XI. *Double Refraction and Dispersion in Iceland Spar: an Experimental Investigation, with a comparison with* HUYGHEN'S *Construction for the Extraordinary Wave.*

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SECTION I.

Preliminary.

IN a paper read before the Royal Society, June 20, 1878, the results of an investigation into the truth of FRESNEL'S theory of double refraction in a biaxal crystal were stated. The comparison between theory and experiment was made by a method suggested by Professor STOKES (British Association Report, 1862), according to which the reciprocal of the velocity of wave propagation was determined by experiment and also on FRESNEL'S theory. The greatest difference between the two amounted to ·0009, and there appeared to be some connexion between the differences and the wave length of the light used. In the endeavour to follow up this connexion I undertook a series of similar experiments with light of different wave lengths, using three lines of the hydrogen spectrum and the sodium line. The extreme smallness of the arragonite prisms I had previously worked with led me to use, at first at least, Iceland spar, which could be obtained in large pieces with ease, and for which the theoretical calculations were greatly more simple. Professor STOKES had already made a series of experiments by the same method with this substance (Proceedings of the Royal Society, vol. 20, p. 443) and arrived at results confirming HUYGHEN'S construction. The details of his experiments are as yet unpublished, and I venture to think it might be useful to have arranged in tabular form a series of results, to serve in the future as a test of any theory of double refraction which might be proposed. The method of the experiments, as suggested by Professor STOKES (British Association Report, 1862), is as follows: A prism is cut from a piece of spar, and the position of its faces with reference to the cleavage faces carefully determined. The prism is mounted on a spectrometer, and the collimator adjusted so that the rays of a definite wave length falling on the prism are parallel, the edge of the prism being parallel to the axis of revolution of the reading telescope. The deviation of the light passing through the prism in any position is observed, also the position of the image of the

slit formed by reflexion at the face of incidence. From this and the known direction of the incident light we can calculate the angle of incidence.

Let this be ϕ. Let the deviation be D and the angle of the prism i. Let V be velocity of the light in air, v in the crystal. Let ψ be the angle of emergence, ϕ' ψ' the angles which the wave normal in the crystal makes with the faces of the prism.

Then we have

$$\left.\begin{aligned}\frac{\sin\phi}{\mathrm{V}}&=\frac{\sin\phi'}{v}\\ \frac{\sin\psi}{\mathrm{V}}&=\frac{\sin\psi'}{v}\end{aligned}\right\} \quad \ldots\ldots\ldots\ldots \quad (1)$$

$$\left.\begin{aligned}\phi'+\psi'&=i\\ \phi+\psi&=\mathrm{D}+i\end{aligned}\right\} \quad \ldots\ldots\ldots\ldots \quad (2)$$

$$\frac{\sin\phi}{\sin\psi}=\frac{\sin\phi'}{\sin\psi'}$$

$$\therefore \frac{\sin\phi+\sin\psi}{\sin\phi-\sin\psi}=\frac{\sin\phi'+\sin\psi'}{\sin\phi'-\sin\psi'}$$

$$\tan\frac{\phi'-\psi'}{2}=\tan\frac{\phi'+\psi'}{2}.\ \tan\frac{\phi-\psi}{2}.\ \cot\frac{\phi+\psi}{2} \quad \ldots\ldots \quad (3)$$

whence we can find $\phi'-\psi'$, and since $\phi'+\psi'$ is known, we can get at once ϕ' and ψ', and then v is given by either of the formulæ

$$\frac{\mathrm{V}}{v}=\frac{\sin\phi}{\sin\phi'}=\frac{\sin\psi}{\sin\psi'} \quad \ldots\ldots\ldots\ldots \quad (4)$$

But since we know the position of the faces of the prism with reference to the optic axis, we can find the angle between the wave normal and the optic axis, and if μ_1, μ_2 be the reciprocals of the principal velocities, μ that of a velocity in a direction making an angle θ with the optic axis, we have by HUYGHEN's construction,

$$\frac{1}{\mu^2}=\frac{\cos^2\theta}{\mu_1^{\,2}}+\frac{\sin^2\theta}{\mu_2^{\,2}} \quad \ldots\ldots\ldots\ldots \quad (5)$$

and from this μ_1, μ_2, θ being known, we can find μ.

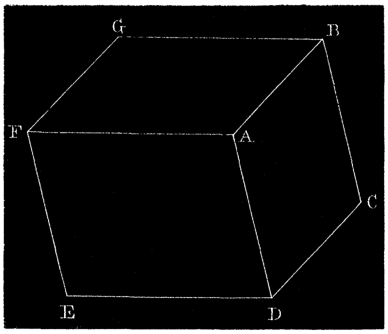
SECTION II.

I. *Description of crystal.*
II. *Account of experiments with the results.*

It was my object in carrying out the work to secure a series of observations for values of θ from 0° to 90°, differing by about 1° 30′ or rather less. This I found could be obtained by the use of four prisms of 44° or thereabouts, each having its edge perpendicular to the optic axis, which would therefore lie in the principal plane of each prism, the prisms being so cut that the optic axis made angles of −32°, 14°, 38°, and 64°, with the outward drawn normal to one of the faces; the angles are considered positive when the optic axis falls on the same side of the normal as the edge of the prism. Prisms cut in this manner would, I found, enable me to work over a range extending from about 5° on one side of the optic axis to about 100° on the other.

Iceland spar, as is well known, cleaves readily so as to form an oblique rhombohedron.

Fig. 1.



Let A B C D E F G, fig. 1, represent a rhomb of spar, and let A be a solid angle, such that each of the three plane angles B A D, D A F, F A B is obtuse. The optic axis is equally inclined to each of the faces B A D, D A F, F A B, the angle of inclination being 26° 15′ 30″ about. It is, therefore, perpendicular to the interior bisectors of the acute angles G F A, G B A. I procured a large rhomb of spar, which was cut by A. HILGER, 196, Tottenham Court Road, into four prisms, the edge of each being nearly parallel to the interior bisectors of the acute angle of the same rhombic face. The angle of each prism was about 44°, and the faces were cut so as to be inclined to the optic axis as stated above.

We proceed now to describe the experiments and give the results for each of these four prisms numbered I., II., III., and IV. In each case let P, Q denote the faces of the prism, i the angle between them, ϕ' ψ' the angles which the wave normal in the prism makes with the normals to P, Q respectively, ϕ ψ the corresponding angles in air; ϕ is the angle of incidence or emergence according as the light is incident on P or Q, and *vice versâ* for ψ.

The values of the angle of incidence on one face extend from nearly grazing incidence

to the position of minimum deviation, forming an arithmetic progression of which the common difference is 4°. The prism was then reversed so that the face of incidence became that of emergence, and another set of results obtained, extending from minimum deviation to nearly grazing incidence on that face.

Each set of experiments was taken twice, and only in two or three cases were the differences between the results of the two measurements, usually made on different days, greater than 20″. In about 18 per cent. of the measurements the differences amounted to 20″, in the rest it was less, so that in comparatively few cases is the difference between the mean and an extreme observation as great as 10″.

The spectrometer was the same as that used in the experiments with arragonite, and was kindly lent me by Professor STOKES. The method of taking the measurements and the means adopted to secure the parallelism of the edge of the prism and the axis of rotation of the telescope are described at length (Phil. Trans., 1879). The collimator and telescope were focused for parallel rays by means of a method suggested by Dr. SCHUSTER (Phil. Mag., February, 1879).

The focusing was done once for each prism, and remained untouched during the experiments with that prism. All the adjustments were made for the red hydrogen line C. When the rays from this line were parallel no appreciable alteration was required to render the sodium rays parallel.

The other hydrogen rays F and g were very nearly parallel, but probably not quite so.

The experiments were performed in the spectroscope room at the Cavendish Laboratory, which was kindly placed at my disposal by Professor MAXWELL during February, March, and April of the present year.

The value given for the angle of the prism is in each case the mean of 10 measures, no two of which differed by more than 20″.

In the course of the preliminary work I found that variations in temperature of 5° or 6° C., to which the room was subject during the months of February and March, produced a very appreciable effect in the value of the angles between some of the faces. In making the final measurements, therefore, I was careful to keep the room at a nearly constant temperature of about 13° C. by means of a gas stove.

For each position of the prism an observation of the deviation of each of the four rays C, D, F, g was taken so that there are four values of deviation, corresponding respectively to these four rays, to each angle of incidence.

Tables I., II., III., and IV. give the results of experiment for the red line C of the hydrogen spectrum in the four prisms.

The error in the result, due to an error in one of the observed quantities, is greatest near the position of minimum deviation. If we assume an error of 10″ in the values of the angle of incidence and the deviation taken so as to produce the maximum error in the result, that error amounts to about ·00005 when a maximum. The probable error of the experiments is considerably less than this.

TABLE I.—Prism I., Ray C.

$i=43°\ 56'\ 20''$.			
D+i.	ϕ.	ϕ'.	μ.
° ′ ″	° ′	° ′ ″	
89 25 5	76 8	35 57 5	1·65367
86 48 5	72 8	35 7 56	1·65393
84 30 20	68 8	34 7 39	1·65416
82 31 0	64 8	32 56 58	1·65438
80 48 40	60 8	31 36 52	1·65431
79 22 35	56 8	30 8 6	1·65395
78 12 30	52 8	28 31 16	1·65335
77 17 15	48 8	26 47 25	1·65223
76 37 25	44 8	24 56 58	1·65078
76 12 35	40 8	23 0 48	1·64873
76 4 0	36 8	20 59 21	1·64623
D+i.	ψ.	ψ'.	μ.
° ′ ″	° ′ ″	° ′ ″	
76 4 1	39 51 48	22 54 48	1·64627
76 12 26	43 51 48	24 56 23	1·64335
76 37 36	47 51 48	26 52 44	1·64021
77 18 41	51 51 48	28 43 11	1·63684
78 15 56	55 51 48	30 26 47	1·63341
79 29 26	59 51 48	32 2 45	1·62991
81 0 36	63 51 48	33 29 49	1·62669
82 49 46	67 51 48	34 47 10	1·62356
84 58 36	71 51 48	35 53 35	1·62095
87 27 31	75 51 48	36 48 20	1·61862
90 17 56	79 51 48	37 30 24	1·61680

TABLE II.—Prism II., Ray C.

$i=43°\ 36'\ 19''$.			
D+i.	ϕ.	ϕ'.	μ.
° ′ ″	° ′ ″	° ′ ″	
84 39 59	71 51 30	35 46 5	1·62580
82 16 14	67 51 30	34 47 11	1·62352
80 9 4	63 51 30	33 38 11	1·62064
78 18 19	59 51 30	32 19 22	1·61734
76 42 24	55 51 30	30 51 43	1·61345
75 21 29	51 51 30	29 15 29	1·60919
74 14 19	47 51 30	27 31 37	1·60438
73 20 59	43 51 30	25 40 30	1·59912
72 41 14	39 51 30	23 42 54	1·59350
72 15 24	35 51 30	21 39 16	1·58745
72 4 29	31 51 30	19 30 7	1·58107
D+i.	ψ.	ψ'.	μ.
° ′ ″	° ′ ″	° ′ ″	
72 8 56	44 8 16	26 15 2	1·57449
72 30 6	48 8 16	28 21 12	1·56820
73 7 36	52 8 16	30 21 6	1·56239
74 1 1	56 8 16	32 13 56	1·55692
75 11 1	60 8 16	33 58 32	1·55187
76 38 41	64 8 16	35 33 35	1·54733
78 25 21	68 8 16	36 57 46	1·54350
80 31 41	72 8 16	38 10 8	1·54019

TABLE III.—Prism III., Ray C.

$i=43° 53' 57''$.

D+i.	ψ.	ψ'.	μ.
° ' "	° ' "	° ' "	
93 2 0	86 6 24	39 29 24	1·56883
89 31 5	82 6 24	39 10 47	1·56789
86 19 45	78 6 24	38 39 37	1·56640
83 27 15	74 6 24	37 56 21	1·56431
80 52 50	70 6 24	37 1 26	1·56163
78 35 20	66 6 24	35 55 34	1·55827
76 35 5	62 6 24	34 38 42	1·55468
74 50 15	58 6 24	33 11 58	1·55058
73 20 10	54 6 24	31 35 58	1·54608
72 3 55	50 6 24	29 51 24	1·54116
71 1 50	46 6 24	27 58 36	1·53619
70 13 0	42 6 24	25 58 29	1·53093
69 37 35	38 6 24	23 51 34	1·52567
69 16 5	34 6 24	21 38 30	1·52042
69 9 10	30 6 24	19 19 56	1·51523
D+i.	ϕ.	ϕ'.	μ.
° ' "	° ' "	° ' "	
69 11 4	36 53 26	23 17 35	1·51805
69 11 24	40 53 26	25 38 16	1·51293
69 27 29	44 53 26	27 53 54	1·50833
69 59 9	48 53 26	30 4 0	1·50435
70 45 59	52 53 26	32 5 50	1·50084
71 48 34	56 53 26	34 0 6	1·49786
73 7 39	60 53 26	35 45 1	1·49537
74 43 49	64 53 26	37 19 39	1·49331
76 38 39	68 53 26	38 42 35	1·49172
78 52 44	72 53 26	39 53 6	1·49044
81 27 44	76 53 26	40 49 54	1·48962
84 23 49	80 53 26	41 32 40	1·48882
87 42 39	84 53 26	42 0 16	1·48841

TABLE IV.—Prism IV., Ray C.

$i=43° 51'$.

D+i.	ϕ.	ϕ'.	μ.
° ' "	° ' "	° ' "	
69 2 48	47 53 9	29 50 17	1·49092
68 16 23	43 53 9	27 44 57	1·48887
67 44 18	39 53 9	25 32 30	1·48726
67 25 43	35 53 9	23 14 7	1·48583
67 22 28	31 53 9	20 50 13	1·48499
D+i.	ψ.	ψ'.	μ.
° ' "	° ' "	° ' "	
67 22 16	33 6 43	21 34 43	1·48534
67 25 16	37 6 43	23 58 43	1·48470
67 43 16	41 6 43	26 17 24	1·48457
68 15 41	45 6 43	28 29 52	1·48491
69 1 41	45 6 43	30 35 34	1·48554

Column 1 gives the value of $D+i$, D being the observed deviation, and i the angle of the prism. (In the calculations D occurs only in the form $D+i$, therefore $D+i$ is given in the tables instead of D.) Column 2 the observed angle of incidence. Column 3 the angle which the wave normal in the crystal makes with the normal to the faces of incidence calculated from the formulæ

$$\psi'+\phi'=i$$

$$\tan\frac{\phi'-\psi'}{2}=\tan\frac{\phi-\psi}{2}\cot\frac{\phi+\psi}{2}\tan\frac{i}{2}$$

already proved, and column 4 the values of μ or $\frac{V}{v}$ calculated from

$$\frac{V}{v}=\frac{\sin\phi}{\sin\phi'}=\frac{\sin\psi}{\sin\psi'}$$

On comparing the results for the ray C with theory I found so close an agreement that I thought it hardly requisite to work out all the calculations for the rays F and g. I therefore completed the calculations for only about a third of the observations, giving a series of values of μ in directions inclined at angles of about 4° to each other, extending in an almost continuous arc from the optic axis to directions perpendicular to it.

These are contained in Tables V. and VI.

The middle column in each case gives the angle of incidence. The columns on the right refer to the ray g, those on the left to the ray F.

For Table V., Prism II., the results for the angle of incidence ϕ have been calculated for the value 46° 36′ 53″ of the angle of the prism instead of 46° 36′ 19″ the value used for the results in which the angle of incidence is denoted by ψ. The reasons for this will be discussed in connexion with the theory.

This closes the experimental part of the work.

TABLE V.—Results of the Experiments.

Ray F.			Angle of incidence.	Ray g.		
Prism I.				i=43° 56′ 20″.		
μ.	ϕ'.	D+i.	ϕ.	D+i.	ϕ'.	μ.
1·66780	32° 39′ 3″	83° 11′ 15″	64° 8′	83° 34′ 30″	32° 28′ 52″	1·67557
1·66663	28 16 25	78 52 40	52 8	..	..	..
1·66385	24 44 25	77 18 15	44 8	77 42 5	24 37 14	1·67143
μ.	ψ'.	D+i.	ψ.	D+i.	ψ'.	μ.
1·65978	22° 42′ 59″	76° 47′ 31″	39 51′ 48″	77° 12′ 6″	22° 36′ 27″	1·66736
1·64996	28 28 13	77 58 16	51 51 48	78 20 6	28 20 6	1·65720
1·64287	31 45 48	80 8 11	59 51 48	80 30 6	31 36 22	1·65022
1·63360	35 34 20	85 36 51	71 51 48	85 58 11	35 23 46	1·64067
1·62930	37 10 12	90 56 6	79 51 48	91 17 56	36 58 48	1·63643
Prism II.			i=43° 36′ 19″.	Prism II.		
μ.	ϕ'.	D+i.	ϕ.	D+i.	ϕ'.	μ.
1·63455	34° 30′ 58″	82° 50′ 18″	67° 51′ 30″	83° 8′ 48″	34° 22′ 2″	1·64087
1·61974	29 2 58	75 53 43	51 51 30	76 11 28	28 55 58	1·62570
1·60336	23 33 38	73 12 43	39 51 30	73 30 18	23 28 21	1·60903
1·59058	19 22 50	72 36 58	31 51 30	72 54 43	19 18 48	1·59590
μ.	ψ'.	D+i.	ψ.	D+i.	ψ'.	μ.
1·58487	26° 3′ 57″	72° 40′ 1″	44° 8′ 16″	72° 57′ 41″	25° 57′ 44″	1·59072
1·56653	32 0 38	74 29 21	56 8 16	74 45 36	31 53 6	1·57205
1·54924	37 54 21	80 59 16	72 8 16	81 15 1	37 45 26	1·55442

TABLE VI.—Results of the Experiments.

Ray F.			Angle of incidence.	Ray g.		
Prism III.				$i=43°\ 53'\ 57''$.		
μ.	ψ'.	D+i.	ψ.	D+i.	ψ'.	μ.
1·57014	36° 47′ 24″	81° 18′ 40″	70° 6′ 24″	81° 33′ 30″	36° 39′ 26″	1·57502
1·55876	33 0 11	75 14 35	58 6 24	75 28 30	32 53 30	1·56345
1·54914	29 41 15	72 27 35	50 6 24	72 41 10	29 35 29	1·55371
1·53312	23 44 11	70 0 25	38 6 24	70 13 40	23 39 56	1·53745
1·52241	19 14 15	69 32 35	30 6 24	69 45 50	19 11 5	1·52644
μ.	ψ'.	D+i.	ϕ.	D+i.	ϕ'.	μ.
1·52573	23° 10′ 8″	69° 34′ 39″	36° 53′ 26″	69° 48′ 14″	23° 5′ 54″	1·53014
1·51571	27 45 2	69 49 24	44 53 26	70 1 34	27 40 10	1·51981
1·50475	33 49 30	72 9 4	56 53 26	72 20 49	33 43 28	1·50870
1·50005	37 7 53	75 4 19	64 53 26	75 15 59	37 1 15	1·50389
1·49610	40 37 1	81 48 14	76 53 26	82 0 4	40 29 39	1·49982
Prism IV.			$i=43°\ 51'$.	Prism IV.		
μ.	ϕ'.	D+i.	ϕ.	D+i.	ϕ'.	μ.
1·49507	27° 37′ 28″	59° 7′ 35″	43° 53′ 9″	58° 57′ 15″	27° 33′ 16″	1·49856
1·49114	20 44 52	60 0 35	31 53 9	59 49 40	20 41 49	1·49460
μ.	ψ'.	D+i.	ψ.	D+i.	ψ'.	μ.
1·49074	26° 10′ 22″	68° 1′ 36″	41° 6′ 43″	68° 12′ 11″	26° 6′ 21″	1·49430

SECTION III.

I. *Determination of the position of the principal plane of the prism.*

II. *Proposition proved.—The principal plane of prisms I., III., and IV. may be treated as if it passed through the optic axis.*

III. *Theoretical calculations for the reciprocal of the wave velocity.*

Our next step will be the determination of the position of the faces of the prisms with reference to the optic axis.

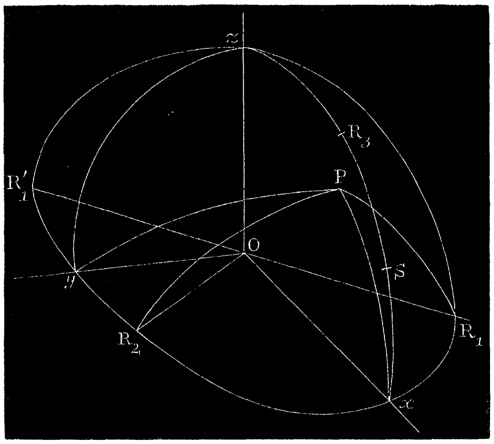
This was accurately determined for each prism by measuring the angles between them and two of the rhombic faces of the crystal.

The angle between these faces and also the angle between the cut faces of each of the prisms were accurately observed.

Let us take point O within the crystal as origin, and from it draw normals to the

faces of the rhomb. Let the normals, drawn in directions making acute angles with each other, meet in R_1 R_2 R_3 a sphere centre O. Then $R_1 R_2 = R_2 R_3 = R_3 R_1$, and if the optic axis meet the sphere in S, $SR_1 = SR_2 = SR_3$.

Fig. 2.



Let P Q be the points in which the normals to the two faces, P Q, of one of the prisms meet the sphere. Let us take the plane $R_1 R_2$ as plane of $x\,y$, the internal and external bisectors of the angle $R_1 OR_2$ as axes of x and y respectively, the axis of z being perpendicular to the plane, $x\,y$. Then R_3 and S lie in the plane, $z\,x$.

Let

$$PR_1 = \theta_1$$
$$PR_2 = \theta_2$$
$$R_1R_2 = 2\mu$$

θ_1 θ_2 μ are known from experiment. Let α β γ be the direction angles of O P. Then from triangles P x R_1 P x R_2

$$\cos\theta_1 = \cos\alpha\cos\mu + \sin\alpha\sin\mu\cos PxR_1$$
$$\cos\theta_2 = \cos\alpha\cos\mu - \sin\alpha\sin\mu\cos PxR_1$$

$$\cos\alpha = \frac{\cos\theta_1 + \cos\theta_2}{2\cos\mu}$$
$$= \frac{\cos\frac{\theta_1+\theta_2}{2}\cos\frac{\theta_1-\theta_2}{2}}{\cos\mu} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

From triangles P y R_1 P y R_2

$$\cos\theta_1 = \cos\beta\cos\left(\frac{\pi}{2}+\mu\right) + \sin\beta\sin\left(\frac{\pi}{2}+\mu\right)\cos PyR_2$$
$$\cos\theta_2 = \cos\beta\cos\left(\frac{\pi}{2}-\mu\right) + \sin\beta\sin\left(\frac{\pi}{2}-\mu\right)\cos PyR_2$$
$$\cos\theta_2 - \cos\theta_1 = 2\cos\beta\sin\mu$$

$$\cos\beta=\frac{\cos\theta_2-\cos\theta_1}{2\sin\mu}$$

$$=\frac{\sin\frac{\theta_2+\theta_1}{2}\sin\frac{\theta_1-\theta_2}{2}}{\sin\mu} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

These formulæ give us the values of α and β.

2μ or the angle between the normals to the rhombic faces was observed in three pieces of the crystal used. The values were

74° 55′ 37″ Mean of four measures. Maximum difference, 10″.
74° 55′ 34″ Mean of five measures.
74° 55′ 35″ Mean of ten measures. Maximum difference, 25″.

We may therefore put with great accuracy

$$2\mu=74^\circ\ 55'\ 35''$$

The temperature indicated by a thermometer placed almost in contact with the crystal, and shaded from the direct radiation of the light used to read the vernier, was from 14° C. to 13° C. Each of the angles θ_1 θ_2 was observed ten times for each face and the mean taken, the temperature being kept as nearly as possible at 13° C. The greatest variation between any two observations never exceeded 40″.

TABLE VII.—The position of the normals to the faces of the prisms.

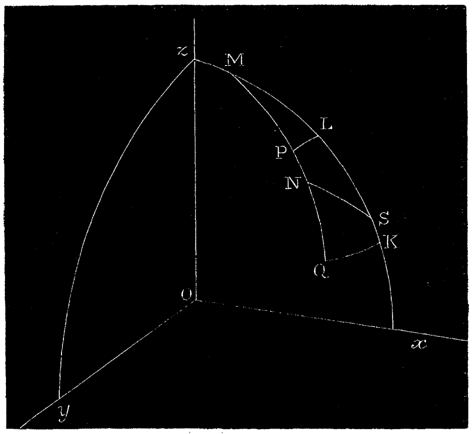
Face and direction of normal.	θ_1	θ_2	α	β
		Prism I.		
	° ′ ″	° ′ ″	° ′ ″	° ′ ″
P outwards. . .	65 48 35	65 41 32	58 50 25	89 54 43
Q inwards . . .	39 50 28	39 59 43	14 55 34	90 4 53
		Prism II.		
	° ′ ″	° ′ ″	° ′ ″	° ′ ″
P inwards . . .	85 48 20	85 39 20	84 37 5·5	89 52 37·7
Q outwards . .	53 31 35	52 53 20	41 0 55	89 34 49
		Prism III.		
	° ′ ″	° ′ ″	° ′ ″	° ′ ″
P inwards . . .	70 14 35	70 10 25	64 44 56	89 56 47
Q outwards . .	104 40 25	104 43 25	108 38 34	90 2 23
		Prism IV.		
	° ′ ″	° ′ ″	° ′ ″	° ′ ″
P inwards . . .	96 40 55	96 38 5	98 23 59	89 57 41
Q outwards . .	128 58 25	128 45 20	142 14 8	89 51 37·5

Table VII. gives the results of these calculations. The first column gives the face to which the normal considered is drawn and its direction with reference to the crystal prism. The next two columns give the values of θ_1 θ_2, the last two those of α β.

The values of β show that the principal plane of the prism which contains the normals to the faces P and Q is nearly coincident with the plane z O x.

We proceed to find the position of the line of junction of these planes and the angle between them.

Fig. 3.



Let P Q (fig. 3) meet z x in M.
Draw Q K, P L arcs perpendicular to z x.
Then from triangle P M L

$$\sin \mathrm{PL} = \sin \mathrm{PM} \sin \mathrm{PML}$$

From triangle Q K M

$$\sin \mathrm{QK} = \sin (\mathrm{PM} + \mathrm{PQ}) \sin \mathrm{PML}$$

Whence

$$\frac{\sin (\mathrm{PM} + \mathrm{PQ})}{\sin \mathrm{PM}} = \frac{\sin \mathrm{QK}}{\sin \mathrm{PL}}$$

and we have

$$\cot \mathrm{PM} \sin \mathrm{PQ} = \frac{\sin \mathrm{KQ}}{\sin \mathrm{LP}} - \cos \mathrm{PQ} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

If P and Q are on opposite sides of z x, we get

$$\cot \mathrm{PM} \sin \mathrm{PQ} = \frac{\sin \mathrm{KQ}}{\sin \mathrm{LP}} + \cos \mathrm{PQ} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

In these formulæ, P Q, P L, Q K being known from Table VII. and the angle of the prism, we can find P M.

Then by substitution in the formula

$$\sin \mathrm{PML} = \frac{\sin \mathrm{PL}}{\sin \mathrm{PM}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (5)$$

we obtain P M L.

If we call P M δ, and the angle P M L χ, we have for the four prisms respectively

TABLE VIII.

	Prism I.	II.	III.	IV.
δ	22° 53′ 23″	14° 23′ 11″	25° 23′ 36″	13° 26′ 29″
χ	0° 13′ 35″	0° 29′ 41·7″	0° 7′ 31″	0° 9′ 57″

We shall now prove that in the case of prisms I., III., and IV. we may neglect the inclination of the plane of the prism to the plane $z\,x$. For S being the optic axis, N the point in which any wave normal meets the sphere, M the intersection of P Q and z O x.

Let

$$\mathrm{NS} = \theta \quad \mathrm{NM} = \psi$$

$$\mathrm{SM} = \lambda \quad \mathrm{SMN} = \chi$$

we have

$$\begin{aligned} \cos\theta &= \cos\lambda\cos\psi + \sin\lambda\sin\psi\cos\chi \\ &= \cos(\lambda - \psi) - 2\sin\lambda\sin\psi\sin^2\frac{\chi}{2} \\ &= \cos(\lambda-\psi) - x \text{ say} \end{aligned}$$

$$\cos^2\theta = \cos^2(\lambda-\psi) - 2x\cos(\lambda-\psi)$$

neglecting x^2.

This we may do, for x^2 is $<(\cdot 004)^4$

$$\begin{aligned} \frac{1}{\mu^2} &= \frac{\cos^2\theta}{\mu_1^{\,2}} + \frac{\sin^2\theta}{\mu_2^{\,2}} \\ &= \frac{1}{\mu_2^{\,2}} - \left(\frac{1}{\mu_2^{\,2}} - \frac{1}{\mu_1^{\,2}}\right)\cos^2\theta \\ &= \frac{1}{\mu_2^{\,2}} - \left(\frac{1}{\mu_2^{\,2}} - \frac{1}{\mu_1^{\,2}}\right)\cos^2(\lambda-\psi) + 2x\left(\frac{1}{\mu_2^{\,2}} - \frac{1}{\mu_1^{\,2}}\right)\cos(\lambda-\psi) \end{aligned}$$

In neglecting the mutual inclination of these planes, *i.e.*, in putting $\chi=0$, we omit a term in $\frac{1}{\mu^2}$ of the value

$$2x\cos(\lambda-\psi)\left(\frac{1}{\mu_2^2}-\frac{1}{\mu_1^2}\right)$$

and in μ of the value

$$\frac{x\cos(\lambda-\psi)\left(\frac{1}{\mu_2^2}-\frac{1}{\mu_1^2}\right)}{\left\{\frac{1}{\mu_2^2}-\left(\frac{1}{\mu_2^2}-\frac{1}{\mu_1^2}\right)\cos^2(\lambda-\psi)\right\}^{\frac{3}{2}}}$$

This is not greater than

$$\mu_1^3\, x\left(\frac{1}{\mu_2^2}-\frac{1}{\mu_1^2}\right)$$

x is not greater than

$$\frac{\chi^2}{2}$$

Term neglected is not greater than

$$\mu_1^3\,\frac{\chi^2}{2}\left(\frac{1}{\mu_2^2}-\frac{1}{\mu_1^2}\right)$$

In the three cases considered χ is less than 14′.

The circular measure of 14′ is ·004.

$$\therefore \frac{\chi^2}{2}=\cdot 000008$$

Using RUDBERG's values of μ_1, μ_2 we have

μ_1 is less than 1·7

$\frac{1}{\mu_2^2}-\frac{1}{\mu_1^2}$ is less than ·1

μ_1^3 is less than 5

Therefore, greatest difference is less than

$$\cdot 5\times\cdot 000008$$
$$\text{or } \cdot 000004$$

Hence neglecting χ or supposing the plane of each of the prisms I., III., and IV. to coincide with the plane $z\,x$ will never produce any change in the fifth decimal figure in the value of μ.

In the case of prism II. the value of χ is nearly 30′, and we may have to take account of the obliquity.

For prisms I., III., and IV. the value of θ is given by formulæ of the form

$$\theta=\lambda\pm\phi'$$

ϕ' having the meaning attached to it in the results of experiment.

To determine λ we require to know the position of the optic axis with reference to x.

The optic axis is equally inclined to R_1 R_2 R_3 (fig. 2). Hence each of the angles subtended at S by the arcs R_1 R_2, R_2 R_3, R_3 R_1 is 120°.

Therefore R_2 S x is 60°

$$\sin R_2x = \sin SR_2 \sin R_2Sx$$

$$\sin SR_2 = \frac{2}{\sqrt{3}} \sin R_2x$$

$$2R_2x = 74^\circ\ 55'\ 35''$$

whence

$$SR_2 = 44^\circ\ 36'\ 57'' \quad \ldots\ldots \quad (6)$$

$$= SR_3 = SR_1$$

Again, from the right-angled triangle R_3 x R_2

$$\cos R_3R_2 = \cos R_3x \cos xR_2$$

and

$$R_3R_2 = 2\mu = 2R_2x$$

$$\therefore \cos R_3x = \cos 2\mu \sec \mu$$

Substituting the value

$$2\mu = 74^\circ\ 55'\ 35''$$

we have

$$R_3x = 70^\circ\ 52'\ 28''$$

$$\therefore Sx = R_3x - R_3S$$

$$= 26^\circ\ 15'\ 31'' \quad \ldots\ldots \quad (7)$$

and

$$\lambda = Sx \sim \alpha$$

where α refers to the face P.

The position of the wave normal is also given by

$$\lambda' \pm \psi'$$

where

$$\lambda' = Sx \sim \alpha'$$

α' being the x direction angle of the face Q.

From these equations we get the following table of values to determine θ the angle between the optic axis and any wave normal.

TABLE IX.

	λ or λ'.	θ.
Prism I. Face P.	32° 36′ 23″	32° 36′ 23″$-\phi'$
" " Q.	11° 19′ 57″	$\psi'-$11° 19′ 57″
Prism III. " P.	38° 29′ 25″	38° 29′ 25″$+\phi'$
" " Q.	82° 23′ 22″	82° 23′ 22″$-\psi'$
Prism IV. " P.	72° 8′ 28″	72° 8′ 28″$+\phi'$
" " Q.	115° 59′ 28″	115° 59′ 28″$-\psi'$

Tables X., XI., XII., and XIII. give the values of θ calculated from these formulæ from the values of ϕ' ψ' given in Tables I., II., III., IV., V., and VI.

For prism II. we must calculate the value of θ from the formula

$$\begin{aligned}\cos\theta &= \cos \mathrm{SM} \cos \mathrm{NM} + \sin \mathrm{SM} \sin \mathrm{NM} - \cos\chi \\ &= \cos \mathrm{SM}\{\cos \mathrm{NM} + \tan \mathrm{SM} \sin \mathrm{NM} \cos\chi\} \\ &= \frac{\cos \mathrm{SM}}{\cos\nu}\cos(\mathrm{NM}-\nu) \quad \ldots\ldots\ldots \quad (8)\end{aligned}$$

if

$$\tan\nu = \tan \mathrm{SM} \cos\chi \quad \ldots\ldots\ldots \quad (9)$$

Now we have from the triangle M Q K (fig. 3)

$$\sin \mathrm{MK} = \tan \mathrm{QK} \cot \mathrm{PML}$$

We know that

$$\mathrm{QK} = 0^\circ\ 25'\ 11''$$

$$\mathrm{PML} = 0^\circ\ 29'\ 41''\cdot 7 \quad \text{[Tables VII. and VIII.]}$$

$$\therefore \mathrm{MK} = 57^\circ\ 59'\ 23'' \quad \ldots\ldots\ldots \quad (10)$$

$$\cos \mathrm{AQ} = \cos \mathrm{AK} \cos \mathrm{KQ}$$

$$\mathrm{AQ} = 41^\circ\ 0'\ 55''$$

$$\therefore \mathrm{AK} = 41^\circ\ 0'\ 49'' \quad \ldots\ldots\ldots \quad (11)$$

$$\therefore \mathrm{AM} = \mathrm{AK} + \mathrm{MK} = 99^\circ\ 0'\ 12''$$

But

$$\mathrm{AS} = 26^\circ\ 15'\ 31'' \quad \text{[From Section III. (7).]}$$

$$\therefore \mathrm{SM} = 72^\circ\ 44'\ 41'' \quad \ldots\ldots\ldots \quad (12)$$

Also from Table VIII.

$$NM=14° \ 23' \ 11''+\phi'$$

Hence

$$\nu=72° \ 44' \ 39'' \quad \text{[From Section III. (9).]}$$

$$\nu-MN=58° \ 21' \ 28''-\phi' \quad . \ . \ . \ . \ . \ . \ . \ . \ . \ . \quad (13)$$

From these values we can obtain the values of θ corresponding to the angles of incidence in Tables I., II., III., and IV.

TABLE X.—Theoretical results for the line C.

	θ.	μ From Theory.	μ From Experiment.	Excess of Experiment.
	° ′ ″			
I.	3 20 42	1·65368	1·65367	− 1
I.	2 31 33	1·65393	1·65393	0
I.	1 31 16	1·65422	1·65416	− 6
I.	0 20 35	1·65435	1·65438	+ 3
I.	0 59 31	1·65430	1·65431	+ 1
I.	2 28 17	1·65399	1·65395	− 4
I.	4 5 7	1·65335	1·65335	0
I.	5 48 58	1·65231	1·65223	− 8
I.	7 39 25	1·65082	1·65078	− 4
I.	9 35 35	1·64883	1·64873	−10
I.	11 34 51	1·64635	1·64627	− 8
I.	11 37 2	1·64631	1·64623	− 8
I.	13 36 26	1·64340	1·64335	− 5
I.	15 32 41	1·64018	1·64021	+ 3
I.	17 23 14	1·63678	1·63684	+ 6
I.	19 6 50	1·63332	1·63341	+ 9
I.	20 42 48	1·62989	1·62991	+ 2
I.	22 9 42	1·62660	1·62669	+ 9
II.	22 35 37	1·62560	1·62580	+20
I.	23 27 13	1·62355	1·62356	+ 1
II.	23 34 31	1·62326	1·62352	+26
I.	24 33 38	1·62085	1·62095	+10
II.	24 43 31	1·62044	1·62064	+20
I.	25 28 23	1·61855	1·61862	+ 7
II.	26 2 20	1·61711	1·61734	+23
I.	26 10 27	1·61676	1·61680	+ 4
II.	27 29 59	1·61329	1·61345	+16
II.	29 6 13	1·60897	1·60919	+22
II.	30 50 5	1·60418	1·60438	+20
II.	32 41 11	1·59893	1·59912	+19
II.	34 38 48	1·59326	1·59350	+24
II.	36 42 26	1·58721	1·58745	+24
II.	38 51 35	1·58078	1·58107	+29
II.	41 0 26	1·57443	1·57449	+ 6
III.	42 53 57	1·56879	1·56883	+ 4
II.	43 6 36	1·56817	1·56820	+ 3
III.	43 12 34	1·56787	1·56789	+ 2
III.	43 43 44	1·56634	1·56640	+ 6
III.	44 27 0	1·56420	1·56431	+11

TABLE X.—continued.

	θ.	μ From Theory.	μ From Experiment.	Excess of Experiment.
II.	45° 6′ 30″	1·56226	1·56239	+13
III.	45 21 55	1·56151	1·56163	+12
III.	46 27 47	1·55830	1·55827	− 3
II.	46 59 20	1·55677	1·55692	+15
III.	47 44 39	1·55459	1·55468	+ 9
II.	48 43 56	1·55175	1·55187	+12
III.	49 11 23	1·55044	1·55058	+14
II.	50 18 59	1·54726	1·54733	+ 7
III.	50 47 23	1·54594	1·54608	+14
II.	51 43 10	1·54334	1·54350	+16
III.	52 31 57	1·54113	1·54116	+ 3
II.	52 55 32	1·54006	1·54019	+13
III.	54 24 45	1·53608	1·53619	+11
III.	56 24 52	1·53087	1·53093	+ 6
III.	58 31 47	1·52560	1·52567	+ 7
III.	60 44 51	1·52033	1·52042	+ 9
III.	61 47 0	1·51797	1·51805	+ 8
III.	63 3 25	1·51516	1·51523	+ 7
III.	64 7 41	1·51288	1·51293	+ 5
III.	66 23 19	1·50830	1·50833	+ 3
III.	68 32 48	1·50428	1·50435	+ 7
III.	70 35 15	1·50079	1·50084	+ 5
III.	72 29 31	1·49782	1·49786	+ 4
III.	74 14 26	1·49534	1·49537	+ 3
III.	75 49 4	1·49331	1·49331	0
III.	77 12 0	1·49170	1·49172	+ 2
IV.	78 1 15	1·49083	1·49092	+ 9
III.	78 22 31	1·49046	1·49044	− 2
III.	79 19 19	1·48955	1·48962	+ 7
IV.	80 2 5	1·48890	1·48882	− 8
IV.	80 6 35	1·48885	1·48887	+ 2
III.	80 29 41	1·48852	1·48841	−11
IV.	82 10 2	1·48715	1·48726	+11
IV.	84 33 25	1·48586	1·48583	− 3
IV.	85 35 15	1·48542	1·48534	− 8
IV.	87 1 19	1·48495	1·48499	+ 4
IV.	87 59 15	1·48474	1·48470	+ 4
IV.	90 17 56	1·48457	1·48457	0
IV.	92 30 24	1·48484	1·48491	+ 7
IV.	94 36 6	1·48549	1·48554	+ 5

θ having been determined μ is given by

$$\begin{aligned}\frac{1}{\mu^2}&=\frac{\cos^2\theta}{\mu_1^2}+\frac{\sin^2\theta}{\mu_2^2}\\&=\frac{1}{\mu_2^2}\left\{1-\mu_2^2\left(\frac{1}{\mu_2^2}-\frac{1}{\mu_1^2}\right)\cos^2\theta\right\}\\&=\frac{1}{\mu_2^2}\left\{1-\frac{\mu_1^2-\mu_2^2}{\mu_1^2}\cos^2\theta\right\}\\&=\frac{1}{\mu_2^2}\cos^2\delta \quad \ldots\ldots\ldots \quad (14)\end{aligned}$$

If

$$\sin^2 \delta = \frac{\mu_1^2 - \mu_2^2}{\mu_1^2} \cos^2 \theta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

$$\therefore \mu = \mu_2 \sec \delta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

and we require to find μ_1, μ_2.

μ_1 is the maximum radius vector of the spheroidal sheet of the surface of wave slowness.

This is given by $\theta = 0$.

From Table X. we have, considering at present the line C, when

$$\theta = 0° \ 20' \ 35'' \qquad \mu = 1 \cdot 65438$$

μ_1 is also the refractive index of the ordinary wave. Its value was determined by observations on the angle of incidence and deviation of the ordinary ray in prisms I., III., and IV.

The values were

$$\begin{array}{ll} 1 \cdot 65438 & \text{Prism I.} \\ \left.\begin{array}{l} 1 \cdot 65438 \\ 1 \cdot 65433 \end{array}\right\} & \text{Prism III.} \\ 1 \cdot 65433 & \text{Prism IV.} \end{array}$$

We take then as the value of μ_1,

$$1 \cdot 65436$$

Observations of the minimum deviation were made to determine μ_1 from the usual formula

$$\mu = \frac{\sin \frac{D+i}{2}}{\sin \frac{i}{2}}$$

D being the minimum deviation.

The mean of these was

$$1 \cdot 65441$$

but the error of this last method is much greater than that in the former, and as any error in the observed value of D would probably increase D, through the prism not being exactly in the position of minimum deviation, we should expect to get a value for μ_1 rather in excess of the true.

The values given by MASCART and RUDBERG are respectively

$$1 \cdot 65452$$

and

$$1 \cdot 65446$$

To determine μ_2 we must consider the minimum radius vector of the spheroidal sheet; this is given by

$$\theta = 90°$$

Now when

$$\theta = 90° \ 17' \ 56''$$

we see from Table X.

$$\mu = 1\cdot48457$$

But

$$\frac{1}{\mu^2} = \frac{\cos^2\theta}{{\mu_1}^2} + \frac{\sin^2\theta}{{\mu_2}^2}$$

$$\therefore \frac{1}{{\mu_2}^2} = \left\{\frac{1}{\mu^2} - \frac{\cos^2\theta}{{\mu_1}^2}\right\} \operatorname{cosec}^2\theta$$

Substituting the values of μ, μ_1, and θ we get

$$\mu_2 = 1\cdot48456$$

The values given by MASCART and RUDBERG are

1·48455
1·48474

The middle column of Table X. gives the values of μ in the directions given by the first column for the values

$$\mu_1 = 1\cdot65436$$
$$\mu_2 = 1\cdot48456$$

The Roman numerals I., II., &c., in the first column refer to the tables of experimental results from which the values of μ in the fourth column are taken. The fifth column gives the excess of experiment over theory.

These differences it will be seen are much greater in the case of prism II. than for any of the others.

They are also greater for the first part of the results in Table II., in which the face of incidence was P, than for the latter, when the light was incident on the face Q.

Postponing, then, for the present the consideration of this point, let us compare the differences between theory and experiment for prisms I., III., and IV. We notice at once their extreme smallness—the greatest is only ·00014, and only in eight out of the sixty measurements taken do they amount to as much as ·0001. The mean irrespective of sign is ·000055. The differences are, on the whole, negative near the major axis. They tend to become least at about 15° away from either axis. From that point they are positive and reach a maximum value at from 45° to 50° away from the major axis. So that the curve given by experiment would, though very nearly coincident with an ellipse, lie inside the ellipse near the major axis, cut it at about 15° from that axis, and lie outside for the rest of its course.

The difference, however, between the radii vectors to the two curves drawn in the same direction would never be greater than $\frac{1}{10000}$th part of either.

My first inference from these results was that HUYGHEN'S construction represented nature to a degree of exactness comparable with the probable error of experiment,

Before considering the results for the rays F and g we must return to the experiments with prism II.

The large differences it gives, overlapping as they do values given by experiments on prism I., in which the differences are small, pointed clearly to errors of experiment. On referring to my note-book containing the direct results of experiment, I found that the observation of deviation and incidence for the face P had been made on March 29, while the observations for the face Q and the angle between the faces were made on April 1.

It seemed possible that the temperature of the prism had been different on the two occasions, and that this was the cause of the error. I therefore proceeded to observe afresh the angle between the faces P Q of the prism. The result differed by 34″ from that found on April 1. I therefore recalculated the experimental results for the prism II. so far as the face P was concerned.

Table X. (A) gives the results of the calculations. In Table X. (B) these are compared with the theory.

TABLE X. (A).—Prism II. Ray C.

$i=43° 36' 53''$			
$D+i$.	ϕ.	ϕ'.	μ.
° ′ ″ 84 39 59	° ′ ″ 71 51 30	° ′ ″ 35 46 16	1·62569
82 16 14	67 51 30	34 47 22	1·62341
80 9 4	63 51 30	33 38 22	1·62053
78 18 19	59 51 30	32 19 32	1·61724
76 42 24	55 51 30	30 51 53	1·61332
75 21 29	51 51 30	29 15 39	1·60906
74 14 19	47 51 30	27 31 46	1·60425
73 20 59	43 51 30	25 40 41	1·59897
72 41 14	39 51 30	23 43 3	1·59336
72 15 24	35 51 30	21 39 24	1·58730
72 4 29	31 51 30	19 30 14	1·58091

TABLE X. (B).—Theory for same.

θ.	μ From Theory.	μ From Experiment.	Excess of Experiment.
° ′ ″ 22 35 27	1·62560	1·62569	+ 9
23 34 21	1·62326	1·62341	+15
24 43 20	1·62043	1·62053	+10
26 2 10	1·61711	1·61724	+13
27 29 48	1·61329	1·61332	+ 3
29 6 1	1·60897	1·60906	+ 9
30 49 54	1·60418	1·60425	+ 7
32 40 59	1·59894	1·59897	+ 3
34 38 36	1·59326	1·59336	+10
36 42 14	1·58721	1·58730	+ 9
38 51 23	1·58083	1·58091	+ 8

Thus this variation has tended to decrease the differences between observation and theory, and has reduced them to almost the same magnitude as those given by the face Q of the prism. They now agree more nearly with the results of prisms I., III., and IV., though even yet the differences observed are greater than in any of the other prisms.

Prism II., however, was at first cut wrongly from the crystal, and when recut it was so small that I formed the intention of not using it at all, and leaving a gap in my series of observations between the values $\theta=27°$ and $\theta=41°$. I found, however, on a second and more careful trial, that the images formed by it were clearer and brighter than I had thought, and so determined to take a series of observations with it. When I observed a second time the angle of prism II., I took a series of measurements of deviation, &c., which lead to results in agreement with Tables X. (A), X. (B), so that on the whole the results given by this prism are in accordance with those already arrived at in prisms I., III., and IV.

Our next step is to consider the theory for the rays F and g.

The position of the plane containing the two normals to the faces of the prism is of course the same, and therefore so also are the formulæ which give the relations between θ and ϕ', θ and ψ'.

The values of the axes of spheroid on HUYGHEN'S theory are, however, different.

Let us take the green line, F, first.

μ_1 is, as before, the value of the ordinary refractive index.

We have as for the line C the four values

$$\mu_1=1{\cdot}66780 \quad \text{Prism I.}$$

$$\left.\begin{array}{l}1{\cdot}66776\\1{\cdot}66778\end{array}\right\}\quad \text{Prism III.}$$

$$1{\cdot}66783 \quad \text{Prism IV.}$$

We take $\mu_1=1{\cdot}66779$. μ_2 is the value of μ when $\theta=90°$ in Table XI.

Now for $\theta=89°\ 49'\ 6''$ experiment gives

$$\mu=1{\cdot}49074$$

we take this as the value of μ_2.

Hence for F we have

$$\mu_1=1{\cdot}66779$$

$$\mu_2=1{\cdot}49074$$

MASCART and RUDBERG give respectively

$$1{\cdot}66802,\ 1{\cdot}49075$$

and

$$1{\cdot}66793,\ 1{\cdot}49084$$

For g we have, as before,

$$\mu_1 = 1{\cdot}67557 \quad \text{Prism I.}$$
$$1{\cdot}67545 \quad \text{Prism III.}$$
$$1{\cdot}67556 \quad \text{Prism IV.}$$

Whence

$$\mu_1 = 1{\cdot}67553$$

and for μ_2 when

$$\theta = 89^\circ\ 53'\ 4'', \text{ Table XII.}$$

we have

$$\mu = 1{\cdot}49430.$$

Whence

$$\mu_2 = 1{\cdot}49430.$$

Thus for g

$$\mu_1 = 1{\cdot}67553$$
$$\mu_2 = 1{\cdot}49430$$

Tables XI. and XII. give the results of the calculations.

TABLE XI.—Results of Theory for F.

	θ.	μ From Theory.	μ From Experiment.	Excess of Experiment.
I.	0° 2′ 40″	1·66779	1·66780	+ 1
II.	4 19 58	1·66660	1·66663	+ 3
I.	7 51 58	1·66387	1·66385	− 2
I.	11 23 12	1·65967	1·65978	+11
I.	17 8 26	1·64987	1·64996	+ 9
I.	20 26 1	1·64279	1·64287	+ 8
II.	23 50 45	1·63451	1·63455	+ 4
I.	24 14 23	1·63351	1·63360	+ 9
I.	25 49 35	1·62934	1·62930	− 4
II.	29 18 42	1·61965	1·61974	+ 9
II.	34 48 0	1·60336	1·60336	0
II.	38 58 47	1·59048	1·59058	+10
II.	40 49 21	1·58478	1·58487	+ 9
III.	45 45 57	1·57000	1·57014	+14
II.	46 46 2	1·56645	1·56653	+ 8
III.	49 23 10	1·55861	1·55876	+15
III.	52 42 6	1·54902	1·54914	+12
III.	58 39 10	1·53303	1·53312	+ 9
III.	61 39 33	1·52570	1·52573	+ 3
III.	63 9 6	1·52228	1·52241	+13
III.	66 14 27	1·51579	1·51571	− 8
III.	72 18 55	1·50476	1·50475	− 1
III.	75 36 18	1·50009	1·50005	− 4
III.	79 6 26	1·49612	1·49610	− 2
IV.	80 14 4	1·49507	1·49507	0
IV.	87 6 40	1·49112	1·49114	+ 2
IV.	89 49 6	1·49074	1·49074	0

TABLE XII.—Results of Theory for g.

	θ.	μ From Theory.	μ From Experiment.	Excess of Experiment.
I.	0° 7′ 31″	1·67553	1·67557	+ 4
		No experiment.		
I.	7 59 9	1·67138	1·67143	+ 5
I.	11 16 40	1·66735	1·66736	+ 1
I.	17 0 9	1·65740	1·65720	−20
I.	20 16 35	1·65023	1·65022	− 1
II.	23 59 41	1·64098	1·64087	−11
I.	24 3 49	1·64080	1·64067	−13
I.	25 39 1	1·63655	1·63643	−12
II.	29 25 42	1·62579	1·62570	− 9
II.	34 53 17	1·60917	1·60903	−14
II.	39 2 49	1·59606	1·59590	+16
II.	40 43 8	1·59071	1·59072	+ 1
III.	45 43 55	1·57486	1·57502	+16
II.	46 38 30	1·57203	1·57205	+ 2
III.	49 29 51	1·56329	1·56345	+16
III.	52 47 52	1·55354	1·55371	+17
III.	58 43 25	1·53729	1·53745	+16
III.	61 35 19	1·53016	1·53014	− 2
III.	63 12 16	1·52637	1·52644	+ 7
III.	66 9 35	1·51992	1·51981	− 9
III.	72 12 53	1·50877	1·50870	− 7
III.	75 30 40	1·50396	1·50387	− 9
III.	78 59 4	1·49991	1·49982	− 9
IV.	80 18 16	1·49865	1·49856	− 9
IV.	87 9 43	1·49467	1·49460	− 7
IV.	89 53 4	1·49430	1·49430	0

TABLE XIII.—Results of Theory for C.

	θ.	μ From Theory.	μ From Experiment.	Excess of Experiment.
I.	0° 20′ 35″	1·65435	1·65438	+ 3
II.	4 5 7	1·65335	1·65335	0
I.	7 39 25	1·65082	1·65078	− 4
I.	11 34 51	1·64635	1·64627	− 8
I.	17 23 14	1·63678	1·63684	+ 6
I.	20 42 48	1·62989	1·62991	+ 2
II.	23 34 21	1·62326	1·62341	+15
I.	24 33 38	1·62085	1·62095	+10
I.	25 28 23	1·61855	1·61862	+ 7
II.	29 6 1	1·60897	1·60906	+11
II.	34 38 36	1·59326	1·59336	+10
II.	38 51 23	1·58083	1·58091	+12
II.	41 0 26	1·57443	1·57449	+ 6
III.	45 51 25	1·56151	1·56163	+12
II.	46 59 20	1·55677	1·55692	+15
III.	49 11 23	1·55044	1·55058	+14
III.	52 31 57	1·54113	1·54116	+ 3
III.	58 31 47	1·52560	1·52567	+ 7
III.	61 47 0	1·51797	1·51805	+ 8
III.	63 3 25	1·51516	1·51523	+ 7
III.	66 23 19	1·50830	1·50833	+ 3
III.	72 29 31	1·49782	1·49786	+ 4
III.	75 49 4	1·49331	1·49331	0
III.	79 19 19	1·48955	1·48962	+ 7
IV.	80 6 35	1·48885	1·48887	+ 2
IV.	87 1 19	1·48495	1·48499	+ 4
IV.	89 42 4	1·48457	1·48457	0

Table XIII. gives the results for the ray C for the same values of the angle of incidence as those given in Tables XI. and XII. for F and *g*. This enables a comparison of the results to be more easily made for the three rays than if it were requisite to refer to X. In each case the results are similar.

The differences are least near the axes, being negative for F near the minor axis, and for *g* near both major and minor.

For C the errors are positive throughout, so that a small increase of the axes of the curve given by theory would, on the whole, bring theory and experiment into closer agreement.

For F the differences near the minor axis being negative, we should require to decrease the minor axis of the ellipse. This would increase slightly the positive errors, and render, on the whole, the variation from FRESNEL'S spheroid more marked, and greater than the variation of the red ray.

While for the violet ray, g, the differences near both axes are negative. To bring the two curves into agreement then we should require to decrease both the axes μ_1, μ_2. This would produce a corresponding increase in all the positive errors and render the variation from FRESNEL'S theory near the middle of the arc more marked than in the case of the red or green rays.

In fact, while for the red, supposing the variations in μ_1, μ_2 contemplated above to have been adopted, the greatest difference between theory and experiment would be about

·0001

for the green ray F it would rise to

·00015

and for the violet, g, to

·0002.

SECTION IV.

I. *Comparison with previous experiments.*
II. *Effect of variation of constants.*

As an additional proof of the accuracy of the experiments it may be worth while giving the results of a series of measurements covering the same ground as the second part, Table I., made some months previously. Since the prism did not occupy exactly the same position relative to the instrument as it did during the experiments in Table I., the values of the angle of incidence, and therefore of ψ', were slightly different to those in Table I.

In making the comparison, therefore, the results of calculation had to be altered by interpolation to give the values of μ corresponding to the values of ψ' in Table I.

The result is contained in Table XIV.

TABLE XIV.

μ from Table I.	μ from Experiments in December, 1878.	Difference.
1·64627	1·64627	0
1·64335	1·64332	3
1·64021	1·64010	11
1·63684	1·63681	3
1·63341	1·63339	3
1·62991	1·62994	3
1·62669	1·62663	6
1·62356	1·62365	9
1·62095	1·62097	2
1·61862	1·61859	3
1·61680	1·61680	0

The agreement between the two results is striking, and seems to show that we may assume the experimental results to be true with an error which is not greater than ·00005.

In my paper on a biaxal crystal I was able to show that the assumption of certain errors in the determination of the position of the plane of the prism with reference to the crystallographic axes led to results rather more in agreement with experiment than those obtained at first.

In the present instance this is impossible, for any change in the position of the plane of the prism would produce effects of almost exactly the same amount in the values of μ for the lines C and g; but the error we wish to correct in C is only half as great as that in g, and hence no change in the position of the plane can produce the required effect.

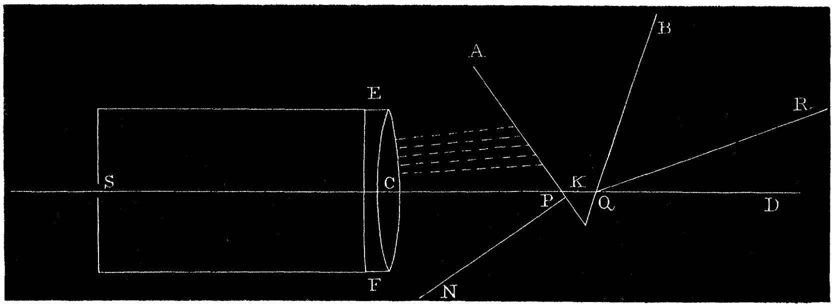
But, again, the telescopes used were not perfectly achromatic for the red and violet rays. I found usually little or no difference in the position of the focus for the lines C and F, but there was an appreciable difference between C and g.

If the collimator be focused so that the rays from the line C emerge parallel, those from g will be divergent. This may produce a variation in the angle of incidence between the waves C and g.

For the prisms were so placed that by turning the table on which they rested without altering the position of the collimator, either face of the prism could be made a face of incidence. To secure this the edge of the prism passed nearly through the axis of the collimator, and in most positions of the prism the light from only about half the collimator lens reached it.

A figure will make this clearer. S C (fig. 5) is the axis of the collimator, A P B the prism, A P being the face of incidence.

Fig. 5.



Almost all the light incident on A P passes through the upper part, C E, of the lens of the collimator. If the prism be turned round K, a point in the axis S C produced, so that B Q becomes the face of incidence, then only the lower portion, C F, of the collimator lens will be used.

Again, since the collimator is focused for red rays, they will be incident on the face A P in direction C P, and if P N be the normal to A P the angle of incidence will be C P N.

The violet rays, however, diverging as they do from a point on C S, will be incident on A P at various angles, most of which, however, will be less than C P N. By assuming then the violet rays to issue parallel from the object glass, we have made the angle of incidence for the violet too great.

Again, if Q R be any emergent ray, we have assumed the deviation to be measured by D Q R.

In the case of the violet rays this again will be too great, and too great by the same amount as the angle of incidence.

We must therefore consider the effect of decreasing the angle of incidence and the deviation by the same amount.

If ϕ be the angle of incidence, ψ the angle of emergence,

$$\psi = D + i - \phi$$
$$\delta\psi = \delta D - \delta\phi = 0$$
$$\delta\psi' + \delta\phi' = 0$$
$$\delta\phi' \text{ is negative since } \delta\phi \text{ is so}$$
$$\therefore \delta\psi' \text{ is positive.}$$

Hence ψ is unchanged, ψ' is increased.

The value of μ will therefore in all cases be decreased.

Now the experimental values of μ are already too great. Hence this alteration will tend to bring them more nearly into agreement with theory. The amount of error introduced depends on the angle of incidence. To find a general expression for it would be a work of difficulty owing to the complicated nature of the formulæ involved.

Let us therefore consider the effect of decreasing the angle of incidence and the deviation by 1′, (*a*) near minimum deviation, (*b*) near grazing incidence for prism I. The effects will be much the same for all the prisms.

We have from Table I.

ϕ	76° 8′	36° 8′
$D+i$	89° 25′ 5″	76° 4′
ϕ'	35° 57′ 5″	20° 59′ 21″
μ	1·65367	1·64623

By decreasing ϕ and $D+i$ each by 1′ we have the following values:—

ϕ'	35° 57′ 3″	20° 59′ 5″
μ	1·65355	1·64591

Thus near minimum deviation the change produced in μ amounts to about ·0003 while at grazing incidence it is only about ·0001.

Of course an error of the same kind occurs in the values of μ_1, μ_2. They, however, were determined from observations at nearly grazing incidence.

They may then be slightly too great.

To correct them completely for this error we should have to reduce the theoretical values by a small quantity nearly the same for all; while the experimental values require reducing by quantities which are greatest near minimum deviation, and decrease as we approach grazing incidence until they reach about the values of the corrections applied to μ_1 and μ_2.

The greatest error for the ray g does not exceed ·0002, so that the results of theory and experiment for g would be brought into very close agreement by supposing the violet rays of the light emerging from the collimator to be inclined to the red at angles not greater than 45″.

Thus, allowing for this probable divergency of the green and violet rays, it appears that HUYGHEN'S construction represents the result of experiment for the three rays of the hydrogen spectrum to a degree of approximation comparable with the probable error of the experiments.*

* In the abstract printed in the 'Proceedings of the Royal Society,' I had assumed that the violet rays issuing from an achromatic lens, focused so as to make the orange rays parallel, were convergent. From this it followed that the correction for want of parallelism tended to increase the difference between theory and experiment, and led me to the inference that HUYGHEN'S construction might be true for the red rays and yet differ appreciably from the truth for light of shorter wave length. Professor STOKES has since pointed out to me that the violet rays are in reality divergent, and that the error introduced by assuming them to be parallel really tends to correct the differences observed between theory and experiment, and so leads to the inference in the text that HUYGHEN'S construction is true for the three hydrogen rays within the limits of experimental error.