XVI. *On the Motion of Fluid, part of which is moving Rotationally and part Irrotationally.*

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Introduction.

Clebsch has shown that the components of the velocity of a fluid u, v, w, parallel to rectangular axes x, y, z, may always be expressed thus

$$u=\frac{d\chi}{dx}+\lambda\frac{d\psi}{dx},\quad v=\frac{d\chi}{dy}+\lambda\frac{d\psi}{dy},\quad w=\frac{d\chi}{dz}+\lambda\frac{d\psi}{dz};$$

where λ, ψ are systems of surfaces whose intersections determine the vortex lines; and the pressure satisfies an equation which is* equivalent to the following

$$\frac{p}{\rho}+\text{V}=-\frac{d\chi}{dt}-\tfrac{1}{2}\left\{\left(\frac{d\chi}{dx}\right)^2+\left(\frac{d\chi}{dy}\right)^2+\left(\frac{d\chi}{dz}\right)^2\right\}+\tfrac{1}{2}\lambda^2\left\{\left(\frac{d\psi}{dx}\right)^2+\left(\frac{d\psi}{dy}\right)^2+\left(\frac{d\psi}{dz}\right)^2\right\}$$

where p is the pressure, ρ the density, and V the potential of the forces acting on the liquid.

It is shown in this paper that an equation of a complicated nature in λ only can be obtained in the following cases (that is to say, as in cases of irrotational motion, the determination of the motion depends on the solution of a single equation only):—

(1.) Plane motion, referred to rectangular coordinates x, y.

The equation is somewhat simpler when the vortex surfaces are of invariable form, and move parallel to one of the axes of coordinates with arbitrary velocity.

(2.) Plane motion, referred to polar coordinates r, θ.

The equation is somewhat simpler when the vortex surfaces are of invariable form, and rotate about the origin with arbitrary angular velocity.

(3.) Motion symmetrical with regard to the axis of z in planes passing through it, referred to cylindric coordinates r, z.

The equation is somewhat simpler, when the vortex surfaces are of invariable form, and move parallel to the axis of z with arbitrary velocity.

* British Association Report for 1881, p. 62.

Suppose that in any of these cases any particular integral of the equation in λ is taken.

It is shown that the components of the velocity can be expressed in terms of λ and differential coefficients of λ, and that the current function is also known.

In the case of a fluid, part of which is moving rotationally and part irrotationally, the boundary surface separating the rotationally moving fluid from that which is moving irrotationally contains the same vortex lines, and may be taken at the surface $\lambda=0$.

Now, if the integral taken of the equation in λ do actually correspond to a case of fluid motion in which part of the fluid is moving rotationally and part irrotationally, the most obvious way to find the irrotational motion will be to find its current function from the conditions supplied by the fact that the components of the velocity are continuous at the surface $\lambda=0$. Examples I. and III. of this paper have been solved in this manner.

If after taking any integral of the equation in λ it be found theoretically impossible to determine the current function of an irrotational motion outside the surface $\lambda=0$, which shall be continuous with the rotational motion inside it, then the integral in question does not correspond to such a case of fluid motion.

In this method no assumption is made as to the distribution of the vortex lines (as in the method of HELMHOLTZ) before commencing the determination of the irrotational motion.

If, however, the rotational motion be known, the components of the velocity are known for this part of the fluid. Let the components of the velocity be expressed in CLEBSCH'S forms, so that χ, λ, ψ are known.

Moreover, let the forms be so arranged that the surface separating the rotationally moving fluid from that which is moving irrotationally is the surface $\lambda=0$.

Then at this surface the components of the velocity are $\frac{d\chi}{dx}, \frac{d\chi}{dy}, \frac{d\chi}{dz}$.

Now, obtain in any manner a velocity potential ϕ for space outside $\lambda=0$ continuous with χ all over the surface $\lambda=0$. This is theoretically possible always.

If the velocity potential so obtained make the velocity and pressure continuous all over the surface $\lambda=0$, then a possible case of motion will have been obtained.

The conditions to be satisfied in order that the velocity may be continuous at the surface $\lambda=0$ are that there $\frac{d\chi}{dx}=\frac{d\phi}{dx}, \frac{d\chi}{dy}=\frac{d\phi}{dy}, \frac{d\chi}{dz}=\frac{d\phi}{dz}$. In order that the pressure may be also continuous, it is further necessary that $\frac{d\chi}{dt}=\frac{d\phi}{dt}$ all over the surface $\lambda=0$.

The most obvious way of obtaining the velocity potential will be to apply HELMHOLTZ'S method of finding the components of the velocity in terms of the supposed distribution of magnetic matter throughout the space occupied by the rotationally moving fluid.

It must, however, be remembered, as is remarked by Mr. HICKS in his report to the

British Association on "Recent Progress in Hydrodynamics," Part 1,* "That the results refer to the cyclic motion of the fluid as determined by the supposed distribution of magnetic matter, and do not give the most general motion possible." It appears also from Examples I. and III. of this paper that it is not possible to assume arbitrarily the distribution of vortex lines, even when it can be shown that the equations of motion are satisfied at all points where the fluid is moving rotationally, and then to proceed to calculate the irrotational motion by means of the supposed distribution of magnetic matter. For in these examples, values of the components of the velocity of a rotational motion, satisfying the equations of motion throughout a finite portion of the plane of x, y, are found. Thus the distribution of vortex lines, and, therefore, that of the supposed magnetic matter over a finite portion of the plane of x, y is known. The surfaces that always contain the same vortex filaments are found. Inside one of these the supposed magnetic matter is distributed, the current function at external points is calculated by HELMHOLTZ's method, and it is shown that the velocity thence deduced is not continuous with the velocity of the rotational motion at the surface, which separates the rotationally moving liquid from that moving irrotationally.

Another way (suggested by CLEBSCH's forms) of obtaining the velocity potential will be as follows:—

Calculate the quantity $\rho=-\frac{1}{4\pi}\left(\frac{d^2\chi}{dx^2}+\frac{d^2\chi}{dy^2}+\frac{d^2\chi}{dz^2}\right)$.

Treating ρ as the density of a material distribution inside $\lambda=0$, taking no account of the value of ρ outside the surface $\lambda=0$, obtain the potential of this distribution. Let the potential inside $\lambda=0$ be χ', and outside let it be ϕ.

χ' will, in general, differ from χ; first, because χ may contain many-valued terms, which may be denoted by P, satisfying LAPLACE's equation; and, secondly, because $\chi-\mathrm{P}$ may be the potential of a distribution of matter, part of which is outside $\lambda=0$.

Accordingly, it is necessary to examine whether it is possible to find many-valued terms P satisfying LAPLACE's equation such that $\chi'+\mathrm{P}=\chi$.

Then $\phi+\mathrm{P}$ will be the velocity potential of the irrotational motion, provided that it give zero velocity at infinity.

Example II. of this paper is solved in this manner. It might also have been solved by HELMHOLTZ's method.

The few illustrations which follow are a first attempt to apply the theory to particular cases.

Example I. treats of the motion of an elliptic vortex cylinder of invariable form parallel to one of its axes with arbitrary velocity. The irrotational motion outside the cylinder cannot be supposed to extend to an infinite distance.

Example II. treats of KIRCHHOFF's elliptic vortex cylinder, in which the angular velocity of the rotation of the cylinder is a function of the vortex strength, and the axes of the elliptic section of the cylinder.

* Report for 1881, Part I., p. 64.

Example III. treats of the revolution of an elliptic vortex cylinder round its axis, where the angular velocity is not restricted as in the last case. The irrotational motion outside may be supposed to be limited by a smooth rigid confocal elliptic cylindric surface, rotating with the same angular velocity. The last example is the particular case of this, obtained by supposing the elliptic section of the external confocal cylinder to become infinite.

Example IV. treats of the motion of the fluid in a fixed circular cylindric surface, where the vortex strength is any function of the distance from the axis, the irrotational motion continuous therewith being supposed to extend to an infinite distance.

Example V. treats of a possible case of rotational motion inside a certain hollow smooth rigid surface of annular form, which moves parallel to its straight axis with arbitrary velocity.

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1. CLEBSCH'S* forms for the components of the velocity of a liquid u, v, w parallel to fixed rectangular axes x, y, z in space are:—

$$u=\frac{d\chi}{dx}+\lambda\frac{d\psi}{dx}; \quad v=\frac{d\chi}{dy}+\lambda\frac{d\psi}{dy}; \quad w=\frac{d\chi}{dz}+\lambda\frac{d\psi}{dz};$$

where the surfaces $\lambda=$ const., $\psi=$ const. determine by their intersections the vortex lines, and always contain the same particles of liquid.

If $F(\lambda, \psi)$ be an arbitrary function of λ, ψ

$$\begin{aligned}u&=\frac{d\chi}{dx}+\lambda\frac{d\psi}{dx}=\frac{d}{dx}(\chi-F(\lambda, \psi))+\left(\frac{\delta F(\lambda, \psi)}{\delta\psi}+\lambda\right)\frac{d\psi}{dx}+\frac{\delta F(\lambda, \psi)}{\delta\lambda}\frac{d\lambda}{dx}\\&=\frac{d\chi'}{dx}+\lambda'\frac{d\psi'}{dx}\end{aligned}$$

and similar expressions for v, w.

Thus these expressions for the components of the velocity are still in CLEBSCH'S form.

λ', ψ' are each functions of λ, ψ.

χ' satisfies the same equation as χ.

* Taking as independent variables three families of surfaces, always containing the same particles, and the time, the writer obtained independently CLEBSCH'S forms in an article published in the Quarterly Journal of Pure and Applied Mathematics, February, 1880, vol. xvii., entitled "On Some Properties of the Equations of Hydrodynamics."

A demonstration of the same forms for any fluid in which the density is any function of the pressure is contained as a particular case in a paper entitled "On Some General Equations which include the Equations of Hydrodynamics," which is published in the Transactions of the Cambridge Philosophical Society, vol. xiv., part i., the writer having previously seen CLEBSCH'S paper, "Ueber Die Integration der hydrodynamischen Gleichungen," 'Crelle,' Bd. lvi., p. 1.

Moreover, since $F(\lambda, \psi)$ is an arbitrary function of λ, ψ; it can be so chosen that λ' may be any required function of λ, ψ; *i.e.*, any vortex sheet.

Therefore the λ in CLEBSCH's expressions for u, v, w may be considered as the surface of any vortex sheet; and, consequently, as the surface separating the rotationally moving fluid from that which is moving irrotationally.

Therefore, if ϕ be the velocity potential of the irrotationally moving fluid, all over the surface $\lambda=0$, supposing the motion continuous there;

$$\frac{d\chi}{dx}=\frac{d\phi}{dx},\ \frac{d\chi}{dy}=\frac{d\phi}{dy},\ \frac{d\chi}{dz}=\frac{d\phi}{dz}.$$

Since also

$$\frac{p}{\rho}+V=-\frac{d\chi}{dt}-\tfrac{1}{2}\left(\left(\frac{d\chi}{dx}\right)^2+\left(\frac{d\chi}{dy}\right)^2+\left(\frac{d\chi}{dz}\right)^2\right)+\tfrac{1}{2}\lambda^2\left(\left(\frac{d\psi}{dx}\right)^2+\left(\frac{d\psi}{dy}\right)^2+\left(\frac{d\psi}{dz}\right)^2\right)$$

in the rotationally moving fluid, and

$$\frac{p}{\rho}+V=-\frac{d\phi}{dt}-\tfrac{1}{2}\left(\left(\frac{d\phi}{dx}\right)^2+\left(\frac{d\phi}{dy}\right)^2+\left(\frac{d\phi}{dz}\right)^2\right)$$

in the irrotationally moving fluid, it follows that the condition for the continuity of the pressure at the surface separating the rotationally moving fluid from that moving irrotationally is that $\frac{d\phi}{dt}=\frac{d\chi}{dt}$ all over the surface $\lambda=0$.

Now suppose that there exists a solution of the three equations to which χ, λ, ψ are subject as given by CLEBSCH; then to find ϕ, it is necessary to find it so as to satisfy the above surface conditions.

In any case in which χ is the potential of a distribution of matter inside the surface $\lambda=0$, together with many valued terms satisfying LAPLACE's equation, then ϕ is the potential of this distribution calculated for a point outside $\lambda=0$, together with the same many valued terms, provided that it give zero values for the components of the velocity at infinity.

With regard to the supposed distribution of matter, its total mass must be zero, in the case of an incompressible fluid.

For total mass of supposed distribution of matter $=-\frac{1}{4\pi}\iint\frac{d\phi}{dn}\,dS$, the integration being extended over the surface $\lambda=0$.

But this $=-\frac{1}{4\pi}$ (total flux outwards across the surface)$=0$.

Therefore total mass of supposed distribution of matter $=0$.

2. Plane Motion. Rectangular Coordinates.

In the ordinary notation the equations are

$$\frac{du}{dt}+u\frac{du}{dx}+v\frac{du}{dy}=-\frac{d}{dx}\left(\int\frac{dp}{\rho}+\mathrm{V}\right)$$

$$\frac{dv}{dt}+u\frac{dv}{dx}+v\frac{dv}{dy}=-\frac{d}{dy}\left(\int\frac{dp}{\rho}+\mathrm{V}\right)$$

$$\frac{d\rho}{dt}+\frac{d}{dx}(\rho u)+\frac{d}{dy}(\rho v)=0$$

From which can be deduced

$$\left(\frac{d}{dt}+u\frac{d}{dx}+v\frac{d}{dy}\right)\left\{\frac{\frac{dv}{dx}-\frac{du}{dy}}{\rho}\right\}=0$$

Now regarding this as a particular case of motion in three dimensions in which $w=0$, the motion being parallel to the plane $z=0$, it is possible to put by means of CLEBSCH's forms

$$u=\frac{d\chi}{dx}+\lambda\frac{d\psi}{dx};\ v=\frac{d\chi}{dy}+\lambda\frac{d\psi}{dy}.$$

Therefore

$$\left(\frac{d}{dt}+u\frac{d}{dx}+v\frac{d}{dy}\right)\left\{\frac{\frac{d\lambda}{dx}\frac{d\psi}{dy}-\frac{d\lambda}{dy}\frac{d\psi}{dx}}{\rho}\right\}=0$$

This result has been deduced from the three equations of motion, and CLEBSCH's forms for the components of the velocity.

But it can be deduced from the equation of continuity alone and the following equations known to be satisfied by λ, ψ.

$$\frac{d\lambda}{dt}+u\frac{d\lambda}{dx}+v\frac{d\lambda}{dy}=0$$

$$\frac{d\psi}{dt}+u\frac{d\psi}{dx}+v\frac{d\psi}{dy}=0$$

Therefore

$$u=\frac{\begin{vmatrix}-\frac{d\lambda}{dt} & \frac{d\lambda}{dy}\\ -\frac{d\psi}{dt} & \frac{d\psi}{dy}\end{vmatrix}}{\begin{vmatrix}\frac{d\lambda}{dx} & \frac{d\lambda}{dy}\\ \frac{d\psi}{dx} & \frac{d\psi}{dy}\end{vmatrix}}\quad\text{and}\quad v=\frac{\begin{vmatrix}\frac{d\lambda}{dx} & -\frac{d\lambda}{dt}\\ \frac{d\psi}{dx} & -\frac{d\psi}{dt}\end{vmatrix}}{\begin{vmatrix}\frac{d\lambda}{dx} & \frac{d\lambda}{dy}\\ \frac{d\psi}{dx} & \frac{d\psi}{dy}\end{vmatrix}}$$

Therefore

$$\frac{du}{dx}+\frac{dv}{dy}=\frac{\begin{vmatrix}-\frac{d^2\lambda}{dxdt} & \frac{d\lambda}{dy}\\ -\frac{d^2\psi}{dxdt} & \frac{d\psi}{dy}\end{vmatrix}+\begin{vmatrix}-\frac{d\lambda}{dt} & \frac{d^2\lambda}{dxdy}\\ -\frac{d\psi}{dt} & \frac{d^2\psi}{dxdy}\end{vmatrix}}{\begin{vmatrix}\frac{d\lambda}{dx} & \frac{d\lambda}{dy}\\ \frac{d\psi}{dx} & \frac{d\psi}{dy}\end{vmatrix}}-\frac{\begin{vmatrix}-\frac{d\lambda}{dt} & \frac{d\lambda}{dy}\\ -\frac{d\psi}{dt} & \frac{d\psi}{dy}\end{vmatrix}}{\begin{vmatrix}\frac{d\lambda}{dx} & \frac{d\lambda}{dy}\\ \frac{d\psi}{dx} & \frac{d\psi}{dy}\end{vmatrix}^2}\frac{d}{dx}\begin{vmatrix}\frac{d\lambda}{dx} & \frac{d\lambda}{dy}\\ \frac{d\psi}{dx} & \frac{d\psi}{dy}\end{vmatrix}$$

$$+\frac{\begin{vmatrix}\frac{d^2\lambda}{dxdy} & -\frac{d\lambda}{dt}\\ \frac{d^2\psi}{dxdy} & -\frac{d\psi}{dt}\end{vmatrix}+\begin{vmatrix}\frac{d\lambda}{dx} & -\frac{d^2\lambda}{dydt}\\ \frac{d\psi}{dx} & -\frac{d^2\psi}{dydt}\end{vmatrix}}{\begin{vmatrix}\frac{d\lambda}{dx} & \frac{d\lambda}{dy}\\ \frac{d\psi}{dx} & \frac{d\psi}{dy}\end{vmatrix}}-\frac{\begin{vmatrix}\frac{d\lambda}{dx} & -\frac{d\lambda}{dt}\\ \frac{d\psi}{dx} & -\frac{d\psi}{dt}\end{vmatrix}}{\begin{vmatrix}\frac{d\lambda}{dx} & \frac{d\lambda}{dy}\\ \frac{d\psi}{dx} & \frac{d\psi}{dy}\end{vmatrix}^2}\frac{d}{dy}\begin{vmatrix}\frac{d\lambda}{dx} & \frac{d\lambda}{dy}\\ \frac{d\psi}{dx} & \frac{d\psi}{dy}\end{vmatrix}$$

But from the equation of continuity

$$\frac{du}{dx}+\frac{dv}{dy}=-\frac{1}{\rho}\left(\frac{d}{dt}+u\frac{d}{dx}+v\frac{d}{dy}\right)\rho$$

Therefore

$$-\frac{1}{\rho}\left(\frac{d}{dt}+u\frac{d}{dx}+v\frac{d}{dy}\right)\rho$$

$$=\frac{\begin{vmatrix}-\frac{d^2\lambda}{dxdt} & \frac{d\lambda}{dy}\\ -\frac{d^2\psi}{dxdt} & \frac{d\psi}{dy}\end{vmatrix}+\begin{vmatrix}\frac{d\lambda}{dx} & -\frac{d^2\lambda}{dydt}\\ \frac{d\psi}{dx} & -\frac{d^2\psi}{dydt}\end{vmatrix}}{\begin{vmatrix}\frac{d\lambda}{dx} & \frac{d\lambda}{dy}\\ \frac{d\psi}{dx} & \frac{d\psi}{dy}\end{vmatrix}}-\frac{\begin{vmatrix}-\frac{d\lambda}{dt} & \frac{d\lambda}{dy}\\ -\frac{d\psi}{dt} & \frac{d\psi}{dy}\end{vmatrix}}{\begin{vmatrix}\frac{d\lambda}{dx} & \frac{d\lambda}{dy}\\ \frac{d\psi}{dx} & \frac{d\psi}{dy}\end{vmatrix}^2}\frac{d}{dx}\begin{vmatrix}\frac{d\lambda}{dx} & \frac{d\lambda}{dy}\\ \frac{d\psi}{dx} & \frac{d\psi}{dy}\end{vmatrix}$$

$$-\frac{\begin{vmatrix}\frac{d\lambda}{dx} & -\frac{d\lambda}{dt}\\ \frac{d\psi}{dx} & -\frac{d\psi}{dt}\end{vmatrix}}{\begin{vmatrix}\frac{d\lambda}{dx} & \frac{d\lambda}{dy}\\ \frac{d\psi}{dx} & \frac{d\psi}{dy}\end{vmatrix}^2}\frac{d}{dy}\begin{vmatrix}\frac{d\lambda}{dx} & \frac{d\lambda}{dy}\\ \frac{d\psi}{dx} & \frac{d\psi}{dy}\end{vmatrix}$$

Therefore

$$-\frac{1}{\rho}\left(\frac{d}{dt}+u\frac{d}{dx}+v\frac{d}{dy}\right)\rho=-\frac{1}{\begin{vmatrix}\frac{d\lambda}{dx} & \frac{d\lambda}{dy}\\ \frac{d\psi}{dx} & \frac{d\psi}{dy}\end{vmatrix}}\left(\frac{d}{dt}+u\frac{d}{dx}+v\frac{d}{dy}\right)\begin{vmatrix}\frac{d\lambda}{dx} & \frac{d\lambda}{dy}\\ \frac{d\psi}{dx} & \frac{d\psi}{dy}\end{vmatrix}$$

Therefore

$$\left(\frac{d}{dt}+u\frac{d}{dx}+v\frac{d}{dy}\right)\frac{\frac{d\lambda}{dx}\frac{d\psi}{dy}-\frac{d\lambda}{dy}\frac{d\psi}{dx}}{\rho}=0 \text{ as before.}$$

But since also

$$\left(\frac{d}{dt}+u\frac{d}{dx}+v\frac{d}{dy}\right)\lambda=0$$

and

$$\left(\frac{d}{dt}+u\frac{d}{dx}+v\frac{d}{dy}\right)\psi=0$$

Therefore

$$\frac{\frac{d\lambda}{dx}\frac{d\psi}{dy}-\frac{d\lambda}{dy}\frac{d\psi}{dx}}{\rho}=\text{ some function of } \lambda,\ \psi=f(\lambda,\ \psi)$$

3. In what immediately follows ρ will be supposed constant. Using suffixes to denote differential coefficients

$$\lambda_x\psi_y-\lambda_y\psi_x=f(\lambda,\ \psi)$$

Now let $g(\lambda,\ \psi)$ be a function of λ, ψ such that

$$\frac{\delta g(\lambda,\ \psi)}{\delta\psi}=\frac{1}{f(\lambda,\ \psi)}.$$

Therefore

$$\lambda_x\left(\frac{\delta g(\lambda,\ \psi)}{\delta\psi}\psi_y\right)-\lambda_y\left(\frac{\delta g(\lambda,\ \psi)}{\delta\psi}\psi_x\right)=1$$

therefore

$$\lambda_x\left(\frac{\delta g(\lambda,\ \psi)}{\delta\psi}\psi_y+\frac{\delta g(\lambda,\ \psi)}{\delta\lambda}\lambda_y\right)-\lambda_y\left(\frac{\delta g(\lambda,\ \psi)}{\delta\psi}\psi_x+\frac{\delta g(\lambda,\ \psi)}{\delta\lambda}\lambda_x\right)=1$$

therefore

$$\lambda_x g_y-\lambda_y g_x=1$$

Treating this as a partial differential equation for g, the auxiliary system of equations is

$$\frac{dx}{-\lambda_y}=\frac{dy}{\lambda_x}=\frac{dg}{1}$$

Whence each of these

$$=\frac{\lambda_x dx+\lambda_y dy}{0}$$

Since, in the differential equation, there are no differential coefficients with regard to t, $\lambda=$const. is one integral of the auxiliary system.

By means of the equation $\lambda=$const., y can be expressed as a function of the constant, x, and t. As the constant will have to be replaced by λ afterwards, it may be said that y can be expressed as a function of λ, x, t; and when this value of y is substituted in $-\lambda_y$, let the result be denoted by $(-\lambda_y)^{\lambda}_{\substack{x\\t}}$. Let differentials of the variables λ, x, t regarded as independent be denoted by $\delta\lambda$, δx, δt respectively.

Then $g=-\int\frac{\delta x}{(\lambda_y)^{\lambda}_{\substack{x\\t}}}+$ an arbitrary function of λ, t.

It will be convenient to write the arbitrary function in the form $\frac{\delta F(\lambda, t)}{\delta\lambda}$

Therefore

$$g=-\int\frac{\delta x}{(\lambda_y)^{\lambda}_{\substack{x\\t}}}+\frac{\delta F(\lambda, t)}{\delta\lambda}$$

Therefore g will appear in the form $G(\lambda, x, t)+\frac{\delta F(\lambda, t)}{\delta\lambda}$.

The equations of the vortex sheets are functions of ψ, λ and therefore of g, λ.

Now

$$\lambda_t+u\lambda_x+v\lambda_y=0$$

and

$$g_t+ug_x+vg_y=0$$

therefore

$$\frac{\delta G}{\delta t}+\frac{\delta G}{\delta\lambda}\lambda_t+\frac{\delta^2 F(\lambda, t)}{\delta t\delta\lambda}+\frac{\delta^2 F(\lambda, t)}{\delta\lambda^2}\lambda_t$$

$$+u\left(\frac{\delta G}{\delta x}+\frac{\delta G}{\delta\lambda}\lambda_x+\frac{\delta^2 F(\lambda, t)}{\delta\lambda^2}\lambda_x\right)+v\left(\frac{\delta G}{\delta\lambda}\lambda_y+\frac{\delta^2 F(\lambda, t)}{\delta\lambda^2}\lambda_y\right)=0$$

that is

$$\frac{\delta G}{\delta t}+\frac{\delta^2 F(\lambda, t)}{\delta t\delta\lambda}+u\frac{\delta G}{\delta x}+\left(\frac{\delta G}{\delta\lambda}+\frac{\delta^2 F(\lambda, t)}{\delta\lambda^2}\right)(\lambda_t+u\lambda_x+v\lambda_y)=0$$

therefore

$$\frac{\delta G}{\delta t}+\frac{\delta^2 F(\lambda, t)}{\delta t\delta\lambda}+u\frac{\delta G}{\delta x}=0$$

But

$$\frac{\delta G}{\delta x}=-\frac{1}{\lambda_y}$$

therefore

$$u=\lambda_y\left(\frac{\delta G}{\delta t}+\frac{\delta^2 F(\lambda, t)}{\delta t\delta\lambda}\right)$$

substituting this value of u in the equation

$$\lambda_t+u\lambda_x+v\lambda_y=0$$

it may be shown that

$$v=-\frac{\lambda_t}{\lambda_y}-\lambda_x\left(\frac{\delta G}{\delta t}+\frac{\delta^2 F(\lambda, t)}{\delta t\delta\lambda}\right)$$

To determine the current function Λ, there are the equations

$$\frac{d\Lambda}{dy}=\lambda_y\left(-\frac{\delta}{\delta t}\int\frac{\delta x}{(\lambda_y)^{\lambda}_{x\atop t}}+\frac{\delta^2 F(\lambda, t)}{\delta\lambda\delta t}\right)$$

$$-\frac{d\Lambda}{dx}=-\frac{\lambda_t}{\lambda_y}-\lambda_x\left(-\frac{\delta}{\delta t}\int\frac{\delta x}{(\lambda_y)^{\lambda}_{x\atop t}}+\frac{\delta^2 F(\lambda, t)}{\delta\lambda\delta t}\right)$$

From the first of these equations

$$\Lambda=\frac{\delta F(\lambda, t)}{\delta t}-\int\lambda_y dy\left[\frac{\delta}{\delta t}\int\frac{\delta x}{(\lambda_y)^{\lambda}_{x\atop t}}\right]+\psi(x, t)$$

where ψ is the symbol of an arbitrary function.

Now $\frac{\delta}{\delta t}\int\frac{\delta x}{(\lambda_y)^{\lambda}_{x\atop t}}$ is a function of λ, x, t; and y occurs in it only because it is contained in λ, therefore

$$\int\lambda_y dy\left[\frac{\delta}{\delta t}\int\frac{\delta x}{(\lambda_y)^{\lambda}_{x\atop t}}\right]=\int\delta\lambda\left[\frac{\delta}{\delta t}\int\frac{\delta x}{(\lambda_y)^{\lambda}_{x\atop t}}\right]=\frac{\delta}{\delta t}\int\delta\lambda\int\frac{\delta x}{(\lambda_y)^{\lambda}_{x\atop t}}+\text{arbitrary function of } x \text{ and } t$$

but the arbitrary function of x and t may be supposed included in $\psi(x, t)$, therefore

$$\Lambda=\frac{\delta F(\lambda, t)}{\delta t}-\frac{\delta}{\delta t}\int\delta\lambda\int\frac{\delta x}{(\lambda_y)^{\lambda}_{x\atop t}}+\psi(x, t)$$

therefore

$$\frac{d\Lambda}{dx}=\frac{\delta^2 F(\lambda, t)}{\delta\lambda\delta t}\lambda_x-\frac{d}{dx}\frac{\delta}{\delta t}\int\delta\lambda\int\frac{\delta x}{(\lambda_y)^{\lambda}_{x\atop t}}+\frac{d}{dx}\psi(x, t)$$

But

$$\frac{d}{dx}=\frac{\delta}{\delta x}+\lambda_x\frac{\delta}{\delta\lambda}$$

$$\frac{d}{dy}=\lambda_y\frac{\delta}{\delta\lambda}$$

$$\frac{d}{dt}=\frac{\delta}{\delta t}+\lambda_t\frac{\delta}{\delta\lambda}$$

whence

$$\frac{\delta}{\delta t}=\frac{d}{dt}-\frac{\lambda_t}{\lambda_y}\frac{d}{dy}$$

therefore

$$\frac{d\Lambda}{dx}=\frac{\delta^2 \mathrm{F}(\lambda, t)}{\delta\lambda\delta t}\lambda_x-\left(\frac{\delta}{\delta x}+\lambda_x\frac{\delta}{\delta\lambda}\right)\frac{\delta}{\delta t}\int\delta\lambda\int\frac{\delta x}{(\lambda_y)^{\lambda}_{\substack{x\\t}}}+\left(\frac{\delta}{\delta x}+\lambda_x\frac{\delta}{\delta\lambda}\right)\psi(x, t)$$

$$=\frac{\delta^2 \mathrm{F}(\lambda, t)}{\delta\lambda\delta t}\lambda_x-\frac{\delta}{\delta t}\int\frac{\delta\lambda}{(\lambda_y)^{\lambda}_{\substack{x\\t}}}-\lambda_x\frac{\delta}{\delta t}\int\frac{\delta x}{(\lambda_y)^{\lambda}_{\substack{x\\t}}}+\frac{\delta}{\delta x}\psi(x, t)$$

But

$$\frac{\delta}{\delta t}\int\frac{\delta\lambda}{(\lambda_y)^{\lambda}_{\substack{x\\t}}}=\int\delta\lambda\frac{\delta}{\delta t}\left(\frac{1}{(\lambda_y)^{\lambda}_{\substack{x\\t}}}\right)=\int\delta\lambda\left(\frac{d}{dt}-\frac{\lambda_t}{\lambda_y}\frac{d}{dy}\right)\frac{1}{\lambda_y}$$

$$=\int\frac{\delta\lambda}{\lambda_y}\frac{d}{dy}\left(-\frac{\lambda_t}{\lambda_y}\right)+\text{arbitrary function of } x \text{ and } t$$

Before the last integration can be performed $\frac{1}{\lambda_y}\frac{d}{dy}\left(-\frac{\lambda_t}{\lambda_y}\right)$ must be expressed as a function of λ, x, t. If $-\frac{\lambda_t}{\lambda_y}$ be expressed as a function of λ, x, t then y can occur in it only through λ. Therefore

$$\frac{d}{dy}\left(-\frac{\lambda_t}{\lambda_y}\right)=\lambda_y\frac{\delta}{\delta\lambda}\left(-\frac{\lambda_t}{\lambda_y}\right)$$

Therefore

$$\frac{\delta}{\delta t}\int\frac{\delta\lambda}{(\lambda_y)^{\lambda}_{\substack{x\\t}}}=\int\delta\lambda\frac{\delta}{\delta\lambda}\left(-\frac{\lambda_t}{\lambda_y}\right)=-\frac{\lambda_t}{\lambda_y}+\Phi(x, t)$$

where Φ is the symbol of an arbitrary function.
therefore

$$\frac{d\Lambda}{dx}=\lambda_x\left(-\frac{\delta}{\delta t}\int\frac{\delta x}{(\lambda_y)^{\lambda}_{\substack{x\\t}}}+\frac{\delta^2 \mathrm{F}(\lambda, t)}{\delta\lambda\delta t}\right)+\frac{\lambda_t}{\lambda_y}-\Phi(x, t)+\frac{\delta}{\delta x}\psi(x, t)$$

Hence choosing the arbitrary function $\psi(x, t)$ so that

$$\frac{\delta}{\delta x}\psi(x, t)=\Phi(x, t)$$

this value of $\frac{d\Lambda}{dx}$ agrees with its known value.

And

$$\psi(x, t)=\int\delta x\Phi(x, t)=\int\delta x\frac{\delta}{\delta t}\int\frac{\delta\lambda}{(\lambda_y)^{\lambda}_{\substack{x\\t}}}+\int\left(\frac{\lambda_t}{\lambda_y}\right)^{\lambda}_{\substack{x\\t}}.\ \delta x=\int\delta x\left(\frac{\lambda_t}{\lambda_y}\right)^{\lambda}_{\substack{x\\t}}+\frac{\delta}{\delta t}\int\delta\lambda\int\frac{\delta x}{(\lambda_y)^{\lambda}_{\substack{x\\t}}}+\mathrm{P}(\lambda, t)+\mathrm{Q}(x, t)$$

where P and Q are arbitrary functions introduced in consequence of the change in the order of integration.

Therefore

$$\Lambda=\frac{\delta F(\lambda, t)}{\delta t}+\int \delta x\left(\frac{\lambda_t}{\lambda_y}\right)_{x\,t}^{\lambda}+P(\lambda, t)+Q(x, t).$$

Comparing now $\frac{d\Lambda}{dx}$ and $\frac{d\Lambda}{dy}$ with their known values, it will follow that $\frac{dQ(x, t)}{dx}=0$, therefore $Q(x, t)$ is a function of t only and may be considered to be included in $P(\lambda, t)$.

As $\frac{\delta F(\lambda, t)}{\delta t}$ may also be considered to be included in it, it follows* that

$$\Lambda=K(\lambda, t)+\int \delta x\left(\frac{\lambda_t}{\lambda_y}\right)_{x\,t}^{\lambda}$$

[This form of Λ may be obtained much more conveniently thus.

Since $\lambda_t+u\lambda_x+v\lambda_y=0$ and $u=\frac{d\Lambda}{dy}$, $v=-\frac{d\Lambda}{dx}$ it follows that

$$\lambda_t+\frac{d\Lambda}{dy}\lambda_x-\frac{d\Lambda}{dx}\lambda_y=0.$$

therefore

$$\Lambda=K(\lambda, t)+\int \delta x\left(\frac{\lambda_t}{\lambda_y}\right)_{x\,t}^{\lambda}.$$

The same way of obtaining Λ is applicable to Arts. 5 and 8.—August 30th, 1884.]

* The form in which Λ appears does not appear to be related to y and $-x$ in the same way, as would be expected.

But denoting differentials of λ, y, t by $\delta\lambda$, δy, δt, it may be shown that

$$\Lambda=M(\lambda, t)-\int \delta y\left(\frac{\lambda_t}{\lambda_x}\right)_{y\,t}^{\lambda}$$

The two forms of Λ will agree if

$$\int \delta x\left(\frac{\lambda_t}{\lambda_y}\right)_{x\,t}^{\lambda}+\int \delta y\left(\frac{\lambda_t}{\lambda_x}\right)_{y\,t}^{\lambda}=\text{a function of } \lambda, t.$$

Suppose that

$$\int \delta y\left(\frac{\lambda_t}{\lambda_x}\right)_{y\,t}^{\lambda}=R, \text{ then } \frac{\delta R}{\delta y}=\frac{\lambda_t}{\lambda_x}$$

But

$$\frac{d}{dx}=\lambda_x\frac{\delta}{\delta\lambda};\ \frac{d}{dy}=\frac{\delta}{\delta y}+\lambda_y\frac{\delta}{\delta\lambda};\ \frac{d}{dt}=\frac{\delta}{\delta t}+\lambda_t\frac{\delta}{\delta\lambda}, \text{ whence } \frac{\delta}{\delta y}=\frac{d}{dy}-\frac{\lambda_y}{\lambda_x}\frac{d}{dx}$$

therefore

$$\frac{dR}{dy}-\frac{\lambda_y}{\lambda_x}\frac{dR}{dx}=\frac{\lambda_t}{\lambda_x}$$

whence

$$R=-\int \delta x\left(\frac{\lambda_t}{\lambda_y}\right)_{x\,t}^{\lambda}+\Phi(\lambda, t)$$

therefore

$$\int \delta x\left(\frac{\lambda_t}{\lambda_y}\right)_{x\,t}^{\lambda}+\int \delta y\left(\frac{\lambda_t}{\lambda_x}\right)_{y\,t}^{\lambda}=\Phi(\lambda, t)$$

so that the two forms agree.

But $v_x-u_y=f(\lambda, \psi)=$ some function of λ, g,
therefore

$$\frac{d^2\Lambda}{dx^2}+\frac{d^2\Lambda}{dy^2}=\mathrm{H}(\lambda, g)$$

where H is an arbitrary function,
therefore

$$\left(\frac{d^2}{dx^2}+\frac{d^2}{dy^2}\right)\left\{\mathrm{K}(\lambda, t)+\int\delta x\left(\frac{\lambda_t}{\lambda_y}\right)^{\lambda}_{\substack{x\\t}}\right\}=\mathrm{H}\left[\lambda, \left\{\frac{\delta\mathrm{F}(\lambda, t)}{\delta\lambda}-\int\frac{\delta x}{(\lambda_y)^{\lambda}_{\substack{x\\t}}}\right\}\right]$$

The arbitrary functions $\mathrm{K}(\lambda, t)$ and $\frac{\delta\mathrm{F}(\lambda, t)}{\delta\lambda}$ being implicitly contained in the integrals need not be expressed.

4. Now suppose that there exists a vortex of invariable form which moves with arbitrary velocity along the axis of y. Let the equation to its surface be a function of x, and $y-\mathrm{Y}$ only, where Y is an arbitrary function of t only.

Let

$$\lambda=\mathrm{L}(x, y-\mathrm{Y})$$

then

$$\lambda_y=\frac{d\mathrm{L}}{dy}$$

$$\lambda_t=\frac{d\mathrm{L}}{dy}(-\dot{\mathrm{Y}})$$

therefore

$$\frac{\lambda_t}{\lambda_y}=-\dot{\mathrm{Y}}$$

Also $y-\mathrm{Y}$ can be expressed as a function of λ, x only from the equation $\lambda=\mathrm{L}(x, y-\mathrm{Y})$.

Therefore λ_y can be expressed as a function of λ, x only, not t.

Therefore $\mathrm{G}(\lambda, x, t)=-\int\frac{\delta x}{(\lambda_y)^{\lambda}_{\substack{x\\t}}}$ does not contain t, since $(\lambda_y)^{\lambda}_{\substack{x\\t}}$ does not contain t, and may now be written $(\lambda_y)^{\lambda}_{x}$.

Therefore

$$\frac{\delta\mathrm{G}}{\delta t}=0$$

In this case the equation takes the form

$$\frac{d}{dx}\left(-\dot{\mathrm{Y}}+\lambda_x\frac{\delta\mathrm{K}(\lambda, t)}{\delta\lambda}\right)+\frac{d}{dy}\left(\lambda_y\frac{\delta\mathrm{K}(\lambda, t)}{\delta\lambda}\right)=\mathrm{H}\left(\lambda, -\int\frac{\delta x}{(\lambda_y)^{\lambda}_{x}}+\frac{\delta\mathrm{F}(\lambda, t)}{\delta\lambda}\right)$$

therefore

$$\left(\frac{d^2}{dx^2}+\frac{d^2}{dy^2}\right)\mathrm{K}(\lambda, t)=\mathrm{H}\left(\lambda, -\int\frac{\delta x}{(\lambda_y)^{\lambda}_{x}}+\frac{\delta\mathrm{F}(\lambda, t)}{\delta\lambda}\right)$$

and now

$$u=\lambda_y\frac{\delta K(\lambda, t)}{\delta\lambda}=\frac{d}{dy}\left(K(\lambda, t)-x\dot{Y}\right)$$

$$v=\dot{Y}-\lambda_x\frac{\delta K(\lambda, t)}{\delta\lambda}=-\frac{d}{dx}\left(K(\lambda, t)-x\dot{Y}\right)$$

therefore

$$K(\lambda, t)-x\dot{Y}$$

is the current function.

Of course this could have been directly deduced from the form of the current function in the preceding article by putting $\frac{\lambda_t}{\lambda_y}=-\dot{Y}$.

5. To obtain similar expressions in polar coordinates.

Let R and Θ be the radial and tangential velocities.

Therefore

$$u=R\cos\theta-\Theta\sin\theta,\quad v=R\sin\theta+\Theta\cos\theta$$

$$\lambda_x=\cos\theta\lambda_r-\frac{\sin\theta}{r}\lambda_\theta,\quad \lambda_y=\sin\theta\lambda_r+\frac{\cos\theta}{r}\lambda_\theta$$

$$\lambda_x\psi_y-\lambda_y\psi_x=\frac{\lambda_r\psi_\theta-\lambda_\theta\psi_r}{r}$$

$$\lambda_t+u\lambda_x+v\lambda_y=\lambda_t+R\lambda_r+\frac{\Theta}{r}\lambda_\theta$$

In this case the equation corresponding to the equation in g of Art. 3 is $\lambda_r g_\theta-\lambda_\theta g_r=r$.

The auxiliary system of equations is

$$\frac{d\theta}{\lambda_r}=\frac{dr}{-\lambda_\theta}=\frac{dg}{r}$$

Let differentials of the variables λ, r, t when regarded as independent be denoted by $\delta\lambda$, δr, δt respectively.

Therefore

$$g=-\int\frac{r\delta r}{(\lambda_\theta)_{t}^{\lambda}{}_{r}}+\frac{\delta F(\lambda, t)}{\delta\lambda}=G(\lambda, r, t)+\frac{\delta F(\lambda, t)}{\delta\lambda}$$

Then since

$$\lambda_t+R\lambda_r+\frac{\Theta}{r}\lambda_\theta=0,\text{ and } g_t+Rg_r+\frac{\Theta}{r}g_\theta=0$$

it follows that

$$\frac{\delta G}{\delta t}+\frac{\delta^2 F}{\delta t\delta\lambda}-R\frac{r}{\lambda_\theta}=0$$

therefore

$$R=\frac{\lambda_\theta}{r}\left(\frac{\delta G}{\delta t}+\frac{\delta^2 F}{\delta t\delta\lambda}\right)$$

$$\Theta=-r\frac{\lambda_t}{\lambda_\theta}-\lambda_r\left(\frac{\delta G}{\delta t}+\frac{\delta^2 F}{\delta t\delta\lambda}\right)$$

But

$$v_x-u_y=\frac{d}{dx}(R\sin\theta+\Theta\cos\theta)-\frac{d}{dy}(R\cos\theta-\Theta\sin\theta)$$

$$=r_x\frac{d}{dr}(R\sin\theta+\Theta\cos\theta)+\theta_x\frac{d}{d\theta}(R\sin\theta+\Theta\cos\theta)$$

$$-r_y\frac{d}{dr}(R\cos\theta-\Theta\sin\theta)-\theta_y\frac{d}{d\theta}(R\cos\theta-\Theta\sin\theta)$$

$$=\frac{d\Theta}{dr}-\frac{1}{r}\frac{dR}{d\theta}+\frac{\Theta}{r}$$

therefore

$$v_x-u_y=\frac{1}{r}\left\{\frac{d}{dr}(r\Theta)-\frac{dR}{d\theta}\right\}$$

To determine the current function Λ there are the equations

$$\frac{1}{r}\frac{d\Lambda}{d\theta}=\frac{1}{r}\lambda_\theta\left(\frac{\delta^2 F(\lambda,t)}{\delta\lambda\delta t}-\frac{\delta}{\delta t}\int\frac{r\delta r}{(\lambda_\theta)_{r\,t}^{\lambda}}\right)$$

$$-\frac{d\Lambda}{dr}=-r\frac{\lambda_t}{\lambda_\theta}-\lambda_r\left(\frac{\delta^2 F(\lambda,t)}{\delta\lambda\delta t}-\frac{\delta}{\delta t}\int\frac{r\delta r}{(\lambda_\theta)_{r\,t}^{\lambda}}\right)$$

From the first of these

$$\Lambda=\frac{\delta F(\lambda,t)}{\delta t}-\frac{\delta}{\delta t}\int\delta\lambda\int\frac{r\delta r}{(\lambda_\theta)_{r\,t}^{\lambda}}+\psi(r,t)$$

where ψ is an arbitrary function.

Therefore

$$\frac{d\Lambda}{dr}=\frac{\delta^2 F(\lambda,t)}{\delta\lambda\delta t}\lambda_r-\frac{d}{dr}\cdot\frac{\delta}{\delta t}\int\delta\lambda\int\frac{r\delta r}{(\lambda_\theta)_{r\,t}^{\lambda}}+\frac{d}{dr}\psi(r,t)$$

But

$$\frac{d}{dr}=\frac{\delta}{\delta r}+\lambda_r\frac{\delta}{\delta\lambda}$$

$$\frac{d}{d\theta}=\qquad\lambda_\theta\frac{\delta}{\delta\lambda}$$

$$\frac{d}{dt}=\frac{\delta}{\delta t}+\lambda_t\frac{\delta}{\delta\lambda}$$

whence

$$\frac{\delta}{\delta t}=\frac{d}{dt}-\frac{\lambda_t}{\lambda_\theta}\frac{d}{d\theta}$$

therefore

$$\frac{d\Lambda}{dr}=\frac{\delta^2 F(\lambda, t)}{\delta\lambda\delta t}\lambda_r-\frac{\delta^2}{\delta r\delta t}\int\delta\lambda\int\frac{r\delta r}{(\lambda_\theta)^{\lambda}_{\substack{r\\t}}}-\lambda_r\frac{\delta^2}{\delta\lambda\delta t}\int\delta\lambda\int\frac{r\delta r}{(\lambda_\theta)^{\lambda}_{\substack{r\\t}}}+\frac{\delta}{\delta r}\psi(r, t)$$

$$=\frac{\delta^2 F(\lambda, t)}{\delta\lambda\delta t}\lambda_r-\frac{\delta}{\delta t}\int\frac{r\delta\lambda}{(\lambda_\theta)^{\lambda}_{\substack{r\\t}}}-\lambda_r\frac{\delta}{\delta t}\int\frac{r\delta r}{(\lambda_\theta)^{\lambda}_{\substack{r\\t}}}+\frac{\delta}{\delta r}\psi(r, t)$$

But

$$\frac{\delta}{\delta t}\int\frac{r\delta\lambda}{(\lambda_\theta)^{\lambda}_{\substack{r\\t}}}=\int r\delta\lambda\frac{\delta}{\delta t}\left(\frac{1}{(\lambda_\theta)^{\lambda}_{\substack{r\\t}}}\right)=\int r\delta\lambda\left(\frac{d}{dt}-\frac{\lambda_t}{\lambda_\theta}\frac{d}{d\theta}\right)\frac{1}{\lambda_\theta}=\int\frac{r\delta\lambda}{\lambda_\theta}\frac{d}{d\theta}\left(-\frac{\lambda_t}{\lambda_\theta}\right)$$

$$=\int r\delta\lambda\frac{\delta}{\delta\lambda}\left(-\frac{\lambda_t}{\lambda_\theta}\right)=-r\frac{\lambda_t}{\lambda_\theta}+\Phi(r, t)$$

where Φ is an arbitrary function.

Therefore

$$\frac{d\Lambda}{dr}=\lambda_r\left(\frac{\delta^2 F(\lambda, t)}{\delta\lambda\delta t}-\frac{\delta}{\delta t}\int\frac{r\delta r}{(\lambda_\theta)^{\lambda}_{\substack{r\\t}}}\right)+r\frac{\lambda_t}{\lambda_\theta}-\Phi(r, t)+\frac{\delta}{\delta r}\psi(r, t)$$

Choosing the arbitrary function $\psi(r, t)$ so that $\frac{\delta}{\delta r}\psi(r, t)=\Phi(r, t)$, the value of $\frac{d\Lambda}{dr}$ agrees with its known value.

And since

$$\psi(r, t)=\int\delta r\Phi(r, t)=\int\delta r\frac{\delta}{\delta t}\int\frac{r\delta\lambda}{(\lambda_\theta)^{\lambda}_{\substack{r\\t}}}+\int r\delta r\left(\frac{\lambda_t}{\lambda_\theta}\right)^{\lambda}_{\substack{r\\t}}=\int r\delta r\left(\frac{\lambda_t}{\lambda_\theta}\right)^{\lambda}_{\substack{r\\t}}+\frac{\delta}{\delta t}\int\delta\lambda\int\frac{r\delta r}{(\lambda_\theta)^{\lambda}_{\substack{r\\t}}}+P(\lambda, t)+Q(r, t)$$

therefore

$$\Lambda=\frac{\delta F(\lambda, t)}{\delta t}+\int r\delta r\left(\frac{\lambda_t}{\lambda_\theta}\right)^{\lambda}_{\substack{r\\t}}+P(\lambda, t)+Q(r, t)$$

and reasoning as in Art. 3 it follows that

$$\Lambda=K(\lambda, t)+\int r\delta r\left(\frac{\lambda_t}{\lambda_\theta}\right)^{\lambda}_{\substack{r\\t}}$$

And since v_x-u_y is a function of λ, g, it follows that

$$\frac{d^2\Lambda}{dx^2}+\frac{d^2\Lambda}{dy^2}=H[\lambda, g]$$

therefore

$$\left(\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}+\frac{1}{r^2}\frac{d^2}{d\theta^2}\right)\left\{K(\lambda, t)+\int r\delta r\left(\frac{\lambda_t}{\lambda_\theta}\right)^{\lambda}_{\substack{r\\t}}\right\}=H\left[\lambda, \left\{\frac{\delta F(\lambda, t)}{\delta\lambda}-\int\frac{r\delta r}{(\lambda_\theta)^{\lambda}_{\substack{r\\t}}}\right\}\right]$$

6. Now take the case of a vortex of invariable form rotating about the origin. Let its equation be

$$\lambda=L(r, \theta-\omega)$$

where ω is a function of t only.

Then as before

$$\frac{\lambda_t}{\lambda_\theta}=-\dot{\omega}$$

and

$$\frac{\delta G}{\delta t}=0$$

therefore

$$R=\frac{\lambda_\theta}{r}\frac{\delta K(\lambda, t)}{\delta\lambda}=\frac{1}{r}\frac{d}{d\theta}\left(K(\lambda, t)-\frac{r^2}{2}\dot{\omega}\right)$$

$$\Theta=r\dot{\omega}-\lambda_r\frac{\delta K(\lambda, t)}{\delta\lambda}=-\frac{d}{dr}\left(K(\lambda, t)-\frac{r^2}{2}\dot{\omega}\right)$$

and the equation in λ may be expressed in the form

$$\left(\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}+\frac{1}{r^2}\frac{d^2}{d\theta^2}\right)\left(K(\lambda, t)-\frac{r^2}{2}\dot{\omega}\right)=H\left(\lambda, -\int\frac{r\delta r}{(\lambda_\theta)_r^\lambda}+\frac{\delta F(\lambda, t)}{\delta\lambda}\right)$$

The current function is

$$K(\lambda, t)-\frac{r^2}{2}\dot{\omega}$$

To obtain this directly from the foregoing article, put for $\frac{\lambda_t}{\lambda_\theta}$ its value $-\dot{\omega}$

7. To obtain similar expressions when the motion of every element of the fluid is in planes passing through the axis of z, the motion being the same in all such planes.

Let τ be the velocity away from the axis of z, and w the velocity parallel axis of z.

The equations of motion are

$$\frac{d\tau}{dt}+\tau\frac{d\tau}{dr}+w\frac{d\tau}{dz}=-\frac{d}{dr}\left(\int\frac{dp}{\rho}+V\right)$$

$$\frac{dw}{dt}+\tau\frac{dw}{dr}+w\frac{dw}{dz}=-\frac{d}{dz}\left(\int\frac{dp}{\rho}+V\right)$$

$$\frac{1}{\rho}\left(\frac{d}{dt}+\tau\frac{d}{dr}+w\frac{d}{dz}\right)\rho+\frac{\tau}{r}+\frac{d\tau}{dr}+\frac{dw}{dz}=0$$

Differentiating the first equation with regard to z, the second with regard to r and subtracting, it can be shown that

$$\left(\frac{d\tau}{dr}+\frac{dw}{dz}\right)\left(\frac{dw}{dr}-\frac{d\tau}{dz}\right)+\left(\frac{d}{dt}+\tau\frac{d}{dr}+w\frac{d}{dz}\right)\left(\frac{dw}{dr}-\frac{d\tau}{dz}\right)=0$$

therefore

$$\left(\frac{d}{dt}+\tau\frac{d}{dr}+w\frac{d}{dz}\right)\left(\frac{dw}{dr}-\frac{d\tau}{dz}\right)=\left\{\frac{1}{\rho}\left(\frac{d}{dt}+\tau\frac{d}{dr}+w\frac{d}{dz}\right)\rho+\frac{\tau}{r}\right\}\left(\frac{dw}{dr}-\frac{d\tau}{dz}\right)$$

therefore

$$\frac{1}{r}\left(\frac{d}{dt}+\tau\frac{d}{dr}+w\frac{d}{dz}\right)\left(\frac{dw}{dr}-\frac{d\tau}{dz}\right)-\frac{\tau}{r^2}\left(\frac{dw}{dr}-\frac{d\tau}{dz}\right)=\frac{1}{r\rho}\left(\frac{dw}{dr}-\frac{d\tau}{dz}\right)\left\{\left(\frac{d}{dt}+\tau\frac{d}{dr}+w\frac{d}{dz}\right)\rho\right\}$$

Observing that

$$\left(\frac{d}{dt}+\tau\frac{d}{dr}+w\frac{d}{dz}\right)\frac{1}{r}=-\frac{\tau}{r^2}$$

This may be written

$$\frac{\left(\frac{d}{dt}+\tau\frac{d}{dr}+w\frac{d}{dz}\right)\left(\frac{1}{r}\left(\frac{dw}{dr}-\frac{d\tau}{dz}\right)\right)}{\frac{1}{r}\left(\frac{dw}{dr}-\frac{d\tau}{dz}\right)}=\frac{\left(\frac{d}{dt}+\tau\frac{d}{dr}+w\frac{d}{dz}\right)\rho}{\rho}$$

therefore

$$\left(\frac{d}{dt}+\tau\frac{d}{dr}+w\frac{d}{dz}\right)\left\{\frac{\frac{dw}{dr}-\frac{d\tau}{dz}}{r\rho}\right\}=0$$

This becomes, on putting in CLEBSCH's forms, $\tau=\frac{d\chi}{dr}+\lambda\frac{d\psi}{dr}$, $w=\frac{d\chi}{dz}+\lambda\frac{d\psi}{dz}$

$$\left(\frac{d}{dt}+\tau\frac{d}{dr}+w\frac{d}{dz}\right)\left\{\frac{1}{r\rho}\left(\frac{d\lambda}{dr}\frac{d\psi}{dz}-\frac{d\psi}{dr}\frac{d\lambda}{dz}\right)\right\}=0$$

And as in Art. 2 this result may be deduced from the equation of continuity and the two equations $\frac{d\lambda}{dt}+\tau\frac{d\lambda}{dr}+w\frac{d\lambda}{dz}=0$, and $\frac{d\psi}{dt}+\tau\frac{d\psi}{dr}+w\frac{d\psi}{dz}=0$ only.

8. Hence, supposing ρ constant

$$\frac{\lambda_r\psi_z-\lambda_z\psi_r}{r}=f(\lambda,\psi)=\frac{1}{\frac{\delta g(\lambda,\psi)}{\delta\psi}}$$

$$\lambda_r g_z-\lambda_z g_r=r$$

Let differentials of the variables λ, r, t, when regarded as independent be denoted by $\delta\lambda$, δr, δt respectively, then

$$g=-\int\frac{r\delta r}{(\lambda_z)^{\lambda}_{\substack{r\\t}}}+\frac{\delta F(\lambda,t)}{\delta\lambda}$$

$$=G(\lambda,r,t)+\frac{\delta F(\lambda,t)}{\delta\lambda}$$

and

$$\tau=\frac{1}{r}\lambda_z\left\{\frac{\delta G}{\delta t}+\frac{\delta^2F(\lambda,t)}{\delta t\delta\lambda}\right\}$$

$$w=-\frac{\lambda_t}{\lambda_z}-\frac{1}{r}\lambda_r\left\{\frac{\delta G}{\delta t}+\frac{\delta^2F(\lambda,t)}{\delta t\delta\lambda}\right\}$$

To find the current function Λ there are the equations

$$\frac{1}{r}\lambda_z\left(\frac{\delta^2F(\lambda,t)}{\delta\lambda\delta t}-\frac{\delta}{\delta t}\int\frac{r\delta r}{(\lambda_z)^{\lambda}_{\substack{r\\t}}}\right)=\frac{1}{r}\frac{d\Lambda}{dz}$$

$$-\frac{\lambda_t}{\lambda_z}-\frac{1}{r}\lambda_r\left(\frac{\delta^2F(\lambda,t)}{\delta\lambda\delta t}-\frac{\delta}{\delta t}\int\frac{r\delta r}{(\lambda_z)^{\lambda}_{\substack{r\\t}}}\right)=-\frac{1}{r}\frac{d\Lambda}{dr}$$

From the first

$$\Lambda=\frac{\delta F(\lambda, t)}{\delta t}-\frac{\delta}{\delta t}\int\delta\lambda\int\frac{r\delta r}{(\lambda_z)^{\lambda}_{\substack{r\\t}}}+\psi(r, t)$$

therefore

$$\frac{d\Lambda}{dr}=\frac{\delta^2 F(\lambda, t)}{\delta\lambda\delta t}\lambda_r-\frac{d}{dr}\frac{\delta}{\delta t}\int\delta\lambda\int\frac{r\delta r}{(\lambda_z)^{\lambda}_{\substack{r\\t}}}+\frac{d}{dr}\psi(r, t)$$

But

$$\frac{d}{dr}=\frac{\delta}{\delta r}+\lambda_r\frac{\delta}{\delta\lambda}$$

$$\frac{d}{dz}=\lambda_z\frac{\delta}{\delta\lambda}$$

$$\frac{d}{dt}=\frac{\delta}{\delta t}+\lambda_t\frac{\delta}{\delta\lambda}$$

whence

$$\frac{\delta}{\delta t}=\frac{d}{dt}-\frac{\lambda_t}{\lambda_z}\frac{d}{dz}$$

therefore

$$\frac{d\Lambda}{dr}=\frac{\delta^2 F(\lambda, t)}{\delta\lambda\delta t}\lambda_r-\frac{\delta}{\delta t}\int\frac{r\delta\lambda}{(\lambda_z)^{\lambda}_{\substack{r\\t}}}-\lambda_r\frac{\delta}{\delta t}\int\frac{r\delta r}{(\lambda_z)^{\lambda}_{\substack{r\\t}}}+\frac{\delta}{\delta r}\psi(r, t)$$

But

$$\frac{\delta}{\delta t}\int\frac{r\delta\lambda}{(\lambda_z)^{\lambda}_{\substack{r\\t}}}=\int r\delta\lambda\,\frac{\delta}{\delta t}\left(\frac{1}{(\lambda_z)^{\lambda}_{\substack{r\\t}}}\right)=\int r\delta\lambda\left(\frac{d}{dt}-\frac{\lambda_t}{\lambda_z}\frac{d}{dz}\right)\frac{1}{\lambda_z}$$

$$=\int r\delta\lambda\left(-\frac{\lambda_{tz}}{\lambda_z^2}+\frac{\lambda_t\lambda_{zz}}{\lambda_z^3}\right)=\int\frac{r\delta\lambda}{\lambda_z}\frac{d}{dz}\left(-\frac{\lambda_t}{\lambda_z}\right)$$

$$=\int r\delta\lambda\,\frac{\delta}{\delta\lambda}\left(-\frac{\lambda_t}{\lambda_z}\right)=-r\frac{\lambda_t}{\lambda_z}+\Phi(r, t)$$

therefore

$$\frac{d\Lambda}{dr}=\lambda_r\left(\frac{\delta^2 F(\lambda, t)}{\delta\lambda\delta t}-\frac{\delta}{\delta t}\int\frac{r\delta r}{(\lambda_z)^{\lambda}_{\substack{r\\t}}}\right)+r\frac{\lambda_t}{\lambda_z}-\Phi(r, t)+\frac{\delta}{\delta r}\psi(r, t)$$

Hence, choosing $\psi(r, t)$ so that

$$\Phi(r, t)=\frac{\delta}{\delta r}\psi(r, t)$$

and therefore

$$\psi(r, t)=\int\Phi(r, t)\delta r=\int r\delta r\left(\frac{\lambda_t}{\lambda_z}\right)^{\lambda}_{\substack{r\\t}}+\int\delta r\,\frac{\delta}{\delta t}\int\frac{r\delta\lambda}{(\lambda_z)^{\lambda}_{\substack{r\\t}}}=\int r\delta r\left(\frac{\lambda_t}{\lambda_z}\right)^{\lambda}_{\substack{r\\t}}+\frac{\delta}{\delta t}\int\delta\lambda\int\frac{r\delta r}{(\lambda_z)^{\lambda}_{\substack{r\\t}}}+P(\lambda, t)+Q(r, t)$$

the required form for $\frac{d\Lambda}{dr}$ is obtained.

$$\Lambda=\frac{\delta F(\lambda, t)}{\delta t}+\int r\delta r\left(\frac{\lambda_t}{\lambda_z}\right)^{\lambda}_{\substack{r\\t}}+P(\lambda, t)+Q(r, t)$$

and reasoning as in Art. 3 it follows that

$$\Lambda = \mathrm{K}(\lambda, t) + \int r\delta r \left(\frac{\lambda_t}{\lambda_z}\right)_{r\ t}^{\lambda}$$

But $\frac{1}{r}\left(\frac{dw}{dr} - \frac{d\tau}{dz}\right)$ is a function of λ, g.

Therefore

$$\frac{1}{r^2}\left(\frac{d^2\Lambda}{dr^2} - \frac{1}{r}\frac{d\Lambda}{dr} + \frac{d^2\Lambda}{dz^2}\right) = \mathrm{H}[\lambda, g]$$

therefore

$$\frac{1}{r^2}\left(\frac{d^2}{dr^2} - \frac{1}{r}\frac{d}{dr} + \frac{d^2}{dz^2}\right)\left\{\mathrm{K}(\lambda, t) + \int r\delta r \left(\frac{\lambda_t}{\lambda_z}\right)_{r\ t}^{\lambda}\right\} = \mathrm{H}\left[\lambda, \frac{\delta \mathrm{F}(\lambda, t)}{\delta\lambda} - \int \frac{r\delta r}{(\lambda_z)_{r\ t}^{\lambda}}\right]$$

9. For a vortex of invariable form which moves parallel to the axis of z.

$$\lambda = \mathrm{L}(r, z - \mathrm{Z})$$

where Z is a function of t only.

As before

$$\frac{\lambda_t}{\lambda_z} = -\dot{\mathrm{Z}} \text{ and } \frac{\delta \mathrm{G}}{\delta t} = 0$$

$$\tau = \frac{1}{r}\lambda_z \frac{\delta \mathrm{K}(\lambda, t)}{\delta\lambda} = \frac{1}{r}\frac{d}{dz}\left(\mathrm{K}(\lambda, t) - \frac{r^2}{2}\dot{\mathrm{Z}}\right)$$

$$w = \dot{\mathrm{Z}} - \frac{1}{r}\lambda_r \frac{\delta \mathrm{K}(\lambda, t)}{\delta\lambda} = -\frac{1}{r}\frac{d}{dr}\left(\mathrm{K}(\lambda, t) - \frac{r^2}{2}\dot{\mathrm{Z}}\right)$$

Therefore the equation in λ becomes

$$\frac{1}{r}\frac{d}{dr}\left\{\left(\frac{1}{r}\frac{d}{dr}\right)\left(\mathrm{K}(\lambda, t) - \frac{r^2}{2}\dot{\mathrm{Z}}\right)\right\} + \frac{1}{r^2}\frac{d^2}{dz^2}\left(\mathrm{K}(\lambda, t) - \frac{r^2}{2}\dot{\mathrm{Z}}\right) = \mathrm{H}\left(\lambda, -\int \frac{r\delta r}{(\lambda_z)_r^{\lambda}} + \frac{\delta \mathrm{F}(\lambda, t)}{\delta\lambda}\right)$$

i.e.,

$$\left\{\left(\frac{1}{r}\frac{d}{dr}\right)^2 + \frac{1}{r^2}\frac{d^2}{dz^2}\right\}\left(\mathrm{K}(\lambda, t) - \frac{r^2}{2}\dot{\mathrm{Z}}\right) = \mathrm{H}\left(\lambda, \frac{\delta \mathrm{F}(\lambda, t)}{\delta\lambda} - \int \frac{r\delta r}{(\lambda_z)_r^{\lambda}}\right)$$

therefore

$$\frac{1}{r^2}\left(\frac{d^2}{dz^2} + \frac{d^2}{dr^2} - \frac{1}{r}\frac{d}{dr}\right)(\mathrm{K}(\lambda, t)) = \mathrm{H}\left(\lambda, \frac{\delta \mathrm{F}(\lambda, t)}{\delta\lambda} - \int \frac{r\delta r}{(\lambda_z)_r^{\lambda}}\right)$$

The current function is

$$\mathrm{K}(\lambda, t) - \frac{r^2}{2}\dot{\mathrm{Z}}$$

To obtain it directly from the preceding article, it would only have been necessary to put

$$\frac{\lambda_t}{\lambda_z} = -\dot{\mathrm{Z}}$$

Illustrations.

10. Example I. Take the simplest case of the equation in λ given in Art. 4, viz.,

$$\left(\frac{d^2}{dx^2}+\frac{d^2}{dy^2}\right)\lambda=c$$

where c is a constant.

It is required to examine whether $\lambda=(f)\left(\frac{x^2}{a^2}+\frac{(y-\mathrm{Y})^2}{b^2}\right)$ can represent vortex sheets in a motion, part of which is rotational and part irrotational (f, a, b being constants).

Substituting in the equation

$$(2f)\left(\frac{1}{a^2}+\frac{1}{b^2}\right)=c$$

Also

$$u=f.\frac{2(y-\mathrm{Y})}{b^2}$$

$$v=\dot{\mathrm{Y}}-f.\frac{2x}{a^2}$$

To find ψ it is necessary to solve the equation

$$\frac{d\psi}{dt}+f.\frac{2(y-\mathrm{Y})}{b^2}\frac{d\psi}{dx}+\left(\dot{\mathrm{Y}}-f.\frac{2x}{a^2}\right)\frac{d\psi}{dy}=0$$

The auxiliary system of equations is

$$\frac{dt}{1}=\frac{dx}{\frac{2f(y-\mathrm{Y})}{b^2}}=\frac{dy}{\dot{\mathrm{Y}}-\frac{2fx}{a^2}}=\frac{d\psi}{0}$$

One integral of which is

$$(f)\left(\frac{x^2}{a^2}+\frac{(y-\mathrm{Y})^2}{b^2}\right)=\text{ const. }=m$$

and the other

$$t-\frac{ab}{2f}\sin^{-1}\frac{x}{a}\sqrt{\frac{f}{m}}=\text{ const. }=n$$

where for m must be substituted its value.

Hence

$$\lambda=(f)\left(\frac{x^2}{a^2}+\frac{(y-\mathrm{Y})^2}{b^2}\right)$$

$$\psi=t-\frac{ab}{2f}\sin^{-1}\frac{\frac{x}{a}}{\sqrt{\frac{x^2}{a^2}+\frac{(y-\mathrm{Y})^2}{b^2}}}$$

Whence

$$\left(\frac{x^2}{a^2}+\frac{(y-\mathrm{Y})^2}{b^2}\right)\frac{d\psi}{dx}=-\frac{y-\mathrm{Y}}{2f}$$

$$\left(\frac{x^2}{a^2}+\frac{(y-\mathrm{Y})^2}{b^2}\right)\frac{d\psi}{dy}=\frac{x}{2f}$$

Substituting for u and v in the dynamical equations

$$\frac{p}{\rho}+\mathrm{V}=\frac{2f^2}{a^2b^2}(x^2+(y-\mathrm{Y})^2)-y\ddot{\mathrm{Y}}+\text{ an arbitrary function of } t \text{ which need not be considered.}$$

Now the equation determining χ which is

$$\frac{d\chi}{dt}+u\frac{d\chi}{dx}+v\frac{d\chi}{dy}=\tfrac{1}{2}(u^2+v^2)-\left(\frac{p}{\rho}+\mathrm{V}\right)$$

becomes

$$\frac{d\chi}{dt}+\frac{2f.(y-\mathrm{Y})}{b^2}\frac{d\chi}{dx}+\left(\dot{\mathrm{Y}}-\frac{2fx}{a^2}\right)\frac{d\chi}{dy}$$

$$=2f^2\left(\frac{1}{a^2}-\frac{1}{b^2}\right)\left(\frac{x^2}{a^2}-\frac{(y-\mathrm{Y})^2}{b^2}\right)-\frac{2f}{a^2}x\dot{\mathrm{Y}}+(y-\mathrm{Y})\ddot{\mathrm{Y}}+\text{ a function of } t.$$

The integrals of the auxiliary system are

$$(f)\left(\frac{x^2}{a^2}+\frac{(y-\mathrm{Y})^2}{b^2}\right)=\text{ const. }=m$$

$$t-\frac{ab}{2f}\sin^{-1}\frac{x}{a}\sqrt{\frac{f}{m}}=\text{ const. }=n$$

$$\chi=\frac{m}{2}\cdot\frac{a^2-b^2}{ab}\sin\frac{4f}{ab}(t-n)+b\sqrt{\frac{m}{f}}\dot{\mathrm{Y}}\cos\frac{2f}{ab}(t-n)+\text{ a function of } t.$$

Hence one value of the integral of the equation in χ, which may be called χ', is obtained by substituting for m and n their values, and is

$$=\frac{a^2-b^2}{a^2b^2}fx(y-\mathrm{Y})+\dot{\mathrm{Y}}(y-\mathrm{Y})$$

Whence

$$\chi'_x=\frac{a^2-b^2}{a^2b^2}f.(y-\mathrm{Y})$$

$$\chi'_y=\frac{a^2-b^2}{a^2b^2}f.x+\dot{\mathrm{Y}}$$

But

$$u=f.\frac{2(y-\mathrm{Y})}{b^2}$$

$$v=\dot{\mathrm{Y}}-\frac{2f.x}{a^2}$$

If now e be some constant, then

$$u-e\left(\frac{x^2}{a^2}+\frac{(y-\mathrm{Y})^2}{b^2}-1\right)\psi_x-\chi'_x=(y-\mathrm{Y})\left(\frac{e}{2f}+f.\frac{a^2+b^2}{a^2b^2}\right)+e\psi_x.$$

$$v-e\left(\frac{x^2}{a^2}+\frac{(y-\mathrm{Y})^2}{b^2}-1\right)\psi_y-\chi'_y=-x\left(\frac{e}{2f}+f.\frac{a^2+b^2}{a^2b^2}\right)+e\psi_y$$

Choosing the constant $e=-\frac{2f^2(a^2+b^2)}{a^2b^2}$, the right hand sides of these equations become $e\psi_x$, $e\psi_y$ respectively. Therefore

$$u=\frac{d}{dx}\left(\chi'-2f^2.\frac{a^2+b^2}{a^2b^2}\psi\right)-2f^2\frac{a^2+b^2}{a^2b^2}\left(\frac{x^2}{a^2}+\frac{(y-\mathrm{Y})^2}{b^2}-1\right)\frac{d\psi}{dx}$$

$$v=\frac{d}{dy}\left(\chi'-2f^2.\frac{a^2+b^2}{a^2b^2}\psi\right)-2f^2\frac{a^2+b^2}{a^2b^2}\left(\frac{x^2}{a^2}+\frac{(y-\mathrm{Y})^2}{b^2}-1\right)\frac{d\psi}{dy}$$

Putting in for χ' and ψ their values, it will be seen that for the χ of CLEBSCH's forms of expression for the components of the velocity, it is necessary to take

$$\frac{a^2-b^2}{a^2b^2}f.x(y-\mathrm{Y})+\dot{\mathrm{Y}}(y-\mathrm{Y})-2f^2\frac{a^2+b^2}{a^2b^2}\left(t-\frac{ab}{2f}\sin^{-1}\frac{\frac{x}{a}}{\sqrt{\frac{x^2}{a^2}+\frac{(y-\mathrm{Y})^2}{b^2}}}\right)$$

The terms, containing t only, may be omitted.

Thus if ρ be the density of a distribution of matter of which χ is the potential

$$\rho=-\frac{1}{4\pi}\left(\frac{d^2\chi}{dx^2}+\frac{d^2\chi}{dy^2}\right)=-\frac{1}{4\pi}.\frac{2f(a^4-b^4)}{a^4b^4}.\frac{x(y-\mathrm{Y})}{\left(\frac{x^2}{a^2}+\frac{(y-\mathrm{Y})^2}{b^2}\right)^2}$$

therefore

$$\rho=-\frac{c}{4\pi}.\frac{a^2-b^2}{a^2b^2}.\frac{x(y-\mathrm{Y})}{\left(\frac{x^2}{a^2}+\frac{(y-\mathrm{Y})^2}{b^2}\right)^2}$$

The potential of this density (see Art. 15) at an internal point is

$$\frac{c}{2}.\frac{a-b}{a+b}.x(y-\mathrm{Y})+\frac{cab}{2}\left(\sin^{-1}\frac{\frac{x}{a}}{\sqrt{\frac{x^2}{a^2}+\frac{(y-\mathrm{Y})^2}{b^2}}}-\sin^{-1}\frac{x}{\sqrt{x^2+(y-\mathrm{Y})^2}}\right)$$

Adding to this the cyclic term

$$\frac{cab}{2}\sin^{-1}\frac{x}{\sqrt{x^2+(y-\mathrm{Y})^2}}$$

$\left(\text{giving a cyclic constant } -\pi abc=\pi ab(+2\zeta) \text{ where } 2\zeta=\frac{dv}{dx}-\frac{du}{dy}\right)$ the expression obtained is

$$\frac{c}{2}\cdot\frac{a-b}{a+b}x(y-\mathrm{Y})+\frac{cab}{2}\sin^{-1}\frac{\frac{x}{a}}{\sqrt{\frac{x^2}{a^2}+\frac{(y-\mathrm{Y})^2}{b^2}}}$$

This will not agree with the value of χ, unless

$$\frac{c}{2}\cdot\frac{a-b}{a+b}=\frac{a^2-b^2}{a^2b^2}f$$

$$\frac{cab}{2}=\frac{f(a^2+b^2)}{ab}$$

$$\dot{\mathrm{Y}}=0$$

The first and second equation require $a=b$.

Hence this method will not lead to a determination of the irrotational motion outside the cylinder. It does not prove that there is no irrotational motion outside continuous with the rotational motion inside the cylinder.

Supposing HELMHOLTZ's method applied to this case, it would be necessary to find a value of Λ which is the potential of a distribution of matter of density $-\frac{c}{4\pi}$ inside the surface

$$\frac{x^2}{a^2}+\frac{(y-\mathrm{Y})^2}{b^2}=1.$$

The result is that inside,

$$\Lambda=\frac{cab}{2}\left[\mathrm{C}+\frac{x^2}{a(a+b)}+\frac{(y-\mathrm{Y})^2}{b(a+b)}\right]$$

and outside,

$$\Lambda=\frac{cab}{2}\left[\mathrm{C}'+\log\left(\sqrt{a^2+\epsilon}+\sqrt{b^2+\epsilon}\right)+\frac{x^2-(y-\mathrm{Y})^2}{a^2-b^2}-\frac{\sqrt{(a^2+\epsilon)(b^2+\epsilon)}}{a^2-b^2}\left(\frac{x^2}{a^2+\epsilon}-\frac{(y-\mathrm{Y})^2}{b^2+\epsilon}\right)\right]$$

where

$$\frac{x^2}{a^2+\epsilon}+\frac{(y-\mathrm{Y})^2}{b^2+\epsilon}=1, \text{ and C and C}' \text{ are constants.}$$

If the constants C and C′ be properly determined, these expressions will be continuous at the surface $\frac{x^2}{a^2}+\frac{(y-\mathrm{Y})^2}{b^2}=1$. Their differential coefficients are also continuous.

Now $\frac{d\Lambda}{dy}=\frac{ca(y-\mathrm{Y})}{a+b}$, $-\frac{d\Lambda}{dx}=\frac{-cbx}{a+b}$ inside the cylinder.

At the surface of the cylinder, $\frac{d\Lambda}{dx}$ and $\frac{d\Lambda}{dy}$ have the same values whether calculated from the value of Λ inside it or outside it.

In order to have the rotational motion continuous with the irrotational motion, it is necessary that all over the surface of the cylinder

$$f.\frac{2(y-\mathrm{Y})}{b^2}=\frac{ca(y-\mathrm{Y})}{a+b}$$

$$\dot{\mathrm{Y}}-\frac{2fx}{a^2}=-\frac{cbx}{a+b}.$$

But these equations cannot be satisfied unless $a=b$, $\dot{\mathrm{Y}}=0$.

The solution may, however, be completed for a finite portion of the plane of x, y outside the cylinder by means of Example III. which follows :—

In Example III., put $\dot{\omega}=0$, this will make

$$\lambda=\Lambda \text{ of Example III.}=f.\frac{a^2+b^2}{ab}\log\left(\sqrt{a^2+\epsilon}+\sqrt{b+\epsilon}\right)+\frac{2f}{a^2-b^2}(x^2-(y-\mathrm{Y})^2)$$
$$-\frac{f(a^2+b^2)}{ab(a^2-b^2)}\sqrt{(a^2+\epsilon)(b^2+\epsilon)}\left(\frac{x^2}{a^2+\epsilon}-\frac{(y-\mathrm{Y})^2}{b^2+\epsilon}\right)$$

where

$$\frac{x^2}{a^2+\epsilon}+\frac{(y-\mathrm{Y})^2}{b^2+\epsilon}=1$$

Therefore the current function Λ of this example is

$$\lambda-x\dot{\mathrm{Y}}=f.\frac{a^2+b^2}{ab}\log\left(\sqrt{a^2+\epsilon}+\sqrt{b^2+\epsilon}\right)+\frac{2f}{a^2-b^2}(x^2-(y-\mathrm{Y})^2)$$
$$-\frac{f(a^2+b^2)}{ab(a^2-b^2)}\sqrt{(a^2+\epsilon)(b^2+\epsilon)}\left(\frac{x^2}{a^2+\epsilon}-\frac{(y-\mathrm{Y})^2}{b^2+\epsilon}\right)-x\dot{\mathrm{Y}}$$

This is equivalent to the form in the abstract printed in the Proceedings.

For the motion to be possible, it must be supposed to be confined to a cylinder of finite section, appropriate surface conditions being supplied at the surface of the bounding cylinder.

11. Example II. In the preceding example it was shown that none of the methods would apply for the whole of space surrounding the rotationally moving liquid. Knowing KIRCHHOFF'S investigation of the rotating elliptic vortex cylinder, let the form of the equation given in Art. 6 be considered, and the following simple case of it taken.

$$\left(\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}+\frac{1}{r^2}\frac{d^2}{d\theta^2}\right)\left(\lambda-\frac{r^2}{2}\dot{\omega}\right)=c$$

A particular integral is

$$\lambda=(f)\left\{\frac{r^2\cos^2(\theta-\omega)}{a^2}+\frac{r^2\sin^2(\theta-\omega)}{b^2}\right\}$$

if $2f\left(\frac{1}{a^2}+\frac{1}{b^2}\right)-2\dot{\omega}=c(=-2\zeta$ of KIRCHHOFF's "Vorlesungen über Mathematische Physik." Zwanzigste Vorlesung). Therefore $\dot{\omega}$ is constant.

Let x', y' be the coordinates of the point x, y if referred to the principal axes of the ellipse which rotates with uniform angular velocity $\dot{\omega}$; u' v' the components of the velocity parallel to these moving axes.

Thus

$$x'=x\cos\dot{\omega}t+y\sin\dot{\omega}t\ ;\ y'=-x\sin\dot{\omega}t+y\cos\dot{\omega}t$$

The velocities along and perpendicular to the radius vector are

$$\mathrm{R}=\frac{1}{r}\frac{d}{d\theta}\left(\lambda-\frac{r^2\dot{\omega}}{2}\right)=-(f)\left(\frac{1}{a^2}-\frac{1}{b^2}\right)r\sin 2(\theta-\omega)$$

$$\Theta=-\frac{d}{dr}\left(\lambda-\frac{r^2\dot{\omega}}{2}\right)=-\left(\frac{f}{a^2}+\frac{f}{b^2}-\dot{\omega}\right)r-\left(\frac{f}{a^2}-\frac{f}{b^2}\right)r\cos 2(\theta-\omega)$$

$$u'=\mathrm{R}\cos(\theta-\omega)-\Theta\sin(\theta-\omega)=\left(\frac{2f}{b^2}-\dot{\omega}\right)y'$$

$$v'=\mathrm{R}\sin(\theta-\omega)+\Theta\cos(\theta-\omega)=-\left(\frac{2f}{a^2}-\dot{\omega}\right)x'$$

$$u=u'\cos\dot{\omega}t-v'\sin\dot{\omega}t$$

$$v=u'\sin\dot{\omega}t+v'\cos\dot{\omega}t$$

$$\lambda=(f)\left(\frac{x'^2}{a^2}+\frac{y'^2}{b^2}\right)$$

To find ψ it is necessary to solve the equation

$$\frac{d\psi}{dt}+u\frac{d\psi}{dx}+v\frac{d\psi}{dy}=0$$

The auxiliary system is

$$\frac{dt}{1}=\frac{dx}{u}=\frac{dy}{v}=\frac{d\psi}{0}$$

It will be more convenient to obtain ψ in terms of x', y'.

Since

$$dx'=\quad\cos\dot{\omega}t.dx+\sin\dot{\omega}t.dy+\dot{\omega}y'.dt$$

$$dy'=-\sin\dot{\omega}t.dx+\cos\dot{\omega}t.dy-\dot{\omega}x'.dt,$$

therefore

$$\frac{dt}{1}=\frac{dx'}{u\cos\dot{\omega}t+v\sin\dot{\omega}t+\dot{\omega}y'}=\frac{dy'}{-u\sin\dot{\omega}t+v\cos\dot{\omega}t-\dot{\omega}x'}=\frac{d\psi}{0}$$

therefore

$$\frac{dt}{1}=\frac{dx'}{u'+\dot{\omega}y'}=\frac{dy'}{v'-\dot{\omega}x'}=\frac{d\psi}{0}$$

therefore

$$\frac{dt}{1}=\frac{dx'}{2f.\frac{y'}{b^2}}=\frac{dy'}{-2f.\frac{x'}{a^2}}=\frac{d\psi}{0}$$

The integrals are

$$(f)\left(\frac{x'^2}{a^2}+\frac{y'^2}{b^2}\right)=m$$

$$t-\frac{ab}{2f}\sin^{-1}\left(\frac{x'}{a}\sqrt{\frac{f}{m}}\right)=n$$

therefore

$$\lambda=(f)\left(\frac{x'^2}{a^2}+\frac{y'^2}{b^2}\right)$$

$$\psi=t-\frac{ab}{2f}\sin^{-1}\frac{\frac{x'}{a}}{\sqrt{\frac{x'^2}{a^2}+\frac{y'^2}{b^2}}}$$

The current function $\Lambda=\lambda-\frac{1}{2}\dot{\omega}r^2$

$$=(f)\left(\frac{x'^2}{a^2}+\frac{y'^2}{b^2}\right)-\frac{\dot{\omega}}{2}(x'^2+y'^2)$$

To find $\frac{p}{\rho}+\mathrm{V}$

First express $\left(\frac{d}{dt}+u\frac{d}{dx}+v\frac{d}{dy}\right)$ in a form in which x', y', t are independent variables.

$$\left(\frac{d}{dt}+u\frac{d}{dx}+v\frac{d}{dy}\right)=\frac{d}{dt}+\frac{dx'}{dt}\,\frac{d}{dx'}+\frac{dy'}{dt}\,\frac{d}{dy'}+u\left(\frac{dx'}{dx}\,\frac{d}{dx'}+\frac{dy'}{dx}\,\frac{d}{dy'}\right)+v\left(\frac{dx'}{dy}\,\frac{d}{dx'}+\frac{dy'}{dy}\,\frac{d}{dy'}\right)$$

$$=\frac{d}{dt}+(\dot{\omega}y'+u\cos\dot{\omega}t+v\sin\dot{\omega}t)\frac{d}{dx'}+(-\dot{\omega}x'-u\sin\dot{\omega}t+v\cos\dot{\omega}t)\,\frac{d}{dy'}$$

$$=\frac{d}{dt}+(\dot{\omega}y'+u')\frac{d}{dx'}+(-\dot{\omega}x'+v')\frac{d}{dy'}$$

$$=\frac{d}{dt}+\frac{2fy'}{b^2}\,\frac{d}{dx'}-\frac{2fx'}{a}\,\frac{d}{dy'}$$

therefore the equations

$$\frac{du}{dt}+u\frac{du}{dx}+v\frac{du}{dy}=-\frac{d}{dx}\left(\frac{p}{\rho}+\mathrm{V}\right)$$

$$\frac{dv}{dt}+u\frac{dv}{dx}+v\frac{dv}{dy}=-\frac{d}{dy}\left(\frac{p}{\rho}+\mathrm{V}\right)$$

become

$$\left(\frac{d}{dt}+\frac{2fy'}{b^2}\frac{d}{dx'}-\frac{2fx'}{a^2}\frac{d}{dy'}\right)(u'\cos\dot{\omega}t-v'\sin\dot{\omega}t)=-\frac{d}{dx}\left(\frac{p}{\rho}+\mathrm{V}\right)$$

$$\left(\frac{d}{dt}+\frac{2fy'}{b^2}\frac{d}{dx'}-\frac{2fx'}{a^2}\frac{d}{dy'}\right)(u'\sin\dot{\omega}t+v'\cos\dot{\omega}t)=-\frac{d}{dy}\left(\frac{p}{\rho}+\mathrm{V}\right)$$

therefore

$$\sin\dot{\omega}t.y'\left(\frac{4f^2}{a^2b^2}-\frac{4f\dot{\omega}}{b^2}+\dot{\omega}^2\right)+\cos\dot{\omega}t.x'\left(-\frac{4f^2}{a^2b^2}+\frac{4f\dot{\omega}}{a^2}-\dot{\omega}^2\right)=-\frac{d}{dx}\left(\frac{p}{\rho}+\mathrm{V}\right)$$

$$-\cos\dot{\omega}t.y'\left(\frac{4f^2}{a^2b^2}-\frac{4f\dot{\omega}}{b^2}+\dot{\omega}^2\right)+\sin\dot{\omega}t.x'\left(-\frac{4f^2}{a^2b^2}+\frac{4f\dot{\omega}}{a^2}-\dot{\omega}^2\right)=-\frac{d}{dy}\left(\frac{p}{\rho}+\mathrm{V}\right)$$

therefore

$$x'\left(-\frac{4f^2}{a^2b^2}+\frac{4f\dot{\omega}}{a^2}-\dot{\omega}^2\right)=-\cos\dot{\omega}t\frac{d}{dx}\left(\frac{p}{\rho}+\mathrm{V}\right)-\sin\dot{\omega}t\frac{d}{dy}\left(\frac{p}{\rho}+\mathrm{V}\right)=-\frac{d}{dx'}\left(\frac{p}{\rho}+\mathrm{V}\right)$$

$$y'\left(-\frac{4f^2}{a^2b^2}+\frac{4f\dot{\omega}}{b^2}-\dot{\omega}^2\right)=+\sin\dot{\omega}t\frac{d}{dx}\left(\frac{p}{\rho}+\mathrm{V}\right)-\cos\dot{\omega}t\frac{d}{dy}\left(\frac{p}{\rho}+\mathrm{V}\right)=-\frac{d}{dy'}\left(\frac{p}{\rho}+\mathrm{V}\right)$$

therefore

$$\frac{p}{\rho}+\mathrm{V}=x'^2\left(\tfrac{1}{2}\dot{\omega}^2-\frac{2f\dot{\omega}}{a^2}+\frac{2f^2}{a^2b^2}\right)+y'^2\left(\tfrac{1}{2}\dot{\omega}^2-\frac{2f\dot{\omega}}{b^2}+\frac{2f^2}{a^2b^2}\right)+\text{ an arbitrary function of the time.}$$

To find χ

$$\frac{p}{\rho}+\mathrm{V}=\tfrac{1}{2}(u^2+v^2)-\frac{d\chi}{dt}-u\frac{d\chi}{dx}-v\frac{d\chi}{dy}$$

The auxiliary system of equations to find χ is

$$\frac{dt}{1}=\frac{dx}{u}=\frac{dy}{v}=\frac{d\chi}{\tfrac{1}{2}(u^2+v^2)-\frac{p}{\rho}-\mathrm{V}}$$

Substituting for u, v, $\frac{p}{\rho}+\mathrm{V}$ their values, there may be substituted for these the equations

$$\frac{dt}{1}=\frac{dx'}{2f.\frac{y'}{b^2}}=\frac{dy'}{-2f.\frac{x'}{a^2}}=\frac{d\chi}{\left(\frac{2f}{b^2}-\frac{2f}{a^2}\right)\left(-\frac{fx'^2}{a^2}+\frac{fy'^2}{b^2}\right)}$$

One integral is

$$f\frac{x'^2}{a^2}+f\frac{y'^2}{b^2}=m$$

Another is

$$t-\frac{ab}{2f}\sin^{-1}\left(\frac{x'}{a}\sqrt{\frac{f}{m}}\right)=n$$

To find the third, substituting for y' its value $b\sqrt{\frac{m}{f}-\frac{x'^2}{a^2}}$

$$\frac{dx'}{\frac{2\sqrt{f}}{b}\sqrt{m-\frac{fx'^2}{a^2}}}=\frac{d\chi}{\left(\frac{2f}{b^2}-\frac{2f}{a^2}\right)\left(m-\frac{2fx'^2}{a^2}\right)}$$

therefore

$$\chi=\sqrt{f}.\frac{a^2-b^2}{a^3b}x'\sqrt{ma^2-fx'^2}$$

therefore one value of χ satisfying the partial differential equation, which may be called χ' is

$$\chi'=f.\frac{a^2-b^2}{a^2b^2}x'y'$$

Also

$$\lambda=(f)\left(\frac{x'^2}{a^2}+\frac{y'^2}{b^2}\right)$$

$$\psi=t-\frac{ab}{2f}\sin^{-1}\frac{\frac{x'}{a}}{\sqrt{\frac{x'^2}{a^2}+\frac{y'^2}{b^2}}}$$

Whence

$$\left(\frac{x'^2}{a^2}+\frac{y'^2}{b^2}\right)\frac{d\psi}{dx'}=-\frac{y'}{2f}$$

$$\left(\frac{x'^2}{a^2}+\frac{y'^2}{b^2}\right)\frac{d\psi}{dy'}=\frac{x'}{2f}$$

Thus

$$u'-\frac{d\chi'}{dx'}-e\left(\frac{x'^2}{a^2}+\frac{y'^2}{b^2}-1\right)\frac{d\psi}{dx'}=y'\left(\frac{f}{b^2}+\frac{f}{a^2}-\dot{\omega}+\frac{e}{2f}\right)+e\frac{d\psi}{dx'}$$

$$v'-\frac{d\chi'}{dy'}-e\left(\frac{x'^2}{a^2}+\frac{y'^2}{b^2}-1\right)\frac{d\psi}{dy'}=-x'\left(\frac{f}{b^2}+\frac{f}{a^2}-\dot{\omega}+\frac{e}{2f}\right)+e\frac{d\psi}{dy'}$$

If, therefore, e be put $=2f\left(\dot{\omega}-\frac{f}{a^2}-\frac{f}{b^2}\right)$

$$u'=\frac{d}{dx'}(\chi'+e\psi)+e\left(\frac{x'^2}{a^2}+\frac{y'^2}{b^2}-1\right)\frac{d\psi}{dx'}$$

$$v'=\frac{d}{dy'}(\chi'+e\psi)+e\left(\frac{x'^2}{a^2}+\frac{y'^2}{b^2}-1\right)\frac{d\psi}{dy'}$$

Thus the proper value to take for the χ of CLEBSCH's forms is $\chi'+e\psi$. Omitting, as unnecessary, terms containing t only

$$\chi=f\frac{a^2-b^2}{a^2b^2}x'y'-ab\left(\dot{\omega}-\frac{f}{a^2}-\frac{f}{b^2}\right)\sin^{-1}\frac{\frac{x'}{a}}{\sqrt{\frac{x'^2}{a^2}+\frac{y'^2}{b^2}}}$$

Calculating

$$\rho=-\frac{1}{4\pi}\left(\frac{d^2\chi}{dx^2}+\frac{d^2\chi}{dy^2}\right)=-\frac{1}{4\pi}\left(\frac{d^2\chi}{dx'^2}+\frac{d^2\chi}{dy'^2}\right)$$

$$=-\frac{1}{4\pi}\left(\frac{1}{b^2}-\frac{1}{a^2}\right)\left(\frac{2f}{a^2}+\frac{2f}{b^2}-2\dot{\omega}\right)\frac{x'y'}{\left(\frac{x'^2}{a^2}+\frac{y'^2}{b^2}\right)^2}$$

$$=-\frac{c}{4\pi}\left(\frac{1}{b^2}-\frac{1}{a^2}\right)\frac{x'y'}{\left(\frac{x'^2}{a^2}+\frac{y'^2}{b^2}\right)^2}$$

The value of the potential due to this density is the single-valued expression given in Art. 15, viz. :—

$$\frac{c}{2}\cdot\frac{a-b}{a+b}x'y'+\frac{cab}{2}\left(\sin^{-1}\frac{\frac{x'}{a}}{\sqrt{\frac{x'^2}{a^2}+\frac{y'^2}{b^2}}}-\sin^{-1}\frac{x'}{\sqrt{x'^2+y'^2}}\right)$$

Add to this the term $\frac{cab}{2}\sin^{-1}\frac{x'}{\sqrt{x'^2+y'^2}}$ to give the cyclic constant $\frac{cab}{2}(-2\pi)=\pi ab.2\zeta$ in KIRCHHOFF's notation; then, in order that what is now obtained may be the same as χ, it is necessary that

$$f\frac{a^2-b^2}{a^2b^2}=\frac{c}{2}\cdot\frac{a-b}{a+b}$$

$$-ab\left(\dot{\omega}-\frac{f}{a^2}-\frac{f}{b^2}\right)=\frac{cab}{2}$$

the latter of which is known to be true, but the former will not also be true unless $\dot{\omega}=-\frac{2f}{ab}=-\frac{cab}{(a+b)^2}=2\zeta\cdot\frac{ab}{(a+b)^2}$ in KIRCHHOFF's notation.

Up to this point no relation has been assumed between $\dot{\omega}$ and f.

Supposing, however, this relation satisfied, the velocity potential at an external point is obtained by adding the same term $\frac{cab}{2}\sin^{-1}\frac{x'}{\sqrt{x'^2+y'^2}}$ to the potential found in Art. 15 for an external point.

Thus velocity potential at an external point is

$$\frac{cabx'y'}{a^2-b^2}\left(\frac{a^2+b^2+2\epsilon}{2\sqrt{(a^2+\epsilon)(b^2+\epsilon)}}-1\right)+\frac{cab}{2}\sin^{-1}\frac{x'}{\sqrt{a^2+\epsilon}}$$

where ϵ is a root of the equation $\frac{x'^2}{a^2+\epsilon}+\frac{y'^2}{b^2+\epsilon}=1$, the axes x', y' turning round with angular velocity $\dot{\omega}=\frac{2\zeta ab}{(a+b)^2}$

The expressions for the velocity, which may be deduced from this, might also have been obtained by HELMHOLTZ's Method from the current function Λ which is the

potential of the density $-\frac{c}{4\pi}$ throughout the cylinder $\frac{x'^2}{a^2}+\frac{y'^2}{b^2}=1$, and which is given in Example I.

These values make $\frac{p}{\rho}+V$ continuous at the surface.

12. Example III. Supposing the relation found in the last article between f and $\dot{\omega}$ not satisfied, it is required to find a solution if possible.

Using the elliptic coordinates ϵ, v which satisfy the equations

$$\frac{x'^2}{a^2+\epsilon}+\frac{y'^2}{b^2+\epsilon}=1$$

$$\frac{x'^2}{a^2+v}+\frac{y'^2}{b^2+v}=1$$

where

$$-b^2<\epsilon<\infty$$

$$-a^2<v<-b^2$$

Hence

$$x'^2=\frac{(a^2+\epsilon)(a^2+v)}{a^2-b^2},\ y'^2=\frac{(b^2+\epsilon)(-b^2-v)}{a^2-b^2}$$

and putting

$$\alpha=\log\left(\sqrt{a^2+\epsilon}+\sqrt{b^2+\epsilon}\right)$$

$$\beta=-\tan^{-1}\sqrt{\frac{-b^2-v}{+a^2+v}}$$

LAPLACE'S equation becomes $\frac{d^2V}{d\alpha^2}+\frac{d^2V}{d\beta^2}=0.$

If V be a function of ϵ only, then $V=C\alpha+C'$. Similarly $V=C\beta+C'$ is a solution. But if $V=E.U$ where E is a function of ϵ only, U a function of v only, then

$$U\frac{d^2E}{d\alpha^2}+E\frac{d^2U}{d\beta^2}=0$$

Suppose there exist a value of E such that $\frac{d^2E}{d\alpha^2}=p^2E.$

Then

$$\frac{d^2U}{d\beta^2}+p^2U=0.$$

Therefore if

$$E=Ae^{p\alpha}+Be^{-p\alpha}$$

$$U=A'\cos p\beta+B'\sin p\beta$$

$$E=A\left(\sqrt{a^2+\epsilon}+\sqrt{b^2+\epsilon}\right)^p+B\left(\sqrt{a^2+\epsilon}+\sqrt{b^2+\epsilon}\right)^{-p}$$

$$U=A'\cos\left\{p\tan^{-1}\sqrt{\frac{-b^2-v}{a^2+v}}\right\}-B'\sin\left\{p\tan^{-1}\sqrt{\frac{-b^2-v}{a^2+v}}\right\}$$

When

$$p=1,\ \mathrm{E}=\mathrm{C}\sqrt{a^2+\epsilon}+\mathrm{D}\sqrt{b^2+\epsilon}$$

$$\mathrm{U}=\mathrm{C}'\sqrt{a^2+v}+\mathrm{D}'\sqrt{-b^2-v}$$

$$p=2,\ \mathrm{E}=\mathrm{C}\left(\epsilon+\frac{a^2+b^2}{2}\right)+\mathrm{D}\sqrt{(a^2+\epsilon)(b^2+\epsilon)}$$

$$\mathrm{U}=\mathrm{C}'\left(v+\frac{a^2+b^2}{2}\right)+\mathrm{D}'\sqrt{(a^2+v)(-b^2-v)}$$

and so on.

Now suppose it required to express Λ as the current function of an irrotational motion under the condition that

$$\Lambda=f\frac{x'^2}{a^2}+f\frac{y'^2}{b^2}-\frac{\dot{\omega}}{2}(x'^2+y'^2) \text{ wherever } \frac{x'^2}{a^2}+\frac{y'^2}{b^2}=$$

Assume if possible

$$f\frac{x'^2}{a^2}+f\frac{y'^2}{b^2}-\frac{\dot{\omega}}{2}(x'^2+y'^2)$$
$$=\mathrm{A}+\mathrm{B}\log\left(\sqrt{a^2+\epsilon}+\sqrt{b^2+\epsilon}\right)+\left[\mathrm{C}\left(\epsilon+\frac{a^2+b^2}{2}\right)+\mathrm{D}\sqrt{(a^2+\epsilon)(b^2+\epsilon)}\right]\left(v+\frac{a^2+b^2}{2}\right)$$

The form of the expression on the right-hand side shows that it is the current function of an irrotational motion.

But

$$\left(\epsilon+\frac{a^2+b^2}{2}\right)\left(v+\frac{a^2+b^2}{2}\right)=\frac{a^2-b^2}{2}(x'^2-y'^2)-\left(\frac{a^2-b^2}{2}\right)^2$$

$$v+\frac{a^2+b^2}{2}=\frac{a^2-b^2}{2}\left(\frac{x'^2}{a^2+\epsilon}-\frac{y'^2}{b^2+\epsilon}\right)$$

therefore the assumption is

$$f\frac{x'^2}{a^2}+f\frac{y'^2}{b^2}-\frac{\dot{\omega}}{2}(x'^2+y'^2)$$
$$=\mathrm{A}+\mathrm{B}\log\left(\sqrt{a^2+\epsilon}+\sqrt{b^2+\epsilon}\right)+\mathrm{C}\frac{a^2-b^2}{2}(x'^2-y'^2)-\mathrm{C}\left(\frac{a^2-b^2}{2}\right)^2$$
$$+\mathrm{D}\frac{a^2-b^2}{2}\sqrt{(a^2+\epsilon)(b^2+\epsilon)}\left(\frac{x'^2}{a^2+\epsilon}-\frac{y'^2}{b^2+\epsilon}\right)$$

If this is to be satisfied when $\frac{x'^2}{a^2}+\frac{y'^2}{b^2}=1$, and therefore when $\epsilon=0$, it is possible to add $k\left(\frac{x'^2}{a^2}+\frac{y'^2}{b^2}-1\right)$ to either side and then equate the coefficients of x'^2, y'^2. The con-

stant A can always be chosen so as to make the absolute terms equal. The elimination of k from the two equations for C, D in which it occurs will leave the one relation between them which must be satisfied.

Adding $k\left(\frac{x'^2}{a^2}+\frac{y'^2}{b^2}-1\right)$ to the right-hand side, the conditions are

$$\frac{f}{a^2}-\frac{\dot\omega}{2}=\mathrm{C}\frac{a^2-b^2}{2}+\mathrm{D}.ab.\frac{a^2-b^2}{2a^2}+\frac{k}{a^2}$$

$$\frac{f}{b^2}-\frac{\dot\omega}{2}=-\mathrm{C}.\frac{a^2-b^2}{2}-\mathrm{D}.ab.\frac{a^2-b^2}{2b^2}+\frac{k}{b^2}$$

and an equation to determine A.

The velocity parallel axis of x' inside cylinder $=\frac{d\Lambda}{dy'}=\left(\frac{2f}{b^2}-\dot\omega\right)y'$.

The velocity parallel axis of y' outside cylinder $=-\frac{d\Lambda}{dx'}=-\left(\frac{2f}{a^2}-\dot\omega\right)x$.

From the assumed value of Λ for space outside cylinder

$$\frac{d\Lambda}{dy'}=-\mathrm{C}(a^2-b^2)y'-\mathrm{D}(a^2-b^2)y'\sqrt{\frac{a^2+\epsilon}{b^2+\epsilon}}$$

$$+\frac{\frac{d\epsilon}{dy'}}{2\sqrt{(a^2+\epsilon)(b^2+\epsilon)}}\left\{\mathrm{B}+\mathrm{D}\frac{(a^2-b^2)^2}{2}\left(\frac{x'^2}{a^2+\epsilon}+\frac{y'^2}{b^2+\epsilon}\right)\right\}$$

$$\frac{d\Lambda}{dx'}=-\mathrm{C}(a^2-b^2)x'-\mathrm{D}(a^2-b^2)x'\sqrt{\frac{b^2+\epsilon}{a^2+\epsilon}}$$

$$-\frac{\frac{d\epsilon}{dx'}}{2\sqrt{(a^2+\epsilon)(b^2+\epsilon)}}\left\{\mathrm{B}+\mathrm{D}\frac{(a^2-b^2)^2}{2}\left(\frac{x'^2}{a^2+\epsilon}+\frac{y'^2}{b^2+\epsilon}\right)\right\}$$

For the continuity of the values of $\frac{d\Lambda}{dx'}, \frac{d\Lambda}{dy'}$ at the surface $\epsilon=0$, it is evident that the coefficients of $\frac{d\epsilon}{dx'}, \frac{d\epsilon}{dy'}$ must vanish. Therefore

$$\mathrm{B}+\mathrm{D}\frac{(a^2-b^2)^2}{2}=0$$

Also

$$\frac{2f}{b^2}-\dot\omega=-\mathrm{C}(a^2-b^2)-\mathrm{D}\frac{a}{b}(a^2-b^2)$$

$$\frac{2f}{a^2}-\dot\omega=\ \mathrm{C}(a^2-b^2)+\mathrm{D}\frac{b}{a}(a^2-b^2)$$

These are the same equations as those obtained for the continuity of Λ if $k=0$, whence

$$C=\frac{ab}{(a^2-b^2)^2}\left\{\frac{4f}{ab}-\dot{\omega}\frac{a^2+b^2}{ab}\right\}$$

$$D=-\frac{ab}{(a^2-b^2)^2}\left\{2f\left(\frac{a^2+b^2}{a^2b^2}\right)-2\dot{\omega}\right\}$$

Therefore

$$\Lambda=f\frac{x'^2}{a^2}+f\frac{y'^2}{b^2}-\frac{\dot{\omega}}{2}(x'^2+y'^2)$$

$$=\text{const.}+ab\left\{f\frac{a^2+b^2}{a^2b^2}-\dot{\omega}\right\}\log\left(\sqrt{a^2+\epsilon}+\sqrt{b^2+\epsilon}\right)+\tfrac{1}{2}\cdot\frac{ab}{a^2-b^2}\left(\frac{4f}{ab}-\dot{\omega}\frac{a^2+b^2}{ab}\right)(x'^2-y'^2)$$

$$-\frac{ab}{a^2-b^2}\left(f\left(\frac{1}{a^2}+\frac{1}{b^2}\right)-\dot{\omega}\right)\sqrt{(a^2+\epsilon)(b^2+\epsilon)}\left(\frac{x'^2}{a^2+\epsilon}-\frac{y'^2}{b^2+\epsilon}\right)$$

To examine whether this will make the value of $\frac{p}{\rho}+V$ continuous at the surface.

Inside it is known that

$$\frac{p}{\rho}+V=x'^2\left(\tfrac{1}{2}\dot{\omega}^2-\frac{2f\dot{\omega}}{a^2}+\frac{2f^2}{a^2b^2}\right)+y'^2\left(\tfrac{1}{2}\dot{\omega}^2-\frac{2f\dot{\omega}}{b^2}+\frac{2f^2}{a^2b^2}\right)+\text{ arbitrary function of } t.$$

Outside

$$\frac{p}{\rho}+V=-\frac{d\phi}{dt}-\tfrac{1}{2}\left(\left(\frac{d\phi}{dx'}\right)^2+\left(\frac{d\phi}{dy'}\right)^2\right)+\text{ arbitrary function of } t,$$

where ϕ is the velocity potential.

Now as $\dot{\omega}$ is constant, if ϕ be expressed as a function of x', y' (ϵ is a function of x', y') then t can only occur in ϕ through occurring in x', y'.

Therefore

$$\frac{d\phi}{dt}=\frac{d\phi}{dx'}\cdot\frac{dx'}{dt}+\frac{d\phi}{dy'}\cdot\frac{dy'}{dt}=u'.\dot{\omega}y'-v'.\dot{\omega}x'$$

But it has already been shown that the velocities are continuous at the surface.

Therefore at the surface

$$\frac{d\phi}{dx'}=2y'\left(\frac{f}{b^2}-\tfrac{1}{2}\dot{\omega}\right);\ \frac{d\phi}{dy'}=-2x'\left(\frac{f}{a^2}-\tfrac{1}{2}\dot{\omega}\right)$$

Therefore at the surface

$$\frac{p}{\rho}+V=-2x'^2\left(\frac{f^2}{a^4}-\tfrac{1}{4}\dot{\omega}^2\right)-2y'^2\left(\frac{f^2}{b^4}-\tfrac{1}{4}\dot{\omega}^2\right)+\text{ arbitrary function of } t.$$

The difference of the values of $\frac{p}{\rho}+\mathrm{V}$ as given by the above expressions obtained from the motion inside and outside is

$$\left\{2f^2\left(\frac{1}{a^2}+\frac{1}{b^2}\right)-2f\dot{\omega}\right\}\left(\frac{x'^2}{a^2}+\frac{y'^2}{b^2}\right)+\text{ arbitrary function of } t.$$

As $\frac{x'^2}{a^2}+\frac{y'^2}{b^2}=1$, choosing the difference of the arbitrary functions of t to be constant and equal to $2f\dot{\omega}-2f^2\left(\frac{1}{a^2}+\frac{1}{b^2}\right)$, the value of $\frac{p}{\rho}+\mathrm{V}$ will be continuous at the surface.

The current function of the irrotational motion is Λ.

Therefore the equation to a surface always containing the same particles is

$$\lambda=\frac{\dot{\omega}}{2}(x'^2+y'^2)+\Lambda=\text{const.}$$

That is, in elliptic coordinates :—

$$\frac{\dot{\omega}}{2}(\epsilon+v+a^2+b^2)+ab\left\{f\frac{a^2+b^2}{a^2b^2}-\dot{\omega}\right\}\log\left(\sqrt{a^2+\epsilon}+\sqrt{b^2+\epsilon}\right)$$
$$+\frac{ab}{(a^2-b^2)^2}\left[\left(\frac{4f}{ab}-\dot{\omega}\frac{a^2+b^2}{ab}\right)\left(\epsilon+\frac{a^2+b^2}{2}\right)-2\left(f\frac{a^2+b^2}{a^2b^2}-\dot{\omega}\right)\sqrt{(a^2+\epsilon)(b^2+\epsilon)}\right]\left(v+\frac{a^2+b^2}{2}\right)=\text{const.}$$

Now if ϵ be chosen so as to make the coefficient of v vanish, it will always be possible to choose the constant so that the equation is satisfied. Therefore the elliptic cylinders corresponding to the values of ϵ which make the coefficient of v vanish will be parts of surfaces $\lambda=$const., and will therefore always contain the same particles.

These values of ϵ satisfy the equation

$$\left(\frac{4f}{ab}-\dot{\omega}\frac{a^2+b^2}{ab}\right)\left(\epsilon+\frac{a^2+b^2}{2}\right)+\frac{\dot{\omega}}{2}\cdot\frac{(a^2-b^2)^2}{ab}=2\left(f\cdot\frac{a^2+b^2}{a^2b^2}-\dot{\omega}\right)\sqrt{(a^2+\epsilon)(b^2+\epsilon)}$$

Solving this in the ordinary way, the roots obtained are

$$\epsilon=0,\ \epsilon=\frac{4f\left(f\cdot\frac{a^2+b^2}{a^2b^2}-\dot{\omega}\right)}{\left(\dot{\omega}+\frac{2f}{ab}\right)\left(\dot{\omega}-\frac{2f}{ab}\right)}$$

It is necessary to examine whether these roots satisfy the above equation, or the equation obtained by changing the sign of the radical on the right-hand side.

As positive values of ϵ only are required, it is necessary to examine the relations between $\dot{\omega}$ and f which will make ϵ positive

$$\epsilon = \frac{4\left(\frac{a^2+b^2}{a^2b^2} - \frac{\dot{\omega}}{f}\right)}{\left(\frac{\dot{\omega}}{f} + \frac{2}{ab}\right)\left(\frac{\dot{\omega}}{f} - \frac{2}{ab}\right)}$$

If

$$-\infty < \frac{\dot{\omega}}{f} < -\frac{2}{ab} \qquad \epsilon \text{ is positive.}$$

$$\frac{\dot{\omega}}{f} = -\frac{2}{ab} \qquad \epsilon \text{ is infinite.}$$

$$-\frac{2}{ab} < \frac{\dot{\omega}}{f} < \frac{2}{ab} \qquad \epsilon \text{ is negative.}$$

$$\frac{\dot{\omega}}{f} = \frac{2}{ab} \qquad \epsilon \text{ is infinite.}$$

$$\frac{2}{ab} < \frac{\dot{\omega}}{f} < \frac{a^2+b^2}{a^2b^2} \qquad \epsilon \text{ is positive.}$$

$$\frac{\dot{\omega}}{f} = \frac{a^2+b^2}{a^2b^2} \qquad \epsilon \text{ is zero.}$$

$$\frac{a^2+b^2}{a^2b^2} < \frac{\dot{\omega}}{f} < \infty \qquad \epsilon \text{ is negative.}$$

If the relation between $\dot{\omega}$ and f is such as to make ϵ positive, it is necessary to see whether the equation in ϵ above is satisfied.

$$\epsilon = \frac{4\left(\frac{a^2+b^2}{a^2b^2} - \frac{\dot{\omega}}{f}\right)}{\left(\frac{\dot{\omega}}{f}\right)^2 - \frac{4}{a^2b^2}}$$

$$a^2 + \epsilon = \frac{\left(a\frac{\dot{\omega}}{f} - \frac{2}{a}\right)^2}{\left(\frac{\dot{\omega}}{f}\right)^2 - \frac{4}{a^2b^2}}$$

$$b^2 + \epsilon = \frac{\left(b\frac{\dot{\omega}}{f} - \frac{2}{b}\right)^2}{\left(\frac{\dot{\omega}}{f}\right)^2 - \frac{4}{a^2b^2}}$$

First suppose

$$-\infty < \frac{\dot{\omega}}{f} < -\frac{2}{ab}$$

Then

$$\sqrt{a^2+\epsilon} = \frac{-a\frac{\dot{\omega}}{f}+\frac{2}{a}}{\sqrt{\left(\frac{\dot{\omega}}{f}\right)^2-\frac{4}{a^2b^2}}}$$

$$\sqrt{b^2+\epsilon} = \frac{-b\frac{\dot{\omega}}{f}+\frac{2}{b}}{\sqrt{\left(\frac{\dot{\omega}}{f}\right)^2-\frac{4}{a^2b^2}}}$$

so that

$$\sqrt{(a^2+\epsilon)(b^2+\epsilon)} = \frac{\left(-a\frac{\dot{\omega}}{f}+\frac{2}{a}\right)\left(-b\frac{\dot{\omega}}{f}+\frac{2}{b}\right)}{\left(\frac{\dot{\omega}}{f}\right)^2-\frac{4}{a^2b^2}}$$

and the equation in ϵ is satisfied.

Next suppose

$$\frac{2}{ab} < \frac{\dot{\omega}}{f} < \frac{a^2+b^2}{a^2b^2}$$

Now

$$\sqrt{a^2+\epsilon} = \frac{a\frac{\dot{\omega}}{f}-\frac{2}{a}}{\sqrt{\left(\frac{\dot{\omega}}{f}\right)^2-\frac{4}{a^2b^2}}}$$

$$\sqrt{b^2+\epsilon} = \frac{-b\frac{\dot{\omega}}{f}+\frac{2}{b}}{\sqrt{\left(\frac{\dot{\omega}}{f}\right)^2-\frac{4}{a^2b^2}}}$$

These values, however, do not satisfy the equation in ϵ.

Hence it appears that if

$$-\infty < \frac{\dot{\omega}}{f} < -\frac{2}{ab}$$

the elliptic cylinder for which

$$\epsilon = \frac{4\left(\frac{a^2+b^2}{a^2b^2}-\frac{\dot{\omega}}{f}\right)}{\left(\frac{\dot{\omega}}{f}\right)^2-\frac{4}{a^2b^2}}$$

is part of one of the surfaces $\lambda=$ const., and therefore always contains the same particles.

Hence, if the smooth hollow rigid cylinder $\frac{x'^2}{a^2+\epsilon}+\frac{y'^2}{b^2+\epsilon}=1$ rotate about its axis with uniform angular velocity $\dot{\omega}$, such that

$$\epsilon=\frac{4\left(\frac{a^2+b^2}{a^2b^2}-\frac{\dot{\omega}}{f}\right)}{\left(\frac{\dot{\omega}}{f}\right)^2-\frac{4}{a^2b^2}},\text{ and } -\infty<\frac{\dot{\omega}}{f}<-\frac{2}{ab},$$

it is possible that the fluid inside the geometrical surface $\frac{x'^2}{a^2}+\frac{y'^2}{b^2}=1$ should move rotationally, and that the fluid between the two cylinders should be moving irrotationally, the rotational and irrotational motion being continuous.

The components of the velocity of the rotational motion parallel to the axes of the sections of the cylinder by the plane of x', y' are

$$y'\left(\frac{2f}{b^2}-\dot{\omega}\right)\text{ and }-x'\left(\frac{2f}{a^2}-\dot{\omega}\right)$$

The components of the velocity of the irrotational motion in the same directions are

$$\frac{d\Lambda}{dy'},\ -\frac{d\Lambda}{dx'}$$

i.e.,

$$\frac{aby'}{a^2-b^2}\left[2\left\{f\left(\frac{1}{a^2}+\frac{1}{b^2}\right)-\dot{\omega}\right\}\sqrt{\frac{a^2+\epsilon}{b^2+\epsilon}}-\left\{\frac{4f}{ab}-\dot{\omega}\frac{a^2+b^2}{ab}\right\}\right]$$

and

$$\frac{abx'}{a^2-b^2}\left[2\left\{f\left(\frac{1}{a^2}+\frac{1}{b^2}\right)-\dot{\omega}\right\}\sqrt{\frac{b^2+\epsilon}{a^2+\epsilon}}-\left\{\frac{4f}{ab}-\dot{\omega}\frac{a^2+b^2}{ab}\right\}\right]$$

These expressions for the velocity will not agree with those obtained by HELMHOLTZ'S Method, viz., $u=\frac{caby'}{a^2-b^2}\left(\sqrt{\frac{a^2+\epsilon}{b^2+\epsilon}}-1\right)$ and $v=\frac{cabx'}{a^2-b^2}\left(\sqrt{\frac{b^2+\epsilon}{a^2+\epsilon}}-1\right)$, unless $\dot{\omega}=-\frac{2f}{ab}$, *i.e.*, in the case of Example II., and the irrotational motion as obtained by HELMHOLTZ'S Method is not continuous with the rotational motion. The other method suggested in this paper (in which the potential of the density $-\frac{1}{4\pi}\left(\frac{d^2\chi}{dx^2}+\frac{d^2\chi}{dy^2}\right)$ is calculated) does not lead to a result.

13. Example IV. To consider the case in which the vortex sheets are coaxial circular cylinders, and the molecular rotation is a function of the distance from the axis.

This investigation will illustrate the reduction of the components of the velocity to CLEBSCH'S forms.

In this case $\frac{d^2\lambda}{dr^2}+\frac{1}{r}\frac{d\lambda}{dr}+\frac{1}{r^2}\frac{d^2\lambda}{d\theta^2}=$ a function of r only, and $\lambda=F(r)$.

It is supposed here that the initial line from which θ is measured is fixed.

Let R, Θ be the radial and tangential velocities, so that

$$R=\frac{1}{r}\frac{d\lambda}{d\theta}=0$$

$$\Theta=-\frac{d\lambda}{dr}=-F'(r)$$

ψ satisfies the equation

$$\frac{d\psi}{dt}+R\frac{d\psi}{dr}+\Theta\frac{d\psi}{rd\theta}=0$$

The auxiliary system of equations is

$$\frac{dt}{1}=\frac{dr}{0}=\frac{d\theta}{\frac{-F'(r)}{r}}=\frac{d\psi}{0}$$

One integral is $r=m$, the other is $t+\frac{m}{F'(m)}\theta=n$

Therefore the integrals are $\lambda=F(r)$, $\psi=t+\frac{r}{F'(r)}\theta$

The most general form of the integral will be $\psi=\Psi(m, n)$ where for m and n their values must be substituted.

Substituting for R and Θ in the dynamical equations

$$\frac{p}{\rho}+V=\int\{F'(r)\}^2\frac{dr}{r}$$

To find χ,

$$\frac{d\chi}{dt}+R\frac{d\chi}{dr}+\Theta\frac{d\chi}{rd\theta}=\tfrac{1}{2}(R^2+\Theta^2)-\frac{p}{\rho}-V$$

Therefore

$$\frac{d\chi}{dt}+0\frac{d\chi}{dr}-\frac{F'(r)}{r}\frac{d\chi}{d\theta}=\tfrac{1}{2}(F'(r))^2-\int\{F'(r)\}^2\frac{dr}{r}$$

The auxiliary system is therefore

$$\frac{dt}{1}=\frac{dr}{0}=\frac{d\theta}{\frac{-F'(r)}{r}}=\frac{d\chi}{\frac{1}{2}\{F'(r)\}^2-\int\{F'(r)\}^2\frac{dr}{r}}$$

The integrals are

$$r=m,$$

$$t+\frac{r}{F'(r)}\theta=n,$$

$$\chi-\theta\left\{-\frac{rF'(r)}{2}+\frac{r}{F'(r)}\int\{F'(r)\}^2\frac{dr}{r}\right\}=q.$$

therefore the integral of the original system is

$$\chi=\theta\left\{-\frac{rF'(r)}{2}+\frac{r}{F'(r)}\int\{F'(r)\}^2\frac{dr}{r}\right\}+X(m, n)$$

Take $\lambda=F(r)$, $\psi=\Psi(m, n)$

To reduce the components of the velocity to CLEBSCH'S forms, it is necessary to find X and Ψ so that

$$\frac{d\chi}{dr}+\lambda\frac{d\psi}{dr}=0,\ \frac{d\chi}{rd\theta}+\lambda\frac{d\psi}{rd\theta}=-F'(r)$$

i.e.

$$0=\theta\left\{\frac{F'(r)}{2}-\frac{rF''(r)}{2}+\left(\frac{F'(r)-rF''(r)}{\{F'(r)\}^2}\right)\int\{F'(r)\}^2\frac{dr}{r}\right\}+\frac{\delta X}{\delta m}$$

$$+\frac{\delta X}{\delta n}\theta\frac{F'(r)-rF''(r)}{\{F'(r)\}^2}+F(r)\left\{\frac{\delta\Psi}{\delta m}+\frac{\delta\Psi}{\delta n}\theta\frac{F'(r)-rF''(r)}{\{F'(r)\}^2}\right\}.$$

$$-F'(r)=-\frac{F'(r)}{2}+\frac{1}{F'(r)}\int\{F'(r)\}^2\frac{dr}{r}+\frac{\delta X}{\delta n}\cdot\frac{1}{F'(r)}+\frac{\delta\Psi}{\delta n}\cdot\frac{F(r)}{F'(r)}$$

Whence

$$\frac{\delta X}{\delta m}+F(r)\frac{\delta\Psi}{\delta m}=0$$

$$\frac{\delta X}{\delta n}+F(r)\frac{\delta\Psi}{\delta n}=-\tfrac{1}{2}\{F'(r)\}^2-\int\{F'(r)\}^2\frac{dr}{r}$$

To satisfy these put $\Psi=n\Phi(r)$ in each, then since $m=r$, it follows from the first of these equations that

$$X=-n\int F(r)\Phi'(r)dr+\Theta(n)$$

Substitute in the second equation, therefore,

$$-\int F(r)\Phi'(r)dr+\Theta'(n)+F(r)\Phi(r)=-\tfrac{1}{2}\{F'(r)\}^2-\int\{F'(r)\}^2\frac{dr}{r}$$

It is necessary that $\Theta'(n)$ should be constant, and it will be found that the constant may be taken as zero.

Differentiate the last equation with regard to r.

Therefore

$$\Phi(r) = -\frac{F'(r)}{r} - F''(r)$$

Therefore

$$\chi = \theta\left\{F(r) - rF'(r) + \frac{rF(r)F''(r)}{F'(r)}\right\} + t\left\{F(r)F''(r) - \tfrac{1}{2}\{F'(r)\}^2 + \frac{F(r)F'(r)}{r} - \int\{F'(r)\}^2\frac{dr}{r}\right\}$$

$$\lambda = F(r)$$

$$\psi = -\left(t + \frac{r}{F'(r)}\theta\right)\left(F''(r) + \frac{F'(r)}{r}\right)$$

If the rotationally-moving liquid be bounded by the cylinder $\lambda=0$, and its radius be $r=a$; then $F(a)=0$

Therefore when

$$r=a,\ \chi = \theta(-aF'(a)) + t\left\{-\tfrac{1}{2}(F'(a))^2 - \int(F'(a))^2\frac{da}{a}\right\}$$

A suitable value for the velocity potential at points outside the cylinder $r=a$ is

$$\phi = \theta(-aF'(a)) + t\left\{-\tfrac{1}{2}(F'(a))^2 - \int(F'(a))^2\frac{da}{a}\right\}$$

This will make the velocity and pressure continuous at the surface of the rotationally-moving liquid. Also the velocity at infinity will be infinitely small.

14. Example V. This case is of interest, because one set of the vortex sheets, viz., $ar^2(z-Z)^2+b(r^2-\alpha^2)^2=\text{const.}$, consists in part of ring-shaped surfaces. The results only are given.

If a and b are positive, and the constant $<b\alpha^4$, then this represents ring-shaped surfaces.

The equation in λ of Art. 9, includes as a special case

$$\frac{1}{r^2}\left(\frac{d^2}{dr^2} - \frac{1}{r}\frac{d}{dr} + \frac{d^2}{dz^2}\right)\lambda = \text{const.}$$

A particular integral is

$$\lambda = ar^2(z-Z)^2 + b(r^2-\alpha^2)^2$$

giving

$$\tau = 2ar(z-Z) \text{ and } w = \dot{Z} - 2a(z-Z)^2 - 4b(r^2-\alpha^2)$$

The value of $\frac{p}{\rho}+\mathrm{V}$ is

$$2ab(r^2-\alpha^2)^2-2a^2(z-\mathrm{Z})^4+8ab\alpha^2(z-\mathrm{Z})^2-(z-\mathrm{Z})\ddot{\mathrm{Z}}+\theta'(t)$$

where $\theta'(t)$ is an arbitrary function of t.

The differential equation in ψ has two independent integrals one of which is

$$ar^2(z-\mathrm{Z})^2+b(r^2-\alpha^2)^2=\text{ const.}=e$$

and the other

$$t-\int\frac{dr}{2\sqrt{a}\sqrt{e-b(r^2-\alpha^2)^2}}=\text{ const.}=f$$

where after the integration is performed, the value of e must be substituted for it.

The differential equation in χ when solved will give

$$\chi=ar^2(z-\mathrm{Z})-\tfrac{2}{3}a(z-\mathrm{Z})^3+4b\alpha^2(z-\mathrm{Z})+z\dot{\mathrm{Z}}-\int\tfrac{1}{2}\dot{\mathrm{Z}}^2dt+\phi(t)$$
$$-2b(4b+a)\alpha^4t+\frac{4b^2+ab}{\sqrt{a}}\int\frac{r^4dr}{\sqrt{e-b(r^2-\alpha^2)^2}}+\mathrm{G}(e,f)$$

where G and ϕ are the symbols of arbitrary functions.

Finally, in order to express τ as $\frac{d\chi}{dr}+\lambda\frac{d\psi}{dr}$ and w as $\frac{d\chi}{dz}+\lambda\frac{d\psi}{dz}$, $\mathrm{G}(e,f)$ is taken as $(8b+2a)(b\alpha^4-e_0)f$.

Then $\lambda=(8b+2a)(e_0-e)$, $\psi=f$, $\chi=$ above expression with the value of $\mathrm{G}(e,f)$ taken as $(8b+2a)(b\alpha^4-e_0)f$; where $e=ar^2(z-\mathrm{Z})^2+b(r^2-\alpha^2)^2$ and

$$f=t-\int\frac{dr}{2\sqrt{a}\sqrt{e-b(r^2-\alpha^2)^2}}.$$

χ is therefore

$$ar^2(z-\mathrm{Z})-\tfrac{2}{3}a(z-\mathrm{Z})^3+4b\alpha^2(z-\mathrm{Z})+z\dot{\mathrm{Z}}+\frac{4b^2+ab}{\sqrt{a}}\int\frac{r^4dr}{\sqrt{e-b(r^2-\alpha^2)^2}}$$
$$-(4b+a)\left(\frac{b\alpha^4-e_0}{\sqrt{a}}\right)\int\frac{dr}{\sqrt{e-b(r^2-\alpha^2)^2}}+\text{ an arbitrary function of } t.$$

The value of $-\frac{1}{4\pi}\left(\frac{d^2\chi}{dr^2}+\frac{1}{r}\frac{d\chi}{dr}+\frac{d^2\chi}{dz^2}\right)$ calculated from this is complicated. The writer has not succeeded in applying any of the methods of this paper to complete this example. To complete the solution it would be necessary to find a value of the velocity potential ϕ which is continuous with χ all over the surface $e=e_0$, and then to examine whether the rate of variation of ϕ is equal to the rate of variation of χ normal to surface $e=e_0$. The former part of the work is always theoretically possible, but it may happen that the latter is not.

The values of the components of the velocity found however completely solve the following problem :—

A hollow, smooth, rigid surface of annular form, whose equation is

$$ar^2(z-Z)^2+b(r^2-\alpha^2)^2=\text{const.}$$

moves parallel to the axis of z with arbitrary velocity $\dot{Z}$, to find a possible rotational motion inside it.

Appendix.

15. The density inside the elliptic cylinder $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is

$$-\frac{\frac{c}{4\pi}\left(\frac{1}{b^2}-\frac{1}{a^2}\right)xy}{\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)^2},$$

it is required (on account of Examples I., II., and III.) to calculate the potential inside and outside.

(It may be noticed that although the density is very great near the axis of the elliptic cylinder, yet the total mass of matter inside the cylinder $\frac{x^2}{a^2}+\frac{y^2}{b^2}=\text{const.}$, however small the constant may be, vanishes. Hence it is not singular that the potential should be finite).

The density varies as xy on every ellipse whose equation is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=\text{const.}$

It will be well to commence by finding the potential of a cylindric shell bounded by the cylinders

$$\frac{x^2}{\alpha^2}+\frac{y^2}{\beta^2}=1 \text{ and } \frac{x^2}{(m\alpha)^2}+\frac{y^2}{(m\beta)^2}=1$$

where m is a little >1, and where the density varies as xy.

The following are suitable values for the potential

$$V'=C.\tfrac{1}{2}\frac{\alpha-\beta}{\alpha+\beta}xy \text{ inside}$$

and

$$V=C.\frac{\alpha\beta}{\beta^2-\alpha^2}\left(1-\frac{\alpha^2+\beta^2+2\epsilon}{2\sqrt{(\alpha^2+\epsilon)(\beta^2+\epsilon)}}\right)xy \text{ outside}$$

where

$$\frac{x^2}{\alpha^2+\epsilon}+\frac{y^2}{\beta^2+\epsilon}=1$$

and C has to be suitably determined.

For V′ is finite and continuous inside the cylinder and satisfies LAPLACE's equation. It is continuous with V at the surface of the cylinder.

Also V satisfies LAPLACE's equation outside the cylinder, and $\frac{dV}{dx}, \frac{dV}{dy}$ both vanish at infinity.

To find the volume density ρ of the shell, let δn be the thickness of the shell, dn an element of normal drawn outwards.

Also

$$\frac{dV}{dn}-\frac{dV'}{dn}+4\pi\rho\delta n=0$$

Calculating the value of $\frac{dV}{dn}$ at the surface $\frac{x^2}{\alpha^2}+\frac{y^2}{\beta^2}=1$, *i.e.*, at the surface $\frac{x^2}{\alpha^2+\epsilon}+\frac{y^2}{\beta^2+\epsilon}=1$ where $\epsilon=0$

$$\frac{dV}{dn}=-p.C.\frac{\alpha-\beta}{\alpha+\beta}.\frac{xy}{\alpha\beta},$$

p being the perpendicular from the centre on the tangent to $\frac{x^2}{\alpha^2}+\frac{y^2}{\beta^2}=1$ at the point x, y.

Similarly at the same point

$$\frac{dV'}{dn}=p.C.\tfrac{1}{2}.\frac{\alpha-\beta}{\alpha+\beta}\left(\frac{1}{\alpha^2}+\frac{1}{\beta^2}\right)xy$$

therefore

$$\frac{dV}{dn}-\frac{dV'}{dn}=-\frac{pCxy}{2}.\frac{\alpha^2-\beta^2}{\alpha^2\beta^2}$$

Also $\frac{\delta n}{p}=\frac{\delta\alpha}{\alpha}$ where $\alpha+\delta\alpha$ is semi-major axis of external boundary of shell.

Hence

$$\rho=\frac{Cxy}{8\pi}.\frac{\alpha^2-\beta^2}{\alpha^2\beta^2}.\frac{\alpha}{\delta\alpha}.$$

Now consider the cylindric shell bounded by the two surfaces

$$\frac{x^2}{(ma)^2}+\frac{y^2}{(mb)^2}=1 \text{ and } \frac{x^2}{[(m+\delta m)a]^2}+\frac{y^2}{[(m+\delta m)b]^2}=1$$

The density at the point x, y inside it is

$$-\frac{c}{4\pi}\frac{\left(\frac{1}{b^2}-\frac{1}{a^2}\right)xy}{\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)^2}$$

To find the potential of the shell, put

$$\frac{Cxy}{8\pi}\cdot\frac{\alpha^2-\beta^2}{\alpha^2\beta^2}\cdot\frac{\alpha}{\delta\alpha}=-\frac{1}{4\pi}\frac{cxy\left(\frac{1}{b^2}-\frac{1}{a^2}\right)}{\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)^2}$$

where

$$\alpha=ma,\ \beta=mb,\ \delta\alpha=a\delta m,\ \frac{x^2}{a^2}+\frac{y^2}{b^2}=m^2$$

Hence

$$C=-2c\frac{\delta m}{m^3}$$

Therefore potential inside is

$$-\frac{2c\delta m}{m^3}\cdot\tfrac{1}{2}\cdot\frac{a-b}{a+b}xy$$

and potential outside is

$$-\frac{2c\delta m}{m^3}\cdot\frac{ab}{b^2-a^2}\left(1-\frac{m^2a^2+m^2b^2+2\epsilon}{2\sqrt{(m^2a^2+\epsilon)(m^2b^2+\epsilon)}}\right)xy$$

where ϵ is given by the equation

$$\frac{x^2}{m^2a^2+\epsilon}+\frac{y^2}{m^2b^2+\epsilon}=1$$

To calculate the potential of the whole cylinder, put $\epsilon=m^2P$ so that

$$\frac{x^2}{a^2+P}+\frac{y^2}{b^2+P}=m^2$$

Then the potential of the shell, considered above, at an external point, becomes

$$-\frac{2c\delta m}{m^3}\cdot\frac{ab}{b^2-a^2}\left(1-\frac{a^2+b^2+2P}{2\sqrt{(a^2+P)(b^2+P)}}\right)xy$$

The limits of integration for m are 0 and 1, when integrating to find the potential of the whole cylinder at an external point.

Therefore the limits of P are ∞ and that value of ϵ which satisfies the equation $\frac{x^2}{a^2+\epsilon}+\frac{y^2}{b^2+\epsilon}=1$ and which makes both $a^2+\epsilon$ and $b^2+\epsilon$ positive.

Also

$$2m\delta m=-\left\{\frac{x^2}{(a^2+P)^2}+\frac{y^2}{(b^2+P)^2}\right\}\delta P.$$

Therefore the potential for the whole cylinder at an external point is

$$c\frac{ab}{b^2-a^2}\int_\infty^\epsilon xy\left(1-\frac{a^2+b^2+2\mathrm{P}}{2\sqrt{(a^2+\mathrm{P})(b^2+\mathrm{P})}}\right)\cdot\frac{\left\{\frac{x^2}{(a^2+\mathrm{P})^2}+\frac{y^2}{(b^2+\mathrm{P})^2}\right\}}{\left\{\frac{x^2}{a^2+\mathrm{P}}+\frac{y^2}{b^2+\mathrm{P}}\right\}^2}d\mathrm{P}$$

$$=\left[\frac{cab}{b^2-a^2}xy\frac{1-\frac{a^2+b^2+2\mathrm{P}}{2\sqrt{(a^2+\mathrm{P})(b^2+\mathrm{P})}}}{\frac{x^2}{a^2+\mathrm{P}}+\frac{y^2}{b^2+\mathrm{P}}}+\frac{cab}{2}\tan^{-1}\left\{\frac{x}{y}\sqrt{\frac{b^2+\mathrm{P}}{a^2+\mathrm{P}}}\right\}\right]_\infty^\epsilon$$

$$=\frac{cab}{b^2-a^2}xy\left(1-\frac{a^2+b^2+2\epsilon}{2\sqrt{(a^2+\epsilon)(b^2+\epsilon)}}\right)+\frac{cab}{2}\left(\tan^{-1}\left\{\frac{x}{y}\sqrt{\frac{b^2+\epsilon}{a^2+\epsilon}}\right\}-\tan^{-1}\frac{x}{y}\right)$$

$$=\frac{cab}{b^2-a^2}xy\left(1-\frac{a^2+b^2+2\epsilon}{2\sqrt{(a^2+\epsilon)(b^2+\epsilon)}}\right)+\frac{cab}{2}\left(\sin^{-1}\frac{x}{\sqrt{a^2+\epsilon}}-\sin^{-1}\frac{x}{\sqrt{x^2+y^2}}\right)$$

To find the potential at an internal point x, y.

Suppose that this point lies on the elliptic cylinder $\frac{x^2}{(\mu a)^2}+\frac{y^2}{(\mu b)^2}=1$

Then the potential at x, y = potential of matter inside this cylinder
+ potential of matter outside it.

The potential of the matter inside $\frac{x^2}{(\mu a)^2}+\frac{y^2}{(\mu b)^2}=1$ is obtained by taking the same integral as in finding the potential of the cylinder $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at an external point, but the limits for m are now 0 and μ, *i.e.*, $\mathrm{P}=\infty$ to $\mathrm{P}=$ a root of the equation $\frac{x^2}{a^2+\mathrm{P}}+\frac{y^2}{b^2+\mathrm{P}}=\mu^2$ which makes $a^2+\mathrm{P}$, $b^2+\mathrm{P}$ both positive, but this root is zero, since $\frac{x^2}{(\mu a)^2}+\frac{y^2}{(\mu b)^2}=1$.

This gives

$$\frac{cab}{b^2-a^2}\cdot\frac{xy}{\mu^2}\left(-\frac{(a-b)^2}{2ab}\right)+\frac{cab}{2}\left(\tan^{-1}\frac{bx}{ay}-\tan^{-1}\frac{x}{y}\right)$$

i.e.,

$$\frac{c}{2}\cdot\frac{a-b}{a+b}\cdot\frac{xy}{\mu^2}+\frac{cab}{2}\left\{\sin^{-1}\frac{\frac{x}{a}}{\sqrt{\frac{x^2}{a^2}+\frac{y^2}{b^2}}}-\sin^{-1}\frac{x}{\sqrt{x^2+y^2}}\right\}$$

The potential of the matter outside the cylinder $\frac{x^2}{(\mu a)^2}+\frac{y^2}{(\mu b)^2}=1$ is the integral of $-\frac{2c\delta m}{m^3}\cdot\frac{1}{2}\cdot\frac{a-b}{a+b}xy$ between the limits $m=\mu$ and $m=1$.

This gives

$$c\left(1-\frac{1}{\mu^2}\right)\frac{1}{2}\frac{a-b}{a+b}xy$$

Therefore the potential at the internal point x, y is

$$\frac{c}{2}\cdot\frac{a-b}{a+b}xy+\frac{cab}{2}\left\{\sin^{-1}\frac{\frac{x}{a}}{\sqrt{\frac{x^2}{a^2}+\frac{y^2}{b^2}}}-\sin^{-1}\frac{x}{\sqrt{x^2+y^2}}\right\}$$

Thus if the density at the point x, y inside the elliptic cylinder $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ be $-\frac{c}{4\pi}\left(\frac{1}{b^2}-\frac{1}{a^2}\right)\frac{xy}{\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)^2}$, then the potential at an internal point is

$$\frac{c}{2}\cdot\frac{a-b}{a+b}xy+\frac{cab}{2}\left\{\sin^{-1}\frac{\frac{x}{a}}{\sqrt{\frac{x^2}{a^2}+\frac{y^2}{b^2}}}-\sin^{-1}\frac{x}{\sqrt{x^2+y^2}}\right\},$$

and the potential at an external point is

$$\frac{cab}{a^2-b^2}xy\left(\frac{a^2+b^2+2\epsilon}{2\sqrt{(a^2+\epsilon)(b^2+\epsilon)}}-1\right)+\frac{cab}{2}\left\{\sin^{-1}\frac{x}{\sqrt{a^2+\epsilon}}-\sin^{-1}\frac{x}{\sqrt{x^2+y^2}}\right\}$$

where

$$\frac{x^2}{a^2+\epsilon}+\frac{y^2}{b^2+\epsilon}=1.$$