sin i\sin(\psi+\tfrac{1}{2}\pi),\ \cos i;\ \text{ or since } i \text{ is small, } -i\sin\psi,\ i\cos\psi,\ 1.$$

Now draw a sphere of unit radius, with the origin as centre; draw a tangent plane

* See the foot-note to § 18 for a comparison of these results with those ordinarily given.

to it at the point where the axis of z' meets the sphere, and project on this plane the poles of the lunar orbit and of the earth. We here in fact map the motion of the two poles on a tangent plane to the celestial sphere. Let x', y' be a pair of axes in this plane parallel to our previous x', y'; and let x', y' be the coordinates of the pole of the lunar orbit, and ξ', η' be the coordinates of the earth's pole. Then

$$x'=j\sin N,\ y'=-j\cos N;\ \xi'=-i\sin\psi,\ \eta'=i\cos\psi\,. \quad . \quad . \quad . \quad (123)$$

Let x, y, ξ, η be the coordinates of these same points referred to another pair of rectangular axes in this plane, inclined at an angle ϕ to the axes x', y'.

Then

$$\begin{aligned} x&=\ \ x'\cos\phi+y'\sin\phi\ , & \xi&=\ \ \xi'\cos\phi+\eta'\sin\phi \\ y&=-x'\sin\phi+y'\cos\phi\ , & \eta&=-\xi'\sin\phi+\eta'\cos\phi \end{aligned}$$

From (123) and (118) we have therefore

$$\left.\begin{aligned} x&=\ \ L_1\sin(\kappa_1 t+\mathrm{m}_1-\phi)+L_2\sin(\kappa_2 t+\mathrm{m}_2-\phi) \\ y&=-L_1\cos(\kappa_1 t+\mathrm{m}_1-\phi)-L_2\cos(\kappa_2 t+\mathrm{m}_2-\phi) \\ \xi&=-L_1'\sin(\kappa_1 t+\mathrm{m}_1-\phi)-L_2'\sin(\kappa_2 t+\mathrm{m}_2-\phi) \\ \eta&=\ \ L_1'\cos(\kappa_1 t+\mathrm{m}_1-\phi)+L_2'\cos(\kappa_2 t+\mathrm{m}_2-\phi) \end{aligned}\right\}$$

Now suppose the new axes to rotate with an angular velocity κ_2, and that $\phi=\kappa_2 t+\mathrm{m}_2$.

Then

$$\left.\begin{aligned} x&=L_1\sin[(\kappa_1-\kappa_2)t+\mathrm{m}_1-\mathrm{m}_2] \\ y+L_2&=-L_1\cos[(\kappa_1-\kappa_2)t+\mathrm{m}_1-\mathrm{m}_2] \\ \xi&=-L_1'\sin[(\kappa_1-\kappa_2)t+\mathrm{m}_1-\mathrm{m}_2] \\ \eta-L_2'&=L_1'\cos[(\kappa_1-\kappa_2)t+\mathrm{m}_1-\mathrm{m}_2] \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (124)$$

These four equations represent that each pole describes a circle, relatively to the rotating axes, with a negative angular velocity (because $\kappa_1-\kappa_2$ is negative). The centres of the circles are on the axis of y. The ratio

$$\frac{\text{distance of centre of terrestrial circle}}{\text{distance of centre of lunar circle}}=\frac{L_2'}{-L_2}=\frac{\kappa_2+\alpha}{\mathrm{a}}=\frac{\mathrm{b}}{\kappa_2+\beta} \quad . \quad . \quad . \quad (125)$$

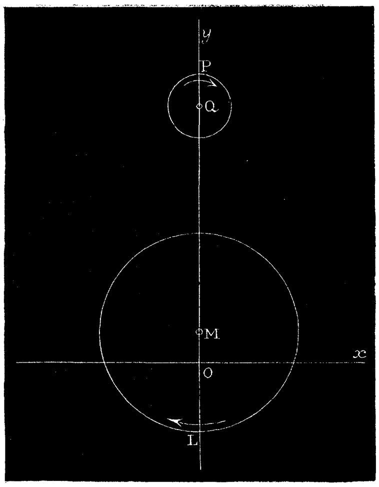
the distances being measured from the pole of the ecliptic. And the ratio

$$\frac{\text{radius of terrestrial circle}}{\text{radius of lunar circle}}=\frac{L_1'}{L_1}=-\frac{\kappa_1+\alpha}{\mathrm{a}}=-\frac{\mathrm{b}}{\kappa_1+\beta} \quad . \quad . \quad . \quad . \quad . \quad (126)$$

According to the definitions adopted in (117) of κ_1 and κ_2, $(\kappa_1+\alpha)/\mathrm{a}$ is negative and $(\kappa_2+\alpha)/\mathrm{a}$ is positive; hence L_1 has the same sign as L_1', and L_2 has the opposite sign from L_2'. When $t=-(\mathrm{m}_1-\mathrm{m}_2)/(\kappa_1-\kappa_2)$, we have

$$x=0,\ y=(-L_2)-L_1,\ \xi=0,\ \eta=L_2'+L_1'$$

Fig. 6.



In fig. 6 let Ox, Oy be the rotating axes, which revolve with a negative rotation equal to κ_2, which is negative. Let M be the centre of the lunar circle, and Q of the terrestrial circle. Then we see that L and P must be simultaneous positions of the two poles, which revolve round their respective circles with an angular velocity $\kappa_2-\kappa_1$, in the direction of the arrows.

M and Q are the poles of two planes, which may be appropriately called the proper planes of the moon and the earth. These proper planes are inclined at a constant angle to one another and to the ecliptic, and have a common node on the ecliptic, and a uniform slow negative precession relatively to the ecliptic.

The lunar orbit and the equator are inclined at constant angles to the lunar and terrestrial proper planes respectively, and the nodes of the orbit, and of the equator regrede uniformly on the respective proper planes.

In the 'Mécanique Céleste' (livre vii., chap. 2, sec. 20) LAPLACE refers to the proper plane of the lunar orbit, but the corresponding inequality of the earth is ordinarily referred to as the 19-yearly nutation. It will be proved later, that the above results are identical with those ordinarily given.

Suppose then that

$$
\left.\begin{array}{l}
\mathrm{I}=\text{the inclination of the earth's proper plane to the ecliptic} \\
\mathrm{J}=\text{the inclination of the lunar orbit to its proper plane} \\
\mathrm{I}_{,}=\text{the inclination of the equator to the earth's proper plane} \\
\mathrm{J}_{,}=\text{the inclination of the moon's proper plane to the ecliptic} \\
\text{Then} \\
\mathrm{J}=L_1,\ \mathrm{I}=L_2',\ \mathrm{I}_{,}=L_1',\ \mathrm{J}_{,}=-L_2 \\
\text{and by (125–6)} \\
\mathrm{I}_{,}=-\frac{\kappa_1+\alpha}{\mathrm{a}}\mathrm{J}=-\frac{\mathrm{b}}{\kappa_1+\beta}\mathrm{J}\ ;\ \mathrm{J}_{,}=\frac{\mathrm{a}}{\kappa_2+\alpha}\mathrm{I}=\frac{\kappa_2+\beta}{\mathrm{b}}\mathrm{I}
\end{array}\right\} \quad . \quad (127)
$$

Thus I and J are the two constants introduced in the integration of the simultaneous differential equations (116).

It is interesting to examine the physical meaning of these results, and to show how the solution degrades into the two limiting cases, viz.: where the planet is spherical, and where the sun's influence is non-existent.

Let $\mathfrak{n}$ be the speed of motion of the nodes, when the ellipticity of the planet is zero.

Let $\mathfrak{l}$ be the purely lunar precession, or the precession when the solar influence is nil.

Let $\mathfrak{m}$ be the ratio of the moment of momentum of the earth's rotation to that of the orbital motion of the two bodies round their common centre of inertia.

Then

$$\mathfrak{n}=\tfrac{1}{2}\frac{\tau'}{\Omega},\quad \mathfrak{l}=\frac{\tau\mathfrak{C}}{n},\quad \mathfrak{m}=\frac{kn}{\xi}$$

Then by (121) and (115) we have

$$\alpha=\mathfrak{m}\mathfrak{l}+\mathfrak{n},\quad \mathrm{a}=\mathfrak{m}\mathfrak{l},\quad \beta=\mathfrak{l}+\frac{\tau'\mathfrak{C}}{n},\quad \mathrm{b}=\mathfrak{l}$$

First suppose that $\mathfrak{n}$ *is large compared with* $\mathfrak{l}$.

This is the case at present with the earth and moon, because the speed of motion of the moon's nodes is very great compared with the speed of the purely lunar precession.

Then a, β, b are small compared with α.

Therefore by (117)

$$\kappa_1-\kappa_2=-\alpha+\beta,\ \kappa_1+\kappa_2=-\alpha-\beta$$

and

$$\kappa_1=-\alpha,\ \kappa_2=-\beta$$

Therefore

$$-\frac{\mathrm{b}}{\kappa_1+\beta}=\frac{\mathrm{b}}{\alpha-\beta}=\frac{\mathfrak{l}}{\mathfrak{n}-(1-\mathfrak{m})\mathfrak{l}-\frac{\tau'\mathfrak{e}}{n}}=\frac{\mathfrak{l}}{\mathfrak{n}}\text{ approximately}$$

$$\frac{\mathrm{a}}{\kappa_2+\alpha}=\frac{\mathrm{a}}{\alpha-\beta}=\mathfrak{m}\frac{\mathfrak{l}}{\mathfrak{n}}\text{ approximately}$$

$$\kappa_2=-\frac{\tau+\tau'}{n}\mathfrak{e},\ \kappa_2-\kappa_1=\mathfrak{n}\text{ approximately}$$

And by (127)

$$\mathrm{I}_{,}=\frac{\mathfrak{l}}{\mathfrak{n}}\mathrm{J},\quad \mathrm{J}_{,}=\mathfrak{m}\frac{\mathfrak{l}}{\mathfrak{n}}\mathrm{I}$$

Now we have shown above that $-\kappa_2$ is the common angular velocity of the pair of proper planes, and the above results show that it is in fact the luni-solar precession.

$\kappa_2-\kappa_1$ is the angular velocity of the two nodes on their proper planes, and it is nearly equal to $\mathfrak{n}$.

The ratio of the amplitude of the 19-yearly nutation to the inclination of the lunar orbit is $\mathfrak{l}/\mathfrak{n}$.

The ratio of the inclination of the lunar proper plane to the obliquity of the ecliptic is $\mathfrak{m}\mathfrak{l}/\mathfrak{n}$.

In this case, therefore, the lunar proper plane is inclined at a small angle to the ecliptic, and if the earth were spherical would be identical with the ecliptic.

Secondly, suppose that $\mathfrak{n}$ *is small compared with* $\mathfrak{l}$.

Then *à fortiori* $\frac{\tau'\mathfrak{e}}{n}$ is small compared with $\mathfrak{l}$. Hence we may put $\beta=\mathrm{b}$.

Therefore

$$\kappa_2-\kappa_1=\sqrt{(\alpha-\beta)^2+4\mathrm{ab}}=\mathrm{a}+\mathrm{b}+\frac{\mathrm{a}-\mathrm{b}}{\mathrm{a}+\mathrm{b}}\mathfrak{n},\text{ nearly}$$

$$=(\mathfrak{m}+1)\mathfrak{l}+\frac{\mathfrak{m}-1}{\mathfrak{m}+1}\mathfrak{n}$$

$$\kappa_2+\kappa_1=-(\mathfrak{m}+1)\mathfrak{l}-\mathfrak{n}$$

$$\kappa_2=-\frac{\mathfrak{n}}{\mathfrak{m}+1},\quad \kappa_1=-(\mathfrak{m}+1)\mathfrak{l}-\frac{\mathfrak{m}}{\mathfrak{m}+1}\mathfrak{n}$$

$$\frac{\kappa_2+\beta}{\mathrm{b}}=1-\frac{1}{\mathfrak{m}+1}\frac{\mathfrak{n}}{\mathfrak{l}};\quad -\frac{\kappa_1+\alpha}{\mathrm{a}}=\frac{1}{\mathfrak{m}}\left(1-\frac{1}{\mathfrak{m}+1}\frac{\mathfrak{n}}{\mathfrak{l}}\right)$$

Therefore

$$\mathrm{I}_{,}=\frac{1}{\mathfrak{m}}\left(1-\frac{1}{\mathfrak{m}+1}\frac{\mathfrak{n}}{\mathfrak{l}}\right)\mathrm{J},\quad \mathrm{J}_{,}=\left(1-\frac{1}{\mathfrak{m}+1}\frac{\mathfrak{n}}{\mathfrak{l}}\right)\mathrm{I}$$

From the last of these,

$$\mathrm{I}-\mathrm{J}_{,}=\frac{1}{\mathfrak{m}+1}\frac{\mathfrak{n}}{\mathfrak{l}}\mathrm{I}$$

$-\kappa_2$ is the precession of the system of proper planes, and the above results show that the solar precession of the planet and satellite together, considered as one system, is one $(\mathfrak{m}+1)^{\text{th}}$ of the angular velocity which the nodes of the satellite would have, if the planet were spherical.

$\kappa_2-\kappa_1$ is the lunar precession of the earth which goes on within the system, and it is approximately the same as though the sun did not exist. (Compare the second and fourth of (107) with $N=\psi$, and use (108)).

It also appears that the lunar proper plane is inclined to the planet's proper plane at a small angle the ratio of which to the inclination of the earth's proper plane to the ecliptic is equal to one $(\mathfrak{m}+1)^{\text{th}}$ part of $\mathfrak{n}/\mathfrak{l}$.

If $\mathfrak{n}$ and $\mathfrak{l}$ are of approximately equal speeds the proper plane of the moon will neither be very near the ecliptic, nor very near the earth's proper plane. The results do not then appear to be reducible to very simple forms; nor are the angular velocities κ_2 and $\kappa_2-\kappa_1$ so easily intelligible, each of them being a sort of compound precession.

If the solar influence were to wane, M and Q, the poles of the proper planes, would approach one another, and ultimately become identical. The two planes would have then become the invariable plane of the system; and the two circles would be concentric and their radii would be inversely proportional to the two moments of momentum (whose ratio is $\mathfrak{m}$).

Now in the problem which is to be here considered the solar influence will in effect wane, because the effect of tidal friction is, in retrospect, to bring the moon nearer and nearer to the earth, and to increase the ellipticity of the earth's figure; hence the relative importance of the solar influence diminishes.

We now see that the problem to be solved is to trace these proper planes, from their present condition when one is nearly identical with the ecliptic and the other is the mean equator, backwards until they are both sensibly coincident with the equator.

We also see that the present angular velocity of the moon's nodes on the ecliptic is analogous to and continuous with the purely lunar precession on the invariable plane of the moon-earth system; and that the present luni-solar precession is analogous to and continuous with a slow precessional motion of the same invariable plane.

Analytically the problem is to trace the secular changes in the constants of integration, when α, a, β, b, instead of being constant, are slowly variable under the influence of tidal friction, and when certain other small terms, also due to tides, are added to the differential equations of motion.

§ 14. *On the small terms in the equations of motion due directly to tidal friction.*

The first step is the formation of the disturbing function.

As we shall want to apply the function both to the case of the earth and to that of the moon, it will be necessary to measure longitudes from a fixed point in the ecliptic; also we must distinguish between the longitude of the equinox and the angle χ, as they enter in the two capacities (viz.: in the X′Y′ and $\mathfrak{X}\mathfrak{Y}$ functions); thus the N and N' of previous developments must become $N-\psi$, $N'-\psi'$; ϵ, ϵ' must become $\epsilon-\psi$, $\epsilon'-\psi'$; and $2(\chi \backsim \chi')$ must be introduced in the arguments of the trigonometrical terms in the semi-diurnal terms, and $\chi \backsim \chi'$ in the diurnal ones.

The disturbing function must be developed so that it may be applicable to the cases either where Diana, the tide-raiser, is or is not identical with the moon; but as we are only going to consider secular inequalities, all those terms which depend on the longitudes of Diana or the moon may be dropped.

In the previous development of Part II. we had terms whose arguments involved $\epsilon-\epsilon'$; in the present case this ought to be written $(\Omega t+\epsilon-\psi)-(\Omega' t+\epsilon'-\psi')$, for which it is, in fact, only an abbreviation.

Now a term involving this expression can only give rise to secular inequalities, in the case where Diana is identical with the moon; and as we shall never want to differentiate the disturbing function with regard to Ω', we may in the present development drop the Ωt and $\Omega' t$.

Having made these preliminary explanations, we shall be able to use previous results for the development of the disturbing function. The work will be much abridged by the treatment of i, j, i', j' as small.

Unaccented symbols refer to the elements of the orbit of the tide-raiser Diana, or (in the case of i, χ, ψ) to the earth as a tidally distorted body; accented symbols refer to the elements of the orbit of the perturbed satellite, or to the earth as a body whose rotation is perturbed.

Then since i, i' and j, j' are to be treated as small, (22) becomes

$$\left.\begin{array}{l} \left.\begin{array}{c}\varpi\\ \underline{\varpi}\end{array}\right\}=Pp-Qqe^{\pm(N-\psi)}=1-\frac{1}{8}i^2-\frac{1}{8}j^2-\frac{1}{4}ije^{\pm(N-\psi)} \\ \left.\begin{array}{c}\kappa\\ \underline{\kappa}\end{array}\right\}=Qp+Pqe^{\pm(N-\psi)}=\frac{1}{2}i+\frac{1}{2}je^{\pm(N-\psi)} \end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (128)$$

The same quantities when accented are equal to the same quantities when i, j, N, ψ are accented.

Then referring to the development in § 5 of the disturbing function, we see that, for the same reasons as before, we need only consider products of terms of the same kind in the sets of products of the type $\mathrm{X'Y'}\times\mathfrak{X}\mathfrak{Y}$. Hence the disturbing function W is the sum of the three expressions (37–9) multiplied by $\tau\tau'/\mathfrak{g}$. Now since we only wish

to develop the expression as far as the squares of i and j, we may at once drop out all those terms in these expressions, in which κ occurs raised to a higher power than the second. This at once relieves us of the sidereal and fast semi-diurnal terms, the fast diurnal and the true fortnightly term. We are, however, left with one part of $\frac{3}{2}(\frac{1}{3}-Z'^2)(\frac{1}{3}-\mathfrak{Z}^2)$, which is independent of the moon's longitude and of the earth's rotation; this part represents the permanent increase of ellipticity of the earth, due to Diana's attraction, and to that part of the tidal action which depends on the longitude of the nodes, in which the tides are assumed to have their equilibrium value. I shall refer to it as the permanent tide.

Then as before, it will be convenient to consider the constituent parts of the disturbing function separately, and to indicate the several parts of W by suffixes as in § 5 and elsewhere; as above explained, we need only consider W_{I}, W_1, W_2, and W_0.

Semi-diurnal term.

From (37) we have

$$W_{\mathrm{I}}\Big/\frac{\tau\tau'}{\mathfrak{g}}=\tfrac{1}{4}[F_1\varpi^4\underline{\varpi}'^4e^{2(\theta'-\theta)-2f_1}+F_1\varpi^4\underline{\varpi}'^4e^{-2(\theta'-\theta)+2f_1}]$$

To the indices of these exponentials we must add $\pm2(\chi-\chi')$, and for θ write $\epsilon-\psi$, and for θ', $\epsilon'-\psi'$.

Then by (128)

$$\varpi^4=1-\tfrac{1}{2}i^2-\tfrac{1}{2}j^2-ije^{(N-\psi)}$$
$$\underline{\varpi}'^4=1-\tfrac{1}{2}i'^2-\tfrac{1}{2}j'^2-i'j'e^{-(N'-\psi')}$$

Hence

$$\begin{aligned}W_{\mathrm{I}}\Big/\frac{\tau\tau'}{\mathfrak{g}}=\tfrac{1}{2}F_1\{&(1-\tfrac{1}{2}i^2-\tfrac{1}{2}j^2-\tfrac{1}{2}i'^2-\tfrac{1}{2}j'^2)\cos[2(\chi-\chi')+2(\epsilon'-\epsilon)-2(\psi'-\psi)-2f_1]\\&-ij\cos[2(\chi-\chi')+2(\epsilon'-\epsilon)-2(\psi'-\psi)+(N-\psi)-2f_1]\\&-i'j'\cos[2(\chi-\chi')+2(\epsilon'-\epsilon)-2(\psi'-\psi)-(N'-\psi')-2f_1]\}\quad.\quad.\quad(129)\end{aligned}$$

Slow diurnal term.

From (38) we have

$$W_1\Big/\frac{\tau\tau'}{\mathfrak{g}}=G_1[\varpi^3\kappa\underline{\varpi}'^3\underline{\kappa}'e^{2(\theta'-\theta)-g_1}+\varpi^3\kappa\underline{\varpi}'^3\underline{\kappa}'e^{-2(\theta'-\theta)+g_1}]$$

To the indices of the exponentials we must add $\pm(\chi-\chi')$; ϖ^3, ϖ'^3 may be obviously put equal to unity, and by (128)

$$\kappa\kappa'=\tfrac{1}{4}[ii'+i'je^{(N-\psi)}+ij'e^{-(N'-\psi')}+jj'e^{(N-N')-(\psi-\psi')}]$$

Hence

$$W_1\Big/\frac{\tau\tau'}{\mathfrak{g}}=\tfrac{1}{2}G_1\{ii'\cos[(\chi-\chi')+2(\epsilon'-\epsilon)-2(\psi'-\psi)-g_1]$$
$$+i'j\cos[(\chi-\chi')+2(\epsilon'-\epsilon)-2(\psi'-\psi)+(N-\psi)-g_1]$$
$$+ij'\cos[(\chi-\chi')+2(\epsilon'-\epsilon)-2(\psi'-\psi)-(N'-\psi')-g_1]$$
$$+jj'\cos[(\chi-\chi')+2(\epsilon'-\epsilon)-2(\psi'-\psi)+(N-N')-(\psi-\psi')-g_1]\}\quad . \quad (130)$$

Sidereal diurnal term.

From (38) we have

$$W_2\Big/\frac{\tau\tau'}{\mathfrak{g}}=G[\varpi\underline{\kappa}(\varpi\underline{\varpi}-\kappa\underline{\kappa})\underline{\varpi}'\kappa'(\varpi'\underline{\varpi}'-\kappa'\underline{\kappa}')e^{-g}+\underline{\varpi}\kappa(\varpi\underline{\varpi}-\kappa\underline{\kappa})\varpi'\underline{\kappa}'(\varpi'\underline{\varpi}'-\kappa'\underline{\kappa}')e^{g}]$$

To the indices of the exponentials must be added $\pm(\chi-\chi')$. ϖ, ϖ' may be treated as unity. Hence the expression becomes $G[\underline{\kappa}\kappa' e^{\chi-\chi'-g}+\kappa\underline{\kappa}' e^{-(\chi-\chi')+g}]$ and

$$W_2\Big/\frac{\tau\tau'}{\mathfrak{g}}=\tfrac{1}{2}G\,\{ii'\cos(\chi-\chi'-g)$$
$$+i'j\cos[(\chi-\chi')-(N-\psi)-g]$$
$$+ij'\cos[(\chi-\chi')+(N'-\psi')-g]$$
$$+jj'\cos[(\chi-\chi')-(N-N')+(\psi-\psi')-g]\}\quad . \quad . \quad . \quad (131)$$

Permanent term.

From (39) we have

$$W_0\Big/\frac{\tau\tau'}{\mathfrak{g}}=\tfrac{3}{2}(\tfrac{1}{3}-2\varpi\underline{\varpi}\kappa\underline{\kappa})(\tfrac{1}{3}-2\varpi'\underline{\varpi}'\kappa'\underline{\kappa}')$$
$$=\tfrac{1}{6}-\kappa\underline{\kappa}-\kappa'\underline{\kappa}' \text{ to our degree of approximation.}$$

Now

$$\kappa\underline{\kappa}=\tfrac{1}{4}(i^2+j^2+ij(e^{N-\psi}+e^{-(N-\psi)}))=\tfrac{1}{4}(i^2+j^2+2ij\cos(N-\psi))$$

Hence

$$W_0\Big/\frac{\tau\tau'}{\mathfrak{g}}=\tfrac{1}{6}-\tfrac{1}{4}(i^2+j^2+2ij\cos(N-\psi))-\tfrac{1}{4}(i'^2+j'^2+2i'j'\cos(N'-\psi'))\quad . \quad (132)$$

W_2 and W_0 are the only terms in W which can contribute anything to the secular inequalities, unless Diana and the satellite are identical; for all the other terms involve $\epsilon-\epsilon'$, and will therefore be periodic however differentiated, unless $\epsilon=\epsilon'$.

We now have to differentiate W with respect to i', χ', ψ', j', ϵ', N'. The results will then have to be applied in the following cases.

For the moon: (i.) When the tide-raiser is the moon.
(ii.) When the tide-raiser is the sun.

For the earth: (iii.) When the tide-raiser is the moon, and the disturber the moon.
(iv.) When the tide-raiser is the sun, and the disturber the sun.
(v.) When the tide-raiser is the moon, and the disturber the sun.
(vi.) When the tide-raiser is the sun, and the disturber the moon.

The sum of the values derived from the differentiations, according to these several hypotheses, will be the complete values to be used in the differential equations (13), (14) and (18) for dj/dt, dN/dt, di/dt, $d\psi/dt$.

A little preliminary consideration will show that the labour of making these differentiations may be considerably abridged.

In the present case i and j are small, and the equations (110) which give the position of the two proper planes, and the inclinations of the orbit and equator thereto, become

$$\left.\begin{aligned}
&\frac{\xi}{k}\frac{dj}{dt}=-\tau\mathfrak{k}i\sin(N-\psi)\\
&\frac{\xi}{k}\sin j\frac{dN}{dt}=-\left(\tau\mathfrak{k}+\tfrac{1}{2}\frac{\tau'}{\Omega}\frac{\xi}{k}\right)j-\tau\mathfrak{k}i\cos(N-\psi)\\
&n\frac{di}{dt}=\tau\mathfrak{k}j\sin(N-\psi)\\
&n\sin i\frac{d\psi}{dt}=-(\tau\mathfrak{k}+\tau'\mathfrak{k})i-\tau\mathfrak{k}j\cos(N-\psi)
\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (133)$$

We are now going to find certain additional terms, depending on frictional tides, to be added to these four equations. These terms will all involve τ^2, τ'^2, or $\tau\tau'$ in their coefficients, and will therefore be small compared with those in (133). If these small terms are of the same types as the terms in (133), they may be dropped; because the only effect of them would be to produce a very small and negligeable alteration in the position of the two proper planes.*

In consequence of this principle, we may entirely drop W_0 from our disturbing function, for W_0 only gives rise to a small permanent alteration of oblateness, and therefore can only slightly modify the positions of the proper planes.

Analytically the same result may be obtained, by observing that W_0 in (132) has the same form as W in (105), when i and j are treated as small.

* For example, we should find the following terms in $\frac{\xi}{k}\sin j\frac{dN}{dt}$, viz.:—

$$-\tfrac{1}{2}j\frac{\tau\tau'}{\mathfrak{g}}-\tfrac{1}{2}i\cos(N-\psi)\sin^2\mathrm{g}\frac{\tau\tau'}{\mathfrak{g}}+\tfrac{1}{2}(j+i\cos(N-\psi))\left[\sin^2 2\mathrm{f}_1-\sin^2\mathrm{g}_1-\sin^2\mathrm{g}\right]\tau^2$$

which may be all coupled up with those in the second of (133).

If the viscosity be small, so that the angles of lagging are small, it will be found that all the terms of this kind vanish in all four equations, excepting the first of those just written down, viz.: $-\frac{1}{2}j\tau\tau'/\mathfrak{g}$.

In each case, after differentiation, the transition will be made to the case of viscosity of the planet, and the proper terms will be dropped out, without further comment.

First take the perturbations of the moon.

For this purpose we have to find dW/dj' and $dW/\sin j'\, dN' + \tan\frac{1}{2}j'\, dW/d\epsilon'$ or $dW/j'dN' + \frac{1}{2}j'dW/d\epsilon'$.

By the above principle, in finding dW/dj' we may drop terms involving j and $i\cos(N-\psi)$, and in finding $dW/j'dN'+\frac{1}{2}j'\,dW/d\epsilon'$, we may drop terms involving $i\sin(N-\psi)$.

We may now suppose $\chi=\chi'$, $\psi=\psi'$.

Take the case (i.), where the tide-raiser is the moon. Then as the perturbed body is also the moon, after differentiation we may drop the accents to all the symbols.

From (129)

$$\frac{dW_1}{dj'}\Big/\frac{\tau^2}{\mathfrak{g}}=\tfrac{1}{2}F_1\{-j\cos 2f_1-i\cos(N-\psi+2f_1)\}$$
$$=\tfrac{1}{4}i\sin(N-\psi)\sin 4f_1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (134)$$

From (130)

$$\frac{dW_1}{dj'}\Big/\frac{\tau^2}{\mathfrak{g}}=\tfrac{1}{2}G_1\{i\cos(N-\psi+g_1)+j\cos g_1\}$$
$$=-\tfrac{1}{4}i\sin(N-\psi)\sin 2g_1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (135)$$

From (131) and symmetry with (135)

$$\frac{dW_2}{dj'}\Big/\frac{\tau^2}{\mathfrak{g}}=\tfrac{1}{4}i\sin(N-\psi)\sin 2g \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (136)$$

Adding these three (134–6) together, we have for the whole effect of the lunar tides on the moon

$$\frac{dW}{dj'}\Big/\frac{\tau^2}{\mathfrak{g}}=\tfrac{1}{4}i\sin(N-\psi)\left[\sin 4f_1-\sin 2g_1+\sin 2g\right] \quad . \quad . \quad . \quad . \quad (137)$$

Now take the case (ii.) where the tide-raiser is the sun.

Here we need only consider W_2, but although we may put $\chi=\chi'$, $\psi=\psi'$, $i=i'$, we must not put $j=j'$, $N=N'$, because the tide-raiser is distinct from the moon.

From (131)

$$\frac{dW_2}{dj'}\Big/\frac{\tau\tau'}{\mathfrak{g}}=\tfrac{1}{2}G\{i\cos(N'-\psi'-g)+j\cos(N-N'+g)\}$$

Here accented symbols refer to the moon (as perturbed), and unaccented to the sun (as tide-raiser). As we refer the motion to the ecliptic $j=0$, and the last term disappears. Also we want accented symbols to refer to the sun and unaccented to

refer to the moon, therefore make τ and τ' interchange their meanings, and drop the accents to N' and ψ'. Thus as far as important

$$\frac{dW_2}{dj'}\bigg/\frac{\tau'\tau}{\mathfrak{g}}=\tfrac{1}{4}i\sin(N-\psi)\sin 2g \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (138)$$

This gives the whole effect of the solar tides on the moon.

Then collecting results from (137–8), we have by (14)

$$\frac{\xi}{k}\sin j\frac{dN}{dt}=\tfrac{1}{4}i\sin(N-\psi)\left[\frac{\tau^2}{\mathfrak{g}}(\sin 4f_1-\sin 2g_1+\sin 2g)+\frac{\tau\tau'}{\mathfrak{g}}\sin 2g\right] \quad . \qquad (139)$$

This gives the required additional terms due to bodily tides in the equation for dN/dt, viz.: the second of (133).

If the viscosity be small

$$\left.\begin{aligned}\sin 4f_1-\sin 2g_1+\sin 2g&=\sin 4f\\ \sin 2g\qquad\qquad&=\tfrac{1}{2}\sin 4f\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (140)$$

Next take the secular change of inclination of the lunar orbit.

For this purpose we have to find $dW/j'dN'+\frac{1}{2}j'dW/d\epsilon'$, and may drop terms in $i\sin(N-\psi)$.

First take the case (i.), where the tide-raiser is the moon.

From (129)

$$\frac{1}{j'}\frac{dW_I}{dN'}\bigg/\frac{\tau^2}{\mathfrak{g}}=\tfrac{1}{2}F_1\,i\sin(N-\psi+2f_1)=\tfrac{1}{4}\,i\cos(N-\psi)\sin 4f_1 \quad . \quad . \quad . \qquad (141)$$

$$\tfrac{1}{2}j'\frac{dW_I}{d\epsilon'}\bigg/\frac{\tau^2}{\mathfrak{g}}=\tfrac{1}{2}F_1 j\sin 2f_1 \qquad\qquad =\tfrac{1}{4}j\sin 4f_1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (142$$

From (130)

$$\frac{1}{j'}\frac{dW_1}{dN'}\bigg/\frac{\tau^2}{\mathfrak{g}}=-\tfrac{1}{2}G_1\{i\sin(N-\psi+g_1)+j\sin g_1\}=-\tfrac{1}{4}(j+i\cos(N-\psi))\sin 2g_1 \quad (143)$$

$$\tfrac{1}{2}j'\frac{dW_1}{d\epsilon'}\bigg/\frac{\tau^2}{\mathfrak{g}}=\quad 0 \text{ to present order of approximation} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (144)$$

From (131)

$$\frac{1}{j'}\frac{dW_2}{dN'}\bigg/\frac{\tau^2}{\mathfrak{g}}=-\tfrac{1}{2}G\{i\sin(N-\psi-g)-j\sin g\}=\tfrac{1}{4}(j+i\cos(N-\psi))\sin 2g \quad . \qquad (145)$$

$$\tfrac{1}{2}j'\frac{dW_2}{d\epsilon'}\bigg/\frac{\tau^2}{\mathfrak{g}}=0 \text{ absolutely.} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (146)$$

Collecting results from the six equations (141–6), we have for the whole perturbation of the moon by the lunar tides

$$\left(\frac{1}{j'}\frac{dW}{dN'}+\tfrac{1}{2}j'\frac{dW}{d\epsilon'}\right)\Big/\frac{\tau^2}{\mathfrak{g}}=\tfrac{1}{4}(j+i\cos(N-\psi))(\sin 4f_1-\sin 2g_1+\sin 2g)\ . \quad (147)$$

Next take the case (ii.), and suppose that the sun is the tide-raiser. Here we need only consider W_2. Then noting that $dW_2/d\epsilon'=0$ absolutely, we have from (131)

$$\left(\frac{1}{j'}\frac{dW_2}{dN'}+\tfrac{1}{2}j'\frac{dW_2}{d\epsilon'}\right)\Big/\frac{\tau\tau'}{\mathfrak{g}}=-\tfrac{1}{2}G\{i\sin(N'-\psi'-g)-j\sin(N-N'+g)\}$$

Accented symbols here refer to the moon (as perturbed), unaccented to the sun (as tide-raiser). Therefore $j=0$. Then reverting to the usual notation by shifting accents and dropping useless terms, this expression becomes

$$+\tfrac{1}{4}i\cos(N-\psi)\sin 2g \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (148)$$

Then collecting results from (147–8), we have by (13)

$$\frac{\xi}{k}\frac{dj}{dt}=-\tfrac{1}{4}(j+i\cos(N-\psi))\frac{\tau^2}{\mathfrak{g}}(\sin 4f_1-\sin 2g_1+\sin 2g)-\tfrac{1}{4}i\cos(N-\psi)\frac{\tau\tau'}{\mathfrak{g}}\sin 2g \quad (149)$$

This gives the additional terms due to bodily tides in the equation for dj/dt, viz.: the first of (133).

If the viscosity be small

$$\left.\begin{aligned}\sin 4f_1-\sin 2g_1+\sin 2g&=\sin 4f\\ \sin 2g&=\tfrac{1}{2}\sin 4f\end{aligned}\right\}\quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (150)$$

Before proceeding further it may be remarked that to the present order of approximation in case (i.)

$$\frac{dW}{d\epsilon'}=\tfrac{1}{2}\sin 4f_1$$

and in case (ii.) it is zero; thus by (11)

$$\frac{1}{k}\frac{d\xi}{dt}=\tfrac{1}{2}\frac{\tau^2}{\mathfrak{g}}\sin 4f_1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (151)$$

We now turn to the perturbations of the earth's rotation.

Here we have to find dW/di' and $\cot i\ dW/d\chi'-dW/\sin i\ d\psi'$ or $(1-\frac{1}{2}i^2)dW/id\chi'-dW/id\psi'$, and in the former may drop terms in i and $i\cos(N-\psi)$, and in the latter terms in $j\sin(N-\psi)$.

First take the case (iii.), where the moon is tide-raiser and disturber. Here we may take $N=N'$, $\epsilon=\epsilon'$, $j=j'$ throughout, and after differentiation may drop the accents to all the symbols.

From (129)

$$\frac{dW_1}{di'}\bigg/\frac{\tau^2}{\mathfrak{g}}=-\tfrac{1}{2}F_1\{i\cos 2f_1+j\cos(N-\psi+2f_1)\}=\ \tfrac{1}{4}j\sin(N-\psi)\sin 4f_1\ . \quad (152)$$

From (130)

$$\frac{dW_1}{di'}\bigg/\frac{\tau^2}{\mathfrak{g}}=\ \tfrac{1}{2}G_1\{i\cos g_1+j\cos(N-\psi-g_1)\}=\ \tfrac{1}{4}j\sin(N-\psi)\sin 2g_1\ . \quad (153)$$

From (131)

$$\frac{dW_2}{di'}\bigg/\frac{\tau^2}{\mathfrak{g}}=\ \tfrac{1}{2}G\{i\cos g+j\cos(N-\psi+g)\}=-\tfrac{1}{4}j\sin(N-\psi)\sin 2g\ . \quad (154)$$

Therefore from (152–4) we have for the whole perturbation of the earth, due to attraction of the moon on the lunar tides,

$$\frac{dW}{di'}\bigg/\frac{\tau^2}{\mathfrak{g}}=\tfrac{1}{4}j\sin(N-\psi)[\sin 4f_1+\sin 2g_1-\sin 2g]\ .\ .\ .\ . \quad (155)$$

The result for case (iv.), where the sun is both tide-raiser and disturber, may be written down by symmetry; and since $j=0$ here, therefore

$$\frac{dW}{di'}\bigg/\frac{\tau'^2}{\mathfrak{g}}=0\ .\ .\ .\ .\ .\ .\ .\ .\ .\ .\ .\ . \quad (156)$$

Next take the cases (v.) and (vi.), where the tide-raiser and disturber are distinct. Here we need only consider W_2.

From (131)

$$\frac{dW_2}{di'}\bigg/\frac{\tau\tau'}{\mathfrak{g}}=\tfrac{1}{2}G\{i\cos g+j\cos(N-\psi+g)\}$$

When the moon is tide-raiser and sun disturber, this becomes

$$-\tfrac{1}{4}j\sin(N-\psi)\sin 2g\ .\ .\ .\ .\ .\ .\ .\ .\ .\ . \quad (157)$$

When sun is tide-raiser and moon disturber it becomes zero.

Then collecting results from (155–7), we have by (18)

$$n \sin i \frac{d\psi}{dt} = \tfrac{1}{4} j \sin (N-\psi)\left[\frac{\tau^2}{\mathfrak{g}}(\sin 4\text{f}_1 + \sin 2\text{g}_1 - \sin 2\text{g}) - \frac{\tau\tau'}{\mathfrak{g}} \sin 2\text{g}\right] \quad . \quad (158)$$

This gives the additional terms due to bodily tides in the equation for $d\psi/dt$, viz.: the last of (133).

If the viscosity be small

$$\left.\begin{aligned} \sin 4\text{f}_1 + \sin 2\text{g}_1 - \sin 2\text{g} &= \sin 4\text{f}(1-2\lambda) \\ \text{where} \qquad \sin 2\text{g} &= \tfrac{1}{2} \sin 4\text{f} \\ \lambda &= \frac{\Omega}{n} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (159)$$

Next consider the change in the obliquity of the ecliptic; for this purpose we must find $(1-\frac{1}{2}i^2)dW/id\chi' - dW/id\psi'$, and may drop terms involving $j \sin (N-\psi)$.

First take the case (iii.), where the moon is both tide-raiser and disturber.

Then from (129)

$$\frac{dW_{\text{I}}}{d\chi'}\Big/\frac{\tau^2}{\mathfrak{g}} = -\text{F}_1\{(1-i^2-j^2)\sin 2\text{f}_1 + ij \sin (N-\psi-2\text{f}_1) - ij \sin (N-\psi+2\text{f}_1)\} \quad . \quad (160)$$

$$-\frac{dW_{\text{I}}}{d\psi'}\Big/\frac{\tau^2}{\mathfrak{g}} = \text{F}_1\{(1-i^2-j^2)\sin 2\text{f}_1 + ij \sin (N-\psi-2\text{f}_1) - \tfrac{1}{2}ij \sin (N-\psi+2\text{f}_1)\}$$

$$-\tfrac{1}{2}i^2\frac{dW_{\text{I}}}{d\chi'}\Big/\frac{\tau^2}{\mathfrak{g}} = \text{F}_1\tfrac{1}{2}i^2 \sin 2\text{f}_1$$

Therefore

$$\begin{aligned}\left[\frac{1}{i}(1-\tfrac{1}{2}i^2)\frac{dW_{\text{I}}}{d\chi'} - \frac{1}{i}\frac{dW_{\text{I}}}{d\psi'}\right]\Big/\frac{\tau^2}{\mathfrak{g}} &= \tfrac{1}{2}\text{F}_1\{i \sin 2\text{f}_1 + j \sin (N-\psi+2\text{f}_1)\} \\ &= \tfrac{1}{4}(i + j \cos (N-\psi)) \sin 4\text{f}_1 \quad . \quad . \quad . \quad . \quad . \quad (161)\end{aligned}$$

From (130)

$$\frac{dW_1}{d\chi'}\Big/\frac{\tau^2}{\mathfrak{g}} = -\tfrac{1}{2}\text{G}_1\{i^2 \sin \text{g}_1 - ij \sin (N-\psi-\text{g}_1) + ij \sin (N-\psi+\text{g}_1) + j^2 \sin \text{g}_1\} \quad . \quad (162)$$

$$-\frac{dW_1}{d\psi'}\Big/\frac{\tau^2}{\mathfrak{g}} = \tfrac{1}{2}\text{G}_1\{2i^2 \sin \text{g}_1 - 2ij \sin (N-\psi-\text{g}_1) + ij \sin (N-\psi+\text{g}_1) + j^2 \sin \text{g}_1\}$$

$$-\tfrac{1}{2}i^2\frac{dW_1}{d\chi'}\Big/\frac{\tau^2}{\mathfrak{g}} = 0$$

Therefore

$$\begin{aligned}\left[\frac{1}{i}(1-\tfrac{1}{2}i^2)\frac{dW_1}{d\chi'} - \frac{1}{i}\frac{dW_1}{d\psi'}\right]\Big/\frac{\tau^2}{\mathfrak{g}} &= \tfrac{1}{2}\text{G}\{i \sin \text{g}_1 - j \sin (N-\psi-\text{g}_1)\} \\ &= \tfrac{1}{4}(i + i \cos (N-\psi)) \sin 2\text{g}_1 \quad . \quad . \quad . \quad . \quad (163)\end{aligned}$$

From (131)

$$\frac{d\mathrm{W}_2}{d\chi'}\bigg/\frac{\tau^2}{\mathfrak{g}}=-\tfrac{1}{2}\mathrm{G}\{i^2\sin\mathrm{g}+ij\sin(N-\psi+\mathrm{g})-ij\sin(N-\psi-\mathrm{g})+j^2\sin\mathrm{g}\}\quad . \quad (164)$$

$$-\frac{d\mathrm{W}_2}{d\psi'}\bigg/\frac{\tau^2}{\mathfrak{g}}=\tfrac{1}{2}\mathrm{G}\{\qquad -ij\sin(N-\psi-\mathrm{g})+j^2\sin\mathrm{g}\}$$

$$-\tfrac{1}{2}i^2\frac{d\mathrm{W}_2}{d\chi'}\bigg/\frac{\tau^2}{\mathfrak{g}}=0$$

Therefore

$$\left[\frac{1}{i}(1-\tfrac{1}{2}i^2)\frac{d\mathrm{W}_2}{d\chi'}-\frac{1}{i}\frac{d\mathrm{W}_2}{d\psi'}\right]\bigg/\frac{\tau^2}{\mathfrak{g}}=-\tfrac{1}{2}\mathrm{G}\{i\sin\mathrm{g}+j\sin(N-\psi+\mathrm{g})\}$$
$$=-\tfrac{1}{4}(i+j\cos(N-\psi))\sin 2\mathrm{g}\quad . \quad . \quad . \quad . \quad (165)$$

Then collecting results from (161–3–5), we have for the whole perturbation of the earth due to the attraction of the moon on the lunar tides,

$$\left[\frac{1}{i}(1-\tfrac{1}{2}i^2)\frac{d\mathrm{W}}{d\chi'}-\frac{1}{i}\frac{d\mathrm{W}}{d\psi'}\right]\bigg/\frac{\tau^2}{\mathfrak{g}}=\tfrac{1}{4}(i+j\cos(N-\psi))(\sin 4\mathrm{f}_1+\sin 2\mathrm{g}_1-\sin 2\mathrm{g}) \qquad (166)$$

The result for case (iv.), where the sun is both tide-raiser and disturber, may be written down by symmetry; and since $j=0$ here, therefore

$$\left[\frac{1}{i}(1-\tfrac{1}{2}i^2)\frac{d\mathrm{W}}{d\chi'}-\frac{1}{i}\frac{d\mathrm{W}}{d\psi'}\right]\bigg/\frac{\tau'^2}{\mathfrak{g}}=\tfrac{1}{4}i\sin 4\mathrm{f}\quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (167)$$

It is here assumed that the solar slow diurnal tide has the same lag as the sidereal diurnal tide, and that the solar slow semi-diurnal tide has the same lag as the sidereal semi-diurnal tide. This is very nearly true, because Ω' is small compared with n.

Next take the cases (v.) and (vi.), where the tide-raiser and disturber are distinct. Here we need only consider W_2

$$\frac{d\mathrm{W}_2}{d\chi'}\bigg/\frac{\tau\tau'}{\mathfrak{g}}=-\tfrac{1}{2}\mathrm{G}\{i^2\sin\mathrm{g}+ij\sin(N-\psi+\mathrm{g})-ij'\sin(N'-\psi'-\mathrm{g})$$
$$+jj'\sin(N-N'+\mathrm{g})\}\quad . \quad (168)$$

$$-\frac{d\mathrm{W}_2}{d\psi'}\bigg/\frac{\tau\tau'}{\mathfrak{g}}=\tfrac{1}{2}\mathrm{G}\{\qquad -ij'\sin(N'-\psi'-\mathrm{g})+jj'\sin(N-N'+\mathrm{g})\}$$

$$-\tfrac{1}{2}i^2\frac{d\mathrm{W}_2}{d\chi'}\bigg/\frac{\tau\tau'}{\mathfrak{g}}=0$$

Therefore

$$\left[\frac{1}{i}(1-\tfrac{1}{2}i^2)\frac{dW_2}{d\chi'}-\frac{1}{i}\frac{dW_2}{d\psi'}\right]\Big/\frac{\tau\tau'}{\mathfrak{g}}=-\tfrac{1}{2}G\{i\sin \mathrm{g}+j\sin(N-\psi+\mathrm{g})\}$$

When the moon is tide-raiser and the sun disturber, this becomes

$$-\tfrac{1}{4}(i+j\cos(N-\psi))\sin 2\mathrm{g} \quad \ldots \ldots \quad (169)$$

When the sun is tide-raiser and the moon disturber, this becomes

$$-\tfrac{1}{4}i\sin 2\mathrm{g} \quad \ldots \ldots \quad (170)$$

Then collecting results from (166–7–9, 170), we have by (18),

$$\left.\begin{aligned} n\frac{di}{dt}&=\tfrac{1}{4}(i+j\cos(N-\psi))\left[\frac{\tau^2}{\mathfrak{g}}(\sin 4\mathrm{f}_1+\sin 2\mathrm{g}_1-\sin 2\mathrm{g})-\frac{\tau\tau'}{\mathfrak{g}}\sin 2\mathrm{g}\right] \\ &+\tfrac{1}{4}i\left[\frac{\tau'^2}{\mathfrak{g}}\sin 4\mathrm{f}-\frac{\tau\tau'}{\mathfrak{g}}\sin 2\mathrm{g}\right] \end{aligned}\right\} \quad . \quad (171)$$

This gives the additional terms due to bodily tides in the equation for di/dt, viz.: the third of (133).

If the viscosity be small

$$\left.\begin{aligned} \sin 4\mathrm{f}_1+\sin 2\mathrm{g}_1-\sin 2\mathrm{g}&=\sin 4\mathrm{f}(1-2\lambda) \\ \sin 2\mathrm{g}&=\tfrac{1}{2}\sin 4\mathrm{f} \\ \text{where}\qquad \lambda&=\frac{\Omega}{n} \end{aligned}\right\} \quad \ldots \ldots \quad (172)$$

Also we have from (160–2–4–8) to the present order of approximation,

$$\frac{dW}{d\chi'}\Big/\frac{\tau^2}{\mathfrak{g}}=-\tfrac{1}{2}\sin 4\mathrm{f}_1$$

and by symmetry,

$$\frac{dW}{d\chi'}\Big/\frac{\tau'^2}{\mathfrak{g}}=-\tfrac{1}{2}\sin 4\mathrm{f}$$

Therefore by (18)

$$-\frac{dn}{dt}=\tfrac{1}{2}\left[\frac{\tau^2}{\mathfrak{g}}\sin 4\mathrm{f}_1+\frac{\tau'^2}{\mathfrak{g}}\sin 4\mathrm{f}\right] \quad \ldots \ldots \quad (173)$$

Now let

$$\left.\begin{aligned}\Gamma &= \tfrac{1}{4}\frac{k}{\xi}\frac{\tau^2}{\mathfrak{g}}(\sin 4\mathrm{f}_1 - \sin 2\mathrm{g}_1 + \sin 2\mathrm{g}) \\ \mathrm{G} &= \tfrac{1}{4}\frac{k}{\xi}\left[\frac{\tau^2}{\mathfrak{g}}(\sin 4\mathrm{f}_1 - \sin 2\mathrm{g}_1 + \sin 2\mathrm{g}) + \frac{\tau\tau'}{\mathfrak{g}}\sin 2\mathrm{g}\right] \\ \Delta &= \frac{1}{4n}\left[\frac{\tau^2}{\mathfrak{g}}(\sin 4\mathrm{f}_1 + \sin 2\mathrm{g}_1 - \sin 2\mathrm{g}) + \frac{\tau'^2}{\mathfrak{g}}\sin 4\mathrm{f} - 2\frac{\tau\tau'}{\mathfrak{g}}\sin 2\mathrm{g}\right] \\ \mathrm{D} &= \frac{1}{4n}\left[\frac{\tau^2}{\mathfrak{g}}(\sin 4\mathrm{f}_1 + \sin 2\mathrm{g}_1 - \sin 2\mathrm{g}) - \frac{\tau\tau'}{\mathfrak{g}}\sin 2\mathrm{g}\right]\end{aligned}\right\} \quad . \quad (174)$$

Then the four equations (139), (149), (158), and (171) may be written

$$\left.\begin{aligned}j\frac{dN}{dt} &= \mathrm{G}i\sin(N-\psi) \\ \frac{dj}{dt} &= -\Gamma j - \mathrm{G}i\cos(N-\psi) \\ i\frac{d\psi}{dt} &= \mathrm{D}j\sin(N-\psi) \\ \frac{di}{dt} &= \Delta i + \mathrm{D}j\cos(N-\psi)\end{aligned}\right\} \quad . \; . \; . \; . \; . \; . \; . \; . \quad (175)$$

Also from (151) and (173)

$$\left.\begin{aligned}\frac{1}{k}\frac{d\xi}{dt} &= \tfrac{1}{2}\frac{\tau^2}{\mathfrak{g}}\sin 4\mathrm{f}_1 \\ -\frac{dn}{dt} &= \tfrac{1}{2}\frac{\tau^2}{\mathfrak{g}}\sin 4\mathrm{f}_1 + \tfrac{1}{2}\frac{\tau'^2}{\mathfrak{g}}\sin 4\mathrm{f}\end{aligned}\right\} \quad . \; . \; . \; . \; . \; . \; . \; . \quad (176)$$

These six equations (175–6) contain all the secular inequalities in the motions of the moon and earth, due to the bodily tides raised by the sun and moon, as far as is material for the present investigation. The terms which are omitted only represent a very small displacement of the proper planes and of the inclinations of the planes of motion of the two parts of the system to those proper planes.

Then reverting to the earlier notation in which

$$\left.\begin{aligned}y &= j\sin N, \; \eta = i\sin\psi \\ z &= j\cos N, \; \zeta = i\cos\psi\end{aligned}\right\} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (177)$$

We easily find

$$\left.\begin{aligned}\frac{dz}{dt}&=-\Gamma z-\mathrm{G}\zeta\\ \frac{dy}{dt}&=-\Gamma y-\mathrm{G}\eta\\ \frac{d\zeta}{dt}&=\quad\Delta\zeta+\mathrm{D}z\\ \frac{d\eta}{dt}&=\quad\Delta\eta+\mathrm{D}y\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (178)$$

These equations contain the additional terms due to tides, which are to be added to the equations (116), in order to find the secular displacements of the proper planes.

The first application, which will be made hereafter, will be to the case where the viscosity is small, and it will be more convenient to make the transition to that hypothesis at present, although the greater part of what follows in this part will be equally applicable whatever may be the viscosity. In the case of small viscosity the functions Γ, Δ, G, D will be indicated by the corresponding small letters γ, δ, g, d. Then by (140), (150), (159), (172) we shall have

$$\left.\begin{aligned}&\gamma=\tfrac{1}{4}\frac{k}{\xi}\frac{\sin 4\mathrm{f}}{\mathfrak{g}}[\tau^2], && \mathrm{g}=\tfrac{1}{4}\frac{k}{\xi}\frac{\sin 4\mathrm{f}}{\mathfrak{g}}[\tau^2+\tfrac{1}{2}\tau\tau']\\ &\delta=\frac{1}{4n}\frac{\sin 4\mathrm{f}}{\mathfrak{g}}[\tau^2(1-2\lambda)+\tau'^2-\tau\tau'], && \mathrm{d}=\frac{1}{4n}\frac{\sin 4\mathrm{f}}{\mathfrak{g}}[\tau^2(1-2\lambda)-\tfrac{1}{2}\tau\tau']\\ &\quad\text{where } \lambda=\frac{\Omega}{n}\end{aligned}\right\} \quad . \quad (179)$$

And in the present case where i and j are small, we have by (112) and (121)

$$\left.\begin{aligned}&\alpha=\frac{k}{\xi}\tau\mathfrak{e}+\tfrac{1}{2}\frac{\tau'}{\Omega}, \qquad \beta=\frac{\tau+\tau'}{n}\mathfrak{e}\\ &\mathrm{a}=\frac{k}{\xi}\tau\mathfrak{e}, \qquad \mathrm{b}=\frac{\tau\mathfrak{e}}{n}\\ &\text{where } \mathfrak{e}=\tfrac{1}{2}\frac{n^2}{\mathfrak{g}}, \text{ the permanent ellipticity of the earth}\end{aligned}\right\} \quad . \quad . \quad (180)$$

These equations (180) are the same whether the viscosity be supposed small or not. Then the complete equations are

$$\left.\begin{aligned}\frac{dz}{dt}&=\alpha y+\mathrm{a}\eta-(\gamma z+\mathrm{g}\zeta)\\ \frac{dy}{dt}&=-(\alpha z+\mathrm{a}\zeta)-(\gamma y+\mathrm{g}\eta)\\ \frac{d\zeta}{dt}&=\beta\eta+\mathrm{b}y+\delta\zeta+\mathrm{d}z\\ \frac{d\eta}{dt}&=-(\beta\zeta+\mathrm{b}z)+\delta\eta+\mathrm{d}y\end{aligned}\right\} \quad \ldots\ldots\ldots \quad (181)$$

If the viscosity be not small we have Γ, G, Δ, D in place of γ, g, δ, d. As it is more convenient to write small letters than capitals, in the whole of the next section the small letters will be employed, although the same investigation would be equally applicable with Γ, G, &c., in place of γ, g, &c.

The terms in γ, g, δ, d are small compared with those in α, a, β, b, and may be neglected as a first approximation. Also α, a, β, b vary slowly in consequence of tidal reaction, tidal friction, and the consequent change of ellipticity of the earth, but as a first approximation they may be treated as constant.

Then if we put

$$\left.\begin{aligned}z_1&=L_1\cos(\kappa_1t+\mathrm{m}_1), & z_2&=L_2\cos(\kappa_2t+\mathrm{m}_2)\\ y_1&=L_1\sin(\kappa_1t+\mathrm{m}_1), & y_2&=L_2\sin(\kappa_2t+\mathrm{m}_2)\\ \zeta_1&=L_1'\cos(\kappa_1t+\mathrm{m}_1), & \zeta_2&=L_2'\cos(\kappa_2t+\mathrm{m}_2)\\ \eta_1&=L_1'\sin(\kappa_1t+\mathrm{m}_1), & \eta_2&=L_2'\sin(\kappa_2t+\mathrm{m}_2)\end{aligned}\right\} \quad \ldots\ldots \quad (182)$$

By (122) or (118) the first approximation is

$$z=z_1+z_2,\ y=y_1+y_2,\ \zeta=\zeta_1+\zeta_2,\ \eta=\eta_1+\eta_2$$

where

$$\left.\frac{L_1'}{L_1}=-\frac{\kappa_1+\alpha}{\mathrm{a}}=-\frac{\mathrm{b}}{\kappa_1+\beta},\ \frac{L_2'}{L_2}=-\frac{\kappa_2+\alpha}{\mathrm{a}}=-\frac{\mathrm{b}}{\kappa_2+\beta}\right\} \quad \ldots\ldots \quad (183)$$

Before considering the secular changes in the constants L of integration, it will be convenient to take one other step.

The equation of tidal friction (173) may be written approximately

$$-\frac{dn}{dt}=\tfrac{1}{2}\frac{\tau^2+\tau'^2}{\mathfrak{g}}\sin 4\mathrm{f}_1 \quad \ldots\ldots\ldots \quad (184)$$

because sin 4f will be nearly equal to sin $4\mathrm{f}_1$ as long as τ'^2 is not small compared with τ^2. (See however § 22, Part IV.)

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Also the equation of tidal reaction (151) is

$$\frac{1}{k}\frac{d\xi}{dt}=\frac{1}{2}\frac{\tau^2}{\mathfrak{g}}\sin 4\mathrm{f}_1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (185)$$

Dividing one by the other and putting $\tau^2=\tau_0{}^2\xi^{-12}$, we have

$$k\frac{dn}{d\xi}=1+\left(\frac{\tau'}{\tau_0}\right)^2\xi^{12}$$

and integrating,

$$\frac{n}{n_0}=1+\frac{1}{kn_0}\left[(1-\xi)+\tfrac{1}{13}\left(\frac{\tau'}{\tau_0}\right)^2(1-\xi^{13})\right] \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (186)$$

This is the equation of conservation of moment of momentum of the moon-earth system, as modified by solar tidal friction. From it we obtain n in terms of ξ.

§ 15. *On the secular changes of the constants of integration.*

It is often found difficult on first reading a long analytical investigation to trace the general method amidst the mass of detail, and it is only at the end that the ruling idea is perceived; in such circumstances it has often appeared to me that a preliminary sketch would be of great service to the reader. I shall act on this idea here, and consider some simple equations analogous to those to be treated.

Let the equations be

$$\frac{dz}{dt}=\alpha y,\ \frac{dy}{dt}=-\alpha z$$

If α be constant, the solution is obviously

$$z=\mathrm{L}\cos(\alpha t+\mathrm{m}),\quad y=-\mathrm{L}\sin(\alpha t+\mathrm{m})$$

Now suppose α to be slowly varying; put therefore $\alpha+\alpha' t$ for α, and treat α, α' as constants.

Then

$$\frac{dz}{dt}=\alpha y+\alpha' ty,\qquad \frac{dy}{dt}=-\alpha z-\alpha' tz$$

Differentiating

$$\frac{d^2z}{dt^2}+\alpha^2 z=-\alpha' t\left(\alpha z-\frac{dy}{dt}\right)+\alpha' y$$

$$\frac{d^2y}{dt^2}+\alpha^2 y=-\alpha' t\left(\alpha y+\frac{dz}{dt}\right)-\alpha' z$$

The terms on the right-hand side of these equations are small, because they involve α', and therefore we may substitute in them from the first approximation.

Hence

$$\frac{d^2z}{dt^2}+\alpha^2 z=-\alpha' L\sin(\alpha t+\mathrm{m})-2\alpha'\alpha t L\cos(\alpha t+\mathrm{m})$$

and a similar equation for y.

The solution of this equation is

$$z=L\cos(\alpha t+\mathrm{m})+\frac{\alpha'}{2\alpha}Lt\cos(\alpha t+\mathrm{m})-\frac{\alpha'}{2\alpha}Lt\cos(\alpha t+\mathrm{m})-\frac{\alpha'}{2}Lt^2\sin(\alpha t+\mathrm{m})$$

The terms depending on t cut one another out, and

$$z=L\cos(\alpha t+\mathrm{m})-\frac{\alpha'}{2}Lt^2\sin(\alpha t+\mathrm{m})$$

Similarly we should find

$$y=-L\sin(\alpha t+\mathrm{m})-\frac{\alpha'}{2}Lt^2\cos(\alpha t+\mathrm{m})$$

The terms in t^2 are obviously equivalent to a change in m, the phase of the oscillation; but the amplitude L is unaffected. We might have arrived at this conclusion about the amplitude if, in solving the differential equations, we had neglected in the solutions the terms depending on t^2, as will be done in considering our equations below. In those equations, however, we shall not find that the terms in t annihilate one another, and thus there will be a change of amplitude.

That this conclusion concerning amplitude is correct, may be seen from the fact that the rigorous solution of the equations

$$\frac{dz}{dt}=\alpha y,\ \frac{dy}{dt}=-\alpha z$$

is

$$\begin{aligned} z&=L\cos(\textstyle\int\alpha dt+\mathrm{m}_0), & y&=-L\sin(\textstyle\int\alpha dt+\mathrm{m}_0)\\ &=L\cos(\alpha t+\mathrm{m}_0-\textstyle\int\alpha' t dt), & &=-L\sin(\alpha t+\mathrm{m}_0-\textstyle\int\alpha' t dt)\end{aligned}$$

Whence L is unaffected, whilst

$$\mathrm{m}=\mathrm{m}_0-\int\alpha' t dt$$

So that

$$\frac{d\mathrm{m}}{dt}=-t\frac{d\alpha}{dt}$$

Next consider the equations

$$\frac{dz}{dt}=\alpha y-\gamma z,\ \frac{dy}{dt}=-\alpha z-\gamma y$$

Where α is constant, but γ is a very small quantity compared with α, but which may vary slowly.

Treat γ as constant, and differentiate, and we have

$$\frac{d^2z}{dt^2}+\alpha^2z=-\gamma\left(\frac{dz}{dt}+\alpha y\right)$$

$$\frac{d^2y}{dt^2}+\alpha^2y=-\gamma\left(\frac{dy}{dt}-\alpha z\right)$$

Then if we neglect γ, we have the first approximation

$$z=L\cos(\alpha t+\mathrm{m}),\ y=-L\sin(\alpha t+\mathrm{m})$$

Substituting these values for z, y on the right, we have

$$\frac{d^2z}{dt^2}+\alpha^2z=\gamma 2\alpha L\sin(\alpha t+\mathrm{m})$$

And a similar equation for y.

The solutions are

$$z=\quad L\cos(\alpha t+\mathrm{m})-\gamma Lt\cos(\alpha t+\mathrm{m})$$

$$y=-L\sin(\alpha t+\mathrm{m})+\gamma Lt\sin(\alpha t+\mathrm{m})$$

From this we see that, if we desire to retain the first approximation as the solution, we must have

$$\frac{1}{L}\frac{dL}{dt}=-\gamma \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (187)$$

This will be true if γ varies slowly; hence

$$L=L_0e^{-\int\gamma dt}$$

and the solution is

$$z=\quad L_0e^{-\int\gamma dt}\cos(\alpha t+\mathrm{m})$$

$$y=-L_0e^{-\int\gamma dt}\sin(\alpha t+\mathrm{m})$$

It is easy to verify that these are the rigorous solutions of the equations, when α is constant but γ varies.

The equation (187) gives the rate of change of amplitude of oscillation.

The cases which we have now considered, by the method of variation of parameters, are closely analogous to those to be treated below, and have been treated in the same way, so that the reader will be able to trace the process.

They are in fact more than simply analogous, for they are what our equations (181) become if the obliquity of the ecliptic be zero and $\zeta=0$, $\eta=0$. In this case $L=j$, and $dj/dt=-j\gamma$.

This shows that the secular change of figure of the earth, and the secular changes in the rate of revolution of the moon's nodes do not affect the rate of alteration of the inclination of the lunar orbit to the ecliptic, so long as the obliquity is zero. This last result contains the implicit assumption that the perturbing influence of the moon on the earth is not so large, but that the obliquity of the equator may always remain small, however the lunar nodes vary. In an exactly similar manner we may show that, if the inclination of the lunar orbit be zero, $di/dt=i\delta$.

This is the result of the previous paper "On the Precession of a Viscous Spheroid," when the obliquity is small.

According to the method which has been sketched, the equations to be integrated are given in (181), when we write $\alpha+\alpha't$ for α, $\mathrm{a}+\mathrm{a}'t$ for a, $\beta+\beta't$ for β, $\mathrm{b}+\mathrm{b}'t$ for b, and then treat α, a, &c., α', a$'$, &c., γ, g, &c., as constants.

Before proceeding to consider the equations, it will be convenient to find certain relations between the quantities α, a, &c., and the two roots κ_1 and κ_2 of the quadratic $(\kappa+\alpha)(\kappa+\beta)=\mathrm{ab}$.

We have supposed the two roots to be such that

$$\left.\begin{aligned}\kappa_1+\kappa_2&=-\alpha-\beta\\ \kappa_1-\kappa_2&=-\sqrt{(\alpha-\beta)^2+4\mathrm{ab}}\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (188)$$

Then

$$\kappa_1\kappa_2=(\alpha\beta-\mathrm{ab}) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (189)$$

$$\left.\begin{aligned}\kappa_1^2+\kappa_2^2&=\alpha^2+\beta^2+2\mathrm{ab}\\ \kappa_1^2\kappa_2^2&=(\alpha^2+\mathrm{ab})(\beta^2+\mathrm{ab})-\mathrm{ab}(\alpha+\beta)^2\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (190)$$

$$\left.\begin{aligned}\beta^2+\mathrm{ab}-\kappa_1^2&=(\kappa_1+\kappa_2)(\kappa_2+\alpha)\\ \beta^2+\mathrm{ab}-\kappa_2^2&=(\kappa_1+\kappa_2)(\kappa_1+\alpha)\\ \alpha^2+\mathrm{ab}-\kappa_1^2&=(\kappa_1+\kappa_2)(\kappa_2+\beta)\\ \alpha^2+\mathrm{ab}-\kappa_2^2&=(\kappa_1+\kappa_2)(\kappa_1+\beta)\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (191)$$

$$\left.\begin{aligned}\kappa_1+\alpha&=-(\kappa_2+\beta)\\ \kappa_2+\alpha&=-(\kappa_1+\beta)\end{aligned}\right\} \quad \ldots \ldots \ldots \quad (192)$$

$$\mathrm{ab}(\alpha+\beta)=(\kappa_1+\alpha)(\kappa_2+\alpha)(\kappa_1+\kappa_2) \quad \ldots \ldots \ldots \quad (193)$$

Now suppose our equations (181) to be written as follows:—

$$\left.\begin{aligned}\frac{dz}{dt}&=\ \ \alpha y+\mathrm{a}\eta+s\\ \frac{dy}{dt}&=-\alpha z-\mathrm{a}\zeta+u\\ \frac{d\zeta}{dt}&=\ \ \beta\eta+\mathrm{b}y+\sigma\\ \frac{d\eta}{dt}&=-\beta\zeta-\mathrm{b}z+v\end{aligned}\right\} \quad \ldots \ldots \ldots \quad (194)$$

Where s, u, σ, v comprise all the terms involving α', a$'$, &c., γ, g, &c.

Then if we write (z) as a type of z, y, ζ, η; (α) as a type of α, a, β, b; (α') as a type of α', a$'$, β', b$'$; (γ) as a type of γ, g, δ, d; and (s) as a type of s, u, σ, v; it is clear that (s) is $(z)(\alpha')t+(\gamma)(z)$.

Differentiate each of the equations (194), and substitute for $\frac{d(z)}{dt}$ after differentiation. Then if we write equations

$$\left.\begin{aligned}\mathrm{S}&=\frac{ds}{dt}+\alpha u+\mathrm{a}v\\ \mathrm{U}&=\frac{du}{dt}-\alpha s-\mathrm{a}\sigma\\ \Sigma&=\frac{d\sigma}{dt}+\beta v+\mathrm{b}u\\ \Upsilon&=\frac{dv}{dt}-\beta\sigma-\mathrm{b}s\end{aligned}\right\} \quad \ldots \ldots \ldots \quad (195)$$

The result is

$$\left.\begin{aligned}\frac{d^2z}{dt^2}&=-(\alpha^2+\mathrm{ab})z-\mathrm{a}(\alpha+\beta)\zeta+\mathrm{S}\\ \frac{d^2y}{dt^2}&=-(\alpha^2+\mathrm{ab})y-\mathrm{a}(\alpha+\beta)\eta+\mathrm{U}\\ \frac{d^2\zeta}{dt^2}&=-(\beta^2+\mathrm{ab})\zeta-\mathrm{b}(\alpha+\beta)z+\Sigma\\ \frac{d^2\eta}{dt^2}&=-(\beta^2+\mathrm{ab})\eta-\mathrm{b}(\alpha+\beta)y+\Upsilon\end{aligned}\right\} \quad \ldots \ldots \quad (196)$$

From the first of these

$$-(\beta^2+\mathrm{ab})\mathrm{a}(\alpha+\beta)\zeta=(\beta^2+\mathrm{ab})\frac{d^2z}{dt^2}+(\alpha^2+\mathrm{ab})(\beta^2+\mathrm{ab})z-\mathrm{S}(\beta^2+\mathrm{ab})$$

Therefore from the third

$$a(\alpha+\beta)\frac{d^2\zeta}{dt^2}=(\beta^2+ab)\frac{d^2z}{dt^2}+\{(\alpha^2+ab)(\beta^2+ab)-ab(\alpha+\beta)^2\}z-S(\beta^2+ab)+\Sigma a(\alpha+\beta)$$

and by (190)

$$\left.\begin{aligned}
a(\alpha+\beta)\frac{d^2\zeta}{dt^2}&=(\beta^2+ab)\frac{d^2z}{dt^2}+\kappa_1^2\kappa_2^2z-S(\beta^2+ab)+\Sigma a(\alpha+\beta)\\
&\text{Similarly}\\
a(\alpha+\beta)\frac{d^2\eta}{dt^2}&=(\beta^2+ab)\frac{d^2y}{dt^2}+\kappa_1^2\kappa_2^2y-U(\beta^2+ab)+\Upsilon a(\alpha+\beta)\\
b(\alpha+\beta)\frac{d^2z}{dt^2}&=(\alpha^2+ab)\frac{d^2\zeta}{dt^2}+\kappa_1^2\kappa_2^2\zeta-\Sigma(\alpha^2+ab)+Sb(\alpha+\beta)\\
b(\alpha+\beta)\frac{d^2y}{dt^2}&=(\alpha^2+ab)\frac{d^2\eta}{dt^2}+\kappa_1^2\kappa_2^2\eta-\Upsilon(\alpha^2+ab)+Ub(\alpha+\beta)
\end{aligned}\right\} \quad . \quad . \quad (197)$$

Differentiate the first of (196) twice, using the first of (197), and we have

$$\frac{d^4z}{dt^4}=-(\alpha^2+ab)\frac{d^2z}{dt^2}-(\beta^2+ab)\frac{d^2z}{dt^2}-\kappa_1^2\kappa_2^2z+\left(\beta^2+ab+\frac{d^2}{dt^2}\right)S-\Sigma a(\alpha+\beta)$$

Therefore by (190)

$$\left[\frac{d^4}{dt^4}+(\kappa_1^2+\kappa_2^2)\frac{d^2}{dt^2}+\kappa_1^2\kappa_2^2\right]z=\left(\beta^2+ab+\frac{d^2}{dt^2}\right)S-\Sigma a(\alpha+\beta)$$

Then writing (S) as a type of S, Σ, U, Υ,—

$$\text{(S) is of the type } (z)(\alpha)(\alpha')t+(\alpha)(\gamma)(z)+(\alpha')(z)+\frac{d(z)}{dt}(\alpha')t+(\gamma)\frac{d(z)}{dt}$$

Hence every term of (S) contains some small term, either (α') or (γ).

Therefore on the right-hand side of the above equation we may substitute for (z) the first approximation, viz.: $(z_1)+(z_2)$ given in (182–3).

When this substitution is carried out, let (S_1), (S_2) be the parts of (S) which contain all terms of the speeds κ_1 and κ_2 respectively.

Then by (191) and (193) the right-hand side in the above equation may be written

$$(\kappa_1+\kappa_2)(\kappa_2+\alpha)S_1-\frac{\Sigma_1}{b}(\kappa_1+\alpha)(\kappa_2+\alpha)(\kappa_1+\kappa_2)+\left(\kappa_1^2+\frac{d^2}{dt^2}\right)S_1$$

$+$ the same with 2 and 1 interchanged.

Now let D^4 stand for the operation $\frac{d^4}{dt^4}+(\kappa_1^2+\kappa_2^2)\frac{d^2}{dt^2}+\kappa_1^2\kappa_2^2$, and we have

$$\left.\begin{aligned}
D^4 z &= (\kappa_1+\kappa_2)(\kappa_2+\alpha)\left\{S_1-\frac{\kappa_1+\alpha}{\mathrm{b}}\Sigma_1\right\}+\left(\kappa_1^2+\frac{d^2}{dt^2}\right)S_1 \\
&\qquad + \text{the same with 2 and 1 reversed} \\
D^4 y &= (\kappa_1+\kappa_2)(\kappa_2+\alpha)\left\{U_1-\frac{\kappa_1+\alpha}{\mathrm{b}}\Upsilon_1\right\}+\left(\kappa_1^2+\frac{d^2}{dt^2}\right)U_1+\text{\&c.} \\
D^4 \zeta &= (\kappa_1+\kappa_2)(\kappa_2+\beta)\left\{\Sigma_1-\frac{\kappa_1+\beta}{\mathrm{a}}S_1\right\}+\left(\kappa_1^2+\frac{d^2}{dt^2}\right)\Sigma_1+\text{\&c.} \\
D^4 \eta &= (\kappa_1+\kappa_2)(\kappa_2+\beta)\left\{\Upsilon_1-\frac{\kappa_1+\beta}{\mathrm{a}}U_1\right\}+\left(\kappa_1^2+\frac{d^2}{dt^2}\right)\Upsilon_1+\text{\&c.}
\end{aligned}\right\} \quad . \quad (198)$$

The last three of these equations are to be found by a parallel process, or else by symmetry.

If the right hand sides of (198) be neglected, we clearly obtain, on integration, the first approximation (183) for z, y, ζ, η. This first approximation was originally obtained by mere inspection.

We now have to consider the effects of the small terms on the right on the constants of integration L_1, L_2, L_1', L_2' introduced in the first approximation.

The small terms on the right are, by means of the first approximation, capable of being arranged in one of the alternative forms

$$\left.\begin{matrix}\cos\\ \sin\end{matrix}\right\}\kappa_1 t+t\left.\begin{matrix}\sin\\ \cos\end{matrix}\right\}\kappa_1 t+\text{ the same with 2 for 1}$$

Now consider the differential equation

$$\frac{d^4x}{dt^4}+(a^2+b^2)\frac{d^2x}{dt^2}+a^2b^2x=A\cos(at+\eta)+Bt\cos(at+\eta) \quad . \quad . \quad . \quad (199)$$

First suppose that B is zero, so that the term in A exists alone.

Assume $x=Ct\sin(at+\eta)$ as the solution.

Then

$$\frac{d^2x}{dt^2}=C\{-a^2t\sin(at+\eta)+2a\cos(at+\eta)\}$$

$$\frac{d^4x}{dt^4}=C\{a^4t\sin(at+\eta)-4a^3\cos(at+\eta)\}$$

By substitution in (199), with $B=0$, we have

$$C\{-4a^3+2a(a^2+b^2)\}=A$$

Therefore the solution is

$$x=-\frac{A}{2a(a^2-b^2)}t\sin(at+\eta)$$

By writing $\eta-\frac{1}{2}\pi$ for η, we see that a term A sin $(at+\eta)$ in the differential equation would generate $\frac{A}{2a(a^2-b^2)}t\cos(at+\eta)$ in the solution.

From this theorem it follows that the solution of the equation

$$D^4z=F_1y_1+F_2y_2$$

is

$$z=\frac{tF_1z_1}{2\kappa_1(\kappa_1^2-\kappa_2^2)}+\text{ the same with 2 and 1 interchanged}$$

and the solution of

$$D^4z=F_1\eta_1+F_2\eta_2$$

is

$$z=\frac{tF_1\zeta_1}{2\kappa_1(\kappa_1^2-\kappa_2^2)}+\text{ the same with 2 and 1 interchanged}$$

Also (writing the two alternatives by means of an easily intelligible notation) the solutions of

$$D^4y=F_1\begin{cases}z_1\\ \zeta_1\end{cases}+F_2\begin{cases}z_2\\ \zeta_2\end{cases}$$

are

$$y=-\frac{tF_1\begin{cases}y_1\\ \eta_1\end{cases}}{2\kappa_1(\kappa_1^2-\kappa_2^2)}-\text{ the same with 2 and 1 interchanged}$$

The similar equations for $D^4\zeta$, $D^4\eta$ may be treated in the same way. The general rule is that

y and η in the differential equations generate in the solution tz and $t\zeta$ respectively; and z and ζ generate $-ty$ and $-t\eta$ respectively; and the terms are to be divided by $2\kappa_1(\kappa_1^2-\kappa_2^2)$ or $2\kappa_2(\kappa_2^2-\kappa_1^2)$ as the case may be.

Next suppose that $A=0$ in the equation (199), and assume as the solution

$$x=Ct^2\sin(at+\eta)+Dt\cos(at+\eta)$$

Then

$$\begin{aligned}\frac{d^2x}{dt^2}=&\ C\{-a^2t^2\sin(at+\eta)+4at\cos(at+\eta)+2\sin(at+\eta)\}\\ &+D\{-a^2t\cos(at+\eta)-2a\sin(at+\eta)\}\\ \frac{d^4x}{dt^4}=&\ C\{a^4t^2\sin(at+\eta)-8a^3t\cos(at+\eta)-12a^2\sin(at+\eta)\}\\ &+D\{a^4t\cos(at+\eta)+4a^3\sin(at+\eta)\}\end{aligned}$$

Then substituting in (199), we must have

$$4aC(a^2+b^2)-8a^3C=B$$

and

$$2(C-aD)(a^2+b^2)-12a^2C+4a^3D=0$$

Whence

$$C=-\frac{B}{4a(a^2-b^2)},\ D=-\frac{5a^2-b^2}{4a^2(a^2-b^2)^2}B$$

Hence the solution of (199), when $A=0$, is

$$x=-\frac{5a^2-b^2}{4a^2(a^2-b^2)^2}Bt\cos(at+\eta)-\frac{1}{4a(a^2-b^2)}Bt^2\sin(at+\eta)$$

If t be very small, the second of these terms may be neglected.

By writing $\eta-\frac{1}{2}\pi$ for η, we see that a term $Bt\sin(at+\eta)$ in the differential equation, would have given rise in the solution to

$$x=-\frac{5a^2-b^2}{4a^2(a^2-b^2)^2}Bt\sin(at+\eta)$$

t being very small.

By this theorem we see that the solutions of the two alternative differential equations

$$\mathrm{D}^4z=t\mathrm{F}_1\begin{cases}z_1\\ \zeta_1\end{cases}+t\mathrm{F}_2\begin{cases}z_2\\ \zeta_2\end{cases}$$

are, when t is very small,

$$z=-\frac{5\kappa_1^2-\kappa_2^2}{4\kappa_1^2(\kappa_1^2-\kappa_2^2)^2}t\mathrm{F}_1\begin{cases}z_1\\ \zeta_1\end{cases}-\text{ the same with 2 and 1 interchanged.}$$

The similar equations for D^4y, $\mathrm{D}^4\eta$, $\mathrm{D}^4\zeta$ may be treated similarly. The general rule is that:—

tz and $t\zeta$ in the differential equations are reproduced, but with an opposite sign in the solution; and similarly ty and $t\eta$ are reproduced with the opposite sign; and in the solution the terms are to be multiplied by

$$\frac{5\kappa_1^2-\kappa_2^2}{4\kappa_1^2(\kappa_1^2-\kappa_2^2)^2}\text{ or }\frac{5\kappa_2^2-\kappa_1^2}{4\kappa_2^2(\kappa_2^2-\kappa_1^2)^2}$$

For the purpose of future developments it will be more convenient to write these factors in the forms

$$\frac{1}{2\kappa_1(\kappa_1^2-\kappa_2^2)}\left\{\frac{2\kappa_1}{\kappa_1^2-\kappa_2^2}+\frac{1}{2\kappa_1}\right\}\text{ and }\frac{1}{2\kappa_2(\kappa_2^2-\kappa_1^2)}\left\{\frac{2\kappa_2}{\kappa_2^2-\kappa_1^2}+\frac{1}{2\kappa_2}\right\}$$

By means of these two rules we see that the solutions of the two alternative differential equations

$$D^4 z = A_1 \begin{Bmatrix} y_1 \\ \eta_1 \end{Bmatrix} + tB_1 \begin{Bmatrix} z_1 \\ \zeta_1 \end{Bmatrix} + \text{the same with 2 for 1} \quad . \quad . \quad . \quad . \quad . \quad (200)$$

are, so long as t is very small,

$$z = z_1 + \frac{tA_1 \begin{Bmatrix} z_1 \\ \zeta_1 \end{Bmatrix}}{2\kappa_1(\kappa_1^2 - \kappa_2^2)} - \frac{tB_1 \begin{Bmatrix} z_1 \\ \zeta_1 \end{Bmatrix}}{2\kappa_1(\kappa_1^2 - \kappa_2^2)} \left[\frac{2\kappa_1}{\kappa_1^2 - \kappa_2^2} + \frac{1}{2\kappa_1} \right]$$

$$+ \text{the same with 2 and 1 interchanged} \quad (201)$$

Then putting for z_1, ζ_1, &c., their values from (182), these solutions may be written,

$$z = \cos(\kappa_1 t + m_1) \left\{ L_1 + \frac{tA_1 \begin{Bmatrix} L_1 \\ L_1' \end{Bmatrix}}{2\kappa_1(\kappa_1^2 - \kappa_2^2)} - \frac{tB_1 \begin{Bmatrix} L_1 \\ L_1' \end{Bmatrix}}{2\kappa_1(\kappa_1^2 - \kappa_2^2)} \left[\frac{2\kappa_1}{\kappa_1^2 - \kappa_2^2} + \frac{1}{2\kappa_1} \right] \right\}$$

$$+ \text{ the same with 2 for 1} \quad (202)$$

Hence we may retain the first approximation

$$z = L_1 \cos(\kappa_1 t + m_1) + L_2 \cos(\kappa_2 t + m_2)$$

as the solution, provided that L_1 and L_2 are no longer constant, but vary in such a way that

$$\left. \begin{array}{l} \dfrac{dL_1}{dt} = \dfrac{A_1 \begin{Bmatrix} L_1 \\ L_1' \end{Bmatrix}}{2\kappa_1(\kappa_1^2 - \kappa_2^2)} - \dfrac{B_1 \begin{Bmatrix} L_1 \\ L_1' \end{Bmatrix}}{2\kappa_1(\kappa_1^2 - \kappa_2^2)} \left[\dfrac{2\kappa_1}{\kappa_1^2 - \kappa_2^2} + \dfrac{1}{2\kappa_1} \right] \\ \text{and a similar equation for } L_2 \end{array} \right\} \quad . \quad . \quad . \quad . \quad (203)$$

It will be found, when we come to apply these results, that the solution of the equation for $D^4 y$ will lead to the same equations for the variation of L_1 and L_2 as are derived from the equation for $D^4 z$.

A similar treatment may be applied to the equations for $D^4 \zeta$ or $D^4 \eta$, and we find similar differential equations for dL_1'/dt and dL_2'/dt.

These equations will be the differential equations for the secular changes in L_1 and L_2', which are the constants of integration in the first approximation.

We will now apply these theorems to the differential equations (181); but as the analysis is rather complex, it will be more convenient to treat the variations of α, a, β, b and the terms in γ, g, δ, d independently.

We will indicate by the symbol Δ the additional terms which arise, and will write

the symbol out of which the term arises as a suffix—*e.g.*, we shall write Δz_α for the additional terms in the complete value of z, which arise from the variation of α. Also $(dL/dt)_\alpha$ will be written for the terms in dL/dt which arise from the variation of α.

Terms depending on the variation of α.

We now put for α in (181) $\alpha+\alpha' t$.

Hence in (194)

$$s=\alpha' ty,\ u=-\alpha' tz,\ \sigma=0,\ v=0$$

Therefore

$$\mathrm{S}=\alpha'\left\{y-t\left(\alpha z-\frac{dy}{dt}\right)\right\},\ \Sigma=-\alpha'\mathrm{b}tz$$

And by substitution from (182–3)

$$\mathrm{S}_1=\alpha'\{y_1+tz_1(\kappa_1-\alpha)\},\quad \Sigma_1=-\alpha'\mathrm{b}tz_1$$

S_2, Σ_2 have similar forms with 2 for 1.

Then

$$\left(\kappa_1^2+\frac{d^2}{dt^2}\right)\mathrm{S}_1=2\alpha'(\kappa_1-\alpha)\frac{dz_1}{dt}=-2\alpha'\kappa_1(\kappa_1-\alpha)y_1$$

$$\mathrm{S}_1-\frac{\kappa_1+\alpha}{\mathrm{b}}\Sigma_1=\alpha'\{y_1+tz_1(\kappa_1-\alpha)\}+\alpha' t(\kappa_1+\alpha)z_1$$

$$=\alpha'\{y_1+2t\kappa_1 z_1\}\quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (204)$$

Hence the equation for z is

$$\frac{1}{\alpha'}\mathrm{D}^4 z=(\kappa_1+\kappa_2)(\kappa_2+\alpha)(y_1+2t\kappa_1 z_1)-2\kappa_1(\kappa_1-\alpha)y_1+\text{the same with 2 for 1.}$$

Hence by the rules found above for the solution of such an equation

$$\frac{1}{\alpha' t}\Delta z_\alpha=\frac{z_1}{2\kappa_1(\kappa_1^2-\kappa_2^2)}\left\{(\kappa_1+\kappa_2)(\kappa_2+\alpha)-2\kappa_1(\kappa_1-\alpha)-2\kappa_1(\kappa_1+\kappa_2)(\kappa_2+\alpha)\left[\frac{2\kappa_1}{\kappa_1^2-\kappa_2^2}+\frac{1}{2\kappa_1}\right]\right\}+\&\text{c.}$$

$$=-\frac{z_1}{\kappa_1^2-\kappa_2^2}\left[\kappa_1-\alpha+\frac{2\kappa_1(\kappa_2+\alpha)}{(\kappa_1-\kappa_2)}\right]+\&\text{c.}$$

$$=-z_1\frac{\kappa_1+\alpha}{(\kappa_1-\kappa_2)^2}-z_2\frac{\kappa_2+\alpha}{(\kappa_1-\kappa_2)^2}$$

Whence

$$\left(\frac{1}{L_1}\frac{dL_1}{dt}\right)_\alpha=-\alpha'\frac{\kappa_1+\alpha}{(\kappa_1-\kappa_2)^2},\quad \left(\frac{1}{L_2}\frac{dL_2}{dt}\right)_\alpha=-\alpha'\frac{\kappa_2+\alpha}{(\kappa_1-\kappa_2)^2}\quad . \quad . \quad . \quad . \quad (205)$$

If we form U and Υ, and solve the equation for D^4y, we obtain the same results. Again

$$\left(\kappa_1^2+\frac{d^2}{dt^2}\right)\Sigma_1=-2\alpha'\mathrm{b}\frac{dz_1}{dt}=2\alpha'\mathrm{b}\kappa_1y_1$$

$$\Sigma_1-\frac{\kappa_1+\beta}{\mathrm{a}}\mathrm{S}_1=\Sigma_1-\frac{\mathrm{b}}{\kappa_1+\alpha}\mathrm{S}_1=\frac{\mathrm{b}}{\kappa_2+\beta}\left\{\mathrm{S}_1-\frac{\kappa_1+\alpha}{\mathrm{b}}\Sigma_1\right\}$$

$$=\alpha'\frac{\mathrm{b}}{\kappa_2+\beta}(y_1+2t\kappa_1z_1)\text{ by (204)}$$

Hence the equation for ζ is

$$\frac{1}{\alpha'\mathrm{b}}D^4\zeta=(\kappa_1+\kappa_2)(y_1+2t\kappa_1z_1)+2\kappa_1y_1+\text{the same with 2 for 1}$$

And by the rules of solution

$$\frac{1}{\alpha'\mathrm{b}t}\Delta\zeta_\alpha=\frac{z_1}{2\kappa_1(\kappa_1^2-\kappa_2^2)}\left[\kappa_1+\kappa_2+2\kappa_1-2\kappa_1(\kappa_1+\kappa_2)\left\{\frac{2\kappa_1}{\kappa_1^2-\kappa_2^2}+\frac{1}{2\kappa_1}\right\}\right]+\&\text{c.}$$

$$=\frac{z_1}{\kappa_1^2-\kappa_2^2}\left[1-\frac{2\kappa_1}{\kappa_1-\kappa_2}\right]+\&\text{c.}$$

$$=-\frac{z_1}{(\kappa_1-\kappa_2)^2}-\frac{z_2}{(\kappa_1-\kappa_2)^2}$$

$$=-\frac{1}{\mathrm{b}}\frac{\kappa_2+\alpha}{(\kappa_1-\kappa_2)^2}\zeta_1-\frac{1}{\mathrm{b}}\frac{\kappa_1+\alpha}{(\kappa_1-\kappa_2)^2}\zeta_2$$

Since

$$z_1=\frac{\kappa_2+\alpha}{\mathrm{b}}\zeta_1,\quad z_2=\frac{\kappa_1+\alpha}{\mathrm{b}}\zeta_2$$

Hence

$$\left(\frac{1}{L_1'}\frac{dL_1'}{dt}\right)_\alpha=-\alpha'\frac{\kappa_2+\alpha}{(\kappa_1-\kappa_2)^2},\quad\left(\frac{1}{L_2'}\frac{dL_2'}{dt}\right)_\alpha=-\alpha'\frac{\kappa_1+\alpha}{(\kappa_1-\kappa_2)^2}\quad\cdot\;\cdot\;\cdot\;\cdot\quad(206)$$

If we form U and Υ, and solve the equation for $D^4\eta$, we obtain the same result.

Terms depending on the variation of β.

The results may be written down by symmetry.

z and y are symmetrical with ζ and η, and therefore unaccented L's are symmetrical with accented ones, and *vice-versâ;* α is symmetrical with β, and *vice-versâ.*

The suffixes 1 and 2 remain unaffected by the symmetry.

Then (by (192)) writing for $\kappa_2+\beta$, $\kappa_1+\beta$; $-(\kappa_1+\alpha)$ and $-(\kappa_2+\alpha)$ respectively, we have by symmetry with (206),

$$\left(\frac{1}{L_1}\frac{dL_1}{dt}\right)_\beta=\beta'\frac{\kappa_1+\alpha}{(\kappa_1-\kappa_2)^2},\quad \left(\frac{1}{L_2}\frac{dL_2}{dt}\right)_\beta=\beta'\frac{\kappa_2+\alpha}{(\kappa_1-\kappa_2)^2} \quad . \quad . \quad . \quad . \quad . \quad (207)$$

And by symmetry with (205),

$$\left(\frac{1}{L_1'}\frac{dL_1'}{dt}\right)_\beta=\beta'\frac{\kappa_2+\alpha}{(\kappa_1-\kappa_2)^2},\quad \left(\frac{1}{L_2'}\frac{dL_2'}{dt}\right)_\beta=\beta'\frac{\kappa_1+\alpha}{(\kappa_1-\kappa_2)^2} \quad . \quad . \quad . \quad . \quad . \quad (208)$$

Terms depending on the variation of a.

We now put for a in (181) $\mathrm{a}+\mathrm{a}'t$.

Then in (194)

$$s=\mathrm{a}'t\eta,\ u=-\mathrm{a}'t\zeta,\ \sigma=0,\ v=0$$

Therefore

$$\mathrm{S}=\mathrm{a}'\eta+\mathrm{a}'t\left(\frac{d\eta}{dt}-\alpha\zeta\right) \qquad \Sigma=-\mathrm{a}'\mathrm{b}t\zeta$$

$$\mathrm{S}_1=\mathrm{a}'\{\eta_1+t\zeta_1(\kappa_1-\alpha)\} \qquad \Sigma_1=-\mathrm{a}'\mathrm{b}t\zeta_1$$

S_2, Σ_2 have similar forms with 2 for 1

$$\left(\kappa_1^2+\frac{d^2}{dt^2}\right)\mathrm{S}_1=2\mathrm{a}'(\kappa_1-\alpha)\frac{d\zeta_1}{dt}=-2\mathrm{a}'\kappa_1(\kappa_1-\alpha)\eta_1$$

$$\mathrm{S}_1-\frac{\kappa_1+\alpha}{\mathrm{b}}\Sigma_1=\mathrm{a}'[\eta_1+t\zeta_1(\kappa_1-\alpha)+t\zeta(\kappa_1+\alpha)]=\mathrm{a}'[\eta_1+2\kappa_1t\zeta_1] \quad . \quad . \quad . \quad (209)$$

Hence the equation for z is

$$\frac{1}{\mathrm{a}'}\mathrm{D}^4z=-2\kappa_1(\kappa_1-\alpha)\eta_1+(\kappa_1+\kappa_2)(\kappa_2+\alpha)(\eta_1+2\kappa_1t\zeta_1)+\text{ the same with 2 for 1}$$

$$\frac{1}{\mathrm{a}'t}\Delta z_{\mathrm{a}}=\frac{\zeta_1}{2\kappa_1(\kappa_1^2-\kappa_2^2)}\Big[-2\kappa_1(\kappa_1-\alpha)+(\kappa_1+\kappa_2)(\kappa_2+\alpha)$$
$$-2\kappa_1(\kappa_1+\kappa_2)(\kappa_2+\alpha)\left(\frac{2\kappa_1}{\kappa_1^2-\kappa_2^2}+\frac{1}{2\kappa_1}\right)\Big]+\&\text{c.}$$
$$=-\frac{\zeta_1}{\kappa_1^2-\kappa_2^2}\left[\kappa_1-\alpha+\frac{2\kappa_1(\kappa_2+\alpha)}{\kappa_1-\kappa_2}\right]-\&\text{c.}$$
$$=-\zeta_1\frac{\kappa_1+\alpha}{(\kappa_1-\kappa_2)^2}-\zeta_2\frac{\kappa_2+\alpha}{(\kappa_1-\kappa_2)^2}$$
$$=-z_1\frac{\mathrm{b}}{(\kappa_1-\kappa_2)^2}\,\frac{\kappa_1+\alpha}{\kappa_2+\alpha}-z_2\frac{\mathrm{b}}{(\kappa_1-\kappa_2)^2}\,\frac{\kappa_2+\alpha}{\kappa_1+\alpha},\text{ since }\zeta_1=z_1\frac{\mathrm{b}}{\kappa_2+\alpha},\ \zeta_2=z_2\frac{\mathrm{b}}{\kappa_1+\alpha}$$

Therefore

$$\left(\frac{1}{L_1}\frac{dL_1}{dt}\right)_{\rm a}=-\frac{\rm a'b}{(\kappa_1-\kappa_2)^2}\frac{\kappa_1+\alpha}{\kappa_2+\alpha},\quad \left(\frac{1}{L_2}\frac{dL_2}{dt}\right)_{\rm a}=-\frac{\rm a'b}{(\kappa_1-\kappa_2)^2}\frac{\kappa_2+\alpha}{\kappa_1+\alpha}\quad . \quad . \quad . \quad (210)$$

Again

$$\Sigma_1-\frac{\kappa_1+\beta}{\rm a}{\rm S}_1=\Sigma_1-\frac{\rm b}{\kappa_1+\alpha}{\rm S}_1=\frac{\rm b}{\kappa_2+\beta}\left({\rm S}_1-\frac{\kappa_1+\alpha}{\rm b}\Sigma_1\right)$$

$$=\frac{\rm a'b}{\kappa_2+\beta}(\eta_1+2\kappa_1 t\zeta_1)\text{ by (209)}$$

Also

$$\left(\kappa_1^2+\frac{d^2}{dt^2}\right)\Sigma_1=-2{\rm a'b}\frac{d\zeta_1}{dt}=2{\rm a'b}\kappa_1\eta_1$$

Therefore the equation for ζ is

$$\frac{1}{\rm a'b}{\rm D}^4\zeta=(\kappa_1+\kappa_2)(\eta_1+2\kappa_1 t\zeta_1)+2\kappa_1\eta_1+\text{ the same with 2 for 1}$$

Therefore

$$\frac{1}{{\rm a'b}t}\Delta\zeta_{\rm a}=\frac{\zeta_1}{2\kappa_1(\kappa_1^2-\kappa_2^2)}\left[(\kappa_1+\kappa_2)+2\kappa_1-2\kappa_1(\kappa_1+\kappa_2)\left(\frac{2\kappa_1}{\kappa_1^2-\kappa_2^2}+\frac{1}{2\kappa_1}\right)\right]+\&c.$$

$$=\frac{\zeta_1}{\kappa_1^2-\kappa_2^2}\left(1-\frac{2\kappa_1}{\kappa_1-\kappa_2}\right)+\&c.$$

$$=-\frac{\zeta_1}{(\kappa_1-\kappa_2)^2}-\frac{\zeta_2}{(\kappa_1-\kappa_2)^2}$$

Therefore

$$\left(\frac{1}{L_1'}\frac{dL_1'}{dt}\right)_{\rm a}=-\frac{\rm a'b}{(\kappa_1-\kappa_2)^2},\quad \left(\frac{1}{L_2'}\frac{dL_2'}{dt}\right)_{\rm a}=-\frac{\rm a'b}{(\kappa_1-\kappa_2)^2}\quad . \quad . \quad . \quad . \quad . \quad (211)$$

The same results might have been obtained from the equations to ${\rm D}^4y$, ${\rm D}^4\eta$.

Terms depending on the variation of b.

By symmetry with (211)

$$\left(\frac{1}{L_1}\frac{dL_1}{dt}\right)_{\rm b}=-\frac{\rm b'a}{(\kappa_1-\kappa_2)^2},\quad \left(\frac{1}{L_2}\frac{dL_2}{dt}\right)_{\rm b}=-\frac{\rm b'a}{(\kappa_1-\kappa_2)^2}\quad . \quad . \quad . \quad . \quad . \quad (212)$$

By symmetry with (210), and putting $-(\kappa_2+\alpha)$ for $(\kappa_1+\beta)$ and $-(\kappa_1+\alpha)$ for $(\kappa_2+\beta)$

$$\left(\frac{1}{L_1'}\frac{dL_1'}{dt}\right)_{\rm b}=-\frac{\rm b'a}{(\kappa_1-\kappa_2)^2}\frac{\kappa_2+\alpha}{\kappa_1+\alpha},\quad \left(\frac{1}{L_2'}\frac{dL_2'}{dt}\right)_{\rm b}=-\frac{\rm b'a}{(\kappa_1-\kappa_2)^2}\frac{\kappa_1+\alpha}{\kappa_2+\alpha}.\quad . \quad . \quad (213)$$

We now come to a different class of terms, viz.: those depending on γ, g, δ, d.

Terms depending on γ.

Here

$$s=-\gamma z, \quad u=-\gamma y, \quad \sigma=0, \quad v=0$$

$$\mathrm{S}=-\gamma\left(\frac{dz}{dt}+\alpha y\right) \qquad \Sigma=-\mathrm{b}\gamma y$$

$$\mathrm{S}_1=\gamma(\kappa_1-\alpha)y_1 \qquad \Sigma_1=-\gamma \mathrm{b} y_1$$

S_2, Σ_2 have similar forms with 2 for 1

Obviously

$$\left(\kappa_1{}^2+\frac{d^2}{dt^2}\right)\mathrm{S}_1=0$$

$$\mathrm{S}_1-\frac{\kappa_1+\alpha}{\mathrm{b}}\Sigma_1=2\gamma\kappa_1 y_1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (214)$$

Hence the equation for z is

$$\frac{1}{\gamma}\mathrm{D}^4 z=2\kappa_1(\kappa_1+\kappa_2)(\kappa_2+\alpha)y_1+\text{ the same with 2 for 1}$$

Therefore

$$\frac{1}{\gamma t}\Delta z_\gamma=z_1\frac{\kappa_2+\alpha}{\kappa_1-\kappa_2}+z_2\frac{\kappa_1+\alpha}{\kappa_2-\kappa_1}$$

And

$$\left(\frac{1}{L_1}\frac{dL_1}{dt}\right)_\gamma=\gamma\frac{\kappa_2+\alpha}{\kappa_1-\kappa_2}, \quad \left(\frac{1}{L_2}\frac{dL_2}{dt}\right)_\gamma=-\gamma\frac{\kappa_1+\alpha}{\kappa_1-\kappa_2} \quad . \quad . \quad . \quad . \quad . \quad . \quad (215)$$

Again

$$\left(\kappa_1{}^2+\frac{d^2}{dt^2}\right)\Sigma_1=0$$

$$\Sigma_1-\frac{\kappa_1+\beta}{\mathrm{a}}\mathrm{S}_1=\frac{\mathrm{b}}{\kappa_2+\beta}\left[\mathrm{S}_1-\frac{\kappa_1+\alpha}{\mathrm{b}}\Sigma_1\right]$$

$$=\mathrm{b}\gamma\frac{2\kappa_1}{\kappa_2+\beta}y_1$$

And the equation for ζ is

$$\frac{1}{\gamma \mathrm{b}}\mathrm{D}^4\zeta=2\kappa_1(\kappa_1+\kappa_2)y_1+\text{ the same with 2 for 1}$$

Therefore

$$\frac{1}{\gamma \mathrm{b} t}\Delta\zeta_\gamma=\frac{z_1}{\kappa_1-\kappa_2}+\frac{z_2}{\kappa_2-\kappa_1}$$

$$=\frac{\zeta_1}{\mathrm{b}}\frac{\kappa_2+\alpha}{\kappa_1-\kappa_2}-\frac{\zeta_2}{\mathrm{b}}\frac{\kappa_1+\alpha}{\kappa_1-\kappa_2}, \text{ since } z_1=\zeta_1\frac{\kappa_2+\alpha}{\mathrm{b}}, \; z_2=\zeta_2\frac{\kappa_1+\alpha}{\mathrm{b}}$$

Hence

$$\left(\frac{1}{L_1'}\frac{dL_1'}{dt}\right)_\gamma=\gamma\frac{\kappa_2+\alpha}{\kappa_1-\kappa_2},\quad \left(\frac{1}{L_2'}\frac{dL_2'}{dt}\right)_\gamma=-\gamma\frac{\kappa_1+\alpha}{\kappa_1-\kappa_2}\quad .\ .\ .\ .\ .\ . \quad (216)$$

Terms depending on δ.

These may be written down by symmetry.

$-\delta$ is symmetrical with γ. Hence writing $-(\kappa_1+\alpha)$ for $\kappa_2+\beta$, and $-(\kappa_2+\alpha)$ for $(\kappa_1+\beta)$, we have by symmetry with (216)

$$\left(\frac{1}{L_1}\frac{dL_1}{dt}\right)_\delta=\delta\frac{\kappa_1+\alpha}{\kappa_1-\kappa_2},\quad \left(\frac{1}{L_2}\frac{dL_2}{dt}\right)_\delta=-\delta\frac{\kappa_2+\alpha}{\kappa_1-\kappa_2}.\quad .\ .\ .\ .\ .\ . \quad (217)$$

And by symmetry with (215)

$$\left(\frac{1}{L_1'}\frac{dL_1'}{dt}\right)_\delta=\delta\frac{\kappa_1+\alpha}{\kappa_1-\kappa_2},\quad \left(\frac{1}{L_2'}\frac{dL_2'}{dt}\right)_\delta=-\delta\frac{\kappa_2+\alpha}{\kappa_1-\kappa_2}\quad .\ .\ .\ .\ . \quad (218)$$

Terms depending on g.

Here

$$s=-\mathrm{g}\zeta,\ u=-\mathrm{g}\eta,\ \sigma=0,\ v=0$$

$$\mathrm{S}=-\mathrm{g}\left(\frac{d\zeta}{dt}+\alpha\eta\right)\qquad \Sigma=-\mathrm{gb}\eta$$

$$\mathrm{S}_1=\ \mathrm{g}(\kappa_1-\alpha)\eta_1\qquad \Sigma_1=-\mathrm{gb}\eta_1$$

S_2, Σ_2 have similar forms with 2 for 1

Clearly

$$\left(\kappa_1^2+\frac{d^2}{dt^2}\right)\mathrm{S}_1=0$$

$$\mathrm{S}_1-\frac{\kappa_1+\alpha}{\mathrm{b}}\Sigma_1=2\mathrm{g}\kappa_1\eta_1.\quad .\ .\ .\ .\ .\ .\ .\ .\ .\ . \quad (219)$$

Therefore the equation for z is

$$\frac{1}{\mathrm{g}}\mathrm{D}^4z=2\kappa_1(\kappa_1+\kappa_2)(\kappa_2+\alpha)\eta_1+\text{ the same with 2 for 1}$$

Thence

$$\frac{1}{\mathrm{g}t}\Delta z_\mathrm{g}=\zeta_1\frac{\kappa_2+\alpha}{\kappa_1-\kappa_2}+\zeta_2\frac{\kappa_1+\alpha}{\kappa_2-\kappa_1}$$

$$=z_1\frac{\mathrm{b}}{\kappa_1-\kappa_2}-z_2\frac{\mathrm{b}}{\kappa_1-\kappa_2},\ \text{since } \zeta_1=z_1\frac{\mathrm{b}}{\kappa_2+\alpha},\ \zeta_2=z_2\frac{\mathrm{b}}{\kappa_1+\alpha}$$

Therefore

$$\left(\frac{1}{L_1}\frac{dL_1}{dt}\right)_{\rm g}={\rm g}\frac{\rm b}{\kappa_1-\kappa_2},\quad \left(\frac{1}{L_2}\frac{dL_2}{dt}\right)_{\rm g}=-{\rm g}\frac{\rm b}{\kappa_1-\kappa_2}\quad . \quad . \quad . \quad . \quad . \quad . \tag{220}$$

Again

$$\left(\kappa_1^{\,2}+\frac{d^2}{dt^2}\right)\Sigma_1=0$$

and

$$\Sigma_1-\frac{\kappa_1+\beta}{\rm a}{\rm S}_1=\frac{\rm b}{\kappa_2+\beta}\left[{\rm S}_1-\frac{\kappa_1+\alpha}{\rm b}\Sigma_1\right]$$

$$={\rm g}\frac{2{\rm b}\kappa_1}{\kappa_2+\beta}\eta_1 \text{ by (219)}$$

Therefore the equation for ζ is

$$\frac{1}{\rm gb}{\rm D}^4\zeta=2\kappa_1(\kappa_1+\kappa_2)\eta_1+\text{ the same with 2 for 1}$$

Hence

$$\frac{1}{{\rm gb}t}\Delta\zeta_{\rm g}=\frac{\zeta_1}{\kappa_1-\kappa_2}+\frac{\zeta_2}{\kappa_2-\kappa_1}$$

Therefore

$$\left(\frac{1}{L_1'}\frac{dL_1'}{dt}\right)_{\rm g}={\rm g}\frac{\rm b}{\kappa_1-\kappa_2},\quad \left(\frac{1}{L_2'}\frac{dL_2'}{dt}\right)_{\rm g}=-{\rm g}\frac{\rm b}{\kappa_1-\kappa_2}\quad . \quad . \quad . \quad . \quad . \tag{221}$$

The same results may be obtained by means of the equations for ${\rm D}^4y$, ${\rm D}^4\eta$.

Terms depending on d.

These may be written down by symmetry.

$-$d is symmetrical with g. Therefore by symmetry with (221)

$$\left(\frac{1}{L_1}\frac{dL_1}{dt}\right)_{\rm d}=-{\rm d}\frac{\rm a}{\kappa_1-\kappa_2},\quad \left(\frac{1}{L_2}\frac{dL_2}{dt}\right)_{\rm d}={\rm d}\frac{\rm a}{\kappa_1-\kappa_2}\quad . \quad . \quad . \quad . \quad . \quad . \tag{222}$$

and by symmetry with (220)

$$\left(\frac{1}{L_1'}\frac{dL_1'}{dt}\right)_{\rm d}=-{\rm d}\frac{\rm a}{\kappa_1-\kappa_2},\quad \left(\frac{1}{L_2'}\frac{dL_2'}{dt}\right)_{\rm d}={\rm d}\frac{\rm a}{(\kappa_1-\kappa_2)}\quad . \quad . \quad . \quad . \quad . \tag{223}$$

This completes the consideration of the effects on the constants of integration L_1, L_2, L_1', L_2' of all the small terms.

Then collecting results from (205–8, 210–13, 215–18, 220–23),

$$\left.\begin{aligned}
\frac{1}{L_1}\frac{dL_1}{dt}&=\frac{1}{(\kappa_1-\kappa_2)^2}\left\{-(\kappa_1+\alpha)(\alpha'-\beta')-\mathrm{a'b}\frac{\kappa_1+\alpha}{\kappa_2+\alpha}-\mathrm{b'a}\right\}\\
&\quad+\frac{1}{(\kappa_1-\kappa_2)}\{\gamma(\kappa_2+\alpha)+\delta(\kappa_1+\alpha)+\mathrm{gb}-\mathrm{da}\}\\
\frac{1}{L_2}\frac{dL_2}{dt}&=\frac{1}{(\kappa_1-\kappa_2)^2}\left\{-(\kappa_2+\alpha)(\alpha'-\beta')-\mathrm{a'b}\frac{\kappa_2+\alpha}{\kappa_1+\alpha}-\mathrm{b'a}\right\}\\
&\quad-\frac{1}{\kappa_1-\kappa_2}\{\gamma(\kappa_1+\alpha)+\delta(\kappa_2+\alpha)+\mathrm{gb}-\mathrm{da}\}\\
\frac{1}{L_1'}\frac{dL_1'}{dt}&=\frac{1}{(\kappa_1-\kappa_2)^2}\left\{-(\kappa_2+\alpha)(\alpha'-\beta')-\mathrm{a'b}-\mathrm{b'a}\frac{\kappa_2+\alpha}{\kappa_1+\alpha}\right\}\\
&\quad+\frac{1}{\kappa_1-\kappa_2}\{\gamma(\kappa_2+\alpha)+\delta(\kappa_1+\alpha)+\mathrm{gb}-\mathrm{da}\}\\
\frac{1}{L_2'}\frac{dL_2'}{dt}&=\frac{1}{(\kappa_1-\kappa_2)^2}\left\{-(\kappa_1+\alpha)(\alpha'-\beta')-\mathrm{a'b}-\mathrm{b'a}\frac{\kappa_1+\alpha}{\kappa_2+\alpha}\right\}\\
&\quad-\frac{1}{\kappa_1-\kappa_2}\{\gamma(\kappa_1+\alpha)+\delta(\kappa_2+\alpha)+\mathrm{gb}-\mathrm{da}\}
\end{aligned}\right\}\quad . \quad . \quad . \quad (224)$$

We shall now show that these four equations are equivalent to two only, and in showing this shall verify the correctness of the results.

To prove that the four equations (224) *are equivalent to two.*

In (118) we showed that

$$\frac{L_1'}{L_1}=-\frac{\kappa_1+\alpha}{\mathrm{a}}$$

Therefore we ought to find that

$$\begin{aligned}
\frac{1}{L_1'}\frac{dL_1'}{dt}-\frac{1}{L_1}\frac{dL_1}{dt}&=\frac{1}{\kappa_1+\alpha}\frac{d}{dt}(\kappa_1+\alpha)-\frac{\mathrm{a'}}{\mathrm{a}}\\
&=\frac{\kappa_1+\beta}{\mathrm{ab}}\frac{d}{dt}(\kappa_1+\alpha)-\frac{\mathrm{a'}}{\mathrm{a}}
\end{aligned}$$

Now by (188)

$$2(\kappa_1+\alpha)=\alpha-\beta-\sqrt{(\alpha-\beta)^2+4\mathrm{ab}}$$

and

$$2\frac{d}{dt}(\kappa_1+\alpha)=\alpha'-\beta'+\frac{(\alpha-\beta)(\alpha'-\beta')+2(\mathrm{a'b}+\mathrm{ab'})}{\kappa_1-\kappa_2}$$

so that

$$\frac{d}{dt}(\kappa_1+\alpha)=\frac{(\alpha'-\beta')(\kappa_1+\alpha)+\mathrm{a'b}+\mathrm{ab'}}{\kappa_1-\kappa_2}$$

And thus we ought to find that,

$$(\kappa_1-\kappa_2)\left[\frac{1}{L_1'}\frac{dL_1'}{dt}-\frac{1}{L_1}\frac{dL_1}{dt}\right]=\alpha'-\beta'-\frac{\mathrm{a'}}{\mathrm{a}}(\kappa_1+\alpha)-\frac{\mathrm{b'}}{\mathrm{b}}(\kappa_2+\alpha)$$

Now if we subtract the first of equations (224) from the third we shall find this relation to be satisfied. Hence the first and third equations are equivalent to only a single one.

Similarly it may be proved that the second and third equations are similarly related.

To prove that the four equations (224) *reduce to those of* § 6, *when the nodes revolve with uniform velocity.*

It appears from § 13 that when a and b are small compared with $\alpha-\beta$, the nodes revolve with approximate uniformity, and the nutations of the system are small.

If this be the case, we have approximately

$$\kappa_1=-\alpha,\ \ \kappa_2=-\beta.$$

It will appear later that $(\alpha'-\beta')/(\alpha-\beta)$, a′/a, b′/b are quantities of the same order of magnitude as γ, g, δ, d.

Now $L_1=\mathrm{J}$, the inclination of the lunar orbit to its proper plane, and $L_2'=\mathrm{I}$, the inclination of the earth's proper plane to the ecliptic.

Therefore, the first and last of equations (224) become

$$\frac{1}{\mathrm{J}}\frac{d\mathrm{J}}{dt}=-\gamma-\frac{\mathrm{gb}-\mathrm{da}}{\alpha-\beta}$$

$$\frac{1}{\mathrm{I}}\frac{d\mathrm{I}}{dt}=\ \ \delta+\frac{\mathrm{gb}-\mathrm{da}}{\alpha-\beta}$$

But since the nodes revolve uniformly, $\mathrm{b}/(\alpha-\beta)$ and $\mathrm{a}/(\alpha-\beta)$ are small, and therefore the latter terms of these equations are negligeable compared with the former.

Hence

$$\frac{1}{\mathrm{J}}\frac{d\mathrm{J}}{dt}=-\gamma,\ \ \frac{1}{\mathrm{I}}\frac{d\mathrm{I}}{dt}=\delta$$

These results in no way depend on the assumption of the smallness of the viscosity of the planet, and therefore we may substitute Γ and Δ (see (174)) for γ and δ.

A comparison of the expressions for Γ and Δ, with those given in Part II. for dj/dt and in my previous paper for di/dt, will show that our present equations for $d\mathrm{J}/dt$ and $d\mathrm{I}/dt$ are what the previous ones reduce to, when i and j are small. But this comparison shows more than this, for it shows that what the equation (61) § 6 really gives is the rate of change of the inclination of the lunar orbit to its proper plane, and that the equation (66) of the paper on "Precession" really gives the rate of change of the inclination of the earth's proper plane (or mean equator) to the ecliptic.

To show how the equations (224) *reduce to those of* § 10.

We now pass to the other extreme, and suppose the solar influence infinitesimal compared with that of oblateness.

Here

$$\alpha=\mathrm{a},\quad \beta=\mathrm{b},\quad \gamma=\mathrm{g},\quad \delta=\mathrm{d}$$
$$\kappa_1=-(\mathrm{a}+\mathrm{b}),\quad \kappa_2=0$$

Then the equations (224) reduce to

$$\left.\begin{aligned}\frac{1}{L_1}\frac{dL_1}{dt}&=-\mathrm{g}+\mathrm{d}+\frac{\mathrm{a}'\mathrm{b}-\mathrm{b}'\mathrm{a}}{\mathrm{a}(\mathrm{a}+\mathrm{b})}\\ \frac{1}{L_1'}\frac{dL_1'}{dt}&=-\mathrm{g}+\mathrm{d}-\frac{\mathrm{a}'\mathrm{b}-\mathrm{b}'\mathrm{a}}{\mathrm{b}(\mathrm{a}+\mathrm{b})}\\ \frac{1}{L_2}\frac{dL_2}{dt}&=0,\quad \frac{1}{L_2'}\frac{dL_2'}{dt}=0\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (225)$$

Therefore L_2 and L_2' are constant. Also from the relationship between them

$$\frac{L_2'}{L_2}=-\frac{(\kappa_2+\alpha)}{\mathrm{a}}=-1$$

Hence it follows that the two proper planes are identical with one another, and are fixed in space. They are, in fact, the invariable plane of the system, as appears as follows :—

If we use the notation of § 10, $L_1=j$, $L_1'=i$, and $L_1'/L_1=-(\kappa_1+\alpha)/\mathrm{a}=\mathrm{b}/\mathrm{a}$; so that $\mathrm{a}i=\mathrm{b}j$.

Now $\mathrm{a}=k\tau\mathfrak{c}/\xi$, $\mathrm{b}=\tau\mathfrak{c}/n$, and i and j are by hypothesis small, therefore we may write the relationship between a, b, i, j in the form

$$\frac{\xi}{k}\sin j=n\sin i.$$

This proves that the two coincident planes fixed in space are identical with the invariable plane of the system (see 108).

But the identity of equations (225) with (71) of § 10 and (29) of the paper on "Precession" remains to be proved.

If i and j be treated as small, those equations are in effect

$$\frac{dj}{dt}=-\mathrm{g}(i+j)$$

$$\frac{di}{dt}=\ \mathrm{d}(i+j)$$

(or with G and D in place of g and d if the viscosity be not small).

Hence if (225) are identical with (71) and (29) of "Precession," we must have

$$-\mathrm{g}\frac{i}{j}=\mathrm{d}+\frac{\mathrm{a'b}-\mathrm{ab'}}{\mathrm{a(a+b)}}$$

$$\mathrm{d}\frac{j}{i}=-\mathrm{g}-\frac{\mathrm{a'b}-\mathrm{ab'}}{\mathrm{b(a+b)}}$$

But $i/j=\mathrm{b/a}$; therefore the condition for the identity of (225) with (71) and (29) of "Precession" is that

$$(\mathrm{a+b})(\mathrm{gb+ad})+\mathrm{a'b}-\mathrm{ab'}=0 \quad \ldots \ldots \quad (226)$$

Or if the viscosity be not small, a similar equation with G and D for g and d.

We cannot prove that this condition is satisfied until a′ and b′ have been evaluated, but it will be proved later in § 16.

This discussion shows that the obliquity of the earth's equator (L_1') to the invariable plane of the moon-earth system, when the solar influence is infinitesimal, degrades into the amplitude of the nineteen-yearly nutation, when the influence of oblateness is infinitesimal. The one quantity is strictly continuous with the other.

This completes the verification of the differential equations (224) in the two extreme cases.

§ 16. *Evaluation of α′, a′ &c., in the case of the earth's viscosity.*

The preceding section does not involve any hypothesis as to the constitution of the earth, but it will now be supposed to be viscous, and the various functions, which occur in (224), will be evaluated.

By (184–5) we have

$$\frac{1}{k}\frac{d\xi}{dt}=\frac{1}{2}\frac{\tau^2}{\mathfrak{g}}\sin 4\mathrm{f}_1 \quad \ldots \ldots \quad (227)$$

$$-\frac{dn}{dt}=\frac{1}{2}\frac{\tau^2}{\mathfrak{g}}\sin 4\mathrm{f}_1\left(1+\left(\frac{\tau'}{\tau}\right)^2\right) \quad \ldots \ldots \quad (228)$$

The last equation is approximate, for by writing it in this form we are neglecting $\tau'^2(\sin 4\mathrm{f}-\sin 4\mathrm{f}_1)/\tau^2 \sin 4\mathrm{f}_1$ compared with unity.

This is legitimate, because when $(\sin 4\mathrm{f}-\sin 4\mathrm{f}_1)/\sin 4\mathrm{f}_1$ is not very small, τ'^2/τ^2 is very small, and *vice-versâ ;* see however § 22.

Hence (228) may be written

$$-\frac{dn}{dt}=\left(\frac{1}{k}\frac{d\xi}{dt}\right)\left(1+\left(\frac{\tau'}{\tau}\right)^2\right) \quad \ldots \ldots \quad (229)$$

Let

$$\mathfrak{m}=\frac{kn}{\xi} \quad \ldots \ldots \quad (230)$$

$\mathfrak{m}$ is the ratio of the moment of momentum of the earth's rotation to that of the orbital motion of moon and earth round their common centre of inertia. (The μ of my paper on "Precession" is equal to the reciprocal of $\mathfrak{m}_0$, where $\mathfrak{m}_0$ is the value of $\mathfrak{m}$ when $t=0$.)

By (121) and (112) we have,

$$\mathrm{a}=\frac{k}{\xi}\tau\mathfrak{e}$$

Now $\mathfrak{e}=\frac{1}{2}n^2/\mathfrak{g}$, the ellipticity of the earth due to rotation; and as $\tau=\frac{3}{2}\mu m/c^3$ and $\xi=\sqrt{c/c_0}$, therefore $\tau=\tau_0/\xi^6$.

Hence

$$\mathrm{a}=\frac{k\tau_0}{2\mathfrak{g}}\frac{n^2}{\xi^7}$$

Differentiating logarithmically

$$\frac{\mathrm{a}'}{\mathrm{a}}=\frac{2}{n}\frac{dn}{dt}-\frac{7}{\xi}\frac{d\xi}{dt}=-\left(\frac{1}{k}\frac{d\xi}{dt}\right)\left\{\frac{2}{n}\left(1+\left(\frac{\tau'}{\tau}\right)^2\right)+\frac{7k}{\xi}\right\}$$

$$\frac{\mathrm{a}'}{\mathrm{a}}=-\left(\frac{1}{k}\frac{d\xi}{dt}\right)\frac{1}{n}\left\{2\left(1+\left(\frac{\tau'}{\tau}\right)^2\right)+7\mathfrak{m}\right\} \quad \ldots \ldots \quad (231)$$

Also since

$$\mathrm{a}=\mathfrak{m}\frac{\tau\mathfrak{e}}{n} \quad \ldots \ldots \quad (232)$$

$$\mathrm{a}'=-\left(\frac{1}{k}\frac{d\xi}{dt}\right)\left(\frac{\tau\mathfrak{e}}{n^2}\right)\mathfrak{m}\left\{2\left(1+\left(\frac{\tau'}{\tau}\right)^2\right)+7\mathfrak{m}\right\} \quad \ldots \ldots \quad (233)$$

Then by (121) and (112)

$$\alpha-\mathrm{a}=\frac{1}{2}\frac{\tau'}{\Omega}$$

Since $\Omega=\Omega_0/\xi^3$, and τ' is constant (or at least varies so slowly that we may neglect its variation), we have

$$\frac{\alpha'-\mathrm{a}'}{\alpha-\mathrm{a}}=\frac{3}{\xi}\frac{d\xi}{dt}=\left(\frac{1}{k}\frac{d\xi}{dt}\right)\frac{1}{n}3\mathfrak{m}$$

Now $\dfrac{\Omega}{n}=\lambda$, hence

$$\alpha-\mathrm{a}=\frac{\tau\mathfrak{e}}{n}\frac{\tau'}{\tau}\left(\frac{1}{2\lambda\mathfrak{e}}\right)$$

Therefore

$$\alpha'-\mathrm{a}'=\left(\frac{1}{k}\frac{d\xi}{dt}\right)\left(\frac{\tau\mathfrak{e}}{n^2}\right)\frac{\tau'}{\tau}\frac{3\mathfrak{m}}{2\lambda\mathfrak{e}} \quad \ldots \ldots \quad (234)$$

From (233–4)

$$\alpha'=\left(\frac{1}{k}\frac{d\xi}{dt}\right)\left(\frac{\tau\mathfrak{k}}{n^2}\right)\mathfrak{m}\left\{\frac{3}{2\lambda\mathfrak{k}}\frac{\tau'}{\tau}-\left[2\left(1+\left(\frac{\tau'}{\tau}\right)^2\right)+7\mathfrak{m}\right]\right\} \quad . \quad . \quad . \quad . \quad (235)$$

Also

$$\alpha=\frac{\tau\mathfrak{k}}{n}\left[\frac{\tau'}{\tau}\frac{1}{2\lambda\mathfrak{k}}+\mathfrak{m}\right] \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (236)$$

By (121) and (112)

$$\mathrm{b}=\frac{\tau\mathfrak{k}}{n}=\frac{\tau_0}{2\mathfrak{g}}\frac{n}{\xi^6} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (237)$$

Therefore

$$\frac{\mathrm{b}'}{\mathrm{b}}=\frac{1}{n}\frac{dn}{dt}-\frac{6}{\xi}\frac{d\xi}{dt}$$

$$=-\left(\frac{1}{k}\frac{d\xi}{dt}\right)\frac{1}{n}\left\{1+\left(\frac{\tau'}{\tau}\right)^2+6\mathfrak{m}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (238)$$

And

$$\mathrm{b}'=-\left(\frac{1}{k}\frac{d\xi}{dt}\right)\left(\frac{\tau\mathfrak{k}}{n^2}\right)\left\{1+\left(\frac{\tau'}{\tau}\right)^2+6\mathfrak{m}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (239)$$

From (231) and (238)

$$\frac{\mathrm{a}'}{\mathrm{a}}-\frac{\mathrm{b}'}{\mathrm{b}}=-\left(\frac{1}{k}\frac{d\xi}{dt}\right)\frac{1}{n}\left\{1+\left(\frac{\tau'}{\tau}\right)^2+\mathfrak{m}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (240)$$

By (121) and (112)

$$\beta-\mathrm{b}=\frac{\tau'\mathfrak{k}}{n}=\frac{\tau'}{2\mathfrak{g}}n$$

$$\beta'-\mathrm{b}'=\frac{\tau'}{2\mathfrak{g}}\frac{dn}{dt}=-\left(\frac{1}{k}\frac{d\xi}{dt}\right)\frac{\tau'}{2\mathfrak{g}}\left(1+\left(\frac{\tau'}{\tau}\right)^2\right)$$

$$=-\left(\frac{1}{k}\frac{d\xi}{dt}\right)\left(\frac{\tau\mathfrak{k}}{n^2}\right)\left(\frac{\tau'}{\tau}\right)\left(1+\left(\frac{\tau'}{\tau}\right)^2\right). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (241)$$

Therefore

$$\beta'=-\left(\frac{1}{k}\frac{d\xi}{dt}\right)\left(\frac{\tau\mathfrak{k}}{n^2}\right)\left\{1+\frac{\tau'}{\tau}+\left(\frac{\tau'}{\tau}\right)^2+\left(\frac{\tau'}{\tau}\right)^3+6\mathfrak{m}\right\} \quad . \quad . \quad . \quad . \quad . \quad (242)$$

Lastly

$$\beta=\frac{\tau\mathfrak{k}}{n}\left(1+\frac{\tau'}{\tau}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (243)$$

By (174), (227), and (230), when the viscosity is not small, we have

$$\left.\begin{aligned}
\Gamma&=\left(\frac{1}{k}\frac{d\xi}{dt}\right)\frac{\mathfrak{m}}{2n}\frac{(\sin 4\mathrm{f}_1-\sin 2\mathrm{g}_1+\sin 2\mathrm{g})}{\sin 4\mathrm{f}_1}\\
\mathrm{G}&=\left(\frac{1}{k}\frac{d\xi}{dt}\right)\frac{\mathfrak{m}}{2n}\frac{(\sin 4\mathrm{f}_1-\sin 2\mathrm{g}_1+\sin 2\mathrm{g})+\frac{\tau'}{\tau}\sin 2\mathrm{g}}{\sin 4\mathrm{f}_1}\\
\Delta&=\left(\frac{1}{k}\frac{d\xi}{dt}\right)\frac{1}{2n}\frac{(\sin 4\mathrm{f}_1+\sin 2\mathrm{g}_1-\sin 2\mathrm{g})-2\frac{\tau'}{\tau}\sin 2\mathrm{g}+\left(\frac{\tau'}{\tau}\right)^2\sin 4\mathrm{f}}{\sin 4\mathrm{f}_1}\\
\mathrm{D}&=\left(\frac{1}{k}\frac{d\xi}{dt}\right)\frac{1}{2n}\frac{(\sin 4\mathrm{f}_1+\sin 2\mathrm{g}_1-\sin 2\mathrm{g})-\frac{\tau'}{\tau}\sin 2\mathrm{g}}{\sin 4\mathrm{f}_1}
\end{aligned}\right\}\quad . \quad . \quad (244)$$

If the viscosity be small we have by (179), (227), and (230)

$$\left.\begin{aligned}
\gamma&=\left(\frac{1}{k}\frac{d\xi}{dt}\right)\frac{\mathfrak{m}}{2n}\frac{1}{1-\lambda}\\
\mathrm{g}&=\left(\frac{1}{k}\frac{d\xi}{dt}\right)\frac{\mathfrak{m}}{2n}\frac{1+\frac{1}{2}\frac{\tau'}{\tau}}{1-\lambda}\\
\delta&=\left(\frac{1}{k}\frac{d\xi}{dt}\right)\frac{1}{2n}\frac{1-2\lambda-\frac{\tau'}{\tau}+\left(\frac{\tau'}{\tau}\right)^2}{1-\lambda}\\
\mathrm{d}&=\left(\frac{1}{k}\frac{d\xi}{dt}\right)\frac{1}{2n}\frac{1-2\lambda-\frac{1}{2}\frac{\tau'}{\tau}}{1-\lambda}
\end{aligned}\right\}\quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (245)$$

I think no confusion will arise between the distinct uses made of the symbol g in (244) and (245); in the first it always must occur with a sine, in the second it never can do so.

[If τ' be zero

$$\mathrm{bG}+\mathrm{aD}=\left(\frac{1}{k}\frac{d\xi}{dt}\right)\left(\frac{\mathfrak{m}}{2n}\right)\left(\frac{\tau\mathfrak{c}}{n}\right)(2)$$

$$\mathrm{a}+\mathrm{b}=\frac{\tau\mathfrak{c}}{n}(1+\mathfrak{m})$$

and by (232), (237), (240)

$$\mathrm{ab}'-\mathrm{ba}'=\left(\frac{1}{k}\frac{d\xi}{dt}\right)\frac{\mathfrak{m}}{n}\left(\frac{\tau\mathfrak{c}}{n}\right)^2(1+\mathfrak{m})$$

Therefore we have

$$(\mathrm{bG}+\mathrm{aD})(\mathrm{a}+\mathrm{b})+\mathrm{a'b}-\mathrm{b'a}=0$$

This was shown in (226) to be the criterion that the differential equations (224) should reduce to those of (71) and of (29) of "Precession," when the solar influence is evanescent, and the above is the promised proof thereof.]

From (244), (237), and (232) we have

$$\mathrm{bG}-\mathrm{aD}=\left(\frac{1}{k}\frac{d\xi}{dt}\right)\left(\frac{\mathfrak{m}}{2n}\right)\left(\frac{\tau\mathfrak{c}}{n}\right)\frac{2\left(1+\frac{\tau'}{\tau}\right)\sin 2\mathrm{g}-2\sin 2\mathrm{g}_1}{\sin 4\mathrm{f}_1} \quad . \quad . \quad . \quad . \quad (246)$$

and similarly

$$\mathrm{bg}-\mathrm{ad}=\left(\frac{1}{k}\frac{d\xi}{dt}\right)\left(\frac{\mathfrak{m}}{2n}\right)\left(\frac{\tau\mathfrak{c}}{n}\right)\frac{2\lambda+\frac{\tau'}{\tau}}{1-\lambda} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (247)$$

§ 17. *Change of independent variable, and formation of equations for integration.*

In the equations (224) the time t is the independent variable, but in order to integrate we shall require ξ to be the variable. It has been shown above that these equations are equivalent to only two of them; henceforth therefore we shall only consider the first and last of them. It will also serve to keep before us the physical meaning of the L's, if the notation be changed; the following notation (which has been already used in (127)) will be adopted :—

$\mathrm{J}=\ \ L_1 =$ the inclination of the lunar orbit to the lunar proper plane.
$\mathrm{I}=\ \ L_2' =$ the inclination of the earth's proper plane to the ecliptic.
$\mathrm{I}_{,}=\ \ L_1' =$ the inclination of the equator to the earth's proper plane.
$\mathrm{J}_{,}=-L_2 =$ the inclination of the lunar proper plane to the ecliptic.

Then since J, I, &c., are small, we may write

$$\frac{dL_1}{L_1}=d.\log\tan\tfrac{1}{2}\mathrm{J},\quad \frac{dL_2'}{L_2'}=d.\log\tan\tfrac{1}{2}\mathrm{I} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (248)$$

This particular transformation is chosen because in Part II., where j and i were not small, $dj/\sin j$ seemed to arise naturally.

Also since

$$\frac{L_1'}{L_1}=-\frac{\kappa_1+\alpha}{\mathrm{a}},\ \frac{L_2'}{L_2}=-\frac{\kappa_2+\alpha}{\mathrm{a}}$$

we have

$$\left.\begin{aligned}\sin\mathrm{I}_{,}&=-\frac{\kappa_1+\alpha}{\mathrm{a}}\sin\mathrm{J}\\ \sin\mathrm{J}_{,}&=\frac{\mathrm{a}}{\kappa_2+\alpha}\sin\mathrm{I}\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (249)$$

These equations will give $\mathrm{I}_{,}$ and $\mathrm{J}_{,}$, when J and I are found.

Now suppose we divide the first and last of (224) by $d\xi/nkdt$, then their left-hand sides may be written

$$nk\frac{d}{d\xi}\log\tan\tfrac{1}{2}\mathrm{J}\quad\text{and}\quad nk\frac{d}{d\xi}\log\tan\tfrac{1}{2}\mathrm{I}$$

In the last section we have determined the functions α, α', &c., and have them in such a form that Γ, G, Δ, D (or γ, g, δ, d) have all a common factor $d\xi/nkdt$.

But this is the expression by which we have to divide the equations in order to change the variable.

Therefore in computing Γ, G, &c. (or γ, g, &c.), we may drop this common factor.

Again α, a, β, b were so written as all to have a common factor $\tau\mathfrak{t}/n$; therefore κ_1 and κ_2 also have the same common factor.

Also α' a′, β', b′ all have a common factor $(d\xi/kdt)(\tau\mathfrak{t}/n^2)$.

From this it follows that when the variable is changed, we may drop the factor $\tau\mathfrak{t}/n$ from α, a, β, b, κ_1, κ_2 and the factor $(d\xi/kdt)(\tau\mathfrak{t}/n^2)$ from α', a′, β', b′.

Hence the differential equations with the new variable become

$$\left.\begin{aligned}
kn\frac{d}{d\xi}\log\tan\tfrac{1}{2}\mathrm{J} &= \frac{1}{(\kappa_1-\kappa_2)^2}\left\{-(\kappa_1+\alpha)(\alpha'-\beta')-\mathrm{a'b}\frac{\kappa_1+\alpha}{\kappa_2+\alpha}-\mathrm{b'a}\right\}\\
&\quad+\frac{1}{(\kappa_1-\kappa_2)}\{\gamma(\kappa_2+\alpha)+\delta(\kappa_1+\alpha)+\mathrm{gb}-\mathrm{da}\}\\
kn\frac{d}{d\xi}\log\tan\tfrac{1}{2}\mathrm{I} &= \frac{1}{(\kappa_1-\kappa_2)^2}\left\{-(\kappa_1+\alpha)(\alpha'-\beta')-\mathrm{a'b}-\mathrm{b'a}\frac{\kappa_1+\alpha}{\kappa_2+\alpha}\right\}\\
&\quad-\frac{1}{(\kappa_1-\kappa_2)}\{\gamma(\kappa_1+\alpha)+\delta(\kappa_2+\alpha)+\mathrm{gb}-\mathrm{da}\}
\end{aligned}\right\} \quad . \quad . \quad (250)$$

or similar equations with Γ, G, Δ, D in place of γ, g, δ, d if the viscosity be not small.

But we now have by (232–3–5–6–7–9, 242–3–4–5–6–7)

$$\left.\begin{aligned}
&\alpha=\mathfrak{m}+\frac{\tau'}{\tau}\frac{1}{2\lambda\mathfrak{t}}, \qquad \mathrm{a}=\mathfrak{m}, \qquad \beta=1+\frac{\tau'}{\tau}, \qquad \mathrm{b}=1\\
&\alpha'=\mathfrak{m}\left\{\frac{\tau'}{\tau}\frac{3}{2\lambda\mathfrak{t}}-\left[2\left(1+\left(\frac{\tau'}{\tau}\right)^2\right)+7\mathfrak{m}\right]\right\}, \ \mathrm{a}'=-\mathfrak{m}\left\{2\left(1+\left(\frac{\tau'}{\tau}\right)^2\right)+7\mathfrak{m}\right\}\\
&\beta'=-\left\{1+\frac{\tau'}{\tau}+\left(\frac{\tau'}{\tau}\right)^2+\left(\frac{\tau'}{\tau}\right)^3+6\mathfrak{m}\right\}, \quad \mathrm{b}'=-\left\{1+\left(\frac{\tau'}{\tau}\right)^2+6\mathfrak{m}\right\}\\
&\Gamma=\tfrac{1}{2}\mathfrak{m}\frac{\sin 4\mathrm{f}_1-\sin 2\mathrm{g}_1+\sin 2\mathrm{g}}{\sin 4\mathrm{f}_1},\\
&\Delta=\frac{(\sin 4\mathrm{f}_1+\sin 2\mathrm{g}_1-\sin 2\mathrm{g})-2\frac{\tau'}{\tau}\sin 2\mathrm{g}+\left(\frac{\tau'}{\tau}\right)^2\sin 4\mathrm{f}}{2\sin 4\mathrm{f}_1}\\
&\gamma=\frac{\mathfrak{m}}{2(1-\lambda)}, \qquad \delta=\frac{1-2\lambda-\frac{\tau'}{\tau}+\left(\frac{\tau'}{\tau}\right)^2}{2(1-\lambda)},\\
&\mathrm{bG}-\mathrm{aD}=\tfrac{1}{2}\mathfrak{m}\frac{2\left(1+\frac{\tau'}{\tau}\right)\sin 2\mathrm{g}-2\sin 2\mathrm{g}_1}{\sin 4\mathrm{f}_1}\\
&\mathrm{bg}-\mathrm{ad}=\frac{\mathfrak{m}\left(2\lambda+\frac{\tau'}{\tau}\right)}{2(1-\lambda)}
\end{aligned}\right\} \quad . \quad (251)$$

In these equations we have, recapitulating the notation

$$\mathfrak{m}=\frac{kn}{\xi},\ \lambda=\frac{\Omega}{n},\ \mathfrak{c}=\frac{1}{2}\frac{n^2}{\mathfrak{g}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (252)$$

Also

$$\left.\begin{array}{l}\kappa_1+\kappa_2=-\alpha-\beta \\ \kappa_1-\kappa_2=-\sqrt{(\alpha-\beta)^2+4\mathrm{ab}}\end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (253)$$

Lastly we have by (186)

$$\frac{n}{n_0}=1+\frac{1}{kn_0}\left\{(1-\xi)+\frac{1}{13}\left(\frac{\tau'}{\tau_0}\right)^2(1-\xi^{13})\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (254)$$

which gives parallel values of n and ξ.

These equations will be solved by quadratures for the case of the moon and earth in Part IV.

If τ'/τ be so small as to be negligeable, and $\tau'/2\lambda\mathfrak{c}\tau$ small compared with unity, then the equations (250) admit of reduction to a simple form.

With this hypothesis it is easy to find approximate values of κ_1 and κ_2, and then by some easy, but rather tedious analysis, it may be shown that (250) reduce to the following—

$$\left.\begin{array}{l} kn\dfrac{d}{d\xi}\log\tan\tfrac{1}{2}\mathrm{J}=-\dfrac{\mathfrak{m}+1}{\mathfrak{m}}\mathrm{G}+\dfrac{\tau'}{\tau}\dfrac{1}{2\lambda\mathfrak{c}}\dfrac{1+11\mathfrak{m}}{(1+\mathfrak{m})^2} \\ kn\dfrac{d}{d\xi}\log\tan\tfrac{1}{2}\mathrm{I}=\dfrac{\tau'}{\tau}\dfrac{1}{2\lambda\mathfrak{c}}\dfrac{1+11\mathfrak{m}}{(1+\mathfrak{m})^2}\end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (255)$$

These equations would give the secular changes of J and I, when the solar influence is very small compared with that of the moon. Of course if G be replaced by g, they are applicable to the case of small viscosity.

It is remarkable that the changes of I are independent of the viscosity; they depend in fact solely on the secular change in the permanent ellipticity of the earth.

IV.

INTEGRATION OF THE DIFFERENTIAL EQUATIONS FOR CHANGES IN THE INCLINATION OF THE ORBIT AND THE OBLIQUITY OF THE ECLIPTIC.

§ 18. *Integration in the case of small viscosity, where the nodes revolve uniformly.*

It is not, even at the present time, rigorously true that the nodes of the lunar orbit revolve uniformly on the ecliptic and that the inclination of the orbit is constant; but it is very nearly true, and the integration may be carried backwards in time for a long way without an important departure from accuracy.

The integrations will be carried out by the method of quadratures, and the process will be divided into a series of "periods of integration," as explained in § 15 and § 17 of the paper on "Precession." These periods will be the same as those in that paper, and the previous numerical work will be used as far as possible. It will be found, however, that it is not sufficiently accurate to assume the uniform revolution of the nodes beyond the first two periods of integration. For these first two periods the equations of § 7, Part II., will be used; but for the further retrospect we shall have to make the transition to the methods of Part III. It is important to defer the transition as long as possible, because Part III. assumes the smallness of i and j, whilst Part II. does not do so.

By (104) and (86) of Part II. we have, when $j'=0$, and Ω'/n is neglected,

$$\frac{di}{dt}=\frac{\sin 4\mathfrak{f}}{n\mathfrak{g}}\frac{1}{4}\sin i\cos i\left\{\tau^2(1-\tfrac{3}{2}\sin^2 j)+\tau'^2-\frac{2\Omega}{n}\tau^2\sec i\cos j-\tau\tau'(1-\tfrac{3}{2}\sin^2 j)\right\}$$

$$-\frac{dn}{dt}=\frac{\sin 4\mathfrak{f}}{n\mathfrak{g}}\left\{(1-\tfrac{1}{2}\sin^2 i)(\tau^2+\tau'^2)-\tfrac{1}{2}(1-\tfrac{3}{2}\sin^2 i)\tau^2\sin^2 j\right.$$
$$\left.-\tau^2\frac{\Omega}{n}\cos i\cos j+\tfrac{1}{2}\tau\tau'\sin^2 i(1-\tfrac{3}{2}\sin^2 j)\right\}$$

If we put $1-\frac{1}{2}\sin^2 i=\cos i$, $1-\frac{3}{2}\sin^2 j=\cos^3 j$, and neglect $\sin^2 i\sin^2 j$, these may be written

$$\left.\begin{aligned}\frac{di}{dt}&=\frac{\sin 4\mathfrak{f}}{n\mathfrak{g}}\frac{1}{4}\sin i\cos i\cos^3 j\left\{\tau^2+\tau'^2\sec^3 j-\tau\tau'-\frac{2\Omega}{n}\tau^2\sec i\sec^2 j\right\}\\ -\frac{dn}{dt}&=\frac{\sin 4\mathfrak{f}}{n\mathfrak{g}}\cos i\cos j\left\{\tau^2+\tau'^2\sec j-\tau^2\frac{\Omega}{n}+\tfrac{1}{2}\tau\tau'\sin i\tan i\cos^2 j\right\}\end{aligned}\right\}\quad(256)$$

If we treat $\sec j$ and $\cos j$ as unity in the small terms in τ'^2, $\tau\tau'$, and Ω/n, (256) only differ from (83) of "Precession" in that di/dt has a factor $\cos^3 j$ and dn/dt has a factor $\cos j$.

Again by (64) and (70)

$$\left.\begin{aligned}-\frac{1}{k}\frac{dj}{dt}&=\frac{1}{\xi}\,\frac{\sin 4\mathfrak{f}}{\mathfrak{g}}\tau^2\,\tfrac{1}{4}\cos i\sin j\\ \frac{1}{k}\frac{d\xi}{dt}&=\frac{\sin 4\mathfrak{f}}{\mathfrak{g}}\tau^2\,\tfrac{1}{2}\cos i\cos j\left(1-\frac{\Omega}{n}\sec i\sec j\right)\end{aligned}\right\}\quad\ldots\ldots\quad(257)$$

If we divide the second of (256) by the second of (257) we get an equation for $dn/d\xi$, which only differs from (84) of "Precession" in the presence of $\sec j$ in place of unity in certain of the small terms. Now j is small for the lunar orbit; hence the equation (88) of "Precession" for the conservation of moment of momentum is very nearly true. The equation is, with present notation,

$$\frac{n}{n_0}=1+\frac{1}{kn_0}\left[1-\xi+\tfrac{1}{13}\left(\frac{\tau'}{\tau_0}\right)^2(1-\xi^{13})+\tfrac{1}{14}\frac{\tau'}{\tau_0}\sin i\tan i\,(1-\xi^7)\right]$$
$$+\tfrac{1}{4}\sin^2 i\frac{\Omega_0}{n_0}\frac{1}{kn_0+1}\left(\frac{1}{\xi}-1\right)\left(\frac{1}{\xi}+\frac{kn_0+3}{kn_0+1}\right)+\tfrac{1}{2}\sin^2 i\frac{\Omega_0}{n_0}\frac{1}{(kn_0+1)^3}\log_e\left(\frac{kn_0+1-\xi}{kn_0\xi}\right) \quad (258)$$

In this equation we attribute to i, as it occurs on the right-hand side, an average value.

By means of this equation, I had already computed a series of values of n corresponding to equidistant values of ξ.

On dividing the first of (256) by the second of (257) we get an expression which differs from the $d\log\tan^2\frac{1}{2}i/d\xi$ of (84) of "Precession" by the presence of a common factor $\cos^2 j$, and by $\sec j$ occurring in some of the small terms. Hence we may, without much error, accept the results of the integration for i in § 17 of "Precession."

Lastly, dividing the first of (257) by the second, we have

$$\frac{d}{d\xi}\log\sin j=\frac{-1}{2\xi\left(1-\frac{\Omega}{n}\sec i\sec j\right)} \quad \ldots\ldots\ldots \quad (259)$$

This equation has now to be integrated by quadratures.

All the numerical values were already computed for § 17 of "Precession," and only required to be combined.

The present mean inclination of the lunar orbit is 5° 9′, so that $j_0=5°\ 9'$. I then conjecture 5° 12′ as a proper mean value to be assigned to j, as it occurs on the right-hand side of (259) for the first period of integration, which extends from $\xi=1$ to ·88.

First period of integration.

From $\xi=1$ to ·88, four equidistant values were computed.

From the computation for § 17 of "Precession" I extract the following.

ξ =	1	·96	·92	·88
$\log\frac{\Omega}{n}\sec i+10=$	8·59979	8·57309	8·56411	8·56746

Then introducing $j=5°\ 12'$, I find

ξ =	1	·96	·92	·88
$\left[2\xi\left(1-\frac{\Omega}{n}\sec i\sec j\right)\right]^{-1}=$	·5208	·5412	·5643	·5901

Combining these four values by the rules of the calculus of finite differences, we have

$$\int_{\cdot 88}^{1} \frac{d\xi}{2\xi\left(1-\frac{\Omega}{n}\sec i \sec j\right)} = \cdot 06641$$

This is equal to $\log_e \sin j - \log_e \sin j_0$. Taking $j_0 = 5° \ 9'$, I find $j = 5° \ 30'$.

Second period of integration.

From $\xi = 1$ to $\cdot 76$, four equidistant values were computed.

From the computation for § 17 "Precession," I extract the following :—

ξ	$= 1$	$\cdot 92$	$\cdot 84$	$\cdot 76$
$\log \frac{\Omega}{n} \sec i + 10$	$= 8 \cdot 56746$	$8 \cdot 59743$	$8 \cdot 65002$	$8 \cdot 72318$

Then assuming $5° \ 55'$ as an average value for j, I find

ξ	$= 1$	$\cdot 92$	$\cdot 84$	$\cdot 76$
$\left[2\xi\left(1-\frac{\Omega}{n}\sec i \sec j\right)\right]^{-1}$	$= \cdot 5193$	$\cdot 5660$	$\cdot 6232$	$\cdot 6948$

Combining these, we have

$$\int_{\cdot 76}^{1} \frac{d\xi}{2\xi\left(1-\frac{\Omega}{n}\sec i \sec j\right)} = \cdot 14345$$

This is equal to $\log_e \sin j - \log_e \sin j_0$. Taking $j_0 = 5° \ 30'$ from the first period, we find $j = 6° \ 21'$.

This completes the integration, as far as it is safe to employ the methods of Part II.

In Part III. it was proved that, in the case where the nodes revolve uniformly, equations (224) reduce to those of Part II. But it was also shown that what the equations of Part II. really give is the change of the inclination of the lunar orbit to the lunar proper plane; also that the equations of "Precession" really give the change of the inclination of the mean equator (that is of the earth's proper plane) to the ecliptic.

The results of the present integration are embodied in the following table, of which the first three columns are taken from the table in § 17 of "Precession."

TABLE I.

Sidereal day in m.s. hours and minutes.			Moon's sidereal period in m.s. days.	Inclination of mean equator to ecliptic.		Inclination of lunar orbit to lunar proper plane.	
	h.	m.	Days.	°	′	°	′
Initial	23	56	27·32	23	28	5	9
	15	28	18·62	20	28	5	30
Final	9	55	8·17	17	4	6	21

We will now consider what amount of oscillation the equator and the plane of the lunar orbit undergo, as the nodes revolve, in the initial and final conditions represented in the above table.

It appears from (119) that $\sin 2j$ oscillates between $\sin 2j_0 \pm \mathrm{a} \sin 2i_0/(\kappa_2+\alpha)$, and that $\sin 2i$ oscillates between $\sin 2i_0 \pm (\kappa_1+\alpha) \sin 2j_0/\mathrm{a}$, where i_0 and j_0 are the mean values of i and j.

With the numerical values corresponding to the initial condition (that is to say in the present configurations of earth, moon, and sun), it will be found on substituting in (115) and (112), with $a_2=\frac{3}{4}\left(\frac{\Omega'}{\Omega}\right)^2\left(1-\frac{3}{8}\frac{\Omega'}{\Omega}\right)\Omega$ instead of simply $\frac{1}{2}\frac{\tau'}{\Omega}$, that

$$\alpha=\cdot 341251,\ \beta=\cdot 000318,\ \mathrm{a}=\cdot 000059,\ \mathrm{b}=\cdot 000150,$$

when the present tropical year is the unit of time.

Since 4ab is very small compared with $(\alpha-\beta)^2$, it follows that we have to a close degree of approximation

$$\kappa_1=-\alpha,\ \kappa_2=-\beta$$

Then since $(\kappa_1+\alpha)/\mathrm{a}=\mathrm{b}/(\kappa_1+\beta)$, it follows that $\sin 2j$ oscillates between $\sin 2j_0 \pm \mathrm{a} \sin 2i_0/(\alpha-\beta)$, and $\sin 2i$ between $\sin 2i_0 \mp \mathrm{b} \sin 2j_0/(\alpha-\beta)$.

Let δj and δi be the oscillations of j and i on each side of the mean, then $\delta \sin 2j=\mathrm{a} \sin 2i/(\alpha-\beta)$ and $\delta \sin 2i=\mathrm{b} \sin 2j/(\alpha-\beta)$.

Hence in seconds of arc

$$\left.\begin{aligned}\delta j&=\frac{648000}{\pi}\,\frac{1}{2}\frac{\mathrm{a}}{\alpha-\beta}\,\frac{\sin 2i}{\cos 2j}\\ \delta i&=\frac{648000}{\pi}\,\frac{1}{2}\frac{\mathrm{b}}{\alpha-\beta}\,\frac{\sin 2j}{\cos 2i}\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (260)$$

Reducing these to numbers with $j=5^\circ\ 9'$, $i=23^\circ\ 28'$. we have $\delta j=13''\cdot 13$, $\delta i=11''\cdot 86$.*

Hence, if the earth were homogeneous, at the present time we should have δj as the inclination of the proper plane of the lunar orbit to the ecliptic, and δi as the amplitude of the 19-yearly nutation. These are very small angles, and therefore initially the method of Part II. was applicable.

* The formulas here used for the amplitude of the 19-yearly nutation and for the inclination of the lunar proper plane to the ecliptic differ so much from those given by other writers that it will be well to prove their identity.

LAPLACE ('Méc. Cél.,' liv. vii., chap. 2) gives as the inclination of the proper plane to the ecliptic

$$\frac{\alpha\rho-\frac{1}{2}\alpha\phi}{g-1}\frac{D^2}{a^2}\sin\lambda\cos\lambda$$

Here $\alpha\rho$ is the earth's ellipticity, and is my $\mathfrak{e}$; $\alpha\phi$ is the ratio of equatorial centrifugal force to gravity, and is my n^2a/g, it is therefore $\frac{4}{5}\mathfrak{e}$ when the earth is homogeneous.

Thus his $\alpha\rho-\frac{1}{2}\alpha\phi=$ my $\frac{3}{5}\mathfrak{e}$. His $g-1$ is the ratio of the angular velocity of the nodes to that of the moon, and is therefore my $(\alpha-\beta)/\Omega$. His D is the earth's mean radius, and is my a. His a is the moon's mean distance, and is my c. His λ is the obliquity, and is my i. Thus his formula is $\frac{3}{5}\frac{\mathfrak{e}\Omega}{\alpha-\beta}\frac{a^2}{c^2}\sin i\cos i$ in my notation.

Now my $\tau=3\mu m/2c^3$, and $\frac{2}{5}a^2=C/M$.

Therefore the formula becomes

$$\tfrac{1}{2}\frac{\tau\mathfrak{e}}{\alpha-\beta}(\Omega c)\frac{C}{\mu Mm}\sin 2i$$

But by (5) $C\Omega c/\mu Mm=k$.

Therefore it becomes

$$\tfrac{1}{2}\frac{k\tau\mathfrak{e}}{\alpha-\beta}\sin 2i$$

Now by (115) and (112), when $\xi=1$, $\mathrm{a}=k\tau\mathfrak{e}\cos j\cos 2j$.

Therefore in my notation LAPLACE'S result for the inclination of the lunar proper plane to the ecliptic is

$$\tfrac{1}{2}\frac{\mathrm{a}}{\alpha-\beta}\frac{\sin 2i}{\cos 2j}\sec j$$

This agrees with the result (260) in the text, from which the amount of oscillation of the lunar orbit was computed, save as to the sec j. Since j is small the discrepancy is slight, and I believe my form to be the more accurate.

LAPLACE states that the inclination is $20''\cdot 023$ (centesimal) if the earth be heterogeneous, and $41''\cdot 470$ (centesimal) if homogeneous. Since $41''\cdot 470$ (centes.)$=13''\cdot 44$, this result agrees very closely with mine. The difference of LAPLACE'S data explains the discrepancy.

If it be desired to apply my formula to the heterogeneous earth we must take $\frac{5}{6}$ of my k, because the $\frac{2}{5}$ of the formula (6) for s will be replaced by $\frac{1}{3}$ nearly. Also $\mathfrak{e}$, which is $\frac{1}{232}$, must be replaced by the precessional constant, which is $\cdot 003272$. Hence my previous result in the text must be multiplied by $\frac{5}{6}$ of $232\times\cdot 003272$ or $\cdot 6326$. This factor reduces the $13''\cdot 13$ of the text to $8''\cdot 31$. LAPLACE'S result ($20''\cdot 023$ centes.) is $6''\cdot 49$. Hence there is a small discrepancy in the results; but it must be remembered that LAPLACE'S value of the actual ellipticity (1/334 instead of 1/295) of the earth was considerably in error. The more correct result is I think $8''\cdot 31$. The amount of this inequality was found by BURG and

Now consider the final condition.

Since the integrations of the two periods have extended from $\xi=1$ to $\cdot 88$, and again from $\xi=1$ to $\cdot 76$,

$$\tau=\tau_0(\cdot 88\times\cdot 76)^{-6},\ \Omega=\Omega_0(\cdot 88\times\cdot 76)^{-3},\ k=k_0(\cdot 88\times\cdot 76)^{-1},$$

also the value of n which gives the day of 9 hrs. 55 m. is given by $\log n=3\cdot 74451$, and $\log \mathfrak{g}+10=1\cdot 21217$, when the year is the unit of time.

We now have $i=17°\ 4'$, $j=6°\ 21'$.

Using these values in (115) and (112), I find

$$\alpha=\cdot 10872,\ \beta=\cdot 00627,\ \mathrm{a}=\cdot 00563,\ \mathrm{b}=\cdot 00510.$$

ab is still small compared with $(\alpha-\beta)$, but not negligeable.

Then by (117)

$$\kappa_1-\kappa_2=-\sqrt{(\alpha-\beta)^2+4\mathrm{ab}}=-(\alpha-\beta)-\frac{2\mathrm{ab}}{\alpha-\beta},\ \text{also}\ \kappa_1+\kappa_2=-(\alpha+\beta)$$

Now $2\mathrm{ab}/(\alpha-\beta)=\cdot 00056$.

Hence we have

$$\left.\begin{aligned}\kappa_1+\kappa_2&=-\cdot 11499\\ \kappa_1-\kappa_2&=-\cdot 10301\end{aligned}\right\}\ \text{whence}\ \begin{aligned}\kappa_1&=-\cdot 10900\\ \kappa_2&=-\cdot 00599\end{aligned}$$

BURCKHARDT from the combined observations of BRADLEY and MASKELYNE to be 8″ (GRANT's 'Hist. Phys. Astr.,' 1852, p. 65).

For the amplitude of the 19-yearly nutation, AIRY gives ('Math. Tracts,' 1858, article "On Precession and Nutation," p. 214)

$$\frac{6\pi^2 \mathrm{B}}{\mathrm{T}'^2\omega(n+1)}\frac{\tau}{4\pi}\cos \mathrm{I}\sin 2i$$

B is the precess. const. = my $\mathfrak{e}$; his $\mathrm{T}'=$ my $2\pi/\Omega$; his $n=$ my ν; his $\omega=$ my n; his $\mathrm{I}=$ my i; his $i=$ my j; and his τ is the period of revolution of the nodes, and therefore = my $2\pi/(\alpha-\beta)$.

Then since my $\tau=3\Omega^2/2(1+\nu)$, the above in my notation is

$$\frac{1}{2}\frac{\tau\mathfrak{e}}{n}\frac{1}{\alpha-\beta}\cos i\sin 2j$$

Now by (115) and (112) $\mathrm{b}=\frac{\tau\mathfrak{e}}{n}\cos i\cos 2i$, when $\xi=1$.

Therefore his result in my notation is

$$\frac{1}{2}\frac{\mathrm{b}}{\alpha-\beta}\frac{\sin 2j}{\cos 2i}$$

This is the result used above (in 260) for computing the nutations of the earth.

If my formula is to be used for the heterogeneous earth, $\mathfrak{e}$ must be replaced by the precessional constant, and therefore the result in the text must be multiplied by $232\times\cdot 003272$ or $\cdot 759$. Hence for the heterogeneous earth the $11''\cdot 86$ must be reduced to $9''\cdot 01$. AIRY computes it as $10''\cdot 33$, but says the observed amount is $9''\cdot 6$, but he takes the precessional constant as $\cdot 00317$, and the moon's mass as 1–70th of that of the earth. I believe that $\cdot 00327$ and 1–82nd are more in accordance with the now accepted views of astronomers.

κ_1 and κ_2 have now come to differ a little from $-\alpha$ and $-\beta$, but still not much. With these values I find

$$\log \frac{\mathrm{a}}{\kappa_2+\alpha}+10=8\cdot76472, \quad \log \frac{-\mathrm{b}}{\kappa_1+\beta}+10=8\cdot69606$$

Substituting in the formulas

$$\delta j=\tfrac{1}{2}\frac{\mathrm{a}}{\kappa_2+\alpha}\frac{\sin 2i}{\cos 2j}, \quad \delta i=\tfrac{1}{2}\frac{\mathrm{b}}{\kappa_1+\beta}\frac{\sin 2j}{\cos 2i}$$

I find

$$\delta j=57'\ 31'', \quad \delta i=22'\ 42''$$

Thus the oscillation of the lunar orbit has increased from 13″ to nearly a degree, and that of the equator from 12″ to 23′.

It is clear therefore that we have carried out the integration by the method of Part II., as far back in retrospect as is proper, even for a speculative investigation like the present one.

We shall here then make the transition to the method of Part III.

Henceforth the formulas used regard the inclination and obliquity as small angles; the obliquity is still however so large that this is not very satisfactory.

§ 19. *Secular changes in the proper planes of the earth and moon where the viscosity is small.*

We now take up the integration, at the point where it stops in the last section, by the method of Part III. The viscosity is still supposed to be small, so that γ, δ, g, d (as defined in (251)) must be taken in place of Γ, Δ, G, D, which refer to any viscosity. The equations are ready for the application of the method of quadratures in (250), and the symbols are defined in (251–4).

The method pursued is to assume a series of equidistant values of ξ, and then to compute all the functions (251–4), substitute them in (250), and combine the equidistant values of the functions to be integrated by the rules of the calculus of finite differences.

The preceding integration terminates where the day is 9 hrs. 55 m., and the moon's sidereal period is 8·17 m.s. days. If the present tropical year be the unit of time, we have, at the beginning of the present integration $\log n_0=3\cdot74451$, $\log \Omega_0=2\cdot44836$, and $\log k+10=6\cdot20990$, k being $s\Omega_0^{\frac{1}{3}}$ of (7).

The first step is to compute a series of values of n/n_0, by means of (254). As a fact, I had already computed n/n_0 corresponding to $\xi=1$, ·92, ·84, ·76 for the paper on "Precession," by means of a formula, which took account of the obliquity of the ecliptic; and accordingly I computed n/n_0, by the same formula, for the values of $\xi=$·96, ·88, ·80, instead of doing the whole operation by means of (254). The difference between my results here used and those from (254) would be very small.

The following table exhibits some of the stages of the computation. The results are given just as they were found, but it is probable that the last place of decimals, and perhaps the last but one, are of no value. As however we really only require a solution in round numbers, this is of no importance.

TABLE II.

ξ =	1·	·96	·92	·88	·84	·80	·76
n/n_0=	1·00000	1·04467	1·08931	1·13392	1·17852	1·22308	1·26763
$\log \mathfrak{e}+10$=	8·40016	8·43812	8·47446	8·50932	8·54284	8·57507	8·60614
$\log \tau'/\tau+10$=	8·61867	8·51230	8·40140	8·28557	8·16435	8·03721	7·90356
$\log \lambda+10$=	8·70384	8·73805	8·77533	8·81581	8·85966	8·90712	8·95841
$\tau'/2\lambda\mathfrak{e}\tau$=	16·3546	10·8418	7·0889	4·5647	2·8895	1·7947	1·0914
$\mathfrak{m}=\mathrm{a}$=	·90035	·97976	1·06603	1·16014	1·26320	1·37648	1·50172
$\log \gamma+10$=	9·67591	9·71452	9·75343	9·79287	9·83307	9·87430	9·91693
$\log \delta+10$=	9·65551	9·65745	9·65824	9·65788	9·65631	9·65341	9·64900
$\log(\mathrm{gb}-\mathrm{ad})+10$=	8·83030	8·86665	8·91307	8·96946	9·03549	9·11080	9·19510
α'=	36·696	23·186	12·583	4·144	− 2·747	− 8·605	−13·873
a'=	− 7·4782	− 8·6811	−10·0883	−11·7426	−13·6966	−16·0163	−18·7899
β'=	− 6·4455	− 6·9122	− 7·4220	− 7·9805	− 8·5940	− 9·2699	−10·0184
b'=	− 6·4038	− 6·8796	− 7·3968	− 7·9612	− 8·5794	− 9·2590	−10·0104
$\log -(\kappa_1+\alpha)+10$=	8·74306	8·95453	9·16587	9·37077	9·55751	9·71146	9·82404
$\log(\kappa_2+\alpha)$	1·21135	1·03659	·86190	·69374	·54396	·42731	·35255
$\log(\kappa_2-\kappa_1)$	1·21283	1·04017	·87056	·71393	·58660	·50372	·46520

The further stage in the computation, when these values are used to compute the several terms of the expressions to be integrated, are given in the following table.

TABLE III.

ξ =	1·	·96	·92	·88	·84	·80	·76
$-(\alpha'-\beta')(\kappa_1+\alpha)/kn(\kappa_2-\kappa_1)^2$=	·00995	·02395	·05424	·10413	·13350	·03053	− ·26438
$\mathrm{a}'\mathrm{b}(\kappa_1+\alpha)/kn(\kappa_2+\alpha)(\kappa_2-\kappa_1)^2$=	·00011	·00064	·00376	·02041	·08937	·27505	·57228
$\mathrm{a}'\mathrm{b}/kn(\kappa_2-\kappa_1)^2$=	−·03117	−·07671	−·18670	−·42945	−·86628	−1·42975	−1·93250
$\mathrm{b}'\mathrm{a}(\kappa_1+\alpha)/kn(\kappa_2+\alpha)(\kappa_2-\kappa_1)^2$=	·00008	·00049	·00294	·01606	·07072	·21887	·45786
$\mathrm{b}'\mathrm{a}/kn(\kappa_2-\kappa_1)^2$=	−·02403	−·05970	−·14593	−·33778	−·68546	−1·13770	−1·54610
$\gamma(\kappa_1+\alpha)/kn(\kappa_2-\kappa_1)$=	−·00179	−·00452	−·01141	−·02759	−·06001	− ·10969	− ·16534
$\gamma(\kappa_2+\alpha)/kn(\kappa_2-\kappa_1)$=	·52483	·54645	·56651	·58035	·58167	·57020	·55832
$\delta(\kappa_1+\alpha)/kn(\kappa_2-\kappa_1)$=	−·00170	−·00397	−·00916	−·02022	−·03995	− ·06596	− ·08922
$\delta(\kappa_2+\alpha)/kn(\kappa_2-\kappa_1)$=	·50075	·47916	·45501	·42530	·38719	·34288	·30127
$(\mathrm{bg}-\mathrm{ad})/kn(\kappa_2-\kappa_1)$=	·00460	·00713	·01124	·01764	·02649	·03675	·04704

The method pursued in the integration of the preceding section proceeds virtually on the assumption that the term $\gamma(\kappa_2+\alpha)/kn(\kappa_1-\kappa_2)$ is the only important one in the expression for $d \log \tan \frac{1}{2}\mathrm{J}/d\xi$, and that the term $\delta(\kappa_2+\alpha)/kn(\kappa_2-\kappa_1)$ is the only important one in the expression for $d \log \tan \frac{1}{2}\mathrm{I}/d\xi$.

Now when $\xi=1$, at the beginning of the present integration, we see from Table III. that the said term in γ is about 22 times as large as any other occurring in $d \log \tan \frac{1}{2}\mathrm{J}$, and that the said term in δ is about 16 times as large as any other which occurs in $d \log \tan \frac{1}{2}\mathrm{I}$. Hence the preceding integration must have given fairly satisfactory results. But after the first column these terms in γ and δ fail to maintain their relative importance, so that when $\xi=\cdot 76$, they have both become considerably less important than other terms—notably $\mathrm{b}'\mathrm{a}/kn(\kappa_2-\kappa_1)^2$ and $\mathrm{a}'\mathrm{b}/kn(\kappa_2-\kappa_1)^2$. This is exactly what is to be expected, because the equations are tending towards the form which they would take if the solar influence were nil, and an inspection of (225) shows that these terms would then be prominent.

Now if we combine these values of the several terms together according to (250), we obtain the seven equidistant values of $d \log \tan \frac{1}{2}\mathrm{J}/d\xi$ and $d \log \tan \frac{1}{2}\mathrm{I}/d\xi$ exhibited in the following table:—

TABLE IV.

ξ =	1·	·96	·92	·88	·84	·80	·76
$d \log \tan \frac{1}{2}\mathrm{J}/d\xi=$	$-\cdot 49386$	$-\cdot 46660$	$-\cdot 37218$	$-\cdot 15627$	$+\ \cdot 16138$	$+\ \cdot 35219$	$+\ \cdot 19330$
$d \log \tan \frac{1}{2}\mathrm{I}/d\xi=$	$+\cdot 54460$	$+\cdot 58194$	$+\cdot 69284$	$+\cdot 93287$	$+1\cdot 28273$	$+1\cdot 51135$	$+1\cdot 39323$

By interpolation it appears that $d\mathrm{J}/d\xi$ vanishes when $\xi=\cdot 8603$. This value of ξ corresponds with 8 hrs. 36 m. for the period of the earth's rotation, and 5·20 m. s. days for the period of the moon's revolution.

Since $d\xi$ is negative in our integration, we see from these values that I, the inclination of the earth's proper plane to the ecliptic, will continue diminishing, and with increasing rapidity. On the other hand, the inclination J of the lunar orbit to its proper plane will increase at first, but at a diminishing rate, and will finally diminish. This is a point of the greatest importance in explaining the present inclination of the lunar orbit to the ecliptic, and we shall recur to it later on.

Now combine the first four values by the rule of finite differences, viz.:

$$[u_0+u_3+3(u_1+u_2)]\tfrac{3}{8}h$$

and all seven by WEDDLE's rule, viz.:

$$[u_0+u_2+u_3+u_4+u_6+5(u_1+u_3+u_5)]\tfrac{3}{10}h$$

where h is our $d\xi$, and the u's are the several numbers given in the above Table IV.; then we have, on integration from 1 to ·88,

$$\log_e \tan \tfrac{1}{2}\mathrm{J} = \log_e \tan \tfrac{1}{2}\mathrm{J}_0 + {\cdot}04750$$
$$\log_e \tan \tfrac{1}{2}\mathrm{I} = \log_e \tan \tfrac{1}{2}\mathrm{I}_0 - {\cdot}07953$$

and on integration from 1 to ·76

$$\log_e \tan \tfrac{1}{2}\mathrm{J} = \log_e \tan \tfrac{1}{2}\mathrm{J}_0 + {\cdot}02425$$
$$\log_e \tan \tfrac{1}{2}\mathrm{I} = \log_e \tan \tfrac{1}{2}\mathrm{I}_0 - {\cdot}23972$$

Then if we take $\mathrm{J}_0=6°$, $\mathrm{I}_0=17°$, which are in round numbers the final values of J and I derived from the first method of integration, we easily find,

$$\text{when } \xi={\cdot}88,\ \mathrm{J}=6°\ 17',\ \mathrm{I}=15°\ 43'$$
$$\text{and when } \xi={\cdot}76,\ \mathrm{J}=6°\ 9',\ \mathrm{I}=13°\ 25'$$

Then we have by (249)

$$\sin \mathrm{I}_{,} = -\frac{(\kappa_1+\alpha)}{\mathrm{a}} \sin \mathrm{J} = \frac{\mathrm{b}}{\kappa_2+\alpha} \sin \mathrm{J}$$
$$\sin \mathrm{J}_{,} = \frac{\mathrm{a}}{\kappa_2+\alpha} \sin \mathrm{I} = -\frac{\kappa_1+\alpha}{\mathrm{b}} \sin \mathrm{I}$$

Now b is always unity, and the logarithms of $(\kappa_2+\alpha)$ and $-(\kappa_1+\alpha)$ are given in Table II.; from this we find

$$\text{when } \xi={\cdot}88,\ \mathrm{I}_{,}=1°\ 16',\ \mathrm{J}_{,}=3°\ 39'$$
$$\text{when } \xi={\cdot}76,\ \mathrm{I}_{,}=2°\ 43',\ \mathrm{J}_{,}=8°\ 54'$$

By the same formula, when $\xi=1$ initially, we have $\mathrm{I}_{,}=22'$, $\mathrm{J}_{,}=56'$. These two results ought to be identical with the results from (260) of the last section; and they are so very nearly, for at the end of the integration we had $\delta i=22'\ 43''$, $\delta j=57'\ 31''$. The small discrepancy which exists is partly due to the assumed smallness of i and j in the present investigation, and also to our having taken the values 6° and 17° for J_0 and I_0 instead of 6° 21′, 17° 4′.

The value $\xi={\cdot}88$ gives the length of day as 8 hrs. 45 m., and the moon's sidereal period as 5·57 m. s. days.

The value $\xi={\cdot}76$ gives the day as 7 hrs. 49 m., and the moon's sidereal period as 3·59 m. s. days. This value of ξ brings us to the point specified as the end of the third period of integration in § 17 of the paper on "Precession."

There is one other point which it will be interesting to determine,—it is to find the rate of the precessional motion of the node of the two proper planes on the ecliptic, and the rate of the motion of the nodes of the equator and orbit upon their respective proper planes. By means of the preceding numerical values, it will be easy to find these quantities at the epochs specified by $\xi=1$, ·88, ·76.

The period of the precession of the two proper planes is $-2\pi/\kappa_2$, and that of the precession of the two nodes on their proper planes is $2\pi/(\kappa_2-\kappa_1)$.

In the preceding computations we omitted a common factor $\tau\mathfrak{k}/n$ from α, β, a, b, κ_1, κ_2; this factor must now be reintroduced. τ' is a constant and $\log\tau'=1{\cdot}77242$, then by means of the numerical values given in the first table I find

	$\xi=1{\cdot}$	$\cdot 88$	$\cdot 76$
	$\log\tau\mathfrak{k}/n+10=7{\cdot}80940$	$8{\cdot}19708$	$8{\cdot}62750$
Also			
	$\log-\kappa_2+10=9{\cdot}99401$	$9{\cdot}89462$	$9{\cdot}53295$

$\log(\kappa_2-\kappa_1)$ is given before in Table II. Then introducing the omitted factor $\tau\mathfrak{k}/n$, I find

$\xi=1{\cdot}$	$\cdot 88$	$\cdot 76$
$-2\pi/\kappa_2=988$ yrs.	509 yrs.	434 yrs.
$2\pi/(\kappa_2-\kappa_1)=$ 60 yrs.	77 yrs.	51 yrs.

Thus both precessional movements on the whole increase in rapidity (because of the increasing value of $\tau\mathfrak{k}/n$), but the rate of the precession of the pair of proper planes increases all through, whilst that of the precession on the proper planes diminishes and then increases. It was pointed out towards the end of § 13 that κ_2 is, so to speak, the ancestor of the luni-solar precession, and $\kappa_2-\kappa_1$ the ancestor of the revolution of the moon's nodes. Hence the 988 years has bred (to continue the metaphor) the present 26,000 years of the precessional period, and the 60 years has bred the present $18\frac{1}{2}$ years of the revolution of the moon's nodes.

We see that the $\kappa_2-\kappa_1$ precession attains a minimum at a certain period being more rapid, both earlier and later.

All the above results will be collected and arranged in a tabular form, after further results have been obtained by means of an integration, carrying out the investigation into the more remote past.

The tidal and precessional effects of the sun's influence have now become exceedingly small, and the only way in which the sun continues to exert a sensible effect is in its tendency to make the nodes of the lunar orbit revolve on the ecliptic. In the analysis therefore we may now treat τ' as zero everywhere, except where it occurs in the form $\tau'/\lambda\mathfrak{k}\tau$. Since λ and $\mathfrak{k}$ are both pretty small, these terms in τ'/τ rise in importance.

The equation of conservation of moment of momentum now becomes

$$\frac{n}{n_0}=1+\frac{1}{kn_0}(1-\xi)$$

Here kn_0 is equal to the value of $\mathfrak{m}$ in the preceding integration when $\xi=\cdot 76$; and hence $1/kn_0=\cdot 665903$.

Then we now have $\beta=$b, $\gamma=$g, $\delta=$d, $\beta'=$b$'$, $\gamma'=$g$'$, $\delta'=$d$'$, but α and α' are not equal to a and a$'$.

It is proposed to carry the new integration over the field defined by $\xi=1$ to ·88, and to compute four equidistant values.

The following tables give the results of the computation, as in the previous case.

TABLE V.

$\xi =$	1·	·96	·92	·88
$n/n_0=$	1·00000	1·02664	1·05327	1·07991
$\log \mathfrak{t}+10=$	8·60614	8·62898	8·65122	8·67292
$\log \tau'/\tau+10=$	7·90356	7·79718	7·68628	7·57045
$\log \lambda+10=$	8·95841	9·00018	9·04451	9·09157
$\log \tau'/2\lambda\mathfrak{t}\tau+10=$	10·03798	9·86699	9·68952	9·50493
$\mathfrak{m}=\mathrm{a}=$	1·5017	1·6060	1·7193	1·8429
$\log \mathrm{g}+10=$	9·91693	9·95049	9·98531	10·02170
$\log \mathrm{d}+10=$	9·65322	9·64780	9·64118	9·63303
$\alpha'=$	−13·873	−17·719	−21·607	−25·692
$\mathrm{a}'=$	−18·790	−21·266	−24·130	−27·460
$\beta'=\mathrm{b}'=$	−10·010	−10·636	−11·316	−12·057
$\log-(\kappa_1+\alpha)+10=$	9·82285	9·88247	9·92401	9·95203
$\log (\kappa_2+\alpha)+10=$	·35374	·32327	·31133	·31347
$\log (\kappa_2-\kappa_1)+10=$	·46586	·45758	·46052	·47035

TABLE VI.

$\xi =$	1·	·96	·92	·88
$-(\alpha'-\beta')(\kappa_1+\alpha)/kn(\kappa_2-\kappa_1)^2=$	− ·1998	− ·4261	− ·6551	− ·8630
$\mathrm{a}'\mathrm{b}(\kappa_1+\alpha)/kn(\kappa_2+\alpha)(\kappa_2-\kappa_1)^2=$	·4312	·6077	·7500	·8445
$\mathrm{a}'\mathrm{b}/kn(\kappa_2-\kappa_1)^2=$	−1·4643	−1·6770	−1·8297	−1·9410
$\mathrm{b}'\mathrm{a}(\kappa_1+\alpha)/kn(\kappa_2+\alpha)(\kappa_2-\kappa_1)^2=$	·3450	·4882	·6047	·6833
$\mathrm{b}'\mathrm{a}/kn(\kappa_2-\kappa_1)^2=$	−1·1714	−1·3469	−1·4752	−1·5706
$\mathrm{g}(\kappa_1+\alpha)/kn(\kappa_2-\kappa_1)=$	− ·1251	− ·1540	− ·1777	− ·1965
$\mathrm{g}(\kappa_2+\alpha)/kn(\kappa_2-\kappa_1)=$	·4248	·4248	·4335	·4517
$\mathrm{d}(\kappa_1+\alpha)/kn(\kappa_2-\kappa_1)=$	− ·0682	− ·0767	− ·0805	− ·0803
$\mathrm{d}(\kappa_2+\alpha)/kn(\kappa_2-\kappa_1)=$	·2315	·2116	·1963	·1846
$(\mathrm{bg}-\mathrm{ad})/kn(\kappa_2-\kappa_1)=$	·0342	·0404	·0469	·0542

Combining these terms according to the formulas (250), we have

TABLE VII.

$\xi =$	$1\cdot$	$\cdot 96$	$\cdot 92$	$\cdot 88$
$d \log \tan \frac{1}{2}\mathrm{J}/d\xi=$	$+\cdot 1496$	$-\cdot 0754$	$-\cdot 3298$	$-\cdot 5625$
$d \log \tan \frac{1}{2}\mathrm{I}/d\xi=$	$+1\cdot 0601$	$+\cdot 8607$	$+\cdot 6354$	$+\cdot 4370$

By interpolation it appears that $d\mathrm{J}/d\xi$ vanishes when $\xi=\cdot 9679$. This value of ξ corresponds with 7 hrs. 47 m. for the period of the earth's rotation, and 3·25 m. s. days for the period of the moon's revolution.

By the rules of the calculus of finite differences, integrating from $\xi=1$ to ·88,

$$\log_e \tan \tfrac{1}{2}\mathrm{J} = \log_e \tan \tfrac{1}{2}\mathrm{J}_0 + \cdot 0244$$
$$\log_e \tan \tfrac{1}{2}\mathrm{I} = \log_e \tan \tfrac{1}{2}\mathrm{I}_0 - \cdot 0898$$

Then with $\mathrm{J}_0=6^\circ\ 9'$, $\mathrm{I}_0=13^\circ\ 25'$ from the previous integration, we have $\mathrm{J}=6^\circ\ 18'$, $\mathrm{I}=12^\circ\ 16'$.

When $\xi=\cdot 88$, the length of the day is 7 hrs. 15 m., and the moon's sidereal period is 2·45 m. s. days. Also $\mathrm{I}_{,}=3^\circ\ 3'$, $\mathrm{J}_{,}=10^\circ\ 58'$.

Thus we have traced the changes back until the inclination of the proper planes to one another is only $12^\circ\ 16' - 10^\circ\ 58'$ or $1^\circ\ 18'$.

In the same way as before it may be shown that, when $\xi=\cdot 88$, the period of the precession of the proper planes is 609 years, and the period of the revolution of the two nodes on their moving proper planes is 22 years. The former of the two precessions is therefore at this stage getting slower, whilst the latter goes on increasing in speed.

The physical results of the whole integration of the present section is embodied in the following table.

TABLE VIII.—Results of integration in the case of small viscosity.

Day in m. s. hours and min.	Moon's sidereal period in m. s. days.	Inclination of earth's proper plane to ecliptic.	Inclination of equator to earth's proper plane.	Inclination of moon's proper plane to ecliptic.	Inclination of lunar orbit to moon's proper plane.	Precessional period of the proper planes.	Period of revolution of the two nodes on their moving proper planes.
h. m.	Days.	° ′	° ′	° ′	° ′	Years.	Years.
9 55	8·17	17 0	0 22	0 57	6 0	988	60
8 45	5·57	15 43	1 16	3 39	6 17	509	77
7 49	3·59	13 25	2 43	8 54	6 9	434	51
7 15	2·45	12 16	3 3	10 58	6 18	609	22

If the integration is to be carried still further back, the solar action may henceforth be neglected, and the motion may be referred to the invariable plane of the system. This plane undergoes a precessional motion due to the sun, which will not interfere with the treatment of it as though fixed. It is inclined to the ecliptic at about 11° 45′, because, at the time when we suppose the solar action to cease, the moment of momentum of the earth's rotation is larger than that of orbital motion, and therefore the earth's proper plane represents the invariable plane of the system more nearly than does the moon's proper plane.

The inclination i of the equator to the invariable plane must be taken as about 3°, and j that of the lunar orbit as something like 5° 30′. The ratio of the two angles 5° 30′ and 3° must be equal to 1·84, which is $\mathfrak{m}$, the ratio of the moment of momentum of the earth's rotation to that of orbital motion, at the point where the preceding integration ceases.

Then in the more remote past the angle i will continue to diminish, until the point is reached where the moon's period is about 12 hours and that of the earth's rotation about 6 hours. The angle j will continue increasing at an accelerating rate.

This may be shown as follows :—

The equations of motion are now those of Part II., which may be written

$$kn\frac{dj}{d\xi}=-\mathrm{g}(i+j)$$

$$kn\frac{di}{d\xi}=\quad\mathrm{d}(i+j)$$

But since $i/j=\xi/kn=1/\mathfrak{m}$, they become

$$kn\frac{d}{d\xi}\log\tan\tfrac{1}{2}j=-\frac{1+\mathfrak{m}}{\mathfrak{m}}\mathrm{g}$$

$$kn\frac{d}{d\xi}\log\tan\tfrac{1}{2}i=\quad(1+\mathfrak{m})\mathrm{d}$$

(Compare with the first of equations (255) given in Part III., when $\tau'=0$.)

These equations are not independent, because of the relationship which must always subsist between i and j.

Then substituting from (251) we have for the case of small viscosity

$$kn\frac{d}{d\xi}\log\tan\tfrac{1}{2}j=-\frac{1+\mathfrak{m}}{2(1-\lambda)}$$

$$kn\frac{d}{d\xi}\log\tan\tfrac{1}{2}i=\quad\frac{(1+\mathfrak{m})(1-2\lambda)}{2(1-\lambda)}$$

From this we see that j will always decrease as ξ increases at a rate which tends to become infinite when $\lambda=1$; and i increases as ξ increases so long as λ is less than ·5, but decreases for values of λ between ·5 and unity at a rate which tends to become infinite when $\lambda=1$. If we consider the subject retrospectively, ξ decreases, j increases, and i decreases, except for values of λ between ·5 and unity.

This continued increase (in retrospect) of the inclination of the lunar orbit to the invariable plane is certainly not in accordance with what was to be expected, if the moon once formed a part of the earth. For if we continued to trace the changes backwards to the initial condition in which (as shown in "Precession") the two bodies move round one another as parts of a rigid body, we should find the lunar orbit inclined at a considerable angle to the equator; and it is hard to see how a portion detached from the primeval planet could ever have revolved in such an orbit.

These considerations led me to consider whether some other hypothesis than that of infinitely small viscosity of the earth might not modify the above results. I therefore determined to go over the same solution again, but with the hypothesis of very large instead of very small viscosity of the planet.

This investigation is given in the next section, but I shall not retraverse the ground covered by the integration of the first method, but shall merely take up the problem at the point where it was commenced in the present section.

§ 20. *Secular changes in the proper planes of the earth and moon when the viscosity is large.*

Let $\mathfrak{p}=2\,gaw/19v$, where v is the coefficient of viscosity of the earth.

Then by the theory of viscous tides

$$\tan 2\mathrm{f}_1=\frac{2(n-\Omega)}{\mathfrak{p}},\ \tan 2\mathrm{f}=\frac{2n}{\mathfrak{p}},\ \tan \mathrm{g}_1=\frac{n-2\Omega}{\mathfrak{p}},\ \tan \mathrm{g}=\frac{n}{\mathfrak{p}} \quad . \quad . \quad . \quad (261)$$

If the viscosity be very large $\mathfrak{p}$ is very small, and the angles $\frac{1}{2}\pi-2\mathrm{f}_1$, $\frac{1}{2}\pi-2\mathrm{f}$, $\frac{1}{2}\pi-\mathrm{g}_1$, $\frac{1}{2}\pi-\mathrm{g}$ are small, so that their cosines are approximately unity and their sines approximately equal to their tangents. Hence

$$\sin 4\mathrm{f}_1=\frac{\mathfrak{p}}{n-\Omega},\ \sin 4\mathrm{f}=\frac{\mathfrak{p}}{n},\ \sin 2\mathrm{g}_1=\frac{2\mathfrak{p}}{n-2\Omega},\ \sin 2\mathrm{g}=\frac{2\mathfrak{p}}{n}$$

Then introducing $\lambda=\Omega/n$, we have

$$\frac{\sin 4\mathrm{f}}{\sin 4\mathrm{f}_1}=1-\lambda,\ \frac{\sin 2\mathrm{g}_1}{\sin 4\mathrm{f}_1}=\frac{2(1-\lambda)}{1-2\lambda},\ \frac{\sin 2\mathrm{g}}{\sin 4\mathrm{f}_1}=2(1-\lambda) \quad . \quad . \quad . \quad . \quad (262)$$

Introducing the transformations (262) into (251), we have

$$\left.\begin{aligned}\Gamma=\tfrac{1}{2}\mathfrak{m}\left[1-\frac{4\lambda(1-\lambda)}{1-2\lambda}\right],\quad \Delta=\tfrac{1}{2}\left[1+\frac{4\lambda(1-\lambda)}{1-2\lambda}-4(1-\lambda)\frac{\tau'}{\tau}+(1-\lambda)\left(\frac{\tau'}{\tau}\right)^2\right]\\ \mathrm{bG}-\mathrm{aD}=-\mathfrak{m}\left[\frac{4\lambda(1-\lambda)}{1-2\lambda}-2(1-\lambda)\frac{\tau'}{\tau}\right]\end{aligned}\right\} \quad . \quad (263)$$

All the other expressions in (251) remain as they were.

Then the terms in Γ, Δ, G, D in (250) are the only ones which have to be recomputed. And all the other arithmetical work of the last section will be applicable here. Also all the materials for calculating these new terms are ready to hand.

The results of the computation are embodied in the following tables.

TABLE IX.

ξ =	1·	·96	·92	·88	·84	·80	·76
$\log\Gamma+10=$	9·54901	9·57529	9·59914	9·61994	9·63663	9·64791	9·65092
$\log\Delta=$	·52876	·55517	·58023	·60484	·63005	·65708	·68739
$\log(\mathrm{aD}-\mathrm{bG})+10=$	9·08381	9·22356	9·34416	9·45433	9·55931	9·66259	9·76574

TABLE X.

ξ =	1·	·96	·92	·88	·84	·80	·76
$\Gamma(\kappa_1+\alpha)/kn(\kappa_2-\kappa_1)=$	−·00133	−·00328	−·00800	−·01853	−·03818	−·06513	−·08961
$\Gamma(\kappa_2+\alpha)/kn(\kappa_2-\kappa_1)=$	·39185	·39657	·39712	·38973	·37003	·33856	30260
$\Delta(\kappa_1+\alpha)/kn(\kappa_2-\kappa_1)=$	−·00199	−·00485	−·01168	−·02688	−·05553	−·09627	−·13761
$\Delta(\kappa_2+\alpha)/kn(\kappa_2-\kappa_1)=$	·58529	·58541	·57994	·56554	·53826	·50044	·46468
$(\mathrm{bG}-\mathrm{aD})/kn(\kappa_2-\kappa_1)=$	−·00825	−·01622	−·03034	−·05388	−·08850	−·13092	−·17504

Then combining these terms with those given in Table III., according to the formulas (250), (with Γ, &c., in place of γ, &c.), we have the following equidistant values.

TABLE XI.

ξ =	1·	·96	·92	·88	·84	·80	·76
$\log\tan\frac{1}{2}\mathrm{J}/d\xi=$	−·3477	−·2925	−·1587	+ ·1125	+ ·5036	+ ·7818	+ ·7195
$\log\tan\frac{1}{2}\mathrm{I}/d\xi=$	+·6168	+·6661	+·7796	+1·0107	+1·3406	+1·5458	+1·4103

By interpolation it appears that $d\mathrm{J}/d\xi$ vanishes when $\xi=·8966$. This value of ξ corresponds with a period of 8 hrs. 54 m. for the earth's rotation, and 5·89 m. s. days for the moon's revolution.

Integrating as in the last section, from $\xi=1$ to $\cdot 88$, we have

$$\log_e \tan \tfrac{1}{2}J = \log_e \tan \tfrac{1}{2}J_0 + \cdot 0238$$
$$\log_e \tan \tfrac{1}{2}I = \log_e \tan \tfrac{1}{2}I_0 - \cdot 0895$$

Taking $I_0=6°$, $J_0=17°$, we have $I=15° \; 34'$, $J=6° \; 9'$.
These values correspond to $I_{,}=1° \; 15'$, $J_{,}=3° \; 37'$.
Again integrating from $\xi=1$ to $\cdot 76$, we have

$$\log_e \tan \tfrac{1}{2}J = \log_e \tan \tfrac{1}{2}J_0 - \cdot 0461$$
$$\log_e \tan \tfrac{1}{2}I = \log_e \tan \tfrac{1}{2}I_0 - \cdot 2552$$

These give $J=5° \; 44'$, $I=13° \; 13'$, which correspond to $I_{,}=2° \; 33'$, $J_{,}=8° \; 46'$.
The integration will now be continued over another period, as in the last section. The following are the results of the computations.

TABLE XII.

$\xi =$	$1\cdot$	$\cdot 96$	$\cdot 92$	$\cdot 88$
$\log(\Gamma=G)+10=$	9·65092	9·64491	9·62783	9·59299
$\log(\Delta=D)+10=$	9·84629	9·86040	9·87686	9·89622

TABLE XIII.

$\xi =$	$1\cdot$	$\cdot 96$	$\cdot 92$	$\cdot 88$
$G(\kappa_1+\alpha)/kn(\kappa_2-\kappa_1)=$	−·06781	−·07617	−·07802	−·07323
$G(\kappa_2+\alpha)/kn(\kappa_2-\kappa_1)=$	·23026	·21018	·19033	·16832
$D(\kappa_1+\alpha)/kn(\kappa_2-\kappa_1)=$	−·10634	−·12511	−·13843	−·14720
$D(\kappa_2+\alpha)/kn(\kappa_2-\kappa_1)=$	·36106	·34521	·33771	·33835
$(bG-aD)/kn(\kappa_2-\kappa_1)=$	−·13815	−·16352	−·19057	−·35054

Substituting these values in the differential equations (250), we have the following equidistant values :—

TABLE XIV.

ξ =	1·	·96	·92	·88
$d \log \tan \frac{1}{2}J/d\xi=$	$+$ ·5547	$+$·3915	$+$·2088	$+$·1925
$d \log \tan \frac{1}{2}I/d\xi=$	$+$1·0746	$+$·8682	$+$·6391	$+$·3093

Then integrating from $\xi=1$ to ·88 we have

$$\log_e \tan \tfrac{1}{2}J = \log_e \tan \tfrac{1}{2}J_0 - \cdot 0382$$
$$\log_e \tan \tfrac{1}{2}I = \log_e \tan \tfrac{1}{2}I_0 - \cdot 0886$$

Then putting $I_0=13^\circ\ 13'$ and $J_0=5^\circ\ 44'$, from the previous integration, we have $J=5^\circ\ 30'$, $I=12^\circ\ 6'$.

These values of J and I give $J_{,}=10^\circ\ 49'$, $I_{,}=2^\circ\ 40'$.

The physical meaning of the results of the whole integration is embodied in the following table.

TABLE XV.—Results of integration in the case of large viscosity.

Day in m. s. hours and minutes.		Moon's sidereal period in m. s. days.	Inclination of earth's proper plane to ecliptic.		Inclination of equator to earth's proper plane.		Inclination of moon's proper plane to ecliptic.		Inclination of lunar orbit to moon's proper plane.	
h.	m.	Days.	°	′	°	′	°	′	°	′
9	55	8·17	17	0	0	22	0	57	6	0
8	45	5·57	15	34	1	15	3	37	6	9
7	49	3·59	13	13	2	33	8	46	5	44
7	15	2·45	12	6	2	40	10	49	5	30

If we compare these results with those in Table VIII. for the case of small viscosity, we see that the inclinations of the two proper planes to one another and to the ecliptic are almost the same as before, but there is here this important distinction, viz.: that the inclinations of the two moving systems to their respective proper planes is less (compare 5° 30′ with 6° 18′, and 2° 40′ with 3° 3′).

And besides, if we had carried the integration, in the case of small viscosity, further back we should have found the inclination of the lunar orbit increasing.

It will now be shown that, in the present case of large viscosity, the inclinations of

the equator and the orbit to their proper planes will continue to diminish, as the square root of the moon's distance diminishes, and at an increasing rate.

Suppose that, in continuing the integration, the solar influence be entirely neglected, and the motion referred to the invariable plane of the system. This plane will be in some position intermediate between the two proper planes, but a little nearer to the earth's plane, and will therefore be inclined to the ecliptic at about 11° 45′.

The equations of motion are now those of § 10, Part II., which may be written

$$kn\frac{dj}{d\xi}=-\mathrm{G}(i+j)$$

$$kn\frac{di}{d\xi}=\quad \mathrm{D}(i+j)$$

But since $i/j=\xi/kn=1/\mathfrak{m}$, they become

$$kn\frac{d}{d\xi}\log\tan\tfrac{1}{2}j=-\frac{1+\mathfrak{m}}{\mathfrak{m}}\mathrm{G}$$

$$kn\frac{d}{d\xi}\log\tan\tfrac{1}{2}i=\quad (1+\mathfrak{m})\mathrm{D}$$

(compare with the first of equations (255) given in Part III., when $\tau'=0$).

These equations are not independent of one another, because of the relationship which must always subsist between i and j.

Then substituting from (263) (in which τ' is put zero, and G, D written for Γ, Δ) we have for the case of large viscosity

$$kn\frac{d}{d\xi}\log\tan\tfrac{1}{2}j=-\tfrac{1}{2}(1+\mathfrak{m})\left[1-\frac{4\lambda(1-\lambda)}{1-2\lambda}\right]$$

$$kn\frac{d}{d\xi}\log\tan\tfrac{1}{2}i=\quad \tfrac{1}{2}(1+\mathfrak{m})\left[1+\frac{4\lambda(1-\lambda)}{1-2\lambda}\right]$$

When $\lambda=\frac{1}{2}$, $4\lambda(1-\lambda)/(1-2\lambda)$ is infinite, and therefore both $dj/d\xi$ and $di/d\xi$ are infinite. This result is physically absurd.

The absurdity enters by supposing that an infinitely slow tide (viz.: that of speed $n-2\Omega$) can lag in such a way as to have its angle of lagging nearly equal to 90°. The correct physical hypothesis, for values of λ nearly equal to $\frac{1}{2}$, is to suppose the lag small for the tide $n-2\Omega$, but large for the other tides. Hence when λ is nearly $=\frac{1}{2}$, we ought to put

$$\sin 4\mathrm{f}_1=\frac{\mathfrak{p}}{n-\Omega},\ \sin 2\mathrm{g}=\frac{2\mathfrak{p}}{n},\ \text{but}\ \sin 2\mathrm{g}_1=\frac{2(n-2\Omega)}{\mathfrak{p}}$$

Then we should have

$$G=\tfrac{1}{2}\mathfrak{m}\left[1+2(1-\lambda)-\frac{2n^2}{\mathfrak{p}^2}(1-\lambda)(1-2\lambda)\right]$$

$$D=\tfrac{1}{2}\left[1-2(1-\lambda)+\frac{2n^2}{\mathfrak{p}^2}(1-\lambda)(1-2\lambda)\right]$$

The last term in each of these expressions involves a small factor both in numerator and denominator, viz.: $1-2\lambda$ because $\lambda=\frac{1}{2}$ nearly, and $\mathfrak{p}$, because the viscosity is large. The evaluation of these terms depends on the actual degree of viscosity, but all that we are now concerned with is the fact that when $\lambda=\frac{1}{2}$ the true physical result is that D changes sign by passing through zero and not infinity, and that G does the same for some value of λ not far removed from $\frac{1}{2}$.

Now consider the function $\frac{4\lambda(1-\lambda)}{1-2\lambda}-1$. The following results are not stated retrospectively, and when it is said that i or j increase or decrease, it is meant increase or decrease as t or ξ increases.

(i.) From $\lambda=1$ to $\lambda=\cdot5$ the function is negative.

Hence for these values of λ the inclination j decreases, or zero inclination is dynamically stable.

When $\lambda=\cdot5$ it is infinite; but we have already remarked on this case.

(ii.) From $\lambda=\cdot5$ to $\lambda=\cdot191$ it is positive.

Therefore for these values of λ the inclination j increases, or zero inclination is dynamically unstable. It vanishes when $\lambda=\cdot191$.

(iii.) From $\lambda=\cdot191$ to $\lambda=0$ it is negative.

Therefore for these values of λ the inclination j decreases, or zero inclination is dynamically stable.

Next consider the function $1+\frac{4\lambda(1-\lambda)}{1-2\lambda}$.

(iv.) From $\lambda=1$ to $\lambda=\cdot809$ it is positive.

Therefore for these values of λ the obliquity i increases, or zero obliquity is dynamically unstable. It vanishes when $\lambda=\cdot809$.

(v.) From $\lambda=\cdot809$ to $\lambda=\cdot5$ it is negative.

Therefore for these values of λ the obliquity i decreases, or zero obliquity is dynamically stable.

When $\lambda=\cdot5$ it is infinite; but we have already remarked on this case.

(vi.) From $\lambda=\cdot5$ to $\lambda=0$ it is positive.

Therefore for these values of λ the obliquity i increases, or zero obliquity is dynamically unstable.

Therefore from $\lambda=1$ to $\cdot809$ the inclination j decreases and the obliquity i increases.

From $\lambda=\cdot809$ to $\cdot5$ both inclination and obliquity decrease.

From $\lambda=\cdot5$ to $\cdot191$ both inclination and obliquity increase.

From $\lambda=\cdot191$ to 0 the inclination decreases and the obliquity increases.

Now at the point where the above retrospective integration stopped, the moon's period was 2·45 days or 59 hours, and the day was 7·25 hours; hence at this point $\lambda=\cdot123$, which falls between ·191 and ·5. Hence both inclination and obliquity decrease retrospectively at a rate which tends to become infinite when we approach $\lambda=\cdot5$, if the viscosity be infinitely great. For large, but not infinite, viscosity the rates become large and then rapidly decrease in the neighbourhood of $\lambda=\cdot5$.

From this it follows that by supposing the viscosity large enough, the obliquity and inclination may be made as small as we please, when we arrive at the point where $\lambda=\cdot5$.

It was shown in § 17 of "Precession" that $\lambda=\cdot5$ corresponds to a month of 12 hours and a day of 6 hours.

Between the values $\lambda=\cdot5$ and $\cdot809$ the solutions for both the cases of small and of large viscosity concur in showing zero obliquity and inclination as dynamically stable. But between $\lambda=\cdot809$ and 1 the obliquity is dynamically unstable for infinitely large, stable for infinitely small viscosity; for these values of λ zero inclination is dynamically stable both for large and small viscosity.

From this it seems probable that for some large but finite viscosity, both zero inclination and zero obliquity would be dynamically stable for values of λ between ·809 and unity.

It appears to me therefore that we have only to accept the hypothesis that the viscosity of the earth has always been pretty large, as it certainly is at present, to obtain a satisfactory explanation of the obliquity of the ecliptic and of the inclination of the lunar orbit. This subject will be again discussed in the summary of Part VII.

§ 21. *Graphical illustration of the preceding integrations.*

A graphical illustration will much facilitate the comprehension of the numerical results of the last two sections.

The integrations which have been carried out by quadratures are of course equivalent to finding the areas of certain curves, and these curves will afford a convenient illustration of the nature of those integrations.

In §§ 19, 20 two separate points of departure were taken, the first proceeding from $\xi=1$ to ·76, and the second from $\xi=1$ to ·88. It is obvious that ξ was referred to different initial values c_0 in the two integrations.

In order therefore to illustrate the rates of increase of $\log \tan \frac{1}{2}J$ and $\log \tan \frac{1}{2}I$ from the preceding numerical results, we must either refer the second sets of ξ's to the same initial value c_0 as the first set, or (which will be simpler) we may take $\sqrt{c}$ as the independent variable.

Then for the values between $\xi=1$ and $\cdot 76$, the ordinates of our curves will be the numerical values given in Tables IV. and XI., each divided by $\sqrt{c_0}$. By the choice of a proper scale of length, c_0 may be taken as unity.

For the values in the second integration from $\xi=1$ to $\cdot 88$, the $\sqrt{c_0}$ is the final value of $\sqrt{c}$ in the first integration. Hence in order to draw the ordinates in the second part of the curve to the same scale as those of the first, the numbers in Tables VII. and XIV. must be divided by $\cdot 76$.

Also the second set of ordinates are not spaced out at the same intervals as the first set, for the $d\sqrt{c}$ of the second integration is $\cdot 76$ of the $d\sqrt{c}$ of the first integration.

Hence the ordinates given in the four Tables, IV., VII., XI., and XIV., are to be drawn corresponding to the abscissæ

$$0, 1, 2, 3, 4, 5, 6, 6\cdot 76, 7\cdot 52, 8\cdot 28.$$

In fig. 7 these abscissæ are marked off on the horizontal axis.

The first integration corresponds to the part OO′, and the marked points correspond to the seven values of ξ from 1 to $\cdot 76$ inclusive. The second integration corresponds to the part O′O″, and the values computed in Tables VII. and XIV. were divided by $\cdot 76$ to give the ordinates.

The value for $\xi=\cdot 76$ of the first integration is identical with that for $\xi=1$ of the second.

The integrations, which have been carried out, correspond to the determination of the areas lying between these curves and the horizontal axis, areas below being esteemed negative.

The two curves for $d \log \tan \frac{1}{2}I/d\sqrt{c}$ lie very close together, and we thus see that the motion of the earth's proper plane is almost independent of the degree of viscosity.

On the other hand, the two curves for $d \log \tan \frac{1}{2}J/d\sqrt{c}$ differ considerably. For large viscosity the positive area is much larger than the negative, whilst for small viscosity the positive area is a little smaller than the negative.

If the figure were extended further to the right, the two curves for the variation of I would become identical, and the ordinates would become very small. The two curves for the variation of J would separate widely. That for large viscosity would go upwards in the positive direction, so that its ordinates would be infinite at the point corresponding to $\lambda=\frac{1}{2}$; the curve for small viscosity would go downwards in the negative direction, and the ordinates would be infinite at the point where $\lambda=1$.

In this figure OO′ is 6 centimeters, OO″ is $8\cdot 28$ centimeters, and the point

corresponding to $\lambda = \frac{1}{2}$ would be 15·2 centimeters from O, and the point corresponding to $\lambda = 1$ would be 17·4 centimeters from O.

We thus see that the degree of viscosity makes an enormous difference in the results.

In the figure, portions of these further parts of the two curves for the variation of J are continued conjecturally by a line of dashes.

The whole figure is to be read from left to right for a retrospective solution, and from right to left if we advance with the time.

Fig. 7.

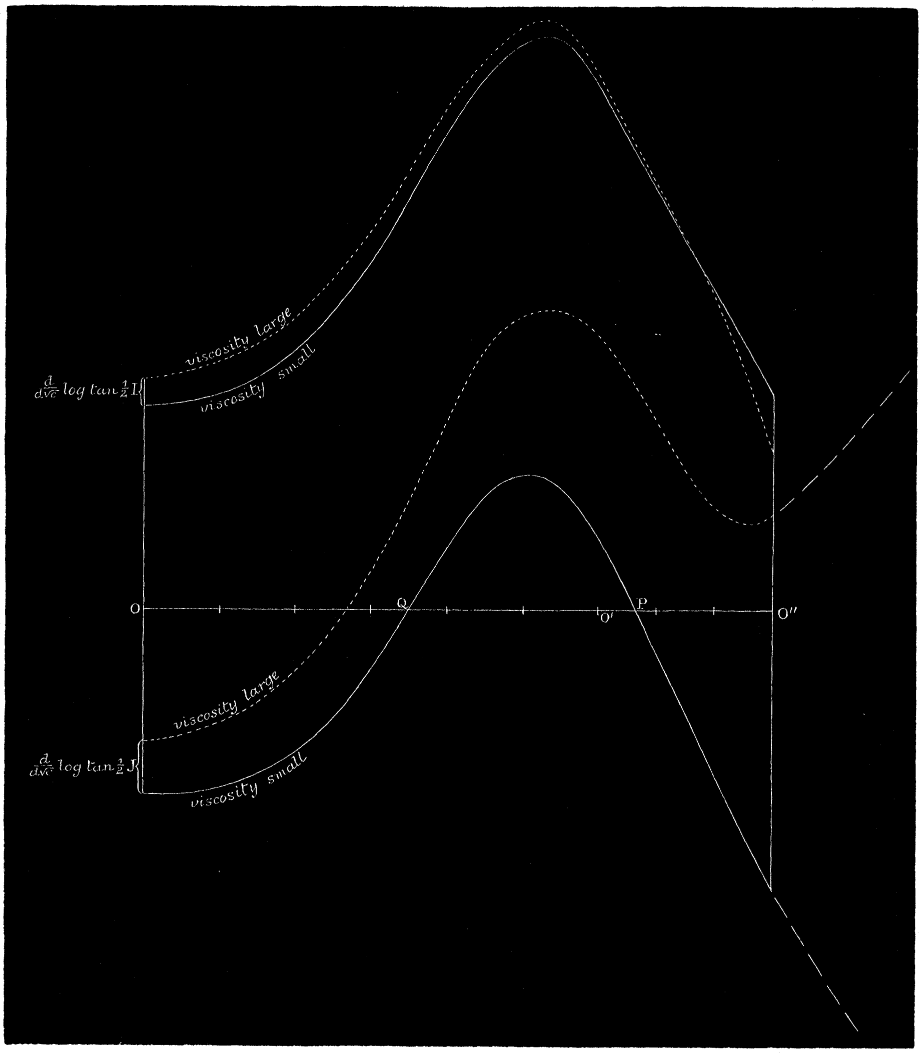


Diagram to illustrate the motion of the proper planes of the moon and earth.

§ 22. *The effects of solar tidal friction on the primitive condition of the earth and moon.*

In the paper on "Precession," § 16, I found, by the solution of a biquadratic equation, the primitive condition in which the earth and moon moved round together as a rigid body.

Since writing that paper certain additional considerations have occurred to me, which seem to be important in regard to the origin of the moon.

It was there remarked that, as we approach that critical condition of dynamical instability, the effects of solar tidal friction must have become sensible, because of the slow relative motion of the moon and earth. I did not at that time perceive the full significance of this, and I will now consider it further.

Suppose the moon to be moving orbitally nearly as fast as the earth rotates. Then the tidal reaction, which depends on the lunar tides alone, must be very small, and therefore the moon's orbital motion increases retrospectively very slowly. On the other hand, the relative motion of the earth and sun is great, and therefore if we approach the critical condition close enough, the solar tidal friction must have been greater than the lunar, however great the viscosity of the planet. The manner in which this will affect the solution of the previous paper may be shown analytically as follows.

If we neglect the obliquity, and divide the equation of tidal friction by that of tidal reaction, and suppose the viscosity small, we have from (176)

$$-k\frac{dn}{d\xi}=1+\left(\frac{\tau'}{\tau}\right)^2\frac{n}{n-\Omega}=1+\left(\frac{\tau'}{\tau_0}\right)^2\xi^{12}+\left(\frac{\tau'}{\tau}\right)^2\frac{\Omega}{n-\Omega}$$

Then integrating we have

$$n=n_0+\frac{1}{k}\left[(1-\xi)+\tfrac{1}{13}\left(\frac{\tau'}{\tau_0}\right)^2(1-\xi^{13})\right]+\frac{1}{k}\int_{\xi}^{1}\left(\frac{\tau'}{\tau}\right)^2\frac{\Omega}{n-\Omega}d\xi$$

If we do not carry the integration to near the critical phase, where n is equal to Ω, the last integral is small, but it tends to become large as n becomes nearly equal to Ω; it has always been neglected in our integration. When however we wish to apply this equation to find the values for which n is equal to Ω, it cannot be neglected.

Suppose the integral to be equal to K. Then in the first part of the above expression we may put $n=\Omega=x^3$ and we may neglect $\frac{1}{13}(\tau'/\tau_0)^2(1-\xi^{13})$. Hence the equation for finding the angular velocity of the two bodies at the critical phase, when $n=\Omega$, is

$$x^3 = n_0 + \frac{1}{k} - \frac{1}{sx} + \mathrm{K}$$

or

$$x^4 - \left(n_0 + \frac{1}{k} + \mathrm{K}\right)x + \frac{1}{s} = 0$$

The root of this equation, which gives the required phase, is nearly equal to the cube-root of the second coefficient, hence

$$x^3 = n = \Omega = \left(n_0 + \frac{1}{k} + \mathrm{K}\right) \text{ nearly.}$$

Now in the paper on "Precession" we found the initial condition, on the hypothesis that K was zero. Hence the effect of solar tidal friction is to increase the angular velocity of the two bodies when their relative motion is zero. Since K may be large, it follows that the disturbance of the solution of § 16 of "Precession" may be considerable.

This therefore shows that it is probable that an accurate solution of our problem would differ considerably from that found in "Precession," and that the common angular velocity of the two bodies might have been very great.

If KEPLER's law holds good, then the periodic time of the moon about the earth, when their centres are 6,000 miles apart, is 2 hrs. 36 m., and when 5,000 miles apart is 1 hr. 57 m.; hence when the two spheroids are just in contact, the time of revolution of the moon would be between 2 hrs. and $2\frac{1}{2}$ hrs.

Now it is a remarkable fact that the most rapid rate of revolution of a mass of fluid, of the same mean density as the earth, which is consistent with an ellipsoidal form of equilibrium, is 2 hrs. 24 m. Is this a mere coincidence, or does it not rather point to the break-up of the primæval planet into two masses in consequence of a too rapid rotation?

*It is not possible to make an adequate consideration of the subject of this section without a treatment of the theory of the tidal friction of a planet attended by a pair of satellites.

It was shown above that if the moon were to move orbitally nearly as fast as the earth rotates, the solar tidal friction would be more important than the lunar, however near the moon might be to the earth. I now (September, 1880) find that the consequence of this is that the earth's rotation continues to increase retrospectively, and the moon's orbital motion does the same; but the difference of the rotation and orbital

* From this point to the end has been added, and the section otherwise abridged since the paper was presented.—September, 1880.

motion gets continually less and less. Meanwhile, the earth's orbital motion round the sun is continually increasing, and the distance from the sun decreasing retrospectively. Theoretically this would go on until the sun and moon (treated as particles) revolve as though rigidly connected with the earth and with one another. This is the configuration of maximum energy of the system.

The solution is physically absurd, because the distance of the two bodies from the earth would then be very much less than the earth's radius, and *à fortiori* than the sun's radius.

It must be observed, however, that in the retrospect the relative motion of the moon and earth would already have become almost insensible, before the earth's distance from the sun could be sensibly affected.

V.

SECULAR CHANGES IN THE ECCENTRICITY OF THE ORBIT.

§ 23. *Formation of the disturbing function.*

We will now consider the rate of change in the eccentricity and mean distance of the orbit of a satellite, moving in an elliptic orbit, but always remaining in a fixed plane, namely, the ecliptic; and the rate of change of the obliquity of the planet's equator when perturbed by such a satellite will also be found.

Up to the end of Part I. the investigation for the formation of the disturbing function was quite general, and we therefore resume the thread at that point.

In the present problem the inclination of the satellite's orbit to the ecliptic is zero, and we have

$$\varpi=\underline{\varpi}=P=\cos\tfrac{1}{2}i,\quad \kappa=\underline{\kappa}=Q=\sin\tfrac{1}{2}i$$

We thus get rid of the ϖ and κ functions, and henceforth ϖ will indicate the longitude of the perigee.

Then by equations (24–8),

$$\mathrm{M}_1{}^2-\mathrm{M}_2{}^2=P^4\cos 2(\chi-\theta)+2P^2Q^2\cos 2\chi+Q^4\cos 2(\chi+\theta)$$

$$-2\mathrm{M}_1\mathrm{M}_2=\text{The same with sines for cosines}$$

$$\mathrm{M}_2\mathrm{M}_3=-P^3Q\cos(\chi-2\theta)+PQ(P^2-Q^2)\cos\chi+PQ^3\cos(\chi+2\theta)$$

$$\mathrm{M}_1\mathrm{M}_3=\text{The same with sines for cosines}$$

$$\tfrac{1}{3}-\mathrm{M}_3{}^2=\tfrac{1}{3}(P^4-4P^2Q^2+Q^4)+2P^2Q^2\cos 2\theta$$

By the definitions (29)

$$X=\left[\frac{c(1-e^2)}{r}\right]^{\frac{3}{2}}M_1,\ Y=\left[\frac{c(1-e^2)}{r}\right]^{\frac{3}{2}}M_2,\ Z=\left[\frac{c(1-e^2)}{r}\right]^{\frac{3}{2}}M_3$$

Now let

$$\Phi(\alpha)=\left[\frac{c(1-e^2)}{r}\right]^3\cos(2\theta+\alpha),\ \Psi(\alpha)=\left[\frac{c(1-e^2)}{r}\right]^3\cos\alpha,\ R=\left[\frac{c(1-e^2)}{r}\right]^3 \quad . \quad (264)$$

Then

$$\left.\begin{aligned}X^2-Y^2&=P^4\Phi(-2\chi)+2P^2Q^2\Psi(2\chi)+Q^4\Phi(2\chi)\\ 2XY&=\text{The same when }\chi+\tfrac{1}{4}\pi\text{ is substituted for }\chi\\ YZ&=-P^3Q\Phi(-\chi)+PQ(P^2-Q^2)\Psi(\chi)+PQ^3\Phi(\chi)\\ XZ&=\text{The same when }\chi-\tfrac{1}{2}\pi\text{ is substituted for }\chi\\ \tfrac{1}{3}(X^2+Y^2-2Z^2)&=\tfrac{1}{3}(P^4-P^2Q^2+Q^4)R+2P^2Q^2\Phi(0)\end{aligned}\right\} \quad . \quad . \quad (265)$$

Hence all the terms of the five X-Y-Z functions belong to one of the three types Φ, Ψ, or R.

The equation to the ellipse described by the satellite Diana is

$$\frac{c(1-e^2)}{r}=1+e\cos(\theta-\varpi) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (266)$$

Hence

$$\left.\begin{aligned}R&=1+\tfrac{3}{2}e^2+3e(1+\tfrac{1}{4}e^2)\cos(\theta-\varpi)+\tfrac{3}{2}e^2\cos2(\theta-\varpi)+\tfrac{1}{4}e^3\cos3(\theta-\varpi)\\ \Phi(\alpha)&=R\cos(2\theta+\alpha)=(1+\tfrac{3}{2}e^2)\cos(2\theta+\alpha)\\ &\quad+\tfrac{3}{2}e(1+\tfrac{1}{4}e^2)[\cos(3\theta+\alpha-\varpi)+\cos(\theta+\alpha+\varpi)]\\ &\quad+\tfrac{3}{4}e^2[\cos(4\theta+\alpha-2\varpi)+\cos(\alpha+2\varpi)]\\ &\quad+\tfrac{1}{8}e^3[\cos(5\theta+\alpha-3\varpi)+\cos(\theta-\alpha-3\varpi)]\end{aligned}\right\} (267)$$

and $\Psi(\alpha)=R\cos\alpha$.

Now by the theory of elliptic motion, θ the true longitude may be expressed in terms of $\Omega t+\epsilon$ and ϖ, in a series of ascending powers of e the eccentricity. Hence $\Phi(\alpha)$, R, and $\Psi(\alpha)$ may be expressed as the sum of a number of cosines of angles of the form $l(\Omega t+\epsilon)+m\varpi+n\alpha$, and in using these functions we shall require to make α either a multiple of χ or zero, or to differ from a multiple of χ by a constant. Therefore the X-Y-Z functions are expressible as the sums of a number of sines or cosines of angles of the form $l(\Omega t+\epsilon)+m\varpi+n\chi$.

Now χ increases uniformly with the time (being equal to nt+a constant); hence, if

we regard the elements of the elliptic orbit as constant, the X-Y-Z functions are expressible as a number of simple time-harmonics. But in § 4, where the state of tidal distortion due to Diana was found, they were assumed to be so expressible; therefore that assumption was justifiable, and the remainder of that section concerning the formation of the disturbing function is applicable.

The problem may now be simplified by the following considerations:—The equation (12) for the rate of variation of the ellipticity of the orbit involves only differentials of the disturbing function with regard to epoch and perigee. It is obvious that in the disturbing function the epoch and perigee will only occur in the argument of trigonometrical functions, therefore after the required differentiations they only occur in the like forms. Now the epoch never occurs except in conjunction with the mean longitude, and the longitude of the perigee increases uniformly with the time (or nearly so), either from the action of other disturbing bodies or from the disturbing action of the permanent oblateness of the planet, which causes a progression of the apses. Hence it follows that the only way in which these differentials of the disturbing function can be non-periodic is when the tide-raiser Diana is identical with the moon. Whence we conclude that—

The tides raised by any one satellite can produce no secular change in the eccentricity of the orbit of any other satellite.

The problem is thus simplified by the consideration that Diana and the moon need only be regarded as distinct as far as regards epoch and perigee, and that they are ultimately to be made identical.

Before carrying out the procedure above sketched, it will be well to consider what sort of approximations are to be made, for the subsequent labour will be thus largely abridged.

From the preceding sketch it is clear that all the terms of the X-Y-Z functions corresponding with Diana's tide-generating potential are of the form

$$(a+b\mathrm{e}+c\mathrm{e}^2+d\mathrm{e}^3+f\mathrm{e}^4+\&c.)\cos[l\chi+m(\Omega t+\epsilon)+n\varpi+\delta].$$

From this it follows that all the terms of the $\mathfrak{X}$-$\mathfrak{Y}$-$\mathfrak{Z}$ functions are of the form

$$\mathrm{F}(a+b\mathrm{e}+c\mathrm{e}^2+d\mathrm{e}^3+f\mathrm{e}^4+\&c.)\cos[l\chi+m(\Omega t+\epsilon)+n\varpi+\delta-f].$$

Also by symmetry all the terms of the X′-Y′-Z′ functions are of the form

$$(a+b\mathrm{e}+c\mathrm{e}^2+d\mathrm{e}^3+f\mathrm{e}^4+\&c.)\cos[l\chi'+m(\Omega t+\epsilon')+n\varpi'+\delta],$$

and in the present problem the accent to χ may be omitted.

The products of the $\mathfrak{X}$-$\mathfrak{Y}$-$\mathfrak{Z}$ functions multiplied by the X′-Y′-Z′ functions occur in such a way that when they are added together in the required manner (as for example in $\mathrm{Y}'\mathrm{Z}'\,\mathfrak{Y}\mathfrak{Z}+\mathrm{X}'\mathrm{Z}'\,\mathfrak{X}\mathfrak{Z}$) only differences of arguments occur, and χ disappears from the disturbing function. Also secular changes can only arise in the satellite's eccentricity and mean distance from such terms in the disturbing function as are independent of $\Omega t+\epsilon$ and ϖ, when we put $\epsilon'=\epsilon$ and $\varpi'=\varpi$. Hence we need only select from the

complete products the products of terms of the like argument in the two sets of functions.

Whence it follows that all the part of the disturbing function, which is here important, consists of terms of the form

$$F(a+b\mathrm{e}+c\mathrm{e}^2+d\mathrm{e}^3+f\mathrm{e}^4+\&c.)^2\cos\left[m(\epsilon-\epsilon')+n(\varpi-\varpi')-f\right]$$

or

$$F(a^2+2ab\mathrm{e}+(2ac+b^2)\mathrm{e}^2+(2ad+2bc)\mathrm{e}^3+(2af+2bd+c^2)\mathrm{e}^4+\&c.)$$
$$\cos\left[m(\epsilon-\epsilon')+n(\varpi-\varpi')-f\right]$$

Now it is intended to develop the disturbing function rigorously with respect to the obliquity of the ecliptic, and as far as the fourth power of the eccentricity.

The question therefore arises, what terms will it be necessary to retain in developing the X-Y-Z functions, so as to obtain the disturbing function correct to e^4.

In the X-Y-Z functions (and in their constituent functions $\Phi(\alpha)$, $\Psi(\alpha)$, R) those terms in which a is not zero will be said to be of the order zero; those in which a is zero, but b not zero, of the first order; those in which $a=b=0$, but c not zero, of the second order, and so on.

Then, by considering the typical term in the disturbing function, we have the following—

Rule of approximation for the development of the X-Y-Z functions and of $\Phi(\alpha)$, $\Psi(\alpha)$, R: develop terms of order zero to e^4; terms of the first order to e^3; terms of the second order to e^2; and drop terms of the third and fourth orders.

To obtain further rules of approximation, and for the subsequent developments, we now require the following theorem.

Expansion of cos $(k\theta+\beta)$ *in powers of the eccentricity.*

θ is the true longitude of the satellite, $\Omega t+\epsilon$ the mean longitude, and ϖ the longitude of the perigee. For the present I shall write simply Ω in place of $\Omega t+\epsilon$.

By the theory of elliptic motion

$$\Omega=\theta-2\mathrm{e}\sin(\theta-\varpi)+\tfrac{3}{4}\mathrm{e}^2(1+\tfrac{1}{6}\mathrm{e}^2)\sin 2(\theta-\varpi)-\tfrac{1}{3}\mathrm{e}^3\sin 3(\theta-\varpi)+\tfrac{5}{32}\mathrm{e}^4\sin 4(\theta-\varpi)$$

If this series be inverted, it will be found that*

$$\theta=\Omega+2\mathrm{e}(1-\tfrac{1}{8}\mathrm{e}^2)\sin(\Omega-\varpi)+\tfrac{5}{4}\mathrm{e}^2(1-\tfrac{11}{30}\mathrm{e}^2)\sin 2(\Omega-\varpi)+\tfrac{13}{12}\mathrm{e}^3\sin 3(\Omega-\varpi)$$
$$+\tfrac{103}{96}\mathrm{e}^4\sin 4(\Omega-\varpi)$$

* See TAIT and STEELE'S 'Dynamics,' art. 118, or any other work on elliptic motion.

By differentiation we find that, when e=0,

$$\frac{d\theta}{de}=2\sin(\Omega-\varpi),\quad \frac{d^2\theta}{de^2}=\tfrac{5}{2}\sin 2(\Omega-\varpi),\quad \frac{d^3\theta}{de^3}=-\tfrac{3}{2}\sin(\Omega-\varpi)+\tfrac{13}{2}\sin 3(\Omega-\varpi)$$

$$\frac{d^4\theta}{de^4}=-11\sin 2(\Omega-\varpi)+\tfrac{103}{4}\sin 4(\Omega-\varpi),\quad \left(\frac{d\theta}{de}\right)^2=2-2\cos(\Omega-\varpi)$$

$$\left(\frac{d\theta}{de}\right)^3=6\sin(\Omega-\varpi)-2\sin 3(\Omega-\varpi),\quad \left(\frac{d\theta}{de}\right)^4=6-8\cos 2(\Omega-\varpi)+2\cos 4(\Omega-\varpi)$$

$$\frac{d\theta}{de}\frac{d^2\theta}{de^2}=\tfrac{5}{2}\cos(\Omega-\varpi)-\tfrac{5}{2}\cos 3(\Omega-\varpi),\quad \left(\frac{d\theta}{de}\right)^2\frac{d^2\theta}{de^2}=5\sin 2(\Omega-\varpi)-\tfrac{5}{2}\sin 4(\Omega-\varpi)$$

$$\left(\frac{d^2\theta}{de^2}\right)^2=\tfrac{25}{8}-\tfrac{25}{8}\cos 4(\Omega-\varpi),\quad \frac{d\theta}{de}\frac{d^3\theta}{de^3}=-\tfrac{3}{2}+8\cos 2(\Omega-\varpi)-\tfrac{13}{2}\cos 4(\Omega-\varpi)$$

To expand $\cos(k\theta+\beta)$ by means of MACLAURIN'S theorem, we require the values of the following differentials when e=0 and $\theta=\Omega$:—

$$\frac{d}{de}\cos(k\theta+\beta)=-k\sin(k\theta+\beta)\frac{d\theta}{de}$$

$$\frac{d^2}{de^2}\cos(k\theta+\beta)=-k^2\cos(k\theta+\beta)\left(\frac{d\theta}{de}\right)^2-k\sin(k\theta+\beta)\frac{d^2\theta}{de^2}$$

$$\frac{d^3}{de^3}\cos(k\theta+\beta)=k^3\sin(k\theta+\beta)\left(\frac{d\theta}{de}\right)^3-3k^2\cos(k\theta+\beta)\frac{d\theta}{de}\frac{d^2\theta}{de^2}-k\sin(k\theta+\beta)\frac{d^3\theta}{de^3}$$

$$\frac{d^4}{de^4}\cos(k\theta+\beta)=k^4\cos(k\theta+\beta)\left(\frac{d\theta}{de}\right)^4+6k^3\sin(k\theta+\beta)\left(\frac{d\theta}{de}\right)^2\frac{d^2\theta}{de^2}-3k^2\cos(k\theta+\beta)\left(\frac{d^2\theta}{de^2}\right)^2$$
$$-4k^2\cos(k\theta+\beta)\frac{d\theta}{de}\frac{d^3\theta}{de^3}-k\sin(k\theta+\beta)\frac{d^4\theta}{de^4}$$

Now when e=0, $k\theta+\beta=k\Omega+\beta$, and the values of the differentials and functions of differentials of e are given above. Then if we substitute for these functions their values, and express the products of sines and cosines as the sums of sines and cosines, and introduce the abridged notation in which $k\Omega+\beta+s(\Omega-\varpi)$ is written $(k+s)$, we have

$$\left.\begin{aligned}
\Theta_1 &= \frac{d}{de}\cos(k\theta+\beta) = -k\cos(k-1)+k\cos(k+1)\\
\Theta_2 &= \frac{d^2}{de^2}\cos(k\theta+\beta) = (k^2-\tfrac{5}{4}k)\cos(k-2)-2k^2\cos k+(k^2+\tfrac{5}{4}k)\cos(k+2)\\
\Theta_3 &= \frac{d^3}{de^3}\cos(k\theta+\beta) = -(k^3-\tfrac{15}{4}k^2+\tfrac{13}{4}k)\cos(k-3)+3(k^3-\tfrac{5}{4}k^2+\tfrac{1}{4}k)\cos(k-1)\\
&\qquad -3(k^3+\tfrac{5}{4}k^2+\tfrac{1}{4}k)\cos(k+1)+(k^3+\tfrac{15}{4}k^2+\tfrac{13}{4}k)\cos(k+3)\\
\Theta_4 &= \frac{d^4}{de^4}\cos(k\theta+\beta) = (k^4-\tfrac{15}{2}k^3+\tfrac{75}{16}k^2+13k^2-\tfrac{103}{8}k)\cos(k-4)\\
&\qquad -(4k^4-15k^3+16k^2-\tfrac{11}{2}k)\cos(k-2)\\
&\qquad +3(2k^4-\tfrac{25}{8}k^2+2k^2)\cos(k)-(4k^4+15k^3+16k^2+\tfrac{11}{2}k)\cos(k+2)\\
&\qquad +(k^4+\tfrac{15}{2}k^3+\tfrac{75}{16}k^2+13k^2+\tfrac{103}{8}k)\cos(k+4)
\end{aligned}\right\} \quad (268)$$

where the Θ's are merely introduced as an abbreviation.

Then by MACLAURIN's theorem

$$\cos(k\theta+\beta)=\cos(k\Omega+\beta)+e\Theta_1+\tfrac{1}{2}e^2\Theta_2+\tfrac{1}{6}e^3\Theta_3+\tfrac{1}{24}e^4\Theta_4. \quad . \quad . \quad . \quad (269)$$

In order to obtain further rules of approximation we will now run through the future developments, merely paying attention to the order of the coefficients and to the factors by which $\Omega t+\epsilon$ will be multiplied in the results. From this point of view we may write

$$\begin{aligned}
\Phi(\alpha) &= (e^0)\cos(2\theta)+(e)[\cos(3\theta)+\cos(\theta)]+(e^2)[\cos(4\theta)+\cos(0)]\\
&\qquad\qquad +(e^3)[\cos(5\theta)+\cos(\theta)]
\end{aligned}$$

$$\Psi(\alpha)=R=(e^0)\cos(0)+(e)\cos(\theta)+(e^2)\cos(2\theta)+(e^3)\cos(3\theta)$$

The cosines of the multiples of θ have now to be found by the theorem (269) and substituted in the above equations.

In making the developments the following abridged notation is adopted; a term of the form $\cos[(k+s)\Omega+\beta-s\varpi]$ is written $\{k+s\}$.

Consider the series for $\Phi(\alpha)$ first.

We have by successive applications of (269) with $k=1, 2, 3, 4, 5$.

$$(e^0)\cos(2\theta)=(e^0)\{2\}+(e)[\{1\}+\{3\}]+(e^2)[\{0\}+\{2\}+\{4\}]$$
$$+(e^3)[\{-1\}+\{1\}+\{3\}+\{5\}]+(e^4)[\{-2\}+\{0\}+\{2\}+\{4\}+\{6\}]$$

$$(e)\cos(3\theta)=(e)\{3\}+(e^2)[\{2\}+\{4\}]+(e^3)[\{1\}+\{3\}+\{5\}]$$
$$+(e^4)[\{0\}+\{2\}+\{4\}+\{6\}]$$

$$(e)\cos(\theta)=(e)\{1\}+(e^2)[\{0\}+\{2\}]+(e^3)[\{-1\}+\{1\}+\{3\}]$$
$$+(e^4)[\{-2\}+\{0\}+\{2\}+\{4\}]$$

$$(e^2)\cos(4\theta)=(e^2)\{4\}+(e^3)[\{3\}+\{5\}]+(e^4)[\{2\}+\{4\}+\{6\}]$$

$$(e^2)\cos(0)=(e^2)\{0\}$$

$$(e^3)\cos(5\theta)=(e^3)\{5\}+(e^4)[\{4\}+\{6\}]$$

$$(e^3)\cos(\theta)=(e^3)\{1\}+(e^4)[\{0\}+\{2\}]$$

In these expressions we have no right, as yet, to assume that $\{-2\}$ and $\{-1\}$ are different from $\{2\}$ and $\{1\}$; and in fact we shall find that in the expansion for $\Phi(\alpha)$ they *are* different, but in that for R they are the same.

Then adding up these, and rejecting terms of the third and fourth orders by the first rule of approximation, we have

$$\Phi(\alpha)=[(e^0)+(e^2)+(e^4)]\{2\}+[(e)+(e^3)][\{1\}+\{3\}]+[(e^2)+(e^4)][\{0\}+\{4\}]$$
$$+(e^3)\{-1\}+(e^4)\{-2\}$$

It will be observed that $\{5\}$ and $\{6\}$ are wanting, and might have been dropped from the expansions. Also $\{0\}$ and $\{4\}$ are terms of the second order, therefore wherever they are multiplied by (e^4) they might have been dropped. Hence $(e^3)\cos(5\theta)$ need not have been expanded at all. A little further consideration is required to show that $(e^3)\cos(\theta)$ need not have been expanded.

$(e^3)\cos(\theta)$ is an abbreviation for $\frac{1}{8}e^3\cos(\theta-\alpha-3\varpi)$, and therefore in this case $\{1\}=\cos(\Omega-\alpha-3\varpi)$ and $\{2\}=\cos(2\Omega-\alpha-4\varpi)$; but in every other case $\{1\}=\cos(\Omega+\alpha+\varpi)$ and $\{2\}=\cos(2\Omega+\alpha)$. Hence the terms $\{1\}$ and $\{2\}$ in $(e^3)\cos(\theta)$ are of the third and fourth orders and may be dropped, and $\{0\}$ may also be dropped. Thus the whole of $(e^3)\cos(\theta)$ may be dropped.

With respect to $\{-2\}$ and $\{-1\}$, observe that $\{2\}$ in the expansion of $\cos(k_1\theta+\beta_1)$ stands for $\cos[2\Omega+(k_1-2)\varpi+\beta_1]$; and $\{-2\}$ in the expansion of $\cos(k_2\theta+\beta_2)$ stands for $\cos[2\Omega-(k_2+2)\varpi-\beta_2]$; and k_1, k_2 are either 1, 2, 3, or 4; and β_1, β_2 are multiples of $\chi+$ a constant. Hence $\{2\}$ and $\{-2\}$ are necessarily different, but if β_1 and β_2 were multiples of ϖ they might be the same, and indeed in the expansion of R necessarily are the same.

In the same way it may be shown that $\{-1\}$ and $\{1\}$ are necessarily different.

Therefore $\{-1\}$ and $\{-2\}$ being terms of the third and fourth orders may be dropped.

It follows from this discussion that, as far as concerns the present problem,

$$(e^0)\cos(2\theta)=(e^0)\{2\}+(e)[\{1\}+\{3\}]+(e^2)[\{0\}+\{2\}+\{4\}]+(e^3)[\{1\}+\{3\}]+(e^4)[\{2\}]$$

$$(e)\cos(3\theta)=(e)\{3\}+(e^2)[\{2\}+\{4\}]+(e^3)[\{1\}+\{3\}]+(e^4)\{2\}$$

$$(e)\cos(\theta)=(e^2)\{1\}+(e^2)[\{0\}+\{2\}]+(e^3)[\{1\}+\{3\}]+(e^4)\{2\}$$

$$(e^2)\cos(4\theta)=(e^2)\{4\}+(e^3)\{3\}+(e^4)\{2\}$$

$$(e^2)\cos(0)=(e^2)\{0\}$$

And the sum of these expressions is equal to $\Phi(\alpha)$.

We thus get the following rules for the use of the expansion (269) of $\cos(k\theta+\beta)$ for the determination of $\Phi(\alpha)$:

When $k=2$, omit in Θ_3 terms in $\cos(k-3)$, $\cos(k+3)$
in Θ_4 terms in $\cos(k-4)$, $\cos(k-2)$, $\cos(k+2)$, $\cos(k+4)$

When $k=3$, omit in Θ_2 term in $\cos(k+2)$
in Θ_3 terms in $\cos(k-3)$, $\cos(k+1)$, $\cos(k+3)$
all of Θ_4

When $k=1$, omit in Θ_2 term in $\cos(k-2)$
in Θ_3 term in $\cos(k-3)$, $\cos(k-1)$, $\cos(k+3)$
all of Θ_4

When $k=4$, omit in Θ_1 term in $\cos(k+1)$
in Θ_2 term in $\cos(k)$, $\cos(k+2)$
all of Θ_3, Θ_4

Then following these rules we easily find,

When $k=2$, $\beta=\alpha$

$$\cos(2\theta+\alpha)=(1-4e^2+\tfrac{55}{16}e^4)\cos(2\Omega+\alpha)-2e(1-\tfrac{7}{8}e^2)\cos(\Omega+\alpha+\varpi)$$
$$+2e(1-\tfrac{27}{8}e^2)\cos(3\Omega+\alpha-\varpi)+\tfrac{3}{4}e^2\cos(\alpha+2\varpi)+\tfrac{13}{4}e^2\cos(4\Omega+\alpha-2\varpi). \quad (270)$$

When $k=3$, $\beta=\alpha-\varpi$

$$\cos(3\theta+\alpha-\varpi)=(1-9e^2)\cos(3\Omega+\alpha-\varpi)-3e(1-\tfrac{11}{4}e^2)\cos(2\Omega+\alpha)$$
$$+3e\cos(4\Omega+\alpha-2\varpi)+\tfrac{21}{8}e^2\cos(\Omega+\alpha+\varpi). \quad (271)$$

When $k=1$, $\beta=\alpha+\varpi$

$$\cos(\theta+\alpha+\varpi)=(1-e^2)\cos(\Omega+\alpha+\varpi)+e(1-\tfrac{5}{4}e^2)\cos(2\Omega+\alpha)-e\cos(\alpha+2\varpi)$$
$$+\tfrac{9}{8}e^2\cos(3\Omega+\alpha-\varpi)\,. \quad (272)$$

When $k=4$, $\beta=\alpha-2\varpi$

$$\cos(4\theta+\alpha-2\varpi)=\cos(4\Omega+\alpha-2\varpi)-4e\cos(3\Omega+\alpha-\varpi)+\tfrac{11}{2}e^2\cos(2\Omega+\alpha)\,. \quad (273)$$

These are all the series required for the expression of $\Phi(\alpha)$, since $\cos(\alpha+2\varpi)$ does not involve θ, and by what has been shown above $\cos(5\theta+\alpha-3\varpi)$ and $\cos(\theta-\alpha-3\varpi)$ need not be expanded.

We now return again to the series for R or $\Psi(\alpha)$, and consider the nature of the approximations to be adopted there.

With the same notation

$$(e^0)\cos(0)=(e^0)\{0\}$$
$$(e)\cos(\theta)=(e)\{1\}+(e^2)[\{0\}+\{2\}]+(e^3)[\{-1\}+\{1\}+\{3\}]$$
$$+(e^4)[\{-2\}+\{0\}+\{2\}+\{4\}]$$
$$(e^2)\cos(2\theta)=(e^2)\{2\}+(e^3)[\{1\}+\{3\}]+(e^4)[\{0\}+\{2\}+\{4\}]$$
$$(e^3)\cos(3\theta)=(e^3)\{3\}+(e^4)[\{2\}+\{4\}]$$

Since R is a function of $\theta-\varpi$, therefore after expansion it must be a function of $\Omega-\varpi$, and hence $\{1\}$ must be necessarily identical with $\{-1\}$, and $\{2\}$ with $\{-2\}$.

Adding these up, and dropping terms of the third and fourth orders,

$$R=[(e^0)+(e^2)+(e^4)]\{0\}+[(e)+(e^3)]\{1\}+(e^3)\{-1\}+[(e^2)+(e^4)]\{2\}+(e^4)\{-2\}$$

Here $\{0\}$ is a term of the order zero, $\{1\}$ of the first order, and $\{2\}$ of the second. Therefore by the first rule of approximation $\{2\}$ and $\{-2\}$ may be dropped when multiplied by (e^4).

Also $\{3\}$ and $\{4\}$ may be dropped.

Hence as far as concerns the present problem

$$(e^0)\cos(0)=(e^0)\{0\}$$
$$(e)\cos(\theta)=(e)\{1\}+(e^2)[\{0\}+\{2\}]+(e^3)[\{-1\}+\{1\}]+(e^4)\{0\}$$
$$(e^2)\cos(2\theta)=(e^2)\{2\}+(e^3)\{1\}+(e^4)\{0\}$$

and $(e^3)\cos(3\theta)$ need not be expanded.

And the sum of these expressions is equal to R.

We thus get the following rules for the use of the expansion of $\cos (k\theta+\beta)$ for the determination of R.

When $k=1$, omit in Θ_2 term in $\cos (k+2)$
in Θ_3 terms in $\cos (k-3)$, $\cos (k+1)$, $\cos (k+3)$
all of Θ_4

When $k=2$, omit in Θ_1 term in $\cos (k+1)$
in Θ_2 terms in $\cos (k)$, $\cos (k+2)$
all of Θ_3, Θ_4.

Then following these rules, we find

When $k=1$, $\beta=-\varpi$

$$\cos (\theta-\varpi)=(1-e^2) \cos (\Omega-\varpi)-e+e \cos 2(\Omega-\varpi) \quad . \quad . \quad . \quad . \quad (274)$$

When $k=2$, $\beta=-2\varpi$

$$\cos 2(\theta-\varpi)= \cos 2(\Omega-\varpi)-2e \cos (\Omega-\varpi)+\frac{3e^2}{4} \quad . \quad . \quad . \quad . \quad (275)$$

These are the only series required for the expansion of R or $\Psi(\alpha)$, since by what is shown above, $\cos 3(\theta-\varpi)$ need not be expanded.

Now multiply (270) by $1+\frac{3}{2}e^2$; (271) by $\frac{3}{2}e(1+\frac{1}{4}e^2)$; (272) by $\frac{3}{2}e(1+\frac{1}{4}e^2)$; and (273) by $\frac{3}{4}e^2$; add the four products together, and add $\frac{3}{4}e^2 \cos (\alpha+2\varpi)$, and we find from (267) after reduction

$$\Phi(\alpha)=(1-\tfrac{11}{2}e^2+\tfrac{181}{16}e^4) \cos (2\Omega+\alpha)-\tfrac{1}{2}e(1-\tfrac{25}{8}e^2) \cos (\Omega+\alpha+\varpi)$$
$$+\tfrac{7}{2}e(1-\tfrac{291}{56}e^2) \cos (3\Omega+\alpha-\varpi)+\tfrac{17}{2}e^2 \cos (4\Omega+\alpha-2\varpi) \quad . \quad . \quad . \quad (276)$$

Next multiply (274) by $3e(1+\frac{1}{4}e^2)$; (275) by $\frac{3}{2}e^2$; add the two products, and add $1+\frac{3}{2}e^2$, and we find from (267) after reduction,

$$R=1-\tfrac{3}{2}e^2+\tfrac{3}{8}e^4+3e(1-\tfrac{15}{8}e^2) \cos (\Omega-\varpi)+\tfrac{9}{2}e^2 \cos 2(\Omega-\varpi) \quad . \quad . \quad (277)$$

Now let

$$\left.\begin{array}{l}E_1=-\tfrac{1}{2}e(1-\tfrac{25}{8}e^2);\ E_2=1-\tfrac{11}{2}e^2+\tfrac{181}{16}e^4;\ E_3=\tfrac{7}{2}e(1-\tfrac{291}{56}e^2);\ E_4=\tfrac{17}{2}e^2\\ J_0=1-\tfrac{3}{2}e^2+\tfrac{3}{8}e^4;\ J_1=\tfrac{3}{2}e(1-\tfrac{15}{8}e^2);\ J_2=\tfrac{9}{4}e^2\end{array}\right\} \quad (278)$$

And we have

$$\left.\begin{array}{l}\Phi(\alpha)=E_1\cos(\Omega+\alpha+\varpi)+E_2\cos(2\Omega+\alpha)+E_3\cos(3\Omega+\alpha-\varpi)\\ \qquad\qquad +E_4\cos(4\Omega+\alpha-2\varpi)\\ R=J_0+2J_1\cos(\Omega-\varpi)+2J_2\cos 2(\Omega-\varpi)\\ \text{whence}\\ \Psi(\alpha)=J_0\cos\alpha+J_1[\cos(\Omega+\alpha-\varpi)+\cos(\Omega-\alpha-\varpi)]\\ \qquad\qquad +J_2[\cos(2\Omega+\alpha-2\varpi)+\cos(2\Omega-\alpha-2\varpi)]\end{array}\right\} \quad (279)$$

These three expressions are parts of infinite series which only go as far as terms in e^2, but the terms of the orders e^0 and e have their coefficients developed as far as e^4 and e^3 respectively.

Then substituting from (279) for Φ, Ψ, and R their values in the expressions (265), we find

$$\left.\begin{array}{l}X^2-Y^2=P^4[E_1\cos(2\chi-\Omega-\varpi)+E_2\cos(2\chi-2\Omega)+E_3\cos(2\chi-3\Omega+\varpi)\\ \qquad\qquad +E_4\cos(2\chi-4\Omega+2\varpi)]\\ \qquad +2P^2Q^2[J_0\cos 2\chi+J_1\{\cos(2\chi-\Omega+\varpi)+\cos(2\chi+\Omega-\varpi)\}\\ \qquad\qquad +J_2\{\cos(2\chi-2\Omega+2\varpi)+\cos(2\chi+2\Omega-2\varpi)\}]\\ \qquad +Q^4[E_1\cos(2\chi+\Omega+\varpi)+E_2\cos(2\chi+2\Omega)+E_3\cos(2\chi+3\Omega-\varpi)\\ \qquad\qquad +E_4\cos(2\chi+4\Omega-2\varpi)]\\ -2XY=\text{The same, with sines for cosines}\\ YZ=\text{The same as } X^2-Y^2\text{, but with } -P^3Q \text{ for } P^4,\ PQ(P^2-Q^2) \text{ for}\\ \qquad 2P^2Q^2,\ PQ^3 \text{ for } Q^4 \text{ and with } \chi \text{ for } 2\chi\\ XZ=\text{The same as the last, but with sines for cosines}\\ \tfrac{1}{3}(X^2+Y^2-2Z^2)=\tfrac{1}{3}(P^4-4P^2Q^2+Q^4)[J_0+2J_1\cos(\Omega-\varpi)+2J_2\cos 2(\Omega-\varpi)]\\ \qquad +2P^2Q^2[E_1\cos(\Omega+\varpi)+E_2\cos 2\Omega+E_3\cos(3\Omega-\varpi)\\ \qquad\qquad +E_4\cos(4\Omega-2\varpi)]\end{array}\right\} \quad (280)$$

Then if we regard ϖ as constant, and remember that $\chi=nt$, and that Ω stands for $\Omega t+\epsilon$, and if we look through the above functions we see that there are trigonometrical

terms of 22 different speeds, viz.: 9 in the first pair all involving $2nt$, 9 in the second pair all involving nt, and 4 in the last.

Then since these five functions correspond to Diana's tide-generating potential, therefore we are going to consider the effects of 22 different tides, nine being semi-diurnal, nine diurnal, and the last four may be conveniently called monthly, since their periods are $\frac{1}{4}$, $\frac{1}{3}$, $\frac{1}{2}$ of a month and one month.

We next have to form the $\mathfrak{X}$-$\mathfrak{Y}$-$\mathfrak{Z}$ functions. We found that in the X-Y-Z functions there were terms of 22 different speeds; hence we shall now have to introduce 44 symbols indicating the reduction in the height of tide below its equilibrium height, and the retardation of phase. The notation adopted is analogous to that used in the preceding problem, and the following schedule gives the symbols.

Semi-diurnal tides.

speed	$2n-4\Omega$	$2n-3\Omega$	$2n-2\Omega$	$2n-\Omega$	$2n$	$2n+\Omega$	$2n+2\Omega$	$2n+3\Omega$	$2n+4\Omega$
height	F^{iv}	F^{iii}	F^{ii}	F^{i}	F	F_{i}	F_{ii}	F_{iii}	F_{iv}
lag	$2f^{iv}$	$2f^{iii}$	$2f^{ii}$	$2f^{i}$	$2f$	$2f_{i}$	$2f_{ii}$	$2f_{iii}$	$2f_{iv}$

Diurnal tides.

speed	$n-4\Omega$	$n-3\Omega$	$n-2\Omega$	$n-\Omega$	n	$n+\Omega$	$n+2\Omega$	$n+3\Omega$	$n+4\Omega$
height	G^{iv}	G^{iii}	G^{ii}	G^{i}	G	G_{i}	G_{ii}	G_{iii}	G_{iv}
lag	g^{iv}	g^{iii}	g^{ii}	g^{i}	g	g_{i}	g_{ii}	g_{iii}	g_{iv}

*Monthly tides.**

speed	Ω	2Ω	3Ω	4Ω
height	H^{i}	H^{ii}	H^{iii}	H^{iv}
lag	h^{i}	$2h^{ii}$	$3h^{iii}$	$4h^{iv}$

The $\mathfrak{X}$-$\mathfrak{Y}$-$\mathfrak{Z}$ functions might now be easily written out; for each term of the X-Y-Z functions is to be multiplied, according to its *speed* by the corresponding *height*, and the corresponding *lag* subtracted from the argument of the trigonometrical term. For example, the first term of $\mathfrak{X}^2-\mathfrak{Y}^2$ is $F^{i}E_1P^4\cos(2\chi-\Omega-\varpi-2f^{i})$. It will however be unnecessary to write out these long expressions.

In order to form the disturbing function W, the $\mathfrak{X}$-$\mathfrak{Y}$-$\mathfrak{Z}$ functions have now to be multiplied by the X′-Y′-Z′ functions according to the formula (31). Now the X′-Y′-Z′ functions only differ from the X-Y-Z functions in the accentuation of Ω and ϖ, because Diana is to be ultimately identical with the moon.

Then in the $\mathfrak{X}$-$\mathfrak{Y}$-$\mathfrak{Z}$ functions Ω is an abbreviation for $\Omega t+\epsilon$, and in the X′-Y′-Z′ functions Ω' for $\Omega t+\epsilon'$; hence wherever in the products we find $\Omega-\Omega'$, we may replace it by $\epsilon-\epsilon'$.

* With periods of $\frac{1}{4}$, $\frac{1}{3}$, $\frac{1}{2}$, and one month.

Again, since we are only seeking to find the secular changes in the ellipticity and mean distance, therefore (as before pointed out) we need only multiply together terms whose arguments only differ by the lag. Secular *inequalities*, in the sense in which the term is used in the planetary theory, will indeed arise from the cross-multiplication of certain terms of like *speeds* but of different *arguments*,—for example, the product of the term $\mathrm{F}^{\text{ii}}P^4\mathrm{E}_2\cos(2\chi-2\Omega-2\mathrm{f}^{\text{ii}})$ in $\mathfrak{X}^2-\mathfrak{Y}^2$ multiplied by the term $2P^2Q^2\mathrm{J}_2\cos(2\chi-2\Omega'+2\varpi')$ in $\mathrm{X}'^2-\mathrm{Y}'^2$, when added to the similar cross-product in $4\mathrm{X}'\mathrm{Y}'\mathfrak{X}\mathfrak{Y}$ (which only differs in having sines for cosines) will give a term $2\mathrm{F}^{\text{ii}}P^6Q^2\mathrm{E}_2\mathrm{J}_2\cos[2(\epsilon'-\epsilon)-2\varpi'-2\mathrm{f}^{\text{ii}}]$. This term in the disturbing function will give a long inequality, but it is of no present interest.

The products may now be written down without writing out in full either the $\mathfrak{X}$-$\mathfrak{Y}$-$\mathfrak{Z}$ functions or the X′-Y′-Z′ functions. In order that the results may form the constituent terms of W, the factor $\frac{1}{2}$ is introduced in the first pair of products, the factor 2 in the second pair, and the factor $\frac{3}{2}$ in the last. Then from (280) we have

$$
\begin{aligned}
&2\,\frac{\mathrm{X}'^2-\mathrm{Y}'^2}{2}\,\frac{\mathfrak{X}^2-\mathfrak{Y}^2}{2}+2\mathrm{X}'\mathrm{Y}'\mathfrak{X}\mathfrak{Y}\\
&\quad=\tfrac{1}{2}P^8\{\mathrm{F}^{\text{i}}\mathrm{E}_1^2\cos[(\epsilon'-\epsilon)+(\varpi'-\varpi)-2\mathrm{f}^{\text{i}}]+\mathrm{F}^{\text{ii}}\mathrm{E}_2^2\cos[2(\epsilon'-\epsilon)-2\mathrm{f}^{\text{ii}}]\\
&\qquad\quad+\mathrm{F}^{\text{iii}}\mathrm{E}_3^2\cos[3(\epsilon'-\epsilon)-(\varpi'-\varpi)-2\mathrm{f}^{\text{iii}}]+\mathrm{F}^{\text{iv}}\mathrm{E}_4^2\cos[4(\epsilon'-\epsilon)-2(\varpi'-\varpi)-2\mathrm{f}^{\text{iv}}]\}\\
&\quad+2P^4Q^4\{\mathrm{F}\mathrm{J}_0^2\cos 2\mathrm{f}\\
&\qquad\quad+\mathrm{F}^{\text{i}}\mathrm{J}_1^2\cos[(\epsilon'-\epsilon)-(\varpi'-\varpi)-2\mathrm{f}^{\text{i}}]+\mathrm{F}_{\text{i}}\mathrm{J}_1^2\cos[(\epsilon'-\epsilon)-(\varpi'-\varpi)+2\mathrm{f}^{\text{i}}]\\
&\qquad\quad+\mathrm{F}^{\text{ii}}\mathrm{J}_2^2\cos[2(\epsilon'-\epsilon)-2(\varpi'-\varpi)-2\mathrm{f}^{\text{ii}}]+\mathrm{F}_{\text{ii}}\mathrm{J}_2^2\cos[2(\epsilon'-\epsilon)-2(\varpi'-\varpi)+2\mathrm{f}_{\text{ii}}]\}\\
&\quad+\tfrac{1}{2}Q^8\{\mathrm{F}_{\text{i}}\mathrm{E}_1^2\cos[(\epsilon'-\epsilon)+(\varpi'-\varpi)+2\mathrm{f}_{\text{i}}]+\mathrm{F}_{\text{ii}}\mathrm{E}_2^2\cos[2(\epsilon'-\epsilon)+2\mathrm{f}_{\text{ii}}]\\
&\qquad\quad+\mathrm{F}_{\text{iii}}\mathrm{E}_3^2\cos[3(\epsilon'-\epsilon)-(\varpi'-\varpi)+2\mathrm{f}_{\text{iii}}]+\mathrm{F}_{\text{iv}}\mathrm{E}_4^2\cos[4(\epsilon'-\epsilon)-2(\varpi'-\varpi)+2\mathrm{f}_{\text{iv}}]\}
\end{aligned}
\tag{281}
$$

$2\mathrm{Y}'\mathrm{Z}'\mathfrak{Y}\mathfrak{Z}+2\mathrm{X}'\mathrm{Z}'\mathfrak{X}\mathfrak{Z}=$ the same, when $2P^6Q^2$ replaces $\frac{1}{2}P^8$; $2P^2Q^2(P^2-Q^2)^2$ replaces $2P^4Q^4$; $2P^2Q^6$ replaces $\frac{1}{2}Q^8$; and G's and g's replace F's and 2f's . (282)

$$
\begin{aligned}
&\tfrac{3}{2}\,\frac{\mathrm{X}'^2+\mathrm{Y}'^2-2\mathrm{Z}'^2}{3}\,\frac{\mathfrak{X}^2+\mathfrak{Y}^2-2\mathfrak{Z}^2}{3}\\
&\quad=\tfrac{1}{6}(P^4-4P^2Q^2+Q^4)^2\{\mathrm{J}_0^2+2\mathrm{H}^{\text{i}}\mathrm{J}_1^2\cos[(\epsilon'-\epsilon)-(\varpi'-\varpi)+\mathrm{h}^{\text{i}}]\\
&\qquad\qquad\qquad\qquad+2\mathrm{H}^{\text{ii}}\mathrm{J}_2^2\cos[2(\epsilon'-\epsilon)-2(\varpi'-\varpi)+2\mathrm{h}^{\text{ii}}]\}\\
&\quad+3P^4Q^4\{\mathrm{H}^{\text{i}}\mathrm{E}_1^2\cos[(\epsilon'-\epsilon)+(\varpi'-\varpi)+\mathrm{h}^{\text{i}}]+\mathrm{H}^{\text{ii}}\mathrm{E}_2^2\cos[2(\epsilon'-\epsilon)+2\mathrm{h}^{\text{ii}}]\\
&\qquad+\mathrm{H}^{\text{iii}}\mathrm{E}_3^2\cos[3(\epsilon'-\epsilon)-(\varpi'-\varpi)+3\mathrm{h}^{\text{iii}}]+\mathrm{H}^{\text{iv}}\mathrm{E}_4^2\cos[4(\epsilon'-\epsilon)-2(\varpi'-\varpi)+4\mathrm{h}^{\text{iv}}]\}
\end{aligned}
\tag{283}
$$

The sum of these three last expressions (281–3) when multiplied by $\frac{\tau^2}{\mathfrak{g}}\frac{1}{(1-e^2)^6}$ is equal to W the disturbing function.

§ 24. *Secular changes in eccentricity and mean distance.*

Before proceeding to the differentiation of W, it is well to note the following coincidences between the coefficients and arguments, viz. : E_1^2 occurs with $(\epsilon'-\epsilon)+(\varpi'-\varpi)$, E_2^2 with $2(\epsilon'-\epsilon)$, E_3^2 with $3(\epsilon'-\epsilon)-(\varpi'-\varpi)$, E_4^2 with $4(\epsilon'-\epsilon)-2(\varpi'-\varpi)$, J_1^2 with $(\epsilon'-\epsilon)-(\varpi'-\varpi)$, J_2^2 with $2(\epsilon'-\epsilon)-2(\varpi'-\varpi)$, and the terms in J_0^2 do not involve ϵ, ϵ', ϖ, ϖ'. In consequence of these coincidences it will be possible to arrange the results in a highly symmetrical form.

By equations (11) and (12)

$$-\frac{\xi}{k}\frac{d}{dt}\log\eta=\left(\frac{d}{d\epsilon'}+\gamma\frac{d}{d\varpi'}\right)\text{W, when } \gamma=\frac{1}{\eta}$$

and

$$\frac{1}{k}\frac{d\xi}{dt}=\left(\frac{d}{d\epsilon'}+\gamma\frac{d}{d\varpi'}\right)\text{W, when } \gamma=0$$

Hence the single operation $d/d\epsilon'+\gamma d/d\varpi'$ will enable us by proper choice of the value of γ to find either $\xi d\log\eta/kdt$ or $d\xi/kdt$.

Perform this operation; then putting $\epsilon'=\epsilon$, $\varpi'=\varpi$, and collecting the terms according to their respective E's and J's, we have

$$\begin{aligned}\left(\frac{dW}{d\epsilon'}+\gamma\frac{dW}{d\varpi'}\right)&\div\frac{\tau^2}{\mathfrak{g}}\frac{1}{(1-e^2)^6}\\
=&\,E_1^2(1+\gamma)\{\tfrac{1}{2}P^8F^{i}\sin 2f^{i}+2P^6Q^2G^{i}\sin g^{i}-2P^2Q^6G_{i}\sin g_{i}-\tfrac{1}{2}Q^8F_{i}\sin 2f_{i}\\
&\qquad-3P^4Q^4H^{i}\sin h^{i}\}\\
&+E_2^2(2)\{\text{the same with ii for i, and } 2h^{ii} \text{ for } h^{i}\}\\
&+E_3^2(3-\gamma)\{\text{the same with iii for i, and } 3h^{iii} \text{ for } h^{i}\}\\
&+E_4^2(4-2\gamma)\{\text{the same with iv for i, and } 4h^{iv} \text{ for } h^{i}\}\\
&+J_1^2(1-\gamma)\{2P^4Q^4(F^{i}\sin 2f^{i}-F_{i}\sin 2f_{i})+2P^2Q^2(P^2-Q^2)^2(G^{i}\sin g^{i}-G_{i}\sin g_{i})\\
&\qquad-\tfrac{1}{3}(P^4-4P^2Q^2+Q^4)^2H^{i}\sin h^{i}\}\\
&+J_2^2(2-2\gamma)\{\text{the same with ii for i, and } 2h^{ii} \text{ for } h^{i}\}\quad\ldots\ldots\quad(284)\end{aligned}$$

The functions of P and Q, which appear here, will occur hereafter so frequently that it will be convenient to adopt an abridged notation for them. Let x then represent either i, ii, iii or iv, and let

$$\left.\begin{aligned}\phi(\mathrm{x})&=\tfrac{1}{2}P^8\mathrm{F}^{\mathrm{x}}\sin 2\mathrm{f}^{\mathrm{x}}+2P^6Q^2\mathrm{G}^{\mathrm{x}}\sin \mathrm{g}^{\mathrm{x}}-2P^2Q^6\mathrm{G}_{\mathrm{x}}\sin \mathrm{g}_{\mathrm{x}}-\tfrac{1}{2}Q^8\mathrm{F}_{\mathrm{x}}\sin 2\mathrm{f}_{\mathrm{x}}\\&\qquad-3P^4Q^4\mathrm{H}^{\mathrm{x}}\sin(\mathrm{xh}^{\mathrm{x}})\\\psi(\mathrm{x})&=2P^4Q^4(\mathrm{F}^{\mathrm{x}}\sin 2\mathrm{f}^{\mathrm{x}}-\mathrm{F}_{\mathrm{x}}\sin 2\mathrm{f}_{\mathrm{x}})+2P^2Q^2(P^2-Q^2)^2(\mathrm{G}^{\mathrm{x}}\sin \mathrm{g}^{\mathrm{x}}-\mathrm{G}_{\mathrm{x}}\sin \mathrm{g}_{\mathrm{x}})\\&\qquad-\tfrac{1}{3}(P^4-4P^2Q^2+Q^4)^2\mathrm{H}^{\mathrm{x}}\sin(\mathrm{xh}^{\mathrm{x}})\end{aligned}\right\}\quad(285)$$

And the generalised definition of the F's, G's, H's, &c., is contained in the following schedule

$$\left.\begin{array}{llllll}\text{speed} & 2n-\mathrm{x}\Omega, & n-\mathrm{x}\Omega, & \mathrm{x}\Omega, & n+\mathrm{x}\Omega, & 2n+\mathrm{x}\Omega\\ \text{height} & \mathrm{F}^{\mathrm{x}} & \mathrm{G}^{\mathrm{x}} & \mathrm{H}^{\mathrm{x}} & \mathrm{G}_{\mathrm{x}} & \mathrm{F}_{\mathrm{x}}\\ \text{lag} & 2\mathrm{f}^{\mathrm{x}} & \mathrm{g}^{\mathrm{x}} & (\mathrm{xh}^{\mathrm{x}}) & \mathrm{g}_{\mathrm{x}} & 2\mathrm{f}_{\mathrm{x}}\end{array}\right\}\quad .\ .\ .\ (286)$$

We must now substitute for the E's and J's their values, and as the ellipticity is chosen as the variable they must be expressed in terms of η instead of e. Also each of the E^2's and J^2's must be divided by $(1-\mathrm{e}^2)^6$.

Then since $\sqrt{1-\mathrm{e}^2}=1-\eta$, therefore

$$\mathrm{e}^2=2\eta-\eta^2 \text{ and } (1-\mathrm{e}^2)^{-6}=(1-\eta)^{-12}=1+12\eta+78\eta^2$$

Then by (278)

$$\left.\begin{array}{ll}\mathrm{E}_1^2=\tfrac{1}{4}\mathrm{e}^2(1-\tfrac{25}{4}\mathrm{e}^2)=\tfrac{1}{2}\eta(1-13\eta) & \text{, and } \dfrac{\mathrm{E}_1^2}{(1-\eta)^{12}}=\tfrac{1}{2}\eta(1-\eta)\\ \mathrm{E}_2^2=1-11\mathrm{e}^2+\tfrac{423}{8}\mathrm{e}^4=1-22\eta+\tfrac{445}{2}\eta^2, & \text{and } \dfrac{\mathrm{E}_2^2}{(1-\eta)^{12}}=1-10\eta+\tfrac{73}{2}\eta^2\\ \mathrm{E}_3^2=\tfrac{49}{4}\mathrm{e}^2(1-\tfrac{291}{28}\mathrm{e}^2)=\tfrac{49}{2}\eta(1-\tfrac{149}{7}\eta) & \text{, and } \dfrac{\mathrm{E}_3^2}{(1-\eta)^{12}}=\tfrac{49}{2}\eta(1-\tfrac{65}{7}\eta)\\ \mathrm{E}_4^2=\tfrac{289}{4}\mathrm{e}^4=289\eta^2 & \text{, and } \dfrac{\mathrm{E}_4^2}{(1-\eta)^{12}}=289\eta^2\\ \mathrm{J}_0^2=1-3\mathrm{e}^2+3\mathrm{e}^4=1-6\eta+15\eta^2 & \text{, and } \dfrac{\mathrm{J}_0^2}{(1-\eta)^{12}}=1+6\eta+21\eta^2\\ \mathrm{J}_1^2=\tfrac{9}{4}\mathrm{e}^2(1-\tfrac{15}{4}\mathrm{e}^2)=\tfrac{9}{2}\eta(1-8\eta) & \text{, and } \dfrac{\mathrm{J}_1^2}{(1-\eta)^{12}}=\tfrac{9}{2}\eta(1+4\eta)\\ \mathrm{J}_2^2=\tfrac{81}{16}\mathrm{e}^4=\tfrac{81}{4}\eta^2 & \text{, and } \dfrac{\mathrm{J}_2^2}{(1-\eta)^{12}}=\tfrac{81}{4}\eta^2\end{array}\right\}\quad .\ (287)$$

When γ is put equal to $\dfrac{1}{\eta}$ we shall also require the following :—

$$\left.\begin{array}{ll}\dfrac{\mathrm{E}_1^2(1+\eta)}{\eta(1-\eta)^{12}}=\tfrac{1}{2}; & \dfrac{\mathrm{E}_2^2(2)}{(1-\eta)^{12}}=2(1-10\eta);\\ \dfrac{\mathrm{E}_3^2(3\eta-1)}{\eta(1-\eta)^{12}}=-\tfrac{49}{2}(1-\tfrac{86}{7}\eta); & \dfrac{\mathrm{E}_4^2(4\eta-2)}{\eta(1-\eta)^{12}}=-578\eta;\\ \dfrac{\mathrm{J}_1^2(\eta-1)}{\eta(1-\eta)^{12}}=-\tfrac{9}{2}(1+3\eta); & \dfrac{\mathrm{J}_2^2(2\eta-2)}{\eta(1-\eta)^{12}}=-\tfrac{81}{2}\eta\end{array}\right\}\quad .\ .\ .\ (288)$$

Therefore by putting $\gamma=\frac{1}{\eta}$ in equation (284) we have

$$-\frac{\mathfrak{g}}{\tau^2}\frac{\xi}{k}\frac{d}{dt}\log\eta=\tfrac{1}{2}\phi(\text{i})+2(1-10\eta)\phi(\text{ii})-\tfrac{49}{2}(1-\tfrac{86}{7}\eta)\phi(\text{iii})-578\eta\phi(\text{iv})$$
$$-\tfrac{9}{2}(1+3\eta)\psi(\text{i})-\tfrac{81}{2}\eta\psi(\text{ii})$$

and by putting $\gamma=0$ in (284)

$$\frac{\mathfrak{g}}{\tau^2}\frac{1}{k}\frac{d\xi}{dt}=\tfrac{1}{2}\eta(1-\eta)\phi(\text{i})+2(1-10\eta+\tfrac{73}{2}\eta^2)\phi(\text{ii})+\tfrac{147}{2}\eta(1-\tfrac{65}{7}\eta)\phi(\text{iii})+1156\eta^2\phi(\text{iv})$$
$$+\tfrac{9}{2}\eta(1+4\eta)\psi(\text{i})+\tfrac{81}{2}\eta^2\psi(\text{ii})$$

The equations may be also arranged in the following form :—

$$-\frac{\mathfrak{g}}{\tau^2}\frac{\xi}{k}\frac{d}{dt}\log\eta=\tfrac{1}{2}[\phi(\text{i})+4\phi(\text{ii})-49\phi(\text{iii})-9\psi(\text{i})]$$
$$+\eta[-20\phi(\text{ii})+301\phi(\text{iii})-578\phi(\text{iv})-\tfrac{27}{2}\psi(\text{i})-\tfrac{81}{2}\psi(\text{ii})] \quad . \quad (289)$$

$$\frac{\mathfrak{g}}{\tau^2}\frac{1}{k}\frac{d\xi}{dt}=2\phi(\text{ii})$$
$$+\eta[\tfrac{1}{2}\phi(\text{i})-20\phi(\text{ii})+\tfrac{147}{2}\phi(\text{iii})+\tfrac{9}{2}\psi(\text{i})]$$
$$+\eta^2[-\tfrac{1}{2}\phi(\text{i})+73\phi(\text{ii})-\tfrac{1365}{2}\phi(\text{iii})+1156\phi(\text{iv})+18\psi(\text{i})+\tfrac{81}{2}\psi(\text{ii})] \quad . \quad . \quad (290)$$

The former of these apparently stops with the first power of η, but it will be observed that we have $d\log\eta/dt$ on the left-hand side so that $d\eta/dt$ is developed as far as η^2.

These equations give the required solutions of the problem.

§ 25. *Application to the case where the planet is viscous.*

If the planet or earth be viscous, we have, as in § 7, $\text{F}^{\text{x}}=\cos 2\text{f}^{\text{x}}$, $\text{G}^{\text{x}}=\cos\text{g}^{\text{x}}$, $\text{H}^{\text{x}}=\cos(\text{xh}^{\text{x}})$, $\text{G}_{\text{x}}=\cos\text{g}_{\text{x}}$, $\text{F}_{\text{x}}=\cos 2\text{f}_{\text{x}}$.

When these values are substituted in (289) we have the equation giving the rate of change of ellipticity in the case of viscosity. The equation is however so long and complex that it does not present to the mind any physical meaning, and I shall therefore illustrate it graphically.

The case taken is the same as that in § 7, where the planet rotates 15 times as fast as the satellite revolves.

The eccentricity or ellipticity is supposed to be small, so that only the first line of (289) is taken.

I took as five several standards of viscosity of the planet, such viscosities as would

make the lag f^{ii} of the principal slow semi-diurnal tide, of speed $2n-2\Omega$, equal to 10°, 20°, 30°, 40°, 44°. (The curves thus correspond to the same cases as in §§ 7 and 10). Values of $\sin 4f^{x}$, $\sin 2g^{x}$, $\sin 2xh^{x}$, $\sin 2g_{x}$, $\sin 4f_{x}$, when x=i, ii, iii were then computed, according to the theory of viscous tides.

These values were then taken for computing values of ϕ(i), ϕ(ii), ϕ(iii), ψ(i) with values of $i=0°$, 15°, 30°, 45°, 60°, 75°, 90°. The results were then combined so as to give a series of values of $d\log\eta/dt$ or de/edt, and these values were set out graphically in the accompanying fig. 8.

Fig. 8.

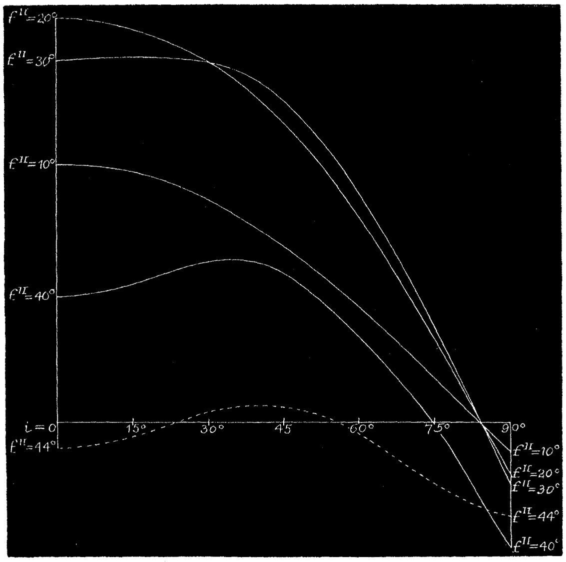


Diagram showing the rate of change in the eccentricity of the orbit of the satellite for various obliquities and viscosities of the planet $\left(\frac{1}{e}\frac{de}{dt}\text{, when e is small}\right)$.

In the figure the ordinates are proportional to de/edt, and the abscissæ to i the obliquity; each curve corresponds to one degree of viscosity.

From the figure we see that, unless the viscosity be so great as to approach rigidity (when $f^{ii}=45°$), the eccentricity will increase for all values of the obliquity, except values approaching 90°.

The rate of increase is greatest for zero obliquity unless the viscosity be very large, and in that case it is a little greater for about 35° of obliquity.

It appears from the paper on "Precession" that if the obliquity be very nearly 90°, the satellite's distance from the planet decreases with the time. Hence it follows from this figure that in general the eccentricity of the orbit increases or diminishes with the mean distance; this is however not true if the viscosity approaches very near rigidity, for then the eccentricity will diminish for zero obliquity, whilst the mean distance will increase.

If the viscosity be very small, the equations (289-90) admit of reduction to very simple forms.

In this case the sines of twice the angles of lagging are proportional to the speeds of the several tides, and we have (as in previous cases)—

$$\frac{\sin 4\mathrm{f}^{\mathrm{x}}}{\sin 4\mathrm{f}}=1-\tfrac{1}{2}\mathrm{x}\lambda,\ \frac{\sin 2\mathrm{g}^{\mathrm{x}}}{\sin 4\mathrm{f}}=\tfrac{1}{2}-\tfrac{1}{2}\mathrm{x}\lambda,\ \frac{\sin 2\mathrm{xh}^{\mathrm{x}}}{\sin 4\mathrm{f}}=\tfrac{1}{2}\mathrm{x}\lambda,\ \frac{\sin 2\mathrm{g}_{\mathrm{x}}}{\sin 4\mathrm{f}_{\mathrm{x}}}=\tfrac{1}{2}+\tfrac{1}{2}\mathrm{x}\lambda,\ \frac{\sin 4\mathrm{f}_{\mathrm{x}}}{\sin 4\mathrm{f}}=1+\tfrac{1}{2}\mathrm{x}\lambda.$$

Therefore

$$\begin{aligned}\phi(\mathrm{x})&=\tfrac{1}{4}\sin 4\mathrm{f}\left[P^8+2P^6Q^2-2P^2Q^6-Q^8-\tfrac{1}{2}\mathrm{x}\lambda(P^8+4P^6Q^2+4P^2Q^6+Q^6+6P^4Q^4)\right]\\&=\tfrac{1}{4}\sin 4\mathrm{f}\,(\cos i-\tfrac{1}{2}\mathrm{x}\lambda)\\ \psi(\mathrm{x})&=\tfrac{1}{2}\sin 4\mathrm{f}\left[-2P^4Q^4\mathrm{x}\lambda-2P^2Q^2(P^2-Q^2)^2\mathrm{x}\lambda-\tfrac{1}{6}\mathrm{x}\lambda(P^4-4P^2Q^2+Q^4)^2\right]\\&=-\tfrac{1}{4}\sin 4\mathrm{f}\,(\tfrac{1}{2}\mathrm{x}\lambda)(\tfrac{2}{3})\end{aligned}$$

And

$$\phi(\mathrm{i})+4\phi(\mathrm{ii})-49\phi(\mathrm{iii})-9\psi(\mathrm{i})=-\sin 4\mathrm{f}(11\cos i-18\lambda)$$

$$-20\phi(\mathrm{ii})+301\phi(\mathrm{iii})-578\phi(\mathrm{iv})-\tfrac{27}{2}\psi(\mathrm{i})-\tfrac{81}{2}\psi(\mathrm{ii})=-\tfrac{1}{4}\sin 4\mathrm{f}(297\cos i-756\lambda)$$

Whence from (289)

$$-\frac{\mathfrak{g}}{\tau^2}\frac{\xi}{k}\frac{d}{dt}\log\eta=-\tfrac{1}{2}\sin 4\mathrm{f}\{11\cos i(1+\tfrac{27}{2}\eta)-18\lambda(1+21\eta)\}$$

or

$$\frac{\xi}{k}\frac{d}{dt}\log\eta=\frac{\tau^2}{\mathfrak{g}}(1+\tfrac{27}{2}\eta)\tfrac{11}{2}\sin 4\mathrm{f}\left\{\cos i-\tfrac{18}{11}\frac{\Omega}{n}(1+\tfrac{15}{2}\eta)\right\}\quad . \quad . \quad . \quad (291)$$

From this we see that, in the case of small viscosity, tidal reaction is in general competent to cause the eccentricity of the orbit of a satellite to increase. But if 18 sidereal days of the planet be greater than 11 sidereal months of the satellite the eccentricity will decrease. Wherefore a circular orbit for the satellite is only dynamically stable provided 18 such days is greater than 11 such months.

Now if we treat the equation (290) for $\frac{d\xi}{dt}$ in the same way, we find—

The first line $=\tfrac{1}{2}\sin 4\mathrm{f}(\cos i-\lambda)$.

The second $=\tfrac{1}{2}\eta\sin 4\mathrm{f}(27\cos i-46\lambda)$.

The third $=\tfrac{1}{2}\eta^2\sin 4\mathrm{f}(273\cos i-697\lambda)$

Therefore

$$\left.\begin{aligned}&\frac{\mathfrak{g}}{\tau^2}\frac{1}{k}\frac{d\xi}{dt}=\tfrac{1}{2}\sin 4\mathrm{f}[(1+27\eta+273\eta^2)\cos i-\lambda(1+46\eta+697\eta^2)]\\ \text{or}\quad&\\ &\frac{1}{k}\frac{d\xi}{dt}=\tfrac{1}{2}\frac{\tau^2}{\mathfrak{g}}(1+27\eta+273\eta^2)\sin 4\mathrm{f}\left[\cos i-\frac{\Omega}{n}(1+19\eta-89\eta^2)\right]\end{aligned}\right\} \quad . \quad . \quad (292)$$

From this it follows that the rate of tidal reaction is greater if the orbit be eccentric than if it be circular. Also for zero obliquity the tidal reaction vanishes when

$$\frac{\Omega}{n}=1-19\eta+450\eta^2$$

Hence if a satellite were to separate from a planet in such a way that, at the moment after separation, its mean motion were equal to the angular velocity of the planet, then if its orbit were eccentric it must fall back into the planet; but if its orbit were circular an infinitesimal disturbance would decide whether it should approach or recede from the planet.*

Now suppose that the viscosity is very large, and that the obliquity is zero.

Then

$$-\frac{\mathfrak{g}}{\tau^2}\frac{\xi}{k}\frac{d}{dt}\log\eta=\tfrac{1}{8}(\sin 4\mathrm{f}^{\mathrm{i}}+4\sin 4\mathrm{f}^{\mathrm{ii}}-49\sin 4\mathrm{f}^{\mathrm{iii}}+6\sin 2\mathrm{h}^{\mathrm{i}})$$

and the sines are reciprocally proportional to the speeds of the tides, from which they take their origin. As to the term in $\sin 2\mathrm{h}^{\mathrm{i}}$, which takes its origin from the elliptic monthly tide, the viscosity must make a close approach to absolute rigidity for this term to be reciprocally proportional to the speed of that tide; for the present, therefore, $\sin 2\mathrm{h}^{\mathrm{i}}$ will be left as it is.

Then the equation becomes, on this hypothesis,

$$-\frac{\mathfrak{g}}{\tau^2}\frac{\xi}{k}\frac{d}{dt}\log\eta=\tfrac{1}{8}\sin 4\mathrm{f}^{\mathrm{ii}}\left[\frac{1-\lambda}{1-\frac{1}{2}\lambda}+4-\frac{49(1-\lambda)}{1-\frac{3}{2}\lambda}\right]+\tfrac{6}{8}\sin 2\mathrm{h}^{\mathrm{i}}$$

$$\frac{\mathfrak{g}}{\tau^2}\frac{\xi}{k}\frac{d}{dt}\log\eta=\tfrac{1}{8}\sin 4\mathrm{f}^{\mathrm{ii}}\frac{44-63\lambda+20\lambda^2}{(1-\frac{1}{2}\lambda)(1-\frac{3}{2}\lambda)}-\tfrac{6}{8}\sin 2\mathrm{h}^{\mathrm{i}}. \quad . \quad . \quad . \quad . \quad . \quad (293)$$

The numerator of the first term on the right is always positive for values of λ less than unity, and the denominator is always positive if λ be less than $\frac{2}{3}$. Hence if the viscosity be not so great but that the last term is small, the eccentricity always increases if λ lies between zero and $\frac{2}{3}$.

* See Appendix (p. 886) for further considerations on this subject.

If however λ be not small, then even though the viscosity be not great enough to approach perfect rigidity, we must have $\sin 2\text{h}^{\text{i}}=2(1-\lambda)\sin 4\text{f}^{\text{ii}}/\lambda$. And of course, by supposing the viscosity great enough, this relation may be fulfilled whatever be λ.

Then our equation becomes

$$\frac{\mathfrak{g}}{\tau^2}\frac{\xi}{k}\frac{d}{dt}\log\eta=-\tfrac{1}{8}\sin 4\text{f}^{\text{ii}}\frac{12-80\lambda+96\lambda^2-29\lambda^3}{\lambda(1-\frac{1}{2}\lambda)(1-\frac{3}{2}\lambda)} \quad . \quad . \quad . \quad . \quad . \quad (294)$$

The numerator on the right-hand side is always positive for values of λ less than unity, and the denominator is positive for values of λ less than $\frac{2}{3}$.

Since

$$\frac{1}{k}\frac{d\xi}{dt}=\tfrac{1}{2}\frac{\tau^2}{\mathfrak{g}}\sin 4\text{f}^{\text{ii}}$$

we have

$$\xi\frac{d}{d\xi}\log\eta=-\tfrac{1}{4}\frac{12-80\lambda+96\lambda^2-29\lambda^3}{\lambda(1-\frac{1}{2}\lambda)(1-\frac{3}{2}\lambda)}$$

From this we see that, for very large viscosity,—

For values of λ between 1 and ·6667, the eccentricity increases per unit increase of ξ, and the rate of increase tends to become infinite when $\lambda=$·6667.

The remarks concerning the physical absurdity of this class of result in § 21 may be repeated in this case.

And for values of λ between ·6667 and 0, the eccentricity diminishes.

A similar treatment of the case of small viscosity shows that—

For values of λ between 1 and ·6111 the eccentricity decreases, and for values of λ between ·6111 and 0 the eccentricity increases.

Thus it is only between $\lambda=$·6111 and ·6667 that the two cases agree.

Hence in the course of evolution of a satellite revolving about a purely viscous planet :—

For small viscosity the orbit will remain circular until 11 months of the satellite are equal to 18 days of the planet, then the eccentricity will increase until this relationship is again fulfilled, when the eccentricity will again diminish.*

And for very large viscosity the orbit will at once become eccentric, and the eccentricity will increase very rapidly until two months of the satellite are equal to three days of the planet. The eccentricity will then diminish until this relationship is again fulfilled, after which the eccentricity will again increase.

We shall consider later which of these views seems the more probable with regard to the history of the moon.

* See "On the Analytical Expressions, &c.," Proc. Roy. Soc., No. 202, 1880.

§ 26. *Secular change in the obliquity and diurnal rotation of the planet, when the satellite moves in an eccentric orbit.*

The method of treating this problem will be the same as that of § 12, to which the reader is referred.

In the complete development of the disturbing function $\chi-\chi'$ would occur wherever the F's and G's occur, but never with the H's.

If we put $\gamma=1$ in (284), we have

$$\frac{dW}{d\epsilon'}+\frac{dW}{d\varpi'}=\frac{2\tau^2}{\mathfrak{g}(1-\eta)^{12}}\Sigma E_x{}^2\phi(x). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (295)$$

Where Σ means summation for i, ii, iii, iv.

This result follows from the fact that in all the E-terms of W, ϵ' and ϖ' enter in the form $l\epsilon'+m\varpi'$, where $l+m=2$.

In the F^x-terms χ' enters in the form $2\chi'$, and is of the opposite sign from $l+m$; in the F_x-terms it enters in the form $2\chi'$, and is of the same sign as $l+m$; in the G^x-terms it enters in the form χ', and is of the opposite sign from $l+m$; in the G_x-terms it enters in the form χ', and is of the same sign as $l+m$.

Hence as far as regards the E-terms of W, we have

$$\text{in the } F^x\text{-terms } \frac{dW}{d\chi'}=-\left(\frac{dW}{d\epsilon'}+\frac{dW}{d\varpi'}\right)$$

$$\text{in the } F_x\text{-terms } \quad = \quad \frac{dW}{d\epsilon'}+\frac{dW}{d\varpi'}$$

$$\text{in the } G^x\text{-terms } \quad =-\tfrac{1}{2}\left(\frac{dW}{d\epsilon'}+\frac{dW}{d\varpi'}\right)$$

$$\text{in the } G_x\text{-terms } \quad = \quad \tfrac{1}{2}\left(\frac{dW}{d\epsilon'}+\frac{dW}{d\varpi'}\right)$$

$$\text{in the H-terms } \quad = \quad 0$$

In the J-terms of W, χ' enters with coefficient 2 in the F^x- and F_x-terms, and with the coefficient 1 in the G^x- and G_x-terms, and is always of the same sign as the corresponding lag.

Hence for the J-terms

$$\frac{dW}{d\chi'}=\Sigma\left(\frac{dW}{df^x}+\frac{dW}{dg^x}\right)$$

Where Σ means summation for the cases where x is zero and both upper and lower i and ii.

From this we have

$$\frac{dn}{dt}=\frac{d\mathrm{W}}{d\chi'}$$

$$=-\frac{\tau^2}{\mathfrak{g}}\frac{1}{(1-\eta)^{12}}[\Sigma \mathrm{E}_x{}^2\{P^8\mathrm{F}^x \sin 2\mathrm{f}^x+2P^6Q^2\mathrm{G}^x \sin \mathrm{g}^x+2P^2Q^6\mathrm{G}_x \sin \mathrm{g}_x+Q^8\mathrm{F}_x \sin 2\mathrm{f}_x\}$$

$$+\mathrm{J}_0{}^2\{4P^4Q^4\mathrm{F} \sin 2\mathrm{f}+2P^2Q^2(P^2-Q^2)^2\mathrm{G} \sin \mathrm{g}\}$$

$$+\Sigma \mathrm{J}_x{}^2\{4P^4Q^4(\mathrm{F}^x \sin 2\mathrm{f}^x+\mathrm{F}_x \sin 2\mathrm{f}_x)+2P^2Q^2(P^2-Q^2)^2(\mathrm{G}^x \sin \mathrm{g}^x+\mathrm{G}_x \sin \mathrm{g}_x)\}] \quad (296)$$

The first Σ being from iv to i, and the last only for ii and i.

This is a partial solution for the tidal friction, and corresponds only to the action of the moon on her own tides; that of the sun on his tides may be obtained by symmetry.

It is easy to see that for the joint effect of the two bodies we have

$$\frac{dn}{dt}=-\frac{2\tau\tau'}{\mathfrak{g}}\frac{1}{(1-\eta)^6(1-\eta')^6}\mathrm{J}_0\mathrm{J}_0{}'\{4P^4Q^4\mathrm{F} \sin 2\mathrm{f}+2P^2Q^2(P^2-Q^2)^2\mathrm{G} \sin \mathrm{g}\} \quad . \quad (297)$$

From (296-7) and (287-8) the complete solution may be collected.

In order to find the secular change in the obliquity, we must consider how ψ' would enter in W.

Now in the development of W, $\Omega' t+\epsilon'$ stands for $\Omega' t+\epsilon'-\psi'$, and ϖ' stands for $\varpi'-\psi'$. Hence from (295)

$$\frac{d\mathrm{W}}{d\psi'}=-\left(\frac{d\mathrm{W}}{d\epsilon'}+\frac{d\mathrm{W}}{d\varpi'}\right)$$

$$=-\frac{2\tau^2}{\mathfrak{g}}\frac{1}{(1-\eta)^{12}}\Sigma \mathrm{E}_x{}^2\phi(\mathrm{x}) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (298)$$

Now by (18)

$$n \sin i\frac{di}{dt}=\frac{d\mathrm{W}}{d\chi'}\cos i-\frac{d\mathrm{W}}{d\psi'}$$

Then substituting for $\frac{d\mathrm{W}}{d\chi'}$ from (296) and for $\frac{d\mathrm{W}}{d\psi'}$ from (298), we find

$$n\frac{di}{dt}=\frac{\tau^2}{\mathfrak{g}}\frac{1}{(1-\eta)^{12}}\{\Sigma \mathrm{E}_x{}^2[P^7Q\mathrm{F}^x \sin 2\mathrm{f}^x+P^5Q(P^2+3Q^2)\mathrm{G}^x \sin \mathrm{g}^x$$

$$-PQ^5(3P^2+Q^2)\mathrm{G}_x \sin \mathrm{g}_x-PQ^7\mathrm{F}_x \sin 2\mathrm{f}_x-3P^3Q^3\mathrm{H}^x \sin (\mathrm{xh}^x)]$$

$$-\mathrm{J}_0{}^2[2P^3Q^3(P^2-Q^2)\mathrm{F} \sin 2\mathrm{f}+PQ(P^2-Q^2)^3\mathrm{G} \sin \mathrm{g}]$$

$$-\Sigma \mathrm{J}_x{}^2[2P^3Q^3(P^2-Q^2)(\mathrm{F}^x \sin 2\mathrm{f}^x+\mathrm{F}_x \sin 2\mathrm{f}_x)$$

$$+PQ(P^2-Q^2)^3(\mathrm{G}^x \sin \mathrm{g}^x+\mathrm{G}_x \sin \mathrm{g}_x)]\} \quad . \quad (299)$$

The first Σ being from iv to i, and the last only for ii and i.

This is only a partial solution, and gives the result of the action of the moon on her own tides; that for the sun on his tides may be obtained by symmetry.

It is easy to see that for the joint effect

$$n\frac{di}{dt}=-\frac{2\tau\tau'}{\mathfrak{g}}\frac{1}{(1-\eta)^6(1-\eta')^6}\mathrm{J}_0\mathrm{J}_0'[2P^3Q^3(P^2-Q^2)\mathrm{F}\sin 2\mathrm{f}+PQ(P^2-Q^2)^3\mathrm{G}\sin\mathrm{g}] \quad . \quad (300)$$

From (299, 300) and (287–8) the complete solution may be collected.

Then if these solutions be applied to the case where the earth is viscous and where the viscosity is small, it will be found after reduction as in previous cases that

$$\begin{aligned}-\frac{dn}{dt}=\frac{\sin 4\mathrm{f}}{2\mathfrak{g}}\Big[&\tau^2(1-\tfrac{1}{2}\sin^2 i)(1+15\eta+\tfrac{195}{2}\eta^2)+\tau'^2(1-\tfrac{1}{2}\sin^2 i)(1+15\eta'+\tfrac{195}{2}\eta'^2)\\&-\tau^2\frac{\Omega}{n}\cos i(1+27\eta+273\eta^2)-\tau'^2\frac{\Omega'}{n}\cos i(1+27\eta'+273\eta'^2)\\&+\tau\tau'\tfrac{1}{2}\sin^2 i(1+3\eta+3\eta'+6\eta^2+9\eta\eta'+6\eta'^2)\Big] \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (301)\end{aligned}$$

$$\begin{aligned}n\frac{di}{dt}=\frac{\sin 4\mathrm{f}}{4\mathfrak{g}}\sin i\cos i\Big[&\tau^2(1+15\eta+\tfrac{195}{2}\eta^2)+\tau'^2(1+15\eta'+\tfrac{195}{2}\eta'^2)\\&-2\tau^2\frac{\Omega}{n}\sec i(1+27\eta+273\eta^2)-2\tau'^2\frac{\Omega'}{n}\sec i(1+27\eta'+273\eta'^2)\\&-\tau\tau'(1+3\eta+3\eta'+6\eta^2+9\eta\eta'+6\eta'^2)\Big] \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (302)\end{aligned}$$

These results give the tidal friction and rate of change of obliquity due both to the sun and moon; η is the ellipticity of the lunar orbit, and η' of the solar (or terrestrial) orbit.

If η and η' be put equal to zero they agree with the results obtained in the paper on "Precession."

§ 27. *Verification of analysis, and effect of evectional tides.*

The analysis of this part of the paper has been long and complex, and therefore a verification is valuable.

The moment of momentum of the orbital motion of the moon and earth round their common centre of inertia is proportional to the square root of the *latus rectum* of the orbit, according to the ordinary theory of elliptic motion. In the present notation this moment of momentum is equal to $\mathrm{C}\xi(1-\eta)/k$. Let us suppose the obliquity of the ecliptic to be zero. Then the whole moment of momentum of the system (supposing only one satellite to exist) is

$$C\left\{\frac{\xi}{k}(1-\eta)+n\right\}$$

Therefore we ought to find, if the analysis has been correctly worked, that

$$\frac{\xi}{k}\frac{d\eta}{dt}=(1-\eta)\frac{1}{k}\frac{d\xi}{dt}+\frac{dn}{dt}$$

This test will be only applied in the case where the viscosity is small, because the analysis is pretty short; but it may also be applied in the general case.

When $i=0$, we have from (292), after multiplying both sides by $1-\eta$,

$$\frac{2}{\sin 4f}\frac{\mathfrak{a}}{\tau^2}(1-\eta)\frac{1}{k}\frac{d\xi}{dt}=1+26\eta+246\eta^2-\lambda(1+45\eta+651\eta^2)$$

And when $i=0$ and $\tau'=0$, from (301)

$$-\frac{2}{\sin 4f}\frac{\mathfrak{g}}{\tau^2}\frac{dn}{dt}=1+15\eta+\tfrac{195}{2}\eta^2-\lambda(1+27\eta+273\eta^2)$$

Hence

$$(1-\eta)\frac{1}{k}\frac{d\xi}{dt}+\frac{dn}{dt}=\tfrac{1}{2}\sin 4f\frac{\tau^2}{\mathfrak{g}}[11\eta(1+\tfrac{27}{2}\eta)-18\lambda\eta(1\cdot+\cdot 21\eta)]$$

$$=\frac{\xi}{k}\frac{d\eta}{dt}\text{ from (291)}$$

Thus the above formulas satisfy the condition of the constancy of the moment of momentum of the system.

The most important lunar inequality after the Equation of the centre is the Evection. The effects of lagging evectional tides may be worked out on the same plan as that pursued above for the Equation of the centre.

I will not give the analysis, but will merely state that, in the case of small viscosity of the earth, the equation for the rate of change of ellipticity, inclusive of the evectional terms, becomes

$$\frac{\xi}{k}\frac{d}{dt}\log\eta=\tfrac{11}{2}(1+\tfrac{27}{2}\eta)\sin 4f\frac{\tau^2}{\mathfrak{g}}\left\{1-\tfrac{18}{11}\frac{\Omega}{n}-\tfrac{675}{352}\left(\frac{\Omega'}{\Omega}\right)^2\right\}$$

where Ω' is the earth's mean motion in its orbit round the sun.

From this we see that, even at the present time, the evectional tides will only reduce the rate of increase of the ellipticity by $\frac{1}{88}$th part of the whole. In the integrations to be carried out in Part VI. this term will sink in importance, and therefore it will be entirely neglected.

The Variation is another lunar inequality of slightly less importance than the Evection; but it may be observed that the Evection was only of any importance because its argument involved the lunar perigee, and its coefficient the eccentricity. Now neither of these conditions are fulfilled in the case of the Variation. Moreover in the retrospective integration the coefficients will degrade far more rapidly than those of the evectional terms, because they will depend on $(\Omega'/\Omega)^4$. Hence the secular effects of the variational tides will not be given, though of course it would be easy to find them if they were required.

VI.

INTEGRATION FOR CHANGES IN THE ECCENTRICITY OF THE ORBIT.

§ 28. *Integration in the case of small viscosity.*

By (291-2), we have approximately

$$\frac{\xi}{k}\frac{d}{dt}\log\eta=\tfrac{11}{2}\sin 4\mathrm{f}\frac{\tau^2}{\mathfrak{g}}(1+\tfrac{27}{2}\eta)[\cos i-\tfrac{18}{11}\lambda]$$

$$\frac{1}{k}\frac{d\xi}{dt}=\tfrac{1}{2}\sin 4\mathrm{f}\frac{\tau^2}{\mathfrak{g}}(1+27\eta)[\cos i-\lambda]$$

Therefore

$$(1+\tfrac{27}{2}\eta)\frac{d}{d\xi}\log\eta=\frac{11}{\xi}\,\frac{1-\frac{18}{11}\lambda\sec i}{1-\lambda\sec i}$$

$$=\frac{11}{\xi}-7\frac{\Omega}{\xi n}\sec i \text{ approximately}$$

The last transformation assumes that λ or Ω/n is small compared with unity; this will be the case in the retrospective integration for a long way back.

Then as a first approximation we have

$$\eta=\eta_0\xi^{11}$$

Therefore

$$\int_1^{\xi}\tfrac{27}{2}\eta d\log\eta=\tfrac{27}{2}(\eta-\eta_0)=-\tfrac{27}{2}\eta_0(1-\xi^{11}) \text{ approximately}$$

And for a second approximation

$$\log_e\left(\frac{\eta}{\eta_0\xi^{11}}\right)=\tfrac{27}{2}\eta_0(1-\xi^{11})-7\Omega_0\int_1^{\xi}\frac{\sec i}{\xi^4 n}d\xi \quad \ldots\ldots \quad (303)$$

The integral in this expression is very small, and therefore to evaluate it we may

assign to i an average value, say I, and neglect the solar tidal friction in assigning a value to n.

Then

$$n=n_0+\frac{1}{k}(1-\xi)$$

Let

$$kn_0+1=\kappa, \text{ so that } n=\frac{1}{k}(\kappa-\xi)$$

Hence the last term in (303) is approximately equal to

$$=-7k\Omega_0 \sec \mathrm{I}\int_1^\xi \frac{d\xi}{\xi^4(\kappa-\xi)}=7k\Omega_0 \sec \mathrm{I}\left[\frac{1}{3\kappa}\left(\frac{1}{\xi^3}-1\right)+\frac{1}{2\kappa^2}\left(\frac{1}{\xi^2}-1\right)+\frac{1}{\kappa^3}\left(\frac{1}{\xi}-1\right)\right]$$
$$-\frac{7k\Omega_0}{\kappa^4}\sec \mathrm{I} \log\left(\frac{\xi n_0}{n}\right)$$

In the last term n has been written for $(\kappa-\xi)/k$.

Now let

$$\mathrm{K}=\left[\frac{1}{3\kappa}\left(\frac{1}{\xi^3}-1\right)+\frac{1}{2\kappa^2}\left(\frac{1}{\xi^2}-1\right)+\frac{1}{\kappa^3}\left(\frac{1}{\xi}-1\right)\right]7k\Omega_0 \sec \mathrm{I}+\tfrac{27}{2}\eta_0(1-\xi^{11})$$

Then

$$\eta=\eta_0\xi^{11}\left(\frac{n}{n_0\xi}\right)^{\frac{7k\Omega_0}{\kappa^4}\sec \mathrm{I}}e^{\mathrm{K}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (304)$$

This formula will now be applied to trace the changes in the eccentricity of the lunar orbit.

The integration will be made over a series of "periods" which cover the same ground as those in the paper on "Precession;" and the numerical results of that paper will be used for assigning the values to n and I.

kn_0 is equal to $1/\mu$ of that paper, and therefore κ is $(1+\mu)/\mu$.

First period of integration.

From $\xi=1$ to ·88.

I is taken as 22°. In "Precession" μ was 4·0074, therefore $kn_0=$·24954 and $\kappa=1$·24954. Also $k\Omega_0=kn_0\Omega_0/n_0$, and $\Omega_0/n_0=1/27$·32.

In computing for § 17 of "Precession" I found at the end of the period $\log n/n_0=$·18971.

Using these values I find

$$\log_{10}\left(\frac{n}{\xi n_0}\right)^{\frac{7k}{\kappa^4}\Omega_0 \sec \mathrm{I}}=\cdot 00692$$

Also

$$\mathrm{K}=\cdot 01980+\tfrac{27}{2}\eta_0(1-\xi^{11})$$

Now e_0, the present eccentricity of the lunar orbit, is ·054908.
Whence

$$\eta_0=1-\sqrt{1-e_0^2}=\cdot 001509$$

And

$$\tfrac{27}{2}\eta_0(1-\xi^{11})=\cdot 015375.$$

Using these values I find

$$\log_{10}\eta=6\cdot 59007-10, \text{ and the first approximation gave } \log_{10}\eta=6\cdot 56788-10$$

Then $\eta=\cdot 00038911$
Whence $e=\cdot 02789$, at the end of the first period of integration.

Second period of integration.

From $\xi=1$ to ·76. I was taken as 18° 45′.
A similar calculation gives

$$\log\left(\frac{n}{n_0\xi}\right)^{\frac{7k\Omega_0}{\kappa^4}\sec I}=\cdot 00817, \text{ the first part of } K=\cdot 06998,\ \tfrac{27}{2}\eta_0(1-\xi^{11})=\cdot 00500$$

Whence

$$\log\eta=5\cdot 31758-10, \text{ and the first approximation gave } \log\eta=5\cdot 27902-10$$

Therefore $\eta=\cdot 000020777$ and $e=\cdot 006446$, at the end of the second period of integration.

Third period of integration.

From $\xi=1$ to 76. I was taken as 16° 13′.
Then a similar calculation gave

$$\log\left(\frac{n}{n_0\xi}\right)^{\frac{7k}{\kappa^4}\Omega_0\sec I}=\cdot 00566, \text{ first part of } K=\cdot 12355,\ \tfrac{27}{2}\eta_0(1-\xi^{11})=\cdot 00027$$

Whence

$$\log\eta=4\cdot 06584-10, \text{ and the first approximation gave } \log\eta=4\cdot 00653-10$$

Therefore $\eta=\cdot 0000011637$, and $e=\cdot 001526$ at the end of the third period of integration.

Fourth period of integration.

The procedure is now changed in the same way, and for the same reason, as in the fourth period of § 17 of "Precession."

Let $N=\frac{n}{n_0}$ (as in that paper). Then the equation of tidal friction is

$$-\frac{dN}{dt}=\tfrac{1}{2}\sin 4\mathrm{f}\frac{\tau^2}{\mathfrak{g}n_0}(1-\lambda)$$

and the equation for the change in η may be written approximately

$$-\frac{d}{dN}\log\eta=\frac{n_0 k}{\xi}\frac{11-18\lambda}{1-\lambda}$$

Since λ or Ω/n is no longer small, this expression will be integrated by quadratures. Using the numerical values given in § 17 of "Precession," I find the following corresponding values.

$N=$	1·000	1·107	1·214	1·321
$\frac{kn_0}{\xi}\frac{11-18\lambda}{1-\lambda}=$	15·469	17·665	19·465	11·994

Then integrating by quadratures with the common difference dN equal to ·107, we find the integral equal to 5·5715.

Whence $\eta=44{\cdot}273\times10^{-10}$, and e$=$·00009411.

The results of the whole integration are given in the following table, of which the first two columns are taken from the paper on "Precession."

TABLE XVI.

Day in m. s. hours and minutes.		Moon's sidereal period in m. s. days.	Eccentricity of lunar orbit.
h.	m.	Days.	
23	56	27·32	·054908
15	28	18·62	·027894
9	55	8·17	·006446
7	49	3·59	·001526
5	55	12 hours	·000094

Beyond this the eccentricity would decrease very little more, because this integration stops where λ is about $\frac{1}{2}$, and the eccentricity ceases to diminish when λ is $\frac{11}{18}$.

The final eccentricity in the above table is only $\frac{1}{580}$th of the initial eccentricity, and the orbit is very nearly circular.

§ 29. *The change of eccentricity when the viscosity is large.*

I shall not integrate the equations in the case where the viscosity is large, because the solution depends so largely on the exact degree of viscosity.

If the viscosity were infinitely large, then in the retrospective integration the eccentricity would be found getting larger and larger and finally would become infinite, when λ is equal to $\frac{2}{3}$. This result is of course physically absurd. If on the other hand the viscosity were large, we might find the eccentricity diminishing, then stationary, and finally increasing until $\lambda=\frac{2}{3}$, after which it would diminish again. Thus by varying the viscosity, supposed always large, we might get considerable diversity of results.

VII.

SUMMARY AND DISCUSSION OF RESULTS.

§ 30. *Explanation of problem.—Summary of Parts I. and II.*

In considering the changes in the orbit of a satellite due to frictional tides, very little interest attaches to those elements of the orbit which are to be specified, in order to assign the position which the satellite would occupy at a given instant of time. We are rather here merely concerned with those elements which contain a description of the nature of the orbit.

These elements are the mean distance, inclination, and eccentricity. Moreover all those inequalities in these three elements, which are periodic in time, whether they fall into the class of "secular" or "periodic" inequalities, have no interest for us, and what we require is to trace their *secular changes.*

Similarly, in the case of the planet we are only concerned to discover the secular changes in the period of its rotation, and in the obliquity of its equator to a fixed plane.

It has unfortunately been found impossible to direct the investigation strictly according to these considerations. Amongst the ignored elements are the longitudes of the nodes of the orbit and equator upon the fixed plane, and it was found in one part of the investigation, viz.: Part III., that secular inequalities (in the ordinary acceptance of the term) had to be taken into consideration both in the five elements which define the nature of the orbit, and the planet's mode of motion, and also in the motion of the two nodes.

In the paper on "Precession" I considered the secular changes in the mean distance of the satellite, and the obliquity and rotation-period of the planet, but the satellite's orbit was there assumed to be circular and confined to the fixed plane. In the present paper the inclination and eccentricity are specially considered, but the introduction of these elements has occasioned a modification of the results attained in the previous

paper. For convenience of diction I shall henceforth speak of the planet as the earth, and of the satellites as the moon and sun; for, as far as regards tides, the sun may be treated as a satellite of the earth. The investigation has been kept as far as possible general, so as to be applicable to any system of tides in the earth; but it has been directed more especially towards the conception of a bodily distortion of the earth's mass, and all the actual applications are made on the hypothesis that the earth is a viscous body. A very slight modification would however make the results applicable to frictional oceanic tides on a rigid nucleus (see § 1 immediately after (15)).

I thought it sufficient to consider the problem as divisible into the two following cases :—

1st. Where the moon's orbit is circular, but inclined to the ecliptic. (Parts I., II., III., IV.)

2nd. Where the orbit is eccentric, but always coincident with the ecliptic. (Parts I., V., VI.)

Now that these problems are solved, it would not be difficult, although laborious, to unite the two investigations into a single one; but the additional interest of the results would hardly repay one for the great labour, and besides this division of the problem makes the formulas considerably shorter, and this conduces to intelligibility.

For the present I only refer to the first of the above problems.

It appears that the problem requires still further subdivision, for the following reasons :—

It is a well-known result of the theory of perturbed elliptic motion, that the orbit of a satellite, revolving about an oblate planet and perturbed by a second satellite, always maintains a constant inclination to a certain plane, which is said to be *proper* to the orbit; the nodes also of the orbit revolve with a uniform motion on that plane, apart from "periodic" inequalities.

If then the moon's proper plane be inclined at a very small angle to the ecliptic, the nodes revolve very nearly uniformly on the ecliptic, and the orbit is inclined at very nearly a constant angle thereto. In this case the equinoctial line revolves also nearly uniformly, and the equator is inclined at nearly a constant angle to the ecliptic.

Here then any inequalities in the motion of the earth and moon, which depend on the longitudes of the nodes or of the equinoctial line, are harmonically periodic in time (although they are "secular inequalities"), and cannot lead to any cumulative effects which will alter the elements of the earth or moon.

Again, suppose that the moon and earth are the only bodies in existence. Here the axis of resultant moment of momentum of the system, or the normal to the invariable plane, remains fixed in space. The component moments of momentum are those of the earth's rotation, and of the moon's and earth's orbital revolution round their common centre of inertia. Hence the earth's axis and the normal to the lunar orbit must always be coplanar with the normal to the invariable plane, and therefore the orbit and equator must have a common node on the invariable plane. This node

revolves with a uniform precessional motion, and (so long as the earth is rigid) the inclinations of the orbit and equator to the invariable plane remain constant.

Here also inequalities, which depend on the longitude of the common node, are harmonically periodic in time, and can lead to no cumulative effects.

But if the lunar proper plane be not inclined at a small angle to the ecliptic, the nodes of the orbit may either revolve with much irregularity, or may oscillate about a mean position* on the ecliptic. In this case the inclinations of the orbit and equator to the ecliptic may oscillate considerably.

Here then inequalities, which depend on the longitudes of the node and of the equinoctial line, are not simply periodic in time, and may and will lead to cumulative effects.

This explains what was stated above, namely, that we cannot entirely ignore the motion of the two nodes.

Our problem is thus divisible into three cases :—

(i.) Where the nodes revolve uniformly on the ecliptic, and where there is a second disturbing satellite, viz. : the sun.

(ii.) Where the earth and moon are the only two bodies in existence.

(iii.) Where the nodes either oscillate, or do not revolve uniformly.

The cases (i.) and (ii.) are distinguished by our being able to ignore the nodes. They afford the subject matter for the whole of Part II.

It is proved in § 5 that the tides raised by any one satellite can produce directly no secular change in the mean distance of any other satellite. This is true for all three of the above cases.

It is also shown that, in cases (i.) and (ii.), the tides raised by any one satellite can produce directly no secular change in the inclination of the orbit of any other satellite to the plane of reference. This is not true for case (iii.).

The change of inclination of the moon's orbit in case (i.) is considered in § 6. The equation expressive of the rate of change of inclination is given in (61) and (62). In § 7 this is applied in the case where the earth is viscous. Fig. 4 illustrates the physical meaning of the equation, and the reader is referred to § 7 for an explanation of the figure. From this figure we learn that the effect of the frictional tides is in general to diminish the inclination of the lunar orbit to the ecliptic, unless the obliquity of the ecliptic be large, when the inclination will increase. The curves also show that for moderate viscosities the rate of decrease of inclination is most rapid when the obliquity of the ecliptic is zero, but for larger viscosities the rate of decrease has a maximum value, when the obliquity is between 30° and 40°.

If the viscosity be small the equation for the rate of decrease of inclination is reducible to a very simple form ; this is given in (64) § 7.

In §§ 8, 9, is found the law of increase of the square root of the moon's distance from the earth under the influence of tidal reaction. The law differs but little from

* It is true that this mean position will itself have a slow precessional motion.

that found and discussed in the paper on "Precession," where the plane of the lunar orbit was supposed to be coincident with the ecliptic. If the viscosity be small the equation reduces to a very simple form; this is given in (70). In § 10 I pass to case (ii.), where the earth and moon are the only bodies. The equation expressive of the rate of change of inclination of the lunar orbit to the invariable plane is given in (71). Fig. 5 illustrates the physical meaning of the equation, and an explanation of it is given in § 10. From it we learn that the effect of the tides is always to cause a diminution of the inclination—at least so long as the periodic time of the satellite, as measured in rotations of the planet, is pretty long. The following considerations show that this must generally be the case. It appears from the paper on "Precession" that the effect of tidal friction is to cause a continual transference of moment of momentum from that of terrestrial rotation to that of orbital motion; hence it follows that the normal to the lunar orbit must continually approach the normal to the invariable plane. It is true that the rate of this approach will be to some extent counteracted by a parallel increase in the inclination of the earth's axis to the same normal. It will appear later that if the moon were to revolve very rapidly round the earth, and if the viscosity of the earth were great, then this counteracting influence might be sufficiently great to cause the inclination to increase.* This possible increase of inclination is not exhibited in fig. 5, because it illustrates the case where the sidereal month is 15 days long.

In § 11 it is shown that, for case (ii.), the rate of variation of the mean distance, obliquity, and terrestrial rotation follow the laws investigated in "Precession," but that the angle, there called the obliquity of the ecliptic, must be interpreted as the angle between the plane of the lunar orbit and the equator.

In § 12 I return again to case (i.) and find the laws governing the rate of increase of the obliquity of the ecliptic, and of decrease of the diurnal rotation of the earth. The results differ so little from those discussed in "Precession" that they need not be further referred to here.

Up to this point no approximation has been admitted with regard to smallness either in the obliquity or the inclination of the orbit, but mathematical difficulties have rendered it expedient to assume their smallness in the following part of the paper.

§ 31. *Summary of Part III.*

Part III. is devoted to case (iii.) of our first problem. It was found necessary in the first instance to consider the theory of the secular inequalities in the motion of a moon revolving about an oblate rigid earth, and perturbed by a second satellite, the sun. The sun being large and distant, the ecliptic is deemed sensibly unaffected, and is taken as the fixed plane of reference.

The proper plane of the lunar orbit has been already referred to, but I was here led

* See the abstract of this paper, Proc. R.S., No. 200, 1879, for certain general considerations bearing on this case.

to introduce a new conception, viz.: that of a second proper plane to which the motion of the earth is referred. It is proved that the motion of the system may then be defined as follows:—

The two proper planes intersect one another on the ecliptic, and their common node regredes on the ecliptic with a slow precessional motion. The lunar orbit and the equator are respectively inclined at constant angles to their proper planes, and their nodes on their respective planes also regrede uniformly and at the same speed. The motions are timed in such a way that when the inclination of the orbit to the ecliptic is at the maximum, the obliquity of the equator to the ecliptic is at the minimum, and *vice versâ*.

Now let us call the angular velocity with which the nodes of the orbit would regrede on the ecliptic, if the earth were spherical, *the nodal velocity.*

And let us call the angular velocity with which the common node of the orbit and equator would regrede on the invariable plane of the system, if the sun did not exist, *the precessional velocity.*

If the various obliquities and inclinations be not large, the precessional velocity is in fact the purely lunar precession.

Then if the nodal velocity be large compared with the precessional velocity, the lunar proper plane is inclined at a small angle to the ecliptic, and the equator is inclined at a small angle to the earth's proper plane.

This is the case with the earth, moon, and sun at present, because the nodal period is about $18\frac{1}{2}$ years, and the purely lunar precession would have a period of between 20,000 and 30,000 years. It is not usual to speak of a proper plane of the earth, because it is more simple to conceive a mean equator, about which the true equator nutates with a period of about $18\frac{1}{2}$ years.

Here the precessional motion of the two proper planes is the whole luni-solar precession, and the regression of the nodes on the proper planes is practically the same as the regression of the lunar nodes on the ecliptic.

A comparison of my result with the formula ordinarily given will be found at the end of § 13, and in a note to § 18.

Secondly, if the nodal velocity be small compared with the precessional velocity, the lunar proper plane is inclined at a small angle to the earth's proper plane.

Also the inclination of the equator to the earth's proper plane bears very nearly the same ratio to the inclination of the orbit to the moon's proper plane as the orbital moment of momentum of the two bodies bears to that of the rotation of the earth.

In the planets of the solar system, on account of the immense mass of the sun, the nodal velocity is never small compared with the precessional velocity, unless the satellite moves with a very short periodic time round its planet, or unless the satellite be very small; and if either of these be the case the ratio of the two moments of momentum is small.

Hence it follows that in our system, if the nodal velocity be small compared with

the precessional velocity, the proper plane of the satellite is inclined at a small angle to the equator of the planet. The rapidity of motion of the satellites of Mars, Jupiter, and of some of the satellites of Saturn, and their smallness compared with their planets, necessitates that their proper planes should be inclined at small angles to the equators of the planets. A system may, however, be conceived in which the two proper planes are inclined at a small angle to one another, but where the satellite's proper plane is not inclined at a small angle to the planet's equator.

In the case now before us the regression of the common node of the two proper planes is a sort of compound solar precession of the planet with its attendant moon, and the regression of the two nodes on their respective proper planes is very nearly the same as the purely lunar precession on the invariable plane of the system. Thus there are two precessions, the first of the system as a whole, and the second going on within the system, almost as though the external precession did not exist.

If the nodal velocity be of nearly equal speed with the precessional velocity, the regression of the proper planes and of the nodes on those planes are each a compound phenomenon, which it is rather hard to disentangle without the aid of analysis. Here none of the angles are necessarily small.

It appears from the investigation in "Precession" that the effect of tidal friction is that, on tracing the changes of the system backwards in time, we find the moon getting nearer and nearer to the earth. The result of this is that the ratio of the nodal velocity to the precessional velocity continually diminishes retrospectively; it is initially very large, it decreases, then becomes equal to unity, and finally is very small. Hence it follows that a retrospective solution will show us the lunar proper plane departing from its present close proximity to the ecliptic, and gradually passing over until it becomes inclined at a small angle to the earth's proper plane.

Therefore the problem, involved in the history of the obliquity of the ecliptic and in the inclination of the lunar orbit, is to trace the secular changes in the pair of proper planes, and in the inclinations of the orbit and equator to their respective proper planes.

The four angles involved in this system are however so inter-related, that it is only necessary to consider the inclination of one proper plane to the ecliptic, and of one plane of motion to its proper plane, and afterwards the other two may be deduced. I chose as the two, whose motions were to be traced, the inclination of the lunar orbit to its proper plane, and the inclination of the earth's proper plane to the ecliptic; and afterwards deduced the inclination of the moon's proper plane to the ecliptic, and the inclination of the equator to the earth's proper plane.

The next subject to be considered (§ 14 to end of Part III.) was the rate of change of these two inclinations, when both moon and sun raise frictional tides in the earth. The change takes place from two sets of causes:—

First because of the secular changes in the moon's distance and periodic time, and in the earth's rotation and ellipticity of figure—for the earth must always remain a figure of equilibrium.

The nodal velocity varies directly as the moon's periodic time, and it will decrease as we look backwards in time.

The precessional velocity varies directly as the ellipticity of the earth's figure (the earth being homogeneous) and inversely as the cube of the moon's distance, and inversely as the earth's diurnal rotation; it will therefore increase retrospectively. The ratio of these two velocities is the quantity on which the position of the proper planes principally depends.

The *second* cause of disturbance is due directly to the tidal interaction of the three bodies.

The most prominent result of this interaction is, that the inclination of the lunar orbit to its proper plane in general diminishes as the time increases, or increases retrospectively. This statement may be compared with the results of Part II., where the ecliptic was in effect the proper plane. The retrospective increase of inclination may be reversed however, under special conditions of tidal disturbance and lunar periodic time.

Also the inclination of the earth's proper plane to the ecliptic in general increases with the time, or diminishes retrospectively. This is exemplified by the results of the paper on "Precession," where the obliquity of the ecliptic was found to diminish retrospectively. This retrospective decrease may be reversed under special conditions.

It is in determining the effects of this second set of causes, that we have to take account of the effects of tidal disturbance on the motions of the nodes of the orbit and equator on the ecliptic.

After a long analytical investigation, equations are found in (224), which give the rate of change of the positions of the proper planes, and of the inclinations thereto.

It is interesting to note how these equations degrade into those of case (i.) when the nodal velocity is very large compared with the precessional velocity, and into those of case (ii.) when the same ratio is very small.

In order completely to define the rate of change of the configuration of the system, there are two other equations, one of which gives the rate of increase of the square root of the moon's distance (which I called in a previous paper the equation of tidal reaction), and the other gives the rate of retardation of the earth's diurnal rotation (which I called before the equation of tidal friction). For the latter of these we may however substitute another equation, in which the time is not involved, and which gives a relationship between the diurnal rotation and the square root of the moon's distance. It is in fact the equation of conservation of moment of momentum of the moon-earth system, as modified by the solar tidal friction. This is the equation which was extensively used in the paper on "Precession."

Except for the solar tidal friction and for the obliquity of the orbit and equator, this equation would be rigorously independent of the kind of frictional tides existing in the earth. If the obliquities are taken as small, they do not enter in the equation, and in the present case the degree of viscosity of the earth only enters to an imperceptible

degree, at least when the day is not very nearly equal to the sidereal month. When that relation between the day and month is very nearly fulfilled, the equation may become largely affected by the viscosity; and I shall return to this point later, while for the present I shall assume the equation to give satisfactory results.

This equation of conservation of moment of momentum enables us to compute as many parallel values of the day and month as may be desired.

Now we have got the time-rates of change of the inclinations of the lunar orbit to its proper plane, and of the earth's proper plane to the ecliptic, and we have also the time-rate of change of the square root of the moon's distance. Hence we may obtain the square-root-of-moon's-distance-rate (or shortly the distance-rate) of change of the two inclinations.

The element of time is thus entirely eliminated; and as the period of time required for the changes has been adequately considered in the paper on "Precession," no further reference will here be made to time.

In a precisely similar manner the equations giving the time-rate in the cases (i.) and (ii.) of our first problem, may be replaced by equations of distance-rate.

Up to this point terrestrial phraseology has been used, but there is nothing which confines the applicability of the results to our own planet and satellite.

§ 32. *Summary of Part IV.*

We now, however, pass to Part IV., which contains a retrospective integration of the differential equations, with special reference to the earth, moon, and sun. The mathematical difficulties were so great that a numerical solution was the only one found practicable.* The computations made for the paper on "Precession" were used as far as possible.

The general plan followed was closely similar to that of the previous paper, and consists in arbitrarily choosing a number of values for the distance of the moon from the earth (or what amounts to the same thing for the sidereal month), and then computing all the other elements of the system by the method of quadratures.

The first case considered is where the earth has a small viscosity. And here it may be remarked that although the solution is only rigorous for infinitely small viscosity, yet it gives results which are very nearly true over a considerable range of viscosity. This may be seen to be true by a comparison of the results of the integrations in §§ 15 and 17 of "Precession," in the first of which the viscosity was not at all small; also by observing that the curves in fig. 2 of "Precession" do not differ materially from the curve of sines until ϵ (the f of this paper) is greater than 25°; also by noting a similar peculiarity in figs. 4 and 5 of this paper. The hypothesis of large viscosity does not cover nearly so wide a field.

* An analytical solution in the case of a single satellite, where the viscosity of the planet is small, is given in Proc. Roy. Soc., No. 202, 1880.

That which we here call a small viscosity is, when estimated by terrestrial standards, very great (see the summary of "Precession").

To return, however, to the case in hand:—We begin with the present configuration of the three bodies, when the moon's proper plane is almost identical with the ecliptic, and when the inclination of the equator to its proper plane is very small. This is the case (i.) of the first problem:—

It appears that the solution of "Precession" is sufficiently accurate for this stage of the solution, and accordingly the parallel values of the day, month, and obliquity of the earth's proper plane (or mean equator) are taken from § 17 of that paper; but the change in the new element, the inclination of the lunar orbit, has to be computed.

The results of the solution are given in Table I., § 18, to which the reader is referred.

This method of solution is not applicable unless the lunar proper plane is inclined at a small angle to the ecliptic, and unless the equator is inclined at a small angle to its proper plane. Now at the beginning of the integration, that is to say with a homogenous earth, and with the moon and sun in their present configuration, the moon's proper plane is inclined to the ecliptic at 13″, and the equator is inclined to the earth's proper plane at 12″ (for the heterogeneous earth these angles are about 8″·3 and 9″·0); and at the end of this integration, when the day is 9 hrs. 55 m. and the month 8·17 m. s. days, the former angle has increased to 57′ 31″, and the latter to 22′ 42″. These last results show that the nutations of the system have already become considerable, and although subsequent considerations show that this method of solution has not been overstrained, yet it here becomes advisable to carry out the solution into the more remote past by the methods of Part III.

It was desirable to postpone the transition as long as possible, because the method used up to this point does not postulate the smallness of the inclinations, whereas the subsequent procedure does make that supposition.

In § 19 the solution is continued by the new method, the viscosity of the earth still being supposed to be small. After laborious computations results are obtained, the physical meaning of which is embodied in Table VIII. The last two columns give the periods of the two precessional motions by which the system is affected. The precession of the pair of proper planes is, as it were, the ancestor of the actual luni-solar precession, and the revolution of the two nodes on their proper planes is the ancestor of the present revolution of the lunar nodes on the ecliptic, and of the 19-yearly nutation of the earth's axis.

This table exhibits a continued approach of the two proper planes to one another, so that at the point where the integration is stopped they are only separated by 1° 18′; at the present time they are of course separated by 23° 28′.

The most remarkable feature in this table is that (speaking retrospectively) the inclination of the lunar orbit to its proper plane first increases, then diminishes, and then increases again.

If it were desired to carry the solution still further back, we might without much

error here make the transition to the method of case (ii.) of the first problem, and neglecting the solar influence entirely, refer the motion to the invariable plane of the moon-earth system. This invariable plane would have to be taken as somewhere between the two proper planes, and therefore inclined to the ecliptic at about $11^\circ\ 45'$; the invariable plane would then really continue to have a precessional motion due to the solar influence on the system formed by the earth and moon together, but this would not much affect the treatment of the plane as though it were fixed in space.

We should then have to take the obliquity of the equator to the invariable plane as about 3°, and the inclination of the lunar orbit to the same plane as about $5^\circ\ 30'$.

In the more remote past the obliquity of the equator to the invariable plane would go on diminishing, but at a slower and slower rate, until the moon's period is 12 hours and the day is 6 hours, when it would no longer diminish; and the inclination of the orbit to the invariable plane would go on increasing, until the day and month come to an identity, and at an ever increasing rate.

It follows from this, that if we continued to trace the changes backwards, until the day and month are identical, we should find the lunar orbit inclined at a considerable angle to the equator. If this were necessarily the case, it would be difficult to believe that the moon is a portion of the primeval planet detached by rapid rotation, or by other causes. But the previous results are based on the hypothesis that the viscosity of the earth is small, and it therefore now became important to consider how a different hypothesis concerning the constitution of the earth might modify the results.

In § 20 the solution of the problem is resumed, at the point where the methods of Part III. were first applied, but with the hypothesis that the viscosity of the earth is very large, instead of very small. The results for any intermediate degree of viscosity must certainly lie between those found before and those to be found now.

Then having retraversed the same ground, but with the new hypothesis, I found the results given in Table XV.

The inclinations of the two proper planes to the ecliptic are found to be very nearly the same as in the case of small viscosity. But the inclination of the lunar orbit to its proper plane increases at first and then continues diminishing, without the subsequent reversal of motion found in the previous solution.

If the solution were carried back into the more remote past, the motion being referred to the invariable plane, we should find both the obliquity of the equator and the inclination of the orbit diminishing at a rate which tends to become *infinite*, if the viscosity is *infinitely* great. Infinite viscosity is of course the same as perfect rigidity, and if the earth were perfectly rigid the system would not change at all. The true interpretation to put on this result is that the rate of change of inclination becomes large, if the viscosity be large. This diminution would continue until the day was 6 hours and the month 12 hours. For an analysis of the state of things further back than this, the reader is referred to § 20.

From this it follows, that by supposing the viscosity large enough we may make the obliquity and inclination to the invariable plane as small as we please, by the time that state is reached in which the month is equal to twice the day.

Hence, on the present hypothesis, we trace the system back until the lunar orbit is sensibly coincident with the equator, and the equator is inclined to the ecliptic at an angle of 11° or 12°.

It is probable that in the still more remote past the plane of the lunar orbit would not have a tendency to depart from that of the equator. It is not, however, expedient to attempt any detailed analysis of the changes further back, for the following reason. Suppose a system to be unstable, and that some infinitesimal disturbance causes the equilibrium to break down; then after some time it is moving in a certain way. Now suppose that from a knowledge of the system we endeavour to compute backwards from the observed mode of its motion at that time, and so find the condition from which the observed state of motion originated. Then our solution will carry us back to a state very near to that of instability, from which the system really departed, but as the calculation can take no account of the infinitesimal disturbance, which caused the equilibrium to break down, it can never bring us back to the state which the system really had. And if we go on computing the preceding state of affairs, the solution will continue to lead us further and further astray from the truth. Now this, I take it, is likely to have been the case with the earth and moon; at a certain period in the evolution (viz.: when the month was twice the day) the system probably became dynamically unstable, and the equilibrium broke down. Thus it seems more likely that we have got to the truth, if we cease the solution at the point where the lunar orbit is nearly coincident with the equator, than by going still further back.

In § 21, fig. 7, is given a graphical illustration of the distance-rate of change in the inclinations of the lunar orbit to its proper plane, and of the earth's proper plane to the ecliptic; the dotted curves refer to the hypothesis of large viscosity, and the firm-curves to that of small viscosity.

The figure is explained and discussed in that section; I will here only draw attention to the wideness apart of the two curves illustrative of the rate of change of the inclination of the lunar orbit. This shows how much influence the degree of viscosity of the earth must have had on the present inclination of the lunar orbit to the ecliptic.

It is particularly interesting to observe that in the case of small viscosity this curve rises above the horizontal axis. If this figure is to be interpreted retrospectively, along with our solution, it must be read from left to right, but if we go with the time, instead of against it, from right to left.

Now if the earth had had in its earlier history infinitely small viscosity, and if the moon had moved primitively in the equator, then until the evolution had reached the point represented by P, the lunar orbit would have always remained sensibly coincident with its proper plane. Then in passing from P to Q the inclination of the

orbit to its proper plane would have increased, but the whole increase could not have amounted to more than a few minutes of arc. At the point P the day is 7 hrs. 47 m. in length, and the month 3·25 m. s. days in length; at the point Q the day is 8 hrs. 36 m., and the month 5·20 m. s. days. From Q down to the present state this small inclination would have always decreased.

If then the earth had had small viscosity throughout its evolution, the lunar orbit would at present be only inclined at a very small angle to the ecliptic. But it is actually inclined at about $5° 9'$, hence it follows that while the hypothesis of small viscosity is competent to explain *some* inclination, it cannot explain the actually existing inclination.

It was shown in the papers on "Tides" and "Precession" that, if the earth be not at present perfectly rigid or perfectly elastic, its viscosity must be very large. And it was shown in "Precession" that if the viscosity be large, the obliquity of the ecliptic must at present be decreasing. Now it will be observed that in resuming the integration with the hypothesis of large viscosity, the solution of the first method with the hypothesis of small viscosity was accepted as the basis for continuing the integration with large viscosity. This appears at first sight somewhat illogical, and to be strictly correct, we ought to have taken as the initial inclination of the earth's proper plane to the ecliptic, at the beginning of the application of the methods of Part III. to the hypothesis of large viscosity, some angle probably a little less than $23\frac{1}{2}°$* instead of 17°. This would certainly disturb the results, but I have not thought it advisable to take this course for the following reasons.

It is probable that at the present time the greater part, if not the whole of the tidal friction is due to oceanic tides, and not to bodily tides. If the ocean were frictionless, it would be low tide under the moon; consequently the effects of fluid friction must be to accelerate, not retard, the ocean tides.† Then in order to apply our present analysis to the case of oceanic tidal friction, that angle which has been called the lag of the tide must be interpreted as the acceleration of the tide.

We know that the actual friction in water is small, and hence the tides of long period will be less affected by friction than those of short period; thus the effects of fluid tidal friction will probably be closely analogous to those resulting from the hypothesis of small viscosity of the whole earth and bodily tides. On the other hand, it is probable that the earth was once more plastic than at present, either superficially or throughout its mass, and therefore it seems probable that the bodily tides, even if small at present, were once more considerable. I think therefore that on the whole

* In the present configuration of the earth, moon, and sun, the obliquity will decrease, if the viscosity be very large. But if we integrate backwards this retrospective increase of obliquity would soon be converted into a decrease. Thus at the end of "the first period of integration," the obliquity would be a little greater than $23\frac{1}{2}°$, but by the end of the "second period" it would probably be a little less than $23\frac{1}{2}°$. It is at the end of the "second period" that the method of Part III. is first applied.

† Otherwise the lunar attraction on the tides would accelerate the earth's rotation—a clear violation of the principles of energy.

we shall be more nearly correct in supposing that the terrestrial nucleus possessed a high degree of stiffness in the earliest times, and that it will be best to apply the hypothesis of small viscosity to the more modern stages of the evolution, and that of large viscosity to the more ancient.

At any rate this appears to be a not improbable theory, and one which accords very well with the present values of the obliquity of the ecliptic, and of the inclination of the lunar orbit.

§ 33. *On the initial condition of the earth and moon.*

It was remarked above that the equation of conservation of moment of momentum, as modified by the effects of solar tidal friction, could only be regarded as practically independent of the degree of viscosity of the earth, so long as the moon's sidereal period was not nearly equal to the day; and that if this relationship were nearly satisfied, the equation which we have used throughout might be considerably in error.

Now in the paper on "Precession" the system was traced backwards, in much the same way as has been done here, until the moon's tide-generating influence was very large compared with that of the sun; the solar influence was then entirely neglected, and the equation of conservation of moment of momentum was used for determining that initial condition, where the month and day were identical, from which the system started its course of development.* The period of revolution of the system in its initial configuration was found to be about $5\frac{1}{2}$ hours. I now however see reason to believe that the solar tidal friction will make the numerical value assigned to this period of revolution considerably in error, whilst the general principle remains almost unaffected. This subject is considered in § 22.

The necessity of correction arises from the assumption that because the moon is retrospectively getting nearer and nearer to the earth, therefore the effects of lunar tidal friction must more and more preponderate over those of solar tidal friction, so that if the solar tidal friction were once negligeable it would always remain so. But tidal friction depends on two elements, viz.: the magnitude of the tide-generating influence, and the relative motion of the two bodies. Now whilst the tide-generating influence of the moon *does* become larger and larger, as we approach the critical state, yet the relative motion of the moon and earth becomes smaller and smaller; on the other hand the tide-generating influence of the sun remains sensibly constant, whilst the relative motion of the earth and sun slightly increases.†

From this it follows that the solar tidal friction must ultimately become actually more important than the lunar, notwithstanding the close proximity of the moon to the earth.

* See also a paper on "The Determination of the Secular Effects of Tidal Friction by a Graphical Method," Proc. Roy. Soc., No. 197, 1879.

† In the paper on "Precession" it was stated in § 18 that this must be the case, but I did not at that time perceive the importance of this consideration

The complete investigation of this subject involves considerations which will require special treatment. In § 22 it is only so far considered as to show that, when there is identity of the periods of revolution of the moon and earth, the angular velocity of the system must be much greater than that given by the solution in § 18 of "Precession."

When the earth rotates in $5\frac{1}{2}$ hours, the motion of the moon relatively to the earth's surface would already be pretty slow. If the system were traced into the more remote past, the earth's rotation would be found getting more and more rapid, and the moon's orbital angular velocity also continually increasing, but ever approximating to identity with the earth's rotation.

When the surfaces of the two bodies are almost in contact, the motion of the moon relatively to the earth's surface would be almost insensible. This appears to point to the break-up of the primeval planet into two parts, in consequence of a rotation so rapid as to be inconsistent with an ellipsoidal form of equilibrium.

Is it then a mere coincidence that the shortest period of revolution, with which a spheroid of the same mean density as the earth could subsist in the ellipsoidal form, is 2 hrs. 24 m.; whilst if KEPLER'S law were to hold true, and if the moon were to revolve round the earth in the same period, the surfaces of the two bodies would just graze one another?

§ 34. *Summary of Parts V. and VI.*

I now come to the second of the two problems, where the moon moves in an eccentric orbit, always coincident with the ecliptic.

In § 23 it is shown that the tides raised by any one satellite can produce no secular change in the eccentricity of the orbit of any other satellite; thus the eccentricity and the mean distance are in this respect on the same footing.

It was found to be more convenient to consider the ellipticity of the orbit instead of the eccentricity. In § 24 (289) and (290), are given the time-rates of increase of the ellipticity and of the square root of mean distance. In § 25 the result for the ellipticity is applied to the case where the earth is viscous, and its physical meaning is graphically illustrated in fig. 8.

This figure shows that in general the ellipticity will increase with the time; but if the obliquity of the ecliptic be nearly 90°, or if the viscosity be so great that the earth is very nearly rigid, the ellipticity will diminish. This last result is due to the rising into prominence of the effects of the elliptic monthly tide.

If the viscosity be very small the equation is reducible to a very simple form, which is given in (291). From (291) we see that if the obliquity of the ecliptic be zero, the ellipticity will either increase or diminish, according as 18 rotations of the planet take a shorter or a longer time than 11 revolutions of the satellite. From this it follows that in the history of a satellite revolving about a planet of small viscosity, the circular orbit is dynamically stable until 11 months of the satellite have become longer than

18 days of the planet. Since the day and month start from equality and end in equality, it follows that the eccentricity will rise to a maximum and ultimately diminish again.

It is also shown that if a satellite be started to move in a circular orbit with the same periodic time as that of the planet's rotation (with maximum energy for given moment of momentum), then if infinitesimal eccentricity be given to the orbit the satellite will ultimately fall into the planet; and if, the orbit being circular, infinitesimal decrease of distance be given the satellite will fall in, whilst if infinitesimal increase of distance be given the satellite will recede from the planet. Thus this configuration, in which the planet and satellite move as parts of a single rigid body, has a complex instability; for there are two sorts of disturbance which cause the satellite to fall in, and one which causes it to recede from the planet.*

If the planet have very large viscosity the case is much more complex, and it is examined in detail in § 25.

It will here only be stated that the eccentricity will diminish if 2 months of the satellite be longer than 3 days of the planet, but will increase if the 2 months be shorter than 3 days; also the rate of increase of eccentricity tends to become infinite, for infinitely great viscosity, if the 2 months are equal to the 3 days.

These results are largely due to the influence of the elliptic monthly tide, and with most of the satellites of the solar system, this is a very slow tide compared with the semi-diurnal tides; therefore it must in general be supposed that the viscosity of the planet makes a close approximation to perfect rigidity, in order that this statement may be true.

The infinite value of the rate of change of eccentricity is due to the speed of the slower elliptic semi-diurnal tide being infinitely slow, when 2 months are equal to 3 days. The result is physically absurd, and its true meaning is commented on in § 25.

In § 26 the time-rate of change of the obliquity of the planet's equator, and of the diurnal rotation is investigated, when the orbits of the tide-raising satellites are eccentric; the only point of general interest in the result is, that the rate of change of obliquity and the tidal friction are both augmented by the eccentricity of the orbit, as was foreseen in the paper on "Precession."

In § 27 it is stated that the effect of the evectional tides is such as to diminish the eccentricity of the orbit, but the formula given shows that the effect cannot have much importance, unless the moon be very distant from the earth.

* Added July, 1880.—This passage appeared to the referee, requested by the R. S. to report on this paper, to be rather obscure, and it has therefore been somewhat modified. To further elucidate the point I have added in an appendix a graphical illustration of the effects of eccentricity, similar to those given in No. 197 of Proc. Roy. Soc., 1879.

See also the abstract of this paper in the Proc. Roy. Soc., No. 200, 1879, for certain general considerations bearing on the problem of the eccentricity.

In Part VI. the equations giving the rate of change of eccentricity are integrated, on the hypothesis that the earth has small viscosity.

The first step is to convert the time-rates of change into distance-rates, and thus to eliminate the time, as in the previous integrations.

The computations made for the paper on "Precession" were here made use of, as far as possible.

The results of the retrospective integration are given in Table XVI., § 28. This table exhibits the eccentricity falling from its present value of $\frac{1}{18}$th down to about $\frac{1}{10600}$th, so that at the end the orbit is very nearly circular.

The integration in the case of large viscosity is not carried out, because the actual degree of viscosity will exercise so very large an influence on the result.

If the viscosity were *infinitely* large, we should find the eccentricity getting larger and larger retrospectively, and ultimately becoming *infinite*, when 2 months were equal to 3 days. This result is of course absurd, and merely represents that the larger the viscosity, the larger would be the eccentricity. On the other hand, if the viscosity were merely large, we might find the eccentricity decreasing at first, then stationary, then increasing until 2 months were equal to 3 days, and then decreasing again.

It follows therefore that various interpretations may be put to the present eccentricity of the lunar orbit.

If, as is not improbable, the more recent changes in the configuration of our system have been chiefly brought about by oceanic tidal friction, whilst the earlier changes were due to bodily tidal friction, with considerable viscosity of the planet, then, supposing the orbit to have been primevally circular, the history of the eccentricity must have been as follows : first an increase to a maximum, then a decrease to a minimum, and finally an increase to the present value. There seems nothing to tell us how large the early maximum, or how small the subsequent minimum of eccentricity may have been.

VIII.

REVIEW OF THE TIDAL THEORY OF EVOLUTION AS APPLIED TO THE EARTH AND THE OTHER MEMBERS OF THE SOLAR SYSTEM.

I will now collect the various results so as to form a sketch of what the previous investigations show as the most probable history of the earth and moon, and in order to indicate how far this history is the result of calculation, references will be given to the parts of my several papers in which each point is especially considered.

We begin with a planet, not very much more than 8,000 miles in diameter,* and probably partly solid, partly fluid, and partly gaseous. This planet is rotating about

* "Precession," § 24.

an axis inclined at about 11° or 12° to the normal to the ecliptic,* with a period of from 2 to 4 hours,† and is revolving about the sun with a period not very much shorter than our present year.‡

The rapidity of the planet's rotation causes so great a compression of its figure that it cannot continue to exist in an ellipsoidal form§ with stability; or else it is so nearly unstable that complete instability is induced by the solar tides.||

The planet then separates into two masses, the larger being the earth and the smaller the moon. I do not attempt to define the mode of separation, or to say whether the moon was initially more or less annular. At any rate it must be assumed that the smaller mass became more or less conglomerated, and finally fused into a spheroid—perhaps in consequence of impacts between its constituent meteorites, which were once part of the primeval planet. Up to this point the history is largely speculative, for although the limiting ellipticity of form of a rotating mass of fluid is known, yet the conditions of its stability, and *à fortiori* of its rupture, have not as yet been investigated.

We now have the earth and the moon nearly in contact with one another, and rotating nearly as though they were parts of one rigid body.

This is the system which has been made the subject of the present dynamical investigation.

As the two masses are not rigid, the attraction of each distorts the other; and if they do not move rigorously with the same periodic time, each raises a tide in the other. Also the sun raises tides in both.

In consequence of the frictional resistance to these tidal motions, such a system is dynamically unstable.¶ If the moon had moved orbitally a little faster than the earth rotates she must have fallen back into the earth; thus the existence of the moon compels us to believe that the equilibrium broke down by the moon revolving orbitally a little slower than the earth rotates. Perhaps the actual rupture into two masses was the cause of this slower motion; for if the detached mass retained the same moment of momentum as it had initially, when it formed a part of the primeval planet, this would, I think, necessarily be the case.

In consequence of the tidal friction the periodic time of the moon (or the month) increases in length, and that of the earth's rotation (or the day) also increases; but the month increases in length at a much greater rate than the day.

* This at least appears to be the obliquity at the earliest stage to which the system has been traced back in detail, but the effect of solar tidal friction would make the obliquity primevally less than this, to an uncertain and perhaps considerable amount.

† "Precession," § 18, and Part IV., § 22.

‡ "Precession," § 19.

§ "Precession," § 18, and Part IV., § 22.

|| Summary of "Precession."

¶ "Secular Effects," &c., Proc. Roy. Soc., 197, 1879; and "Precession," § 18.

At some early stage in the history of the system, the moon has conglomerated into a spheroidal form, and has acquired a rotation about an axis nearly parallel with that of the earth. We will now follow the moon itself for a time.

The axial rotation of the moon is retarded by the attraction of the earth on the tides raised in the moon, and this retardation takes place at a far greater rate than the similar retardation of the earth's rotation.* As soon as the moon rotates round her axis with twice the angular velocity with which she revolves in her orbit, the position of her axis of rotation (parallel with the earth's axis) becomes dynamically unstable.† The obliquity of the lunar equator to the plane of the orbit increases, attains a maximum, and then diminishes. Meanwhile the lunar axial rotation is being reduced towards identity with the orbital motion.

Finally her equator is nearly coincident with the plane of her orbit, and the attraction of the earth on a tide, which degenerates into a permanent ellipticity of the lunar equator, causes her always to show the same face to the earth.‡ LAPLACE has shown that this is a necessary consequence of the elliptic form of the lunar equator.

All this must have taken place early in the history of the earth, to which I now return.

As the month increases in length the lunar orbit becomes eccentric, and the eccentricity reaches a maximum when the month occupies about a rotation and a half of the earth. The maximum of eccentricity is probably not large. After this the eccentricity diminishes.§

The plane of the lunar orbit is at first practically identical with the earth's equator, but as the moon recedes from the earth the sun's attraction begins to make itself felt. Here then we must introduce the conception of the two ideal planes (here called the proper planes), to which the motion of the earth and moon must be referred.|| The lunar proper plane is at first inclined at a very small angle to the earth's proper plane, and the orbit and equator coincide with their respective proper planes.

As soon as the earth rotates with twice the angular velocity with which the moon revolves in her orbit, a new instability sets in. The month is then about 12 of our present hours, and the day is about 6 of our present hours in length.

The inclinations of the lunar orbit and of the equator to their respective proper planes

* "Precession," § 23.

† "Precession," § 17. It is of course possible that the lunar rotation was very rapidly reduced by the earth's attraction on the lagging tides, and was never permitted to be more than twice the orbital motion. In this case the lunar equator has never deviated much from the plane of the orbit.

‡ HELMHOLTZ, I believe, first suggested the reduction of the moon's axial rotation by means of tidal friction.

§ Parts V. and VI. The exact history of the eccentricity is somewhat uncertain, because of the uncertainty as to the degree of viscosity of the earth.

|| See Parts III. and IV. (and the summaries thereof in Part VII.) for this and what follows about proper planes.

increase. The inclination of the lunar orbit to its proper plane increases to a maximum of 6° or 7°,* and ever after diminishes; the inclination of the equator to its proper plane increases to a maximum of about 2° 45′,† and ever after diminishes. The maximum inclination of the lunar orbit to its proper plane takes place when the day is a little less than 9 of our present hours, and the month a little less than 6 of our present days. The maximum inclination of the equator to its proper plane takes place earlier than this.

Whilst these changes have been going on, the proper planes have been themselves changing in their positions relatively to one another and to the ecliptic. At first they were nearly coincident with one another and with the earth's equator, but they then open out, and the inclination of the lunar proper plane to the ecliptic continually diminishes, whilst that of the terrestrial proper plane continually increases.

At some stage the earth has become more rigid, and oceans have been formed, so that it is probable that oceanic tidal friction has come to play a more important part than bodily tidal friction.‡ If this be the case the eccentricity of the orbit, after passing through a stationary phase, begins to increase again.

We have now traced the system to a state in which the day and month are increasing, but at unequal rates; the inclination of the lunar proper plane to the ecliptic and of the orbit to its proper plane are diminishing; the inclination of the terrestrial proper plane to the ecliptic is increasing, and of the equator to its proper plane is diminishing; and the eccentricity of the orbit is increasing.

No new phase now supervenes,§ and at length we have the system in its present configuration. The minimum time in which the changes from first to last can have taken place is 54,000,000 years.||

In a previous paper it was shown that there are other collateral results of the viscosity of the earth; for during this course of evolution the earth's mass must have suffered a screwing motion, so that the polar regions have travelled a little from west to east relatively to the equator. This affords a possible explanation of the north and south trend of our great continents.¶ Also a large amount of heat has been generated by friction deep down in the earth, and some very small part of the observed increase of temperature in underground borings may be attributable to this cause.**

* Table XV., Part IV.

† Found from the values in Table XV., and by a graphical construction.

‡ Compare with "Precession," § 14, where the present secular acceleration of the moon's mean motion is considered.

§ Unless the earth's proper plane (or mean equator) be now slowly diminishing in obliquity, as would be the case if the bodily tides are more potent than the oceanic ones. In any case this diminution must ultimately take place in the far future.

|| "Precession," end of § 18.

¶ "Problems," Part I.

** "Problems," Part II.

The preceding history might vary a little in detail, according to the degree of viscosity which we attribute to the earth's mass, and according as oceanic tidal friction is or is not, now and in the more recent past, a more powerful cause of change than bodily tidal friction.

The argument reposes on the imperfect rigidity of solids, and on the internal friction of semi-solids and fluids; these are *veræ causæ*. Thus changes of the kind here discussed must be going on, and must have gone on in the past. And for this history of the earth and moon to be true throughout, it is only necessary to postulate a sufficient lapse of time, and that there is not enough matter diffused through space to materially resist the motions of the moon and earth in perhaps several hundred million years.

It hardly seems too much to say that granting these two postulates, and the existence of a primeval planet, such as that above described, then a system would necessarily be developed which would bear a strong resemblance to our own.

A theory, reposing on *veræ causæ,* which brings into quantitative correlation the lengths of the present day and month, the obliquity of the ecliptic, and the inclination and eccentricity of the lunar orbit, must, I think, have strong claims to acceptance.

But if this has been the evolution of the earth and moon, then a similar process must have been going on elsewhere. The present investigation has only dealt with a single satellite and the sun, but the theory may of course be extended, with some modification, to planets attended by several satellites. I will now therefore consider some of the other members of the solar system.

A large planet has much more energy of rotation to be destroyed, and moment of momentum to be redistributed than a small one, and therefore a large planet ought to proceed in its evolution more slowly than a small one. Therefore we ought to find the larger planets less advanced than the smaller ones.

The masses of such of the planets as have satellites are, in terms of the earth's mass, as follows: Mars $=\frac{1}{10}$; Jupiter $=301$; Saturn $=90$; Uranus $=14$; Neptune $=16$.

Mars should therefore be furthest advanced in its evolution, and it is here alone in the whole system that we find a satellite moving orbitally faster than the planet rotates. This will also be the ultimate fate of our moon, because, after the moon's orbital motion has been reduced to identity with that of the earth's rotation, solar tidal friction will further reduce the earth's angular velocity, the tidal reaction on the moon will be reversed, and the moon's orbital velocity will increase, and her distance from the earth will diminish. But since the moon's mass is very large, the moon must recede to an enormous distance from the earth, before this reversal will take place. Now the satellites of Mars are very small, and therefore they need only to recede a short distance from the planet before the reversal of tidal reaction.*

* In the graphical method of treating the subject, "the line of momentum" will only just intersect "the curve of rigidity." See Proc. Roy. Soc., No. 197, 1879.

The periodic time of the satellite Deimos is 30 hrs. 18 m.,* and as the period of rotation of Mars is 24hrs. 37m.,† Deimos must be still receding from Mars, but very slowly.

The periodic time of the satellite Phobos in 7 hrs. 39 m.; therefore Phobos must be approaching Mars. It does not seem likely that it has ever been remote from the planet.

The eccentricities of the orbits of both satellites are small, though somewhat uncertain. The eccentricity of the orbit of Phobos appears however to be the larger of the two.

If the viscosity of the planet be small, or if oceanic tidal friction be the principal cause of change, both eccentricities are diminishing; but if the viscosity be large, both are increasing. In any case the rate of change must be excessively slow. As we have no means of knowing whether the eccentricities are increasing or diminishing this larger eccentricity of the orbit of Phobos cannot be a fact of much importance either for or against the present views. But it must be admitted that it is a slightly unfavourable indication.

The position of the proper plane of a satellite is determined by the periodic time of the satellite, the oblateness of the planet, and the sun's distance. The inclination of the orbit of a satellite to its proper plane is not determined by anything in the system. Hence it is only the inclination of the orbit which can afford any argument for or against the theory.

The proper planes of both satellites are necessarily nearly coincident with the equator of the planet; but it is in accordance with the theory that the inclinations of the orbits to their respective proper planes should be small.‡

Any change in the obliquity of the equator of Mars to the plane of his orbit must be entirely due to solar tides. The present obliquity is about 27°, and this points also to an advanced stage of evolution—at least if the axis of the planet was primitively at all nearly perpendicular to the ecliptic.

We now come to the system of Jupiter.

This enormous planet is still rotating in about 10 hours, its axis is nearly perpendicular to the ecliptic, and three of its satellites revolve in 7 days or less, whilst the fourth has a period of 16 days 16 hrs. This system is obviously far less advanced than our own.

The inclinations of the proper planes to Jupiter's equator are necessarily small, but

* 'Observations and Orbits of the Satellites of Mars,' by ASAPH HALL. Washington Government Printing Office, 1878.

† According to KAISER, as quoted by SCHMIDT. 'Ast. Nach.,' vol. 82, p. 333.

‡ For the details of the Martian system, see the paper by Professor ASAPH HALL, above quoted.

With regard to the proper planes, see a paper by Prof. J. C. ADAMS read before the R. Ast. Soc. on Nov. 14, 1879, R. A. S. Month. Not. There is also a paper by Mr. MARTH, 'Ast. Nach.,' No. 2280, vol. 95, Oct., 1879.

the inclinations of the orbits to the proper planes appear to be very interesting from a theoretical point of view. They are as follows:—*

Satellite.	Inclination of orbit to proper plane.
	° ′ ″
First	0 0 0
Second	0 27 50
Third	0 12 20
Fourth	0 14 58

Now we have shown above that the orbit of a satellite is at first coincident with its proper plane, that the inclination afterwards rises to a maximum, and finally declines. If then we may assume, as seems reasonable, that the satellites are in stages of evolution corresponding to their distances from the planet, these inclinations accord well with the theory.

The eccentricities of the orbits of the two inner satellites are insensible, those of the outer two small. This does not tell strongly either for or against the theory, because the history of the eccentricity depends considerably on the degree of viscosity of the planet; yet it on the whole agrees with the theory that the eccentricity should be greater in the more remote satellites. It appears that the satellites of Jupiter always present the same face to the planet, just as does our moon.† This was to be expected.

The case of Saturn is not altogether so favourable to the theory. The extremely rapid rotation, the ring, and the short periodic time of the inner satellites point to an early stage of development; whilst the longer periodic time of the three outer satellites, and the high obliquity of the equator indicate a later stage. Perhaps both views may be more or less correct, for successive shedding of satellites would impart a modern appearance to the system. It may be hoped that the investigation of the effects of tidal friction in a planet surrounded by a number of satellites may throw some light on the subject. This I have not yet undertaken, and it appears to have peculiar difficulties. It has probably been previously remarked, that the Saturnian system bears a strong analogy with the solar system, Titan being analgous to Jupiter, Hyperion and Iapetus to Uranus and Neptune, and the inner satellites being analogous to the inner planets. Thus anything which aids us in forming a theory of the one system will throw light on the other.‡

The details of the Saturnian system seem more or less favourable to the theory.

The proper planes of the orbits (except that of Iapetus) are nearly in the plane of the ring, and the inclinations of all the orbits to their proper planes appear not to be large.

* HERSCHEL'S 'Astron.' Synoptic Tables in appendix.

† HERSCHEL'S 'Astron.' 9th ed., § 546.

‡ An investigation, now (September, 1880) almost completed, seems to show pretty conclusively that tidal friction cannot be in all cases the most important feature in the evolution of such systems as that of Saturn and his satellites, and the solar system itself. I am not however led to reject the views maintained in this paper.

HERSCHEL gives the following eccentricities of orbit:—

Tethys ·04 (?), Dione ·02 (?), Rhea ·02 (?), Titan ·029314, Hyperion "rather large;" and he says nothing of the eccentricities of the orbits of the remaining three satellites. If the dubious eccentricities for the first three of the above are of any value, we seem to have some indication of the early maximum of eccentricity to which the analysis points; but perhaps this is pushing the argument too far. The satellite Iapetus appears always to present the same face to the planet.*

Concerning Uranus and Neptune there is not much to be said, as their systems are very little known; but their masses are much larger than that of the earth, and their satellites revolve with a short periodic time. The retrogade motion and high inclination of the satellites of Uranus are, if thoroughly established, very remarkable.

The above theory of the inclination of the orbit has been based on an assumed smallness of inclination, and it is not very easy to see to what results investigation might lead, if the inclination were large. It must be admitted however that the Uranian system points to the possibility of the existence of a primitive planet, with either retrograde rotation, or at least with a very large obliquity of equator.

It appears from this review that the other members of the solar system present some phenomena which are strikingly favourable to the tidal theory of evolution, and none which are absolutely condemnatory. Perhaps by further investigations some light may be thrown on points which remain obscure.

APPENDIX.

(Added July, 1880.)

A graphical illustration of the effects of tidal friction when the orbit of the satellite is eccentric.

In a previous paper (Proc. Roy. Soc., No. 197, 1879†) a graphical illustration of the effects of tidal friction was given for the case of a circular orbit. As this method makes the subject more easily intelligible than the purely analytical method of the present paper, I propose to add an illustration for the case of the eccentric orbit.

Consider the case of a single satellite, treated as a particle, moving in an elliptic orbit, which is co-planar with the equator of the planet.

Let Ch be the resultant moment of momentum of the system. Then with the notation of the present paper, by § 27 the equation of conservation of moment of momentum is

$$n+\frac{\xi}{k}(1-\eta)=h$$

* HERSCHEL'S 'Astron.' 9th ed., § 547.

† The last sentence of this paper contains an erroneous statement; the line of zero eccentricity on the energy surface is not a ridge as there stated. See the figure on p. 890.

Here Cn is the moment of momentum of the planet's rotation, and $C\xi(1-\eta)/k$ is the moment of momentum of the orbital motion; and the whole moment of momentum is the sum of the two.

By the definitions of ξ and k in § 2, $C\frac{\xi}{k}=\frac{\mu Mm}{\sqrt{\mu(M+m)}}\sqrt{c}$, where μ is the attraction between unit masses at unit distance.

By a proper choice of units we may make $\mu Mm/\sqrt{\mu(M+m)}$ and C equal to unity.*

Then let x be equal to the square root of the satellite's mean distance c, and the equation of conservation of moment of momentum becomes

$$n+x(1-\eta)=h \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\alpha)$$

If in (α) η, the *ellipticity* of the orbit, be zero, we have equation (3) of the previous paper, No. 197, 1879.

It is well known that the sum of the potential and kinetic energies in elliptic motion is independent of the eccentricity of the orbit, and depends only on the mean distance.

Hence if CE be the whole energy of the system, we have (as in equations (2) and (4) of the above paper, No. 197), with the present units

$$2E=n^2-\frac{1}{x^2}$$

Then if z be written for 2E, and if the value of n be substituted from (α), we have

$$z=\{h-x(1-\eta)\}^2-\frac{1}{x^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\beta)$$

This is the equation of energy of the system.

* In the paper above referred to, and in another, Proc. Roy. Soc., No. 202, of 1880, the physical meaning of the units adopted is scarcely adequately explained.

The units are such that C, the planet's moment of inertia, is unity, that $\mu(M+m)$ is unity, and that a quantity called s and defined in (6) of this paper is unity.

From this it may be deduced that the unit length is such a distance that the moment of inertia of planet and satellite when at this distance apart about their common centre of inertia is equal to the moment of inertia of the planet about its own axis. If γ be this unit of length, this condition gives $\frac{Mm}{M+m}\gamma^2=C$, or $\gamma=\sqrt{\frac{C(M+m)}{Mm}}$.

The unit of time is the time taken by the satellite to describe an arc of 57°·3 in a circular orbit at distance γ; it is therefore $\left(\frac{C}{\mu Mm}\right)^{\frac{1}{2}}\left(C\frac{M+m}{Mm}\right)^{\frac{1}{4}}$. The unit of mass is $\frac{Mm}{M+m}$.

From this it follows that the unit of moment of momentum is the moment of momentum of orbital motion when the satellite moves in a circular orbit at distance γ. The critical moment of momentum of the system, referred to in those two papers and below in this appendix, is $4/3^{\frac{3}{4}}$ of this unit of moment of momentum.

In whatever manner the two bodies may interact on one another, the resultant moment of momentum h must remain constant, and therefore (α) will always give one relation between n, x, and η; a second relation would be given by a knowledge of the nature of the interaction between the two bodies.

The equation (α) might be illustrated by taking n, x, η as the three rectangular co-ordinates of a point, and the resulting surface might be called the surface of momentum, in analogy with the "line of momentum" in the above paper.

This surface is obviously a hyperboloid, which cuts the plane of nx in the straight line $n+x=h$; the planes of $n\eta$ and $\eta=1$ in the straight line determined by $n=h$; and the plane of $x\eta$ in the rectangular hyperbola $x(1-\eta)=h$.

The contour lines of this surface for various values of n are a family of rectangular hyperbolas with common asymptotes, viz.: $\eta=1$ and $x=0$. It does not however seem worth while to give a figure of them.

If the satellite raises frictional tides of any kind in the planet, the system is non-conservative of energy, and therefore in equation (β) x and η must so vary that z may always diminish.

Suppose that equation (β) be represented by a surface the points on which have co-ordinates x, η, z, and suppose that the axis of z be vertical. Then each point on the surface represents by the co-ordinates x and η one configuration of the system, with given moment of momentum h. Then since the energy must diminish, it follows that the point which represents the configuration of the system must always move down hill. To determine the exact path pursued by the point it would be necessary to take into consideration the nature of the frictional tides which are being raised by the satellite.

I will now consider the nature of the surface of energy.

It is clear that it is only necessary to consider positive values of η lying between zero and unity, because values of η greater than unity correspond to a hyperbolic orbit; and the more interesting part of the surface is that for which η is a pretty small fraction.

The curves, formed on the surface by the intersection of vertical planes parallel to x, have maxima and minima points determined by $dz/dx=0$.

This condition gives by differentiation of (β)

$$x^4-\frac{h}{1-\eta}x^3+\frac{1}{(1-\eta)^2}=0 \quad \cdots\cdots\cdots \quad (\gamma)$$

From the considerations adduced in previous papers, namely, those in the Proc. Roy. Soc., No. 197, 1879, and No. 202, 1880, it follows that this equation has either two real roots or no real roots.

When $\eta=0$ the equation has real roots provided h be greater than $4/3^{\frac{3}{4}}$, and since this case corresponds to that of all but one of the satellites of the solar system, I shall

henceforth suppose that h is greater than $4/3^{\frac{3}{4}}$. It will be seen presently that in this case every section parallel to x has a maximum and minimum point, and the nature of the sections is exhibited in the curves of energy in the two previous papers.

Now consider the condition $n = \Omega$, which expresses that the planet rotates in the same period as that in which the satellite revolves, so that if the orbit be circular the two bodies revolve like a single rigid body.

With the present units $\Omega = 1/x^3$, and by (α), $n = h - x(1-\eta)$.

Hence the condition $n = \Omega$ leads to the biquadratic

$$x^4 - \frac{1}{1-\eta}x^3 + \frac{1}{1-\eta} = 0 \quad \dots\dots\dots\dots \quad (\delta)$$

If η be zero this equation is identical with (γ), which gives the maxima and minima of energy.

Hence if the orbit be circular the maximum and minimum of energy correspond to two cases in which the system moves as a rigid body. If however the orbit be elliptical, and if $n = \Omega$, there is still relative motion during revolution of the satellite, and the energy must be capable of degradation. The principal object of the present note is to investigate the stability of the circular orbit in these cases, and this question involves a determination of the nature of the degradation when the orbit is elliptical.

In Part V. of the present paper it has been shown that if the planet be a fluid of small viscosity the ellipticity of the satellite's orbit will increase if 18 rotations of the planet be less than 11 revolutions of the satellite, and *vice versâ*. Hence the critical relation between n and Ω is $n = \frac{18}{11}\Omega$. This leads to the biquadratic

$$x^4 - \frac{h}{1-\eta}x^3 + \frac{18}{11}\frac{1}{1-\eta} = 0 \quad \dots\dots\dots\dots \quad (\epsilon)$$

This is an equation with two real roots, and when it is illustrated graphically it will lead to a pair of curves. For configurations of the system represented by points lying between these curves the eccentricity increases, and outside it diminishes,—supposing the viscosity of the planet and the eccentricity of the satellite's orbit to be small.

In order to illustrate the surface of energy (β) and the three biquadratics (γ), (δ), and (ϵ), I chose $h = 3$, which is greater than $4/3^{\frac{3}{4}}$.

By means of a series of solutions, for several values of η, of the equations (γ), (δ), (ϵ), and a method of graphical interpolation, I have drawn the accompanying figure.

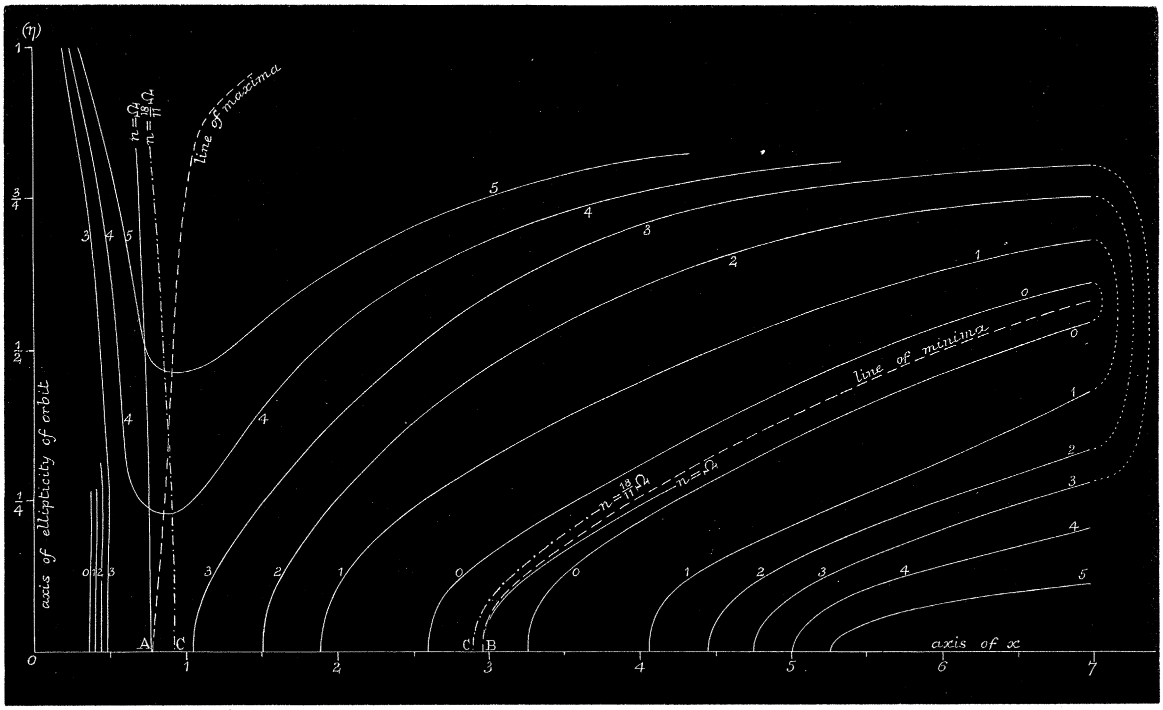
The horizontal axis is that of x, the square root of the satellite's distance, and the numbers written along it are the various values of x. The vertical axis is that of η, and it comprises values of η between 0 and 1. The axis of z is perpendicular to the plane of the paper, but the contour lines for various values of z are projected on to the plane of the paper.

The numbers written on the curves represent the values of z, viz., $z=0, 1, 2, 3, 4, 5$.

The ends of the contour lines on the right are joined by dotted lines, because it would be impossible to draw the curves completely without a very large extension of the figure.

The broken lines (– – –) marked "line of maxima," terminating at A, and "line of minima," terminating at B, represent the two roots of the biquadratic (γ).

The lines marked $n=\Omega$ represent the two roots of (δ), but computation showed that the right-hand branch fell so very near the line of minima, that it was necessary to somewhat exaggerate the divergence in order to show it on the figure.



Contour lines of surface of energy.

The chain-dot lines (– · – · –) C, C, marked $n=\frac{18}{11}\Omega$, represent the two roots of (ϵ). For configurations of the system represented by points lying between these two curves, the ellipticity of orbit will increase; for the regions outside it will decrease. This statement only applies to cases of small ellipticity, and small viscosity of the planet.

Inspection of the figure shows that the line of minima is an infinitely long valley of a hyperbolic sort of shape, with gently sloping hills on each side, and the bed of the valley gently slopes up as we travel away from B.

The line of maxima is a ridge running up from A with an infinitely deep ravine on the left, and the gentle slopes of the valley of minima on the right.

Thus the point B is a true minimum on the surface, whilst the point A is a maximum-minimum, being situated on a saddle-shaped part of the surface.

The lines $n=\Omega$ start from A and B, but one deviates from the ridge of maxima towards the ravine ; and the other branch deviates from the valley of minima by going up the slope on the side remote from the origin.

This surface enables us to perfectly determine the stabilities of the circular orbit, when planet and satellite are moving as parts of a rigid body.

The configuration B is obviously dynamically stable in all respects ; for any configuration represented by a point near B must degrade down to B.

It is also clear that the configuration A is dynamically unstable, but the nature of the instability is complex. A displacement on the right-hand side of the ridge of maxima will cause the satellite to recede from the planet, because x must increase when the point slides down hill.

If the viscosity be small, the ellipticity given to the orbit will diminish, because A is not comprised between the two chain-dot curves. Thus for this class of tide the *circularity* is stable, whilst the configuration is unstable.

A displacement on the left-hand side of the ridge of maxima will cause the satellite to fall into the planet, because the point will slide down into the ravine. But the circularity of the orbit is again stable.

This figure at once shows that if planet and satellite be revolving with maximum energy as parts of a rigid body, and if, without altering the total moment of momentum, or the equality of the two periods, we impart infinitesimal ellipticity to the orbit, the satellite will fall into the planet. This follows from the fact that the line $n=\Omega$ runs on to the slope of the ravine.

If on the other hand without affecting the moment of momentum, or the circularity, we infinitesimally disturb the relation $n=\Omega$, then the satellite will either recede or approach the planet according to the nature of the disturbance.

These two statements are independent of the nature of the frictional interaction of the two bodies.

The only parts of this figure which postulate anything about the nature of the interaction are the curves $n=\frac{18}{11}\Omega$.

I have not thought it worth while to illustrate the case where h is less than $4/3^{\frac{3}{4}}$, or the negative side of the surface of energy ; but both illustrations may easily be carried out.