III. *On the connexion between Electric Current and the Electric and Magnetic Inductions in the surrounding field.**

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Received January 31,—Read February 12, 1885.

IN a paper published in the Philosophical Transactions for 1884 (Part II., pp. 343–361), I have deduced from MAXWELL's equations for the electromagnetic field the mode in which the energy moves in the field. The result there obtained is that the energy moves at any point perpendicularly to the plane containing the directions of the electric and magnetic intensities, and in the direction in which a right-handed screw would move if turned round from the positive direction of the electric intensity to the positive direction of the magnetic intensity. The quantity crossing the plane per unit area per second is equal to the product of the two intensities multiplied by the sine of the included angle divided by 4π.†

Hence it follows that the energy moves along the intersections of the two sets of level surfaces, electric and magnetic, where they both exist, their intersections giving, as it were, the lines of flow. In the particular case of a steady current in a wire

* [Added July 15.—Since the reading of the paper I have found a remarkable passage in FARADAY's 'Experimental Researches,' vol. 1, p. 529, § 1659, which I give below. The words I have put in *italics* might be regarded as the starting point of the views which I have attempted to develop in this paper. "§ 1659. According to the beautiful theory of AMPÈRE, the transverse force of a current may be represented by its attraction for a similar current and its repulsion of a contrary current. May not then the equivalent transverse force of static electricity be represented by that lateral tension or repulsion which the lines of inductive action appear to possess (1304)? Then, again, *when current or discharge occurs between two bodies, previously under inductrical relations to each other, the lines of inductive force will weaken and fade away, and, as their lateral repulsive tension diminishes, will contract and ultimately disappear in the line of discharge.* May not this be an effect identical with the attractions of similar currents, *i.e.*, may not the passage of static electricity into current electricity, and that of the lateral tension of the lines of inductive force into the lateral attraction of lines of similar discharge, have the same relation and dependence, and run parallel to each other?"]

† I here adopt the simpler term "Electric Intensity," denoted by E.I., instead of "Electromotive Intensity," for the force which would act on a small body charged with unit of positive electrification. The magnetic intensity, *i.e.*, the force which would act on a unit north-seeking Pole, will be denoted by M.I.

where the electrical level surfaces cut the wire perpendicularly to the axis, it appears that the energy dissipated in the wire as heat comes in from the surrounding medium, entering perpendicularly to the surface.

In that paper I made no assumption as to the transfer of the electric and magnetic inductions—the electric and magnetic conditions—through the medium, merely considering the movement of energy. I now propose to develop a hypothesis as to the transfer of the inductive condition in the medium, and its movement inwards upon current-bearing wires.

The value of the electric induction at any point in an isotropic medium is equal to $K \times$ E.I.$/4\pi$, and the direction of the induction coincides with that of the intensity. MAXWELL terms this electric induction "displacement," but I think that "induction" is preferable, as it implies no hypothesis beyond that of some alteration in the medium, which can be described by a vector. The value of the magnetic induction is equal to $\mu \times$ M.I., and its direction coincides with that of the magnetic intensity.

If we symbolise the electric and magnetic conditions of the field by induction tubes running in the directions of the intensities, the tubes being supposed drawn in each case so that the total induction over a cross section is unity, then we have reason to suppose that the electric tubes are continuous except where there are electric charges, while the magnetic tubes are probably in all cases continuous and re-entrant.

In the neighbourhood of a wire containing a current, the electric tubes may in general be taken as parallel to the wire while the magnetic tubes encircle it. The hypothesis I propose is that the tubes move in upon the wire, their places being supplied by fresh tubes sent out from the seat of the so-called electromotive force. The change in the point of view involved in this hypothesis consists chiefly in this, that induction is regarded as being propagated sideways rather than along the tubes or lines of induction. This seems natural if we are correct in supposing that the energy is so propagated, and if we therefore cease to look upon current as merely something travelling along the conductor carrying it, and in its passage affecting the surrounding medium. As we have no means of examining the medium, to observe what goes on there, but have to be content with studying what takes place in conductors bounded by the medium, the hypothesis is at present incapable of verification. Its use, then, can only be justified if it accounts for known facts better than any other hypothesis.

The basis of MAXWELL'S *Electromagnetic Theory.*

MAXWELL'S Electromagnetic Theory rests on three general principles.

I. The first principle consists in the assumption that energy has position, *i.e.*, that it occupies space. The electric and magnetic energies of an electromagnetic system reside therefore somewhere in the field. It is an inevitable conclusion that they are present wherever the electric and magnetic intensities can be shown to exist. For

instance, suppose a small electrified body placed in a field where there is electric intensity; then the body will be acted on by force and will receive energy which appears as the energy of motion, the electric energy at the same time decreasing. If energy has position that which is now in the body must have come into it through the surrounding space, or it was present in that space before the body took it up. The alternative that it appeared in the body without passing through the space immediately surrounding the body need not be discussed. Hence the existence of electric intensity implies the existence of electric energy in the place where the electric intensity is capable of manifestation. Similarly magnetic energy accompanies magnetic intensity. The inductive condition of the medium imagined by FARADAY is due then to its modification when containing energy. MAXWELL has shown that all the energy is accounted for on the supposition that the electric energy per unit volume at any point is $K(E.I.)^2/8\pi$, and that the magnetic energy is $\mu(M.I.)^2/8\pi$. He has given in his 'Elementary Treatise on Electricity,' p. 47, another way of describing the distribution of energy which will be more useful for my purpose. If the field be mapped out by unit induction tubes—either electric or magnetic—*i.e.*, tubes drawn so that the total induction over every cross section of a tube is unity, and if these tubes be divided into cells of length such that the difference of potential or the line integral of the intensity between the two ends of each cell is unity, then each cell contains, if electric, half a unit of energy, if magnetic $\frac{1}{8\pi}$ of a unit, the divisor 4π being introduced by the difference in definition of the two inductions. MAXWELL terms these unit cells.

II. The second principle is in part experimental, viz.:—that the line integral of the electric intensity round any closed curve is equal to the rate of decrease of the total magnetic induction through the curve. This is verified by experiment when the curve is drawn through conducting material. MAXWELL supposes it to be true in all cases, that is, he supposes that electric induction can be produced in insulators by means of magnetic changes, without the presence of charges on conductors, and is therefore led to identify the growth and decrease of electric induction with current.

III. The third principle is also in part experimental, viz.:—that the line integral of the magnetic intensity round any closed curve is equal to $4\pi\times$ current through the curve. This is verified by experiment when the current is in a wire, and MAXWELL supposes it to be also true in the case where there is change of electric induction in an insulator. The supposition is justified by Prof. ROWLAND'S well-known experiment.

From these three principles MAXWELL deduces his general equations of the Electromagnetic Field. I have stated them in full as I propose to modify the second and third principles, and I wish to make quite clear the nature of the proposed changes.

Modification of the Second Principle.

I propose to replace the second principle by the following:—*Whenever electromotive force is produced by change in the magnetic field, or by motion of matter through the field, the E.M.F. per unit length or the electric intensity is equal to the number of tubes of magnetic induction cutting or cut by the unit length per second, the E.M.F. tending to produce induction in the direction in which a right-handed screw would move if turned round from the direction of motion relatively to the tubes towards the direction of the magnetic induction.**

In order that the results obtained from this should agree with those obtained from MAXWELL'S statement of the principle, it is necessary that change in the total quantity of magnetic induction passing through a closed curve should always be produced by the passage of induction tubes through the curve inwards or outwards. In some instances this is undoubtedly the case, as, for instance, where a part of a circuit moves so as to cut a fixed magnetic field, or where a magnet moves in the neighbourhood of a circuit. Here the E.M.F. is equal to the number of tubes cut by the wire per second, and its seat is that part of the wire cutting the tubes. In other cases, as, for instance, where the wire is between the poles of an electromagnet whose magnetising current is changing, we have no direct experimental evidence of the movement of the induction in or out. But the induction tubes are closed, and to make them thread a circuit we might expect that they would have to cut through the boundary. The alternative seems to be that they should grow or diminish from within, the change in intensity being propagated *along* the tubes. This would be inconsistent with their closed nature, unless the energy were instantaneously propagated along the whole length, and is further negatived by the theory of the transfer of energy, which implies that the energy flows transversely to the direction of the tubes. I shall suppose, then, that alteration in the quantity of magnetic induction through a closed curve is always produced by motion of induction tubes inwards or outwards through the bounding curve.

* Taking the electric intensity as always perpendicular to the plane of motion of the magnetic tubes through a point, and equal to the number cut per second by unit length of the normal to the plane of motion, we can easily show that the component of the intensity in any other direction will be equal to the number of tubes cut by a line of unit length in that direction. For let OA represent a small length drawn perpendicular to the plane of motion, and let OP represent a line drawn in any direction making θ with OA. Draw AP perpendicular to OA, and meeting OP in P. Then the same number of tubes will cut both OA and OP, since AP is parallel to their plane of motion. If the number cutting OA be $E \times OA$, where E is the number cutting unit length, and therefore equal to the resulting intensity, the number cutting unit length of OP will be $E.\frac{OA}{OP} = E \cos\theta$, or the component of the intensity along OP.

Modification of the Third Principle.

The third principle admits of similar analysis, according to which we may regard the magnetic intensity along a closed curve as due to the cutting of the curve by tubes of electric induction. If we regard the line integral of the magnetic intensity round a tube of induction as measuring the magnetomotive force—employing a useful term suggested by Mr. BOSANQUET—we may put the modification in the following form :—

Whenever magnetomotive force is produced by change in the electric field, or by motion of matter through the field, the magnetomotive force per unit length is equal to $4\pi\times$ *the number of tubes of electric induction cutting or cut by unit length per second, the magnetomotive force tending to produce induction in the direction in which a right-handed screw would move if turned round from the direction of the electric induction towards the direction of motion of the unit length relatively to the tubes of induction.*

This is the most general form of the principle, but we shall only require the more special statement which immediately follows from it : that the line integral of the M.I. round any curve is equal to $4\pi\times$ the number of tubes passing in or out through the curve per second.

We have reasons exactly similar to those given in the last case for supposing that any change in the total electric induction through a curve is caused by the passage of induction tubes in or out across the boundary. The alternative that change takes place by propagation from the ends, seems inconsistent with the theory of the transverse flow of energy.

I shall postpone the discussion of the modifications of the general equations of the electromagnetic field following from these changes in the fundamental principles, and proceed to discuss the bearing which they have upon the nature of currents in conductors.

A straight wire carrying a steady current.

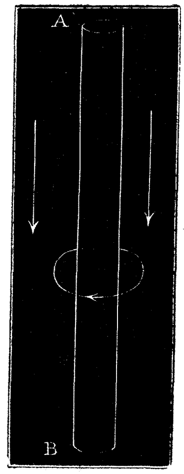
Let AB represent a wire in which is a steady current from A to B. The direction of the electric induction in the surrounding field near the wire, if the field be homogeneous, is parallel to AB.

Let E be the value of the electric intensity, or the difference of potential per unit length perpendicular to the level surfaces, and let R be the resistance of the wire per unit length. Then $C=\frac{E}{R}$ where C is the current, and C is uniform throughout the circuit. The magnetic intensity in the immediate neighbourhood of the wire at a distance r from the axis of the wire is $\frac{2C}{r}$.

The hypothesis proposed as to the nature of the current is that C electric induction tubes close in upon the wire per second. The wire is not capable of bearing a

continually-increasing induction, and breaks the tubes up, as it were, their energy appearing finally as heat.*

Fig. 1.



Let us see how this hypothesis accounts for known facts, when aided by the two principles just laid down.

It accounts at once for the constancy of the current at all parts of the wire in the steady state, in so far as it reduces this constancy to a particular case of the law according to which there is the same total induction over all cross sections of a tube. If, for instance, there were more induction entering at A than at B, then more tubes must be entering at A, and so there would be an increase in the number of tubes left in the medium about B, or the field would not be steady.

Further, if we draw any closed curve embracing the wire once, we may apply the third principle to give us the line integral of the magnetic intensity round the curve. For this is a case where change is certainly going on in the electric field, and the magnetomotive force is due to this change. The field being steady, if C tubes enter the wire and are there broken up, C tubes must cross through any encircling curve to supply their place, or the line integral of the magnetic intensity round the curve is equal to $4\pi\times$ number of tubes passing through the boundary per second, *i.e.*, 4πC. If the curve be a circle of radius r, with its centre in the axis and plane perpendicular thereto, the intensity at any point of this circle will be tangential to it, and equal to

$$\frac{4\pi C}{2\pi r}=\frac{2C}{r}.$$

The known constancy of the line integral of the magnetic intensity round the wire, which the hypothesis thus accounts for, almost seems to force the hypothesis upon us,

* May we not say that the tubes are dissolved. The term seems to suggest that the induction is not destroyed, but only loses its continuity. Probably this is the case; for on the electromagnetic theory of radiant energy, when the wire is heated, it sends out the energy it received, again in the electro-magnetic form.

if we regard the field as caused by the inward flowing of the energy rather than by something propagated out from the wire.

Assuming that the induction tubes bring in their energy, the quantity is easily found. The number of unit cells per unit length is equal to the difference of potential per unit length, or E. Hence the energy per unit length of each tube is $\frac{E}{2}$, since each cell contains a half unit. If C tubes disappear in the wire per second they yield up $\frac{CE}{2}$ of energy per unit length. Now the total energy dissipated per unit length is CE per second. Or the movement inwards of the electric induction will only account for half of the energy. The other half must be accounted for by the movement inwards of the magnetic induction. This movement of the magnetic induction is suggested by the existence of electric induction, which cannot be ascribed to statical charges.

The electric intensity is E. Hence E tubes of magnetic induction must move in per second, cutting unit length parallel to the axis of the wire, in accordance with the second principle, and it will easily be seen that the inward motion gives the right direction of the electric intensity. The line integral of the magnetic intensity round a tube is $4\pi C$, the tubes being closed rings. Hence there are $4\pi C$ unit cells in the length. Since each of these contains $\frac{1}{8\pi}$ of energy the quantity per tube$=\frac{4\pi C}{8\pi}=\frac{C}{2}$. E tubes entering the wire per second will carry in $\frac{CE}{2}$ of energy, the other half to be accounted for.

We can in a similar manner trace the dissipation of the energy, which we must suppose taking place within the wire. The line integral of the magnetic intensity round a circle, with its centre in the axis of the wire, is constant up to the wire, and equal to $4\pi C$. Within the wire it gradually diminishes as the circle contracts. At a distance r from the centre it is $4\pi C.\frac{r^2}{a^2}$ when a is the radius of the wire. If we assume this intensity to be still due to the passage inwards of the tubes of electric induction only, $\frac{Cr^2}{a^2}$ cross inwards per second at a distance r, the difference between this number and the C tubes entering the outer boundary being destroyed and their energy dissipated. The energy thus dissipated per unit length between the outer boundary and a coaxal cylinder of radius r will be $\frac{EC}{2}\left(1-\frac{r^2}{a^2}\right)$ per second. If $r=0$ the whole of the electric energy is dissipated. It would appear, then, that we may represent the dissipation of the electric energy by the total destruction of the tubes all through their length.

The value of the electric intensity being E throughout the wire the number of tubes of magnetic induction cutting unit length parallel to the axis is the same at all parts, viz., E per second. Hence, the magnetic tubes are not destroyed as the electric tubes

are. But the line integral of the magnetic intensity round the tubes diminishes as they approach the axis, being $4\pi\frac{Cr^2}{a^2}$ round that at distance r. The number of unit cells diminishes, and, therefore, the energy per tube is less, the decrease being due to that dissipated. Thus the energy entering in the E tubes at the outer boundary is $\frac{4\pi CE}{8\pi}$ or $\frac{CE}{2}$. That crossing in E tubes at a distance r is $\frac{4\pi}{8\pi}\frac{Cr^2}{a^2}E=\frac{CE}{2}\frac{r^2}{a^2}$. The difference $\frac{CE}{2}\left(1-\frac{r^2}{a^2}\right)$ being dissipated.

Hence it appears that the energy dissipated per second may be represented as half electric half magnetic, the electric energy being dissipated by the breaking up of the tubes, and their disappearance while the magnetic energy is dissipated by the shortening of the tubes, and their final disappearance by contraction to infinitely small dimensions of the diameters of the rings by which we may represent them. At all points therefore outside and inside the energy crossing any surface may be represented as equally divided between the two kinds.

As we know the value of the induction at any point, or the number of tubes passing through unit area, and as we also know the number of tubes cutting the boundary it is easy, on the assumption that the tubes move on unchanged, to calculate their velocity. Of course this velocity is purely hypothetical, as we cannot examine minutely into the medium and observe what goes on there. Probably, if we could observe with sufficient minuteness we should find unevennesses in the induction. If the velocity of the tubes has any physical meaning it is that these unevennesses are carried forward with that velocity. To illustrate this let us suppose that we have water flowing through a glass tube at a steady rate. We have nothing to show that the water is moving past any point in the tube beyond its disappearance at the entrance and its appearance at the exit, but knowing the cross section of the tube, *i.e.*, the quantity of water in any part of it, and the quantity entering and leaving it is easy to assign a velocity to the water in the tube which shall account for the observed amount entering and leaving. This velocity is to a certain extent hypothetical. But if we examine the tube with a sufficient magnifying power to show particles of dust in the water the existence of the velocity receives a more direct proof. I do not know whether we should have any right to expect a similar proof of the motion of induction even if we had the means of observation.

To find the hypothetical velocity of the electric induction tubes let us calculate the number of tubes passing through a circular band with radii r and $r+dr$ and centre in the axis of the wire, and lying in a plane perpendicular to the axis. The intensity being E the induction is $\frac{KE}{4\pi}$, and therefore the area of cross section of each tube is $\frac{4\pi}{KE}$, since area $\times$ induction is unity. The number passing through the circular band is therefore $2\pi r dr\cdot\frac{KE}{4\pi}=\frac{KErdr}{2}$.

Since C tubes move in through the inner circle per second, $\frac{\mathrm{KE}r\,dr}{2}$ tubes move in in $\frac{\mathrm{KE}r\,dr}{2\mathrm{C}}$ of a second, *i.e.*, all the tubes passing through the band will have just moved in in this time. The outermost tubes therefore describe the space dr in time $\frac{\mathrm{KE}r\,dr}{2\mathrm{C}}$, or the velocity is $\frac{2\mathrm{C}}{\mathrm{KE}r}$. Now we know that if R be the resistance per unit length $\mathrm{C}=\frac{\mathrm{E}}{\mathrm{R}}$. Hence we may put the velocity in the form

$$\frac{2}{\mathrm{KR}}\cdot\frac{1}{r}$$

which is independent of the current.

To take a special case let us calculate the velocity just outside the boundary of a copper wire, the specific resistance of copper being 1642 in electromagnetic measure. Then if a be the radius of the wire

$$\mathrm{R}=\frac{1642}{\pi a^2}$$

and $\mathrm{K}=\frac{1}{v^2}$ where v is the ratio of the units, which in air may be taken as 3×10^{10}.

Then the velocity

$$=\frac{2v^2\pi a^2}{1642a}$$

$$=\frac{2\times 9\times 10^{20}\pi a}{1642}$$

$$=345\times 10^{16}a$$

At greater distances the velocity will be less, diminishing according to the inverse distance.

The hypothetical velocity of propagation of the magnetic induction may be calculated in a similar manner. The intensity at a distance r from the axis is $\frac{2\mathrm{C}}{r}$ and the induction is $\frac{2\mu\mathrm{C}}{r}$. The area of each tube is therefore $\frac{r}{2\mu\mathrm{C}}$, and the number lying in a ring of rectangular section with depth unity and internal and external radii r and $r+dr$, will be $1\times dr\div\frac{r}{2\mu\mathrm{C}}=\frac{2\mu\mathrm{C}dr}{r}$.

But E tubes move in per second through the inner face of the ring, so that $\frac{2\mu\mathrm{C}dr}{r}$ tubes move in in time $\frac{2\mu\mathrm{C}dr}{\mathrm{E}r}$, or this is the time taken by the outermost tubes to move across the ring describing a distance dr. The velocity is therefore

$$\frac{\mathrm{E}r}{2\mu\mathrm{C}}=\frac{\mathrm{R}r}{2\mu}$$

which is again independent of the current.

If the current-bearing wire is copper, $R=\frac{1642}{\pi a^2}$, and with $\mu=1$ the velocity becomes

$$\frac{1642r}{2\pi a^2}$$

We cannot assign a velocity to the electric tubes within the wire since the number is diminishing as their energy dissipates. But the magnetic tubes crossing unit length parallel to the axis are still unchanged in number, so that we may assign a velocity to them. This velocity means that with the known value of the magnetic induction this velocity will give the number crossing inwards required to produce electric intensity E.

The velocity will be found equal to $\frac{Ea^2}{2\mu Cr}$ or $\frac{Ra^2}{2\mu r}$.

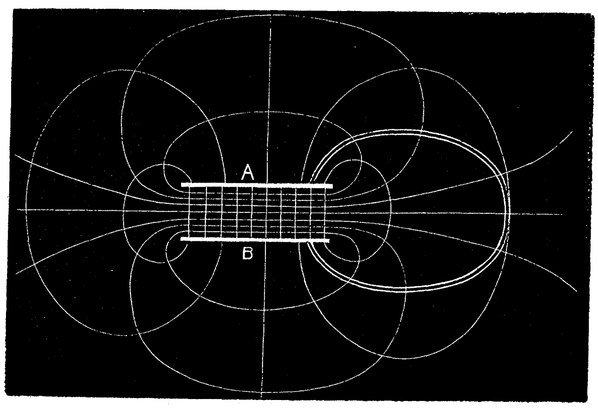
In the case of a copper wire this becomes

$$\frac{1642}{2\mu\pi r}$$

Discharge of a condenser through a fine wire.

Let us suppose that we have a condenser consisting of two parallel plates A and B and charged with equal and opposite charges. Then we know that there will be electric induction between the two plates, and that according to MAXWELL'S theory the energy of the system is stored there. We may form an idea of the distribution of the energy by drawing the unit induction tubes, each starting from and ending in unit quantity of electricity, and dividing these into unit cells by the level surfaces, drawn at unit difference of potential (fig. 2). If the dimensions of the plates be

Fig. 2.



great compared with their distance apart, then nearly all the cells will be between the two plates, and since each cell contains half a unit of energy, nearly all the energy is there. There will, however, be slight induction, and therefore some small quantity of energy in the surrounding space.

Now let the two plates be connected by a wire. Discharge takes place, and we are fairly justified, from the heat in the wire and the transient magnetic effects, in saying that a current has been in the wire from the positive to the negative plate, or the wire was for the time being in the same relation to the surrounding medium as the wire in the case just considered, the condition of affairs, however, not being steady.

Let us suppose the wire to have a very great resistance, in order that, at least in imagination, we may lengthen out the time of discharge. On the ordinary current theory, combined with MAXWELL'S "displacement" theory, the medium between the plates has returned from the strained condition, denoted by "displacement" from the positive to the negative plate, causing displacement through the plates and along the wire, the displacement being in the same direction all round the circuit. This is generally, I think, supposed to take place by the recovery of the medium between the plates causing displacement in the metal immediately in front of it, the displacement being analogous to the forcing of water along a pipe corresponding to the plates and wire, by the recovery from strain of some substance placed in a chamber corresponding to the space between the plates.

According to the hypothesis here advanced we must suppose the lessening of the induction between the plates—induction being used with the same physical meaning as MAXWELL'S displacement—to take place by the divergence outwards of the induction tubes. We may picture them as taking up the positions of successive lines of induction further and further away from the space between the plates, their ends always remaining on the plates. They finally converge on the wire, and are then broken up and their energy dissipated as heat. At the same time some of the energy becomes magnetic, this occurring as the difference of potential between the plates lowers, so that the tubes contain fewer unit cells.

The magnetic energy will be contained in ring-shaped tubes which will expand from between the plates and then contract upon some other part of the circuit. To illustrate the movement of the electric induction tubes let us suppose them to be represented by elastic strings stretched between the two plates. Then the motion of the tubes outwards would be roughly represented by pulling the elastic strings outwards and doubling them back close against the wire, their ends being still attached to the plates. It is evident that if any ring surround the wire each of the strings must break through it in order to reach the wire. Hence the total number of strings cutting any ring surrounding the wire is the same wherever the ring be placed. Similarly the total number of tubes of electric induction cutting any curve encircling the wire is the same, and therefore the line integral of the magnetic intensity round the curve integrated throughout the time of discharge is the same, or the total magnetic effect is the same at all parts of the circuit. It is not necessary to suppose that a tube enters the wire at the same moment throughout its whole length; indeed, the experiments of WHEATSTONE on the so-called velocity of electricity prove clearly that

this is not the case, for in those experiments the tubes reached air breaks near the two ends of the wire before they reached a break in the middle.

We cannot by this general reasoning show that the energy entering any length of the wire will be proportional to the resistance of that length—the result obtained by RIESS. Indeed, this cannot always be the case. For instance, imagine a condenser discharged by two wires connected to the two plates of another condenser of greater capacity, whose plates are again connected by a fine wire of enormous resistance, through which the discharge can only take place slowly. Then the energy dissipated in the wires will not to a first approximation depend on their resistances but on the ratios of the capacities, that in the wire of high resistance bearing to that in the other wires the ratio of the less capacity to the greater. Probably RIESS's results only hold when the discharge takes place in such a way that it may be looked upon at any one moment as approximately in the steady state.

We have shown that the magnetic measure of the total current is the same all along the wire. Probably also the chemical measure is the same—meaning by the chemical measure whatever interchanging or turning round of molecules may occur when induction takes place in a conductor. For even if a tube does not enter the wire at the same time throughout its length, an end part, say, entering first, the point of attachment of the tube to the conductor being transferred from the plate to somewhere along the wire, this transference of the point of attachment from molecule to molecule implies the same amount of chemical change within the wire as if the tube entered all at the same moment. It will not, however, take place equally throughout the cross section as it does in the steady state.

Probably we only have the simultaneous disappearance of all parts of a tube when the wire follows a line of electric induction, and has its resistance per unit length proportional to the intensity which would exist there if the wire were removed.

The hypothesis here advanced is in accordance with MAXWELL's doctrine of closed currents. For the induction dissipated at one part of the circuit has come there from another part where relatively to the circuit it ran in the opposite direction. The total result is equivalent to the addition of so many closed induction tubes to the circuit, the induction running the same way relatively to the circuit throughout.

If the two plates of the condenser are not connected by a wire but are discharged gradually by the imperfect insulation of the dielectric, then we must suppose that the tubes of induction in this case are dissipated *in situ*, the induction simply decaying at a rate depending on its amount and upon the conductivity of the dielectric. We may still represent this process by a closed current by regarding the loss of induction $\left(\text{MAXWELL's } -\frac{df}{dt}\right)$ and the quantity of induction dissipated (MAXWELL's p), as two different quantities. We have then $p+\frac{df}{dt}=0$ or we have two equal and opposite currents. But this seems artificial. It is more natural to look upon the process

merely as a decay of electric induction without movement inwards of fresh induction tubes, and therefore without the formation of magnetic induction.

I have discussed the case of discharge of a condenser at some length, as we can here realise more easily what goes on at the source of energy. The results obtained suggest that a similar action occurs at the source of energy or seat of the electromotive force in other cases where we do not know the distribution of induction, and are obliged to guess at the action.

A circuit containing a voltaic cell.

We may pass on from the discharge of a condenser to the consideration of the current in a circuit containing a voltaic cell. The chemical theory of the cell will be here adopted—in fact, the hypothesis I am endeavouring to set forth has no meaning on the voltaic metal-contact theory.

Let us suppose the cell to consist of zinc and copper plates, a vessel of dilute sulphuric acid, and copper wires attached to each plate which on junction complete the circuit. For simplicity I shall disregard the effect of the air and suppose that it is a neutral gas causing no induction.

We shall begin by supposing the circuit open. Then we know that on immersion there will be temporary currents in the wires, the quantities of these currents depending on the electrostatic capacity of the system composed of the wires. The currents last till the wires have received charges such that they are, say at difference of potential V. If the terminals are connected to a condenser the temporary currents may be easily detected by a galvanometer in the circuit. They are in no way to be distinguished in kind from the permanent current which will be established when the circuit is complete, except that they are of short duration and in general very small. There is no reason then to suppose that the action in the cell is different from that which takes place when the current is permanent, and I think we may safely assume that FARADAY'S law of electrolysis holds according to which the quantity of electricity flowing along either wire is proportional to the quantity of chemical action—or, in the form appropriate here, the number of tubes of induction produced is proportional to the quantity of chemical action.

Let Q be the total quantity of electricity upon the positive terminal; then $\frac{QV}{2}$ is the total energy thrown out into the dielectric.

Let z be the quantity of zinc consumed per unit of electricity, then Qz is the total quantity consumed in the charging of the terminals. Let E be the energy set free by each quantity z of zinc consumed, after all actions in the cell have been provided for. E then is the E.M.F. which the cell will have on the closure of the circuit, as long as the chemical actions remain the same, for z corresponds to the passage of a unit of electricity or the production of one tube, and we know that the energy set free by C units is CE.

Now while the charges are gathering and while the potential difference of the terminals is gradually increasing, the energy required to add equal increments of charge will also increase, and the charging will cease when the amount of energy given up by a given amount of chemical action in the cell is equal to the amount required to add the corresponding charge to the terminals. For to suppose the action to go beyond this is to suppose that the energy thrown out into the space between the terminals is greater than that yielded by the battery.

Let dQ be the last quantity of charge added to the terminals. This requires energy VdQ.

The corresponding quantity of zinc consumed is zdQ, giving up energy EdQ.

The condition of equilibrium is that

$$VdQ = EdQ$$

or

$$V = E$$

which agrees with the result of experiment that the difference of potential of the terminals in open circuit is equal to the E.M.F. of the cell immediately after closure.

It may be noticed that the total quantity of energy extracted from the battery is

$$QE = QV$$

while the electric energy left in the medium is

$$\frac{QV}{2}$$

or half the energy has been converted into heat in the wires.

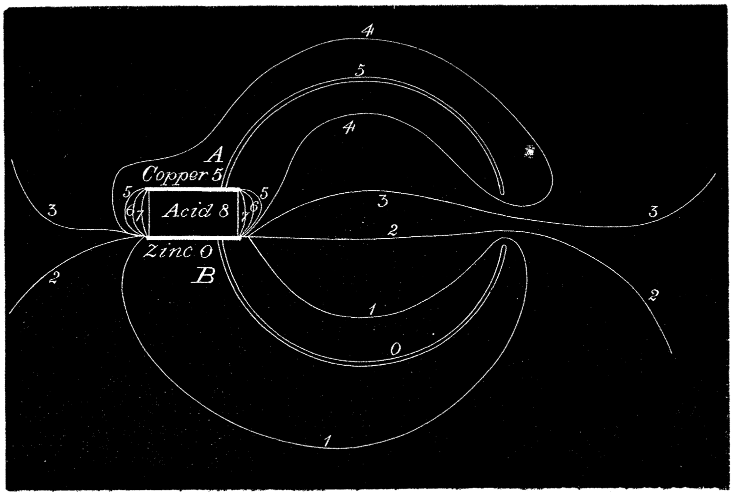
We will now consider the distribution of level surfaces in the field while the circuit is still open. There will be $V-1$ surfaces between the terminals dividing each tube into V cells. None of these will cut the homogeneous parts of the circuit, since the whole of each of these must be at one and the same potential.

They can only cut the circuit by passing through the regions where there is contact of dissimilar bodies. We will neglect the contact of the zinc and copper, as the difference of potential there is insignificant compared with that at the two surfaces, zinc-acid and copper-acid.

Now we know that the energy of the cell is put out at the zinc-acid contact, but the amount is greater than that obtained from a consideration of the E.M.F. of the cell, for some energy is absorbed again, probably, at the copper-acid contact in the evolution of hydrogen. There is probably, then, induction between the acid and the zinc, and between the acid and the copper, these resembling the spaces between the plates of two condensers, the acid being at a higher potential than either. But if a given amount of induction disappears from the zinc-acid contact and appears at the terminals, more energy is lost at the former than appears at the latter. Hence all the cells have not been transferred from one to the other, or the difference of potential

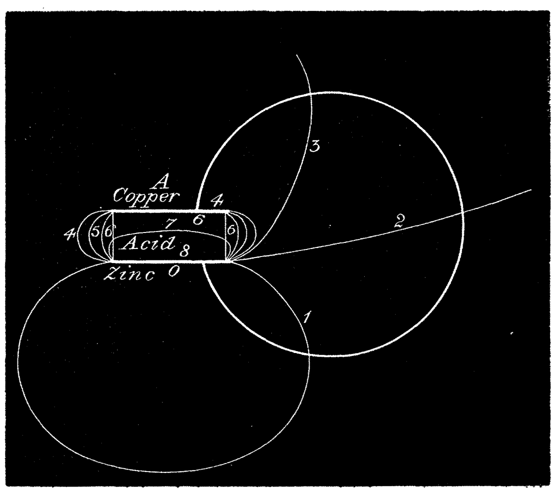
zinc-acid is greater than V. Then more than $V-1$ level surfaces pass between the zinc and the acid, the excess over $V-1$ going round and passing between the copper and the acid, somewhat as in fig. 3, where A, B, are the metal plates. The surfaces

Fig. 3.



are roughly sketched and numbered, on the supposition that the zinc terminal is at 0, the copper at 5, and the acid at 8. They have probably the same shape as those which would be produced by condensers at A and B with the wires attached, respectively, to one terminal of each, the other terminals being connected together and the charges adjusted, so that the difference of potential of the two terminals at A was 3, while that at B was 8.

Fig. 4.



Let us now suppose the circuit closed. Then the level surface will cut the circuit at various points, somewhat as in fig. 4.

The energy being dissipated in the wire, the cell will continually send out fresh energy, the induction tubes, which proceed from the acid to the zinc, diverging outwards in the same way as described in the discharge of a condenser. They bend round, and finally go into the circuit, the energy they carry being used for the necessary molecular changes, and finally appearing as heat in the circuit—except at the copper-acid contact where there is a crowding in of level surfaces, and therefore a convergence of more energy, which is required to set the hydrogen free.

At the same time magnetic ring-shaped tubes will be continually sent out from the zinc-acid contact, expanding for a time and then contracting again on various parts of the circuit and also giving up their energy.

There is, therefore, a convergence of tubes of electric induction on the circuit, running in the same direction throughout, viz., from copper to zinc outside the cell, and from zinc to copper inside, except between the zinc and acid, where there is a divergence of tubes in which the induction runs in the opposite way. But a divergence of negative tubes causes magnetic intensity in the same direction as, and may therefore be considered as equivalent to, a convergence of positive tubes. The current may therefore be said to go round the circuit in the same way throughout.

The tendency to a steady state in which the current or the number of induction tubes broken up per second is the same at all parts of the circuit, admits of simple explanation. We know, as the result of experiment given by OHM's law, that $C=\frac{E}{R}$ where R is the resistance per unit length and E the electric intensity. Until we can explain the molecular working of the current, *i.e.*, the mode in which the induction tubes are broken up, we must accept OHM's law as a simple fact. Let us suppose that we have not yet arrived at the steady state, so that in some part of the circuit the electric intensity is less than in the steady state, while in another part it is equal to it or greater. Let the steady value of the intensity be E, the actual value in the former part E', and in the latter E''. By OHM's law the number of tubes absorbed by the wire per second is given by $C'=E'/R$, and $C''=E''/R$, in the two parts respectively, so that $C'<C''$ since $E'<E''$ or less tubes are being destroyed in the first than in the second part. But all the tubes are sent out from the source of the energy, and are only destroyed in the circuit, being otherwise continuous and with their two ends in the circuit. Hence, if more tubes are destroyed at one part than another, the parts of the tubes not yet destroyed will gather in the medium surrounding the part where fewer are destroyed, increasing the induction there, and so raising the intensity in the wire and therefore the number of tubes destroyed. The field can evidently only be steady when the number of tubes destroyed in all parts of the circuit is the same.

But it does not follow that in the steady state each tube enters the wire along its whole length at the same moment. This would imply that the axis of the wire is a line of electric induction perpendicular everywhere to the level surfaces. If we draw

the level surfaces due to the seats of induction at the contacts of acid and metal, they will probably be somewhat as drawn in fig. 4. If now the wire is not so arranged as to follow with properly adjusted resistances a line of induction for these surfaces, but pursues an irregular course, then the level surfaces will be much distorted, and the distribution of the induction will be greatly altered.

We may ascribe this alteration to a distribution of electricity along the wire, the quantity in any small area on the surface of the wire being equal to the difference between the number of tubes which have entered and the number which have left that area since the beginning of the system. We have a familiar example of this in the charging of deep-sea cables. Another example is afforded by a condenser with terminals connected to two points in the circuit. The plates of the condenser are then virtually parts of the circuit.

The effect of a junction of two wires, say of the same diameter, but of different specific resistances, upon the level surface will resemble that of a charge upon the separating surface. This can be seen in a general way from the fact that the level surfaces must cut the wire with the higher specific resistance at intervals shorter than those at which it cuts the other wire.

If there be an insulated conducting body, say a metal sphere, near the circuit, we know that in the steady state there is no electric intensity, and therefore no current within it; consequently there is no movement of energy and no movement of induction through it. We can see how this condition is arrived at. As the first tubes of electric and magnetic induction come up to the sphere they will enter it, and the parts of the electric induction tubes thus entering will be broken up, causing a transient current in the sphere. The parts of the tubes left in the medium will end on the sphere giving a negative charge on the end nearer the regions of higher potential, and a positive charge on the end nearer the regions of lower potential. This will go on until such charges have accumulated that the sphere becomes itself a level surface. When this point is reached no more energy can enter the sphere, and the parts of the magnetic tubes within it cease to move.

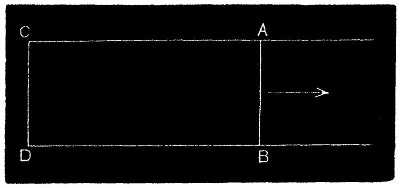
The charges formed on the wire or on neighbouring conductors are to be distinguished from ordinary statical charges in this: that their existence depends on the existence of the current, and therefore on the motion of magnetic induction. If the current is stopped by a break in the circuit, so that the motion of the magnetic induction ceases, the electric induction ceases and the charges are all lost. We should expect, therefore, to find that these charges can be described in terms of the magnetic motions which have occurred and are occurring in the system.

Current produced by motion of a conductor in a magnetic field.

We may explain by general reasoning the production of a current by motion of a part of a circuit so as to cut the tubes of magnetic induction. We will consider the

simple case of a slider AB, fig. 5, running on two parallel rails, AC BD, with a fixed cross piece CD, the tubes of magnetic induction running from above downwards through the paper. Let AB move so as to enlarge the circuit. We know from experiment that this tends to cause a current in the direction AC DB.

Fig. 5.



As AB moves through the field its motion tends to cause electric intensity in the direction BA. At the same time its kinetic energy is being continually converted into electric and magnetic energy which travels to the rest of the circuit there to be dissipated, that is, there must be a divergence of energy from AB. Instead then of a convergence of positive tubes running from B to A, we shall have what is magnetically equivalent—a divergence of negative tubes or tubes running from A to B, their motion outwards being accompanied by tubes of magnetic induction running round in the same way as if there were an ordinary current from B to A. These magnetic tubes must be supposed to move outwards in order to account for the direction of the electric intensity.*

When these electric and magnetic tubes converge upon the rest of the circuit they will evidently form a current running in the direction AC DB. We have here taken, just as in the case of the condenser and the voltaic cell, the lessening of negative induction by its motion outwards, as equivalent to the increase of positive induction by its motion inwards, and we have considered both of them to indicate the application of electric intensity in the same direction in the conductor.

If instead of considering AB as a whole we break it up into elements, each element will be a source of diverging negative tubes, and the remainder of AB will to that element be a part of the rest of the circuit. Hence some of the energy sent out from the element will converge on and be dissipated in AB, or AB will be heated just as the rest of the circuit.

The general equations of the electromagnetic field.

We can easily obtain equations corresponding to and closely resembling those of MAXWELL by means of the principles upon which this paper is founded.

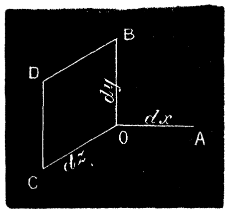
The assumption that if we take any closed curve the number of tubes of magnetic

* [Added July 15.—The above must not be regarded as an attempt to explain the production of electric induction by the motion of a conductor in a magnetic field, but merely as an attempt to show how the induction arising in the moving part of a circuit finds its way into the rest of the circuit.]

induction passing through it is equal to the excess of the number which have moved in over the number which have moved out through the boundary since the beginning of the formation of the field, suggests a historical mode of describing the state of the field at any moment.

Let a, b, c be the components of magnetic induction at any point O. Consider a small area $dy\,dz$ close to the point, then the number of tubes passing through the area $dy\,dz$ will be $a dy\,dz$. This will be equal to the difference between those which have come in and those which have gone out.

Fig. 6.



Let $\mathrm{L}dx$, $\mathrm{M}dy$, $\mathrm{N}dz$ denote the numbers of tubes which have cut the lengths dx, dy, dz since the beginning of the system, those being positive which have tended to produce electric intensity in the positive direction along the axes, and those being negative and therefore subtracted which have tended to produce intensity in the opposite direction. Let us consider the number which have come into the area OBCD$=dy\,dz$ (fig. 6). The number which have come in across OB is$-\mathrm{M}dy$ ($-$because the movement of tubes passing through $dy\,dz$ in the positive direction must be outwards to produce E.I. along OB). The number which has passed out across CD is$-\left(\mathrm{M}+\frac{d\mathrm{M}}{dz}dz\right)dy$. The difference is $\frac{d\mathrm{M}}{dz}dy\,dz$. The number which has come in across OC is$+\mathrm{N}dz$ ($+$ because the movement of tubes passing through $dy\,dz$ in the positive direction must be inwards to produce E.I. along OC). The number which has passed out across BD is $\left(\mathrm{N}+\frac{d\mathrm{N}}{dy}dy\right)dz$. The difference is$-\frac{d\mathrm{N}}{dy}dy\,dz$.

The number still passing through $dy\,dz$ is therefore $\left(\frac{d\mathrm{M}}{dz}-\frac{d\mathrm{N}}{dy}\right)dy\,dz$.

Equating this to the actual induction through the area, viz.,

$$a dy\,dz$$

and performing the same process for the corresponding areas $dz\,dx$, $dx\,dy$, we obtain

$$\left.\begin{aligned} a&=\frac{d\mathrm{M}}{dz}-\frac{d\mathrm{N}}{dy} \\ b&=\frac{d\mathrm{N}}{dx}-\frac{d\mathrm{L}}{dz} \\ c&=\frac{d\mathrm{L}}{dy}-\frac{d\mathrm{M}}{dx} \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots \quad (1)$$

Comparing these with MAXWELL'S equations (vol. ii., p. 216) we see that

$$\frac{dM}{dz}-\frac{dN}{dy}=\frac{dH}{dy}-\frac{dG}{dz}$$

with two similar equations, F, G, H being the components of the vector potential. We should obtain MAXWELL'S equations if we defined F, G, H to be the number of tubes which would cut the axes per unit length if the system were to be allowed to return to its original unmagnetic condition, the tubes now moving in the opposite direction. According to MAXWELL, the vector whose components are F, G, and H "represents the time integral of the electromotive force which a particle placed at the point (x, y, z) would experience if the primary current were suddenly stopped" (vol. ii., 2nd Ed., p. 215). If the electric intensity is produced by the motion of magnetic induction, then our definition of F, G, H will by the second fundamental principle agree with MAXWELL'S statement.

If u, v, w be the components of current—including, of course, under currents, growth of induction—we have from the third principle MAXWELL'S equations E (vol. ii., p. 233), which on multiplying by μ become when μ is constant

$$\left.\begin{aligned}4\pi\mu u&=\frac{dc}{dy}-\frac{db}{dz}\\4\pi\mu v&=\frac{da}{dz}-\frac{dc}{dx}\\4\pi\mu w&=\frac{db}{dx}-\frac{da}{dy}\end{aligned}\right\} \quad \ldots \ldots \ldots \quad (2)$$

Combining these with equations (1) (as in MAXWELL, vol. ii., pp. 236–7), and writing $-\nabla^2$ for $\frac{d^2}{dx^2}+\frac{d^2}{dy^2}+\frac{d^2}{dz^2}$ we obtain

$$\left.\begin{aligned}4\pi\mu u&=-\nabla^2 L-\frac{d}{dx}\left(\frac{dL}{dx}+\frac{dM}{dy}+\frac{dN}{dz}\right)\\4\pi\mu v&=-\nabla^2 M-\frac{d}{dy}\left(\frac{dL}{dx}+\frac{dM}{dy}+\frac{dN}{dz}\right)\\4\pi\mu w&=-\nabla^2 N-\frac{d}{dz}\left(\frac{dL}{dx}+\frac{dM}{dy}+\frac{dN}{dz}\right)\end{aligned}\right\} \quad \ldots \ldots \ldots \quad (3)$$

These equations only differ in sign from MAXWELL'S, and are therefore to be solved in the same way.

It is easy to see by substitution that if we assume

$$\left.\begin{aligned}
\mathrm{L}' &= -\mu\iiint\frac{u}{r}dxdydz \\
\mathrm{M}' &= -\mu\iiint\frac{v}{r}dxdydz \\
\mathrm{N}' &= -\mu\iiint\frac{w}{r}dxdydz \\
\mathrm{H} &= \frac{1}{4\pi}\iiint\left(\frac{d\mathrm{L}}{dx}+\frac{d\mathrm{M}}{dy}+\frac{d\mathrm{N}}{dz}\right)\frac{1}{r}dxdydz
\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

then the following will be solutions

$$\left.\begin{aligned}
\mathrm{L} &= \mathrm{L}' - \frac{d\mathrm{H}}{dx} \\
\mathrm{M} &= \mathrm{M}' - \frac{d\mathrm{H}}{dy} \\
\mathrm{N} &= \mathrm{N}' - \frac{d\mathrm{H}}{dz}
\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

It is evident that we may add to the right-hand side of equations (5) $\frac{d\phi}{dx}, \frac{d\phi}{dy}, \frac{d\phi}{dz}$ respectively, where ϕ is any function of x, y, z, since these will disappear from (3) and also from (1).

The electric intensity, in so far as it depends upon magnetic motions, will consist of two terms, one depending upon the motion of the material at the point (its components being found as in MAXWELL, vol. ii., p. 227, note), the other upon the motion of magnetic induction about the point. We may add a third term, arising from any electrical distribution with a potential ψ.

If there is no material motion we shall have

$$\left.\begin{aligned}
\mathrm{P} &= {}^{d\mathrm{L}} - \frac{d\psi}{dx} \\
\mathrm{Q} &= \frac{d\mathrm{M}}{dt} - \frac{d\psi}{dy} \\
\mathrm{R} &= \frac{d\mathrm{N}}{dt} - \frac{d\psi}{dz}
\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

Substituting from (4) and (5) we get

$$\begin{aligned}
\mathrm{P} &= -\mu\iiint\frac{du}{dt}\cdot\frac{1}{r}dxdydz - \frac{1}{4\pi}\frac{d}{dx}\iiint\left(\frac{d}{dx}\frac{d\mathrm{L}}{dt}+\frac{d}{dy}\frac{d\mathrm{M}}{dt}+\frac{d}{dz}\frac{d\mathrm{N}}{dt}\right)\frac{1}{r}dxdydz - \frac{d\psi}{dx} \\
&= -\mu\iiint\frac{du}{dt}\cdot\frac{1}{r}dxdydz - \frac{1}{4\pi}\frac{d}{dx}\iiint\left(\frac{d\mathrm{P}}{dx}+\frac{d\mathrm{Q}}{dy}+\frac{d\mathrm{R}}{dz}\right)\frac{1}{r}dxdydz \\
&\qquad - \frac{1}{4\pi}\frac{d}{dx}\iiint\left(\frac{d^2\psi}{dx^2}+\frac{d^2\psi}{dy^2}+\frac{d^2\psi}{dz^2}\right)\frac{1}{r}dxdydz - \frac{d\psi}{dx},
\end{aligned}$$

substituting for $\frac{d\mathrm{L}}{dt}$, &c., from (6).

The last two terms cancel each other, and we get

$$\mathrm{P}=-\mu\iiint\frac{du}{dt}\frac{1}{r}dxdydz-\frac{1}{4\pi}\frac{d}{dx}\iiint\left(\frac{d\mathrm{P}}{dx}+\frac{d\mathrm{Q}}{dy}+\frac{d\mathrm{R}}{dz}\right)\frac{1}{r}dxdydz \quad . \quad . \quad . \quad . \quad (7)$$

or if we put

$$\frac{d\mathrm{P}}{dx}+\frac{d\mathrm{Q}}{dy}+\frac{d\mathrm{R}}{dz}=4\pi\rho$$

and

$$\mathrm{V}=\iiint\frac{\rho}{r}dxdydz,$$

then

$$\mathrm{P}=-\mu\iiint\frac{du}{dt}\cdot\frac{1}{r}dxdydz-\frac{d\mathrm{V}}{dx} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

with similar equations for Q and R.

If the system is steady $\frac{du}{dt}, \frac{dv}{dt}, \frac{dw}{dt}$ are all zero, and then

$$\mathrm{P}=-\frac{d\mathrm{V}}{dx}, \quad \mathrm{Q}=-\frac{d\mathrm{V}}{dy}, \quad \mathrm{R}=-\frac{d\mathrm{V}}{dz}.$$

The quantity ρ, of which V is the potential, will be zero within non-conducting homogeneous parts of the field, for there

$$f=\frac{\mathrm{KP}}{4\pi}, \quad g=\frac{\mathrm{KQ}}{4\pi}, \quad h=\frac{\mathrm{KR}}{4\pi},$$

and

$$\frac{d\mathrm{P}}{dx}+\frac{d\mathrm{Q}}{dy}+\frac{d\mathrm{R}}{dz}=\frac{4\pi}{\mathrm{K}}\left(\frac{df}{dx}+\frac{dg}{dy}+\frac{dh}{dz}\right)=0$$

since no charges can reside within a homogeneous non-conducting medium. Or, stating it in another way, all the induction tubes brought into any part of such a medium remain there without dissipation, a charge in a non-homogeneous medium being due to unequal amounts of dissipation of induction in different parts of the medium.

But ρ will have value at surfaces separating dissimilar substances either in the insulating or conducting parts of the medium. For in the former the induction is continuous, while the intensity is discontinuous, and in the latter the current or rate of destruction of induction may be continuous, but the relation between intensity and current changes discontinuously with the conductivity. At surfaces separating insulators from conductors ρ may have value, as, for instance, at the surfaces of the plates of a condenser with its terminals connected with two points in a circuit, or at the surface of an insulated conductor near the circuit. It is also to be noted that ρ will have values at the seat of electromotive force.

The values of the components of magnetic induction a, b, c are not in any way dependent on ρ. For taking the first of equations (1) and substituting from (5) we have

$$a=\frac{dM}{dz}-\frac{dN}{dy}=\frac{dM'}{dz}-\frac{dN'}{dy}-\frac{d^2H}{dydz}+\frac{d^2H}{dzdy}=\frac{dM'}{dz}-\frac{dN'}{dy} \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

where M′ and N′ depend on the currents in the system and not on the charges.

Comparing our equations with MAXWELL'S we see that the important point of difference is that we can no longer put the quantity corresponding to his J equal to zero, J being given by $\frac{dF}{dx}+\frac{dG}{dy}+\frac{dH}{dz}$.

This does not affect the determination of velocity of propagation of disturbance in a homogeneous non-conducting medium.

For in such a medium we shall have

$$u=\frac{df}{dt}=\frac{K}{4\pi}\frac{dP}{dt}$$

with corresponding values for v and w.

Substituting in (3) the first equation becomes

$$K\mu\frac{dP}{dt}=-\nabla^2L-\frac{d}{dx}\left(\frac{dL}{dx}+\frac{dM}{dy}+\frac{dN}{dz}\right)$$

differentiating with respect to t

$$K\mu\frac{d^2P}{dt^2}=-\nabla^2\frac{dL}{dt}-\frac{d}{dx}\left(\frac{d}{dx}\frac{dL}{dt}+\frac{d}{dy}\frac{dM}{dt}+\frac{d}{dz}\frac{dN}{dt}\right)$$

and putting $\frac{dL}{dt}=P+\frac{d\psi}{dx}$

$$K\mu\frac{d^2P}{dt^2}=-\nabla^2P-\frac{d}{dx}\nabla^2\psi-\frac{d}{dx}\left(\frac{dP}{dx}+\frac{dQ}{dy}+\frac{dR}{dz}\right)+\frac{d}{dx}\nabla^2\psi=-\nabla^2P \quad . \quad . \quad . \quad (10)$$

since

$$\frac{dP}{dx}+\frac{dQ}{dy}+\frac{dR}{dz}=\frac{4\pi}{K}\left(\frac{df}{dx}+\frac{dg}{dy}+\frac{dh}{dz}\right)=0$$

within a homogeneous non-conductor.

This gives the velocity of propagation of electric induction equal to $1/\sqrt{K\mu}$.

We can also obtain the corresponding equation for the magnetic induction.

Substituting in (3) for u, v, and w in terms of P, Q, and R, as above, differentiating the second with respect to z, and the third with respect to y, and subtracting

$$K\mu \frac{d}{dt}\left(\frac{dQ}{dz}-\frac{dR}{dy}\right)=-\nabla^2\left(\frac{dM}{dz}-\frac{dN}{dy}\right)$$

then from (6)

$$K\mu \frac{d}{dt}\left(\frac{d}{dt}\frac{dM}{dz}-\frac{d^2\psi}{dzdy}-\frac{d}{dt}\frac{dN}{dy}+\frac{d^2\psi}{dydz}\right)=-\nabla^2\left(\frac{dM}{dz}-\frac{dN}{dy}\right)$$

or

$$K\mu \frac{d^2}{dt^2}\left(\frac{dM}{dz}-\frac{dN}{dy}\right)=-\nabla^2\left(\frac{dM}{dz}-\frac{dN}{dy}\right)$$

or from (1)

$$K\mu \frac{d^2a}{dt^2}=-\nabla^2 a \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

whence the velocity of propagation of magnetic induction is also equal to $1/\sqrt{K\mu}$.

It would seem that in some cases, such as that of the field surrounding a straight wire with a steady current, the electric intensity may be regarded as entirely due to the motion of magnetic induction, and its components will therefore be $\frac{dL}{dt}, \frac{dM}{dt}, \frac{dN}{dt}$.

But in other cases it would seem that the electric induction cannot be wholly due to the motion of magnetic induction, and we must therefore introduce the terms involving ψ. If, for instance, the electric and magnetic intensities were inclined at an angle θ, we should have to suppose the electric intensity E to be produced by the motion of the component of magnetic induction I perpendicular to E, viz., $\mu I \sin\theta$, the other component $\mu I \cos\theta$ being at rest. To produce intensity E, E tubes must cut unit length in the direction of E per second; and since the value of the magnetic induction is $\mu I \sin\theta$, this requires a velocity v, given by $v.\mu I \sin\theta = E$ or $v = E/\mu I \sin\theta$. Now we can easily imagine a case where E and I coincide, as, for instance, a condenser with its planes parallel to the axis of a wire carrying a current, and its terminals connected with two points in the wire. Here $I \sin\theta = 0$, and v is infinite. Or we have to suppose the electric intensity to be produced by the movement of tubes of induction of no intensity with infinite velocity, a statement without physical meaning.

But it is, perhaps, worth noting that if we suppose that the electric intensity is produced by the motion of magnetic induction, and that the magnetic intensity is produced by the motion of the electric induction, each carrying their energy with them, the right quantity of energy crosses the unit area.

For E magnetic tubes, with $I \sin\theta$ unit cells per unit length, will carry across unit area in the plane of E and I a quantity $\frac{EI \sin\theta}{8\pi}$, or half the energy which actually crosses the plane. If $I \sin\theta$ is due to the motion of electric tubes, then $I \sin\theta/4\pi$ tubes must cut unit length in the direction of $I \sin\theta$ per second. The number of unit cells per unit length is E, and therefore the motion of the tubes will carry a quantity of energy $\frac{EI \sin\theta}{8\pi}$, or the other half actually crossing.

The equations which have been obtained in the foregoing manner by the aid of the

hypothesis of movement of magnetic induction, may also be obtained without any special hypothesis as to the motion of the induction tubes, merely assuming that growth of induction through a curve is accompanied by electric intensity round the curve. Instead of connecting L, M, N with the number of tubes which have cut the axes, we start with the following definitions :—

Let L, M, N denote the time integrals of the components of the electric intensity parallel to the axes since the origin of the system, so that

$$L=\int P dt \qquad M=\int Q dt \qquad N=\int R dt,$$

then

$$P=\frac{dL}{dt} \qquad Q=\frac{dM}{dt} \qquad R=\frac{dN}{dt}.$$

If a, b, c be components of magnetic induction, since the growth of induction through a curve is equal to the line integral of the electric intensity round a curve in the negative direction, we have

$$\frac{da}{dt}=\frac{d}{dz}-\frac{dR}{dy}=\frac{d}{dt}\left(\frac{dM}{dz}-\frac{dN}{dy}\right)$$

with corresponding equations for $\frac{db}{dt}$ and $\frac{dc}{dt}$.

Integrating with respect to t from the origin of the system, when all the quantities were zero

$$\left.\begin{aligned} a&=\frac{dM}{dz}-\frac{dN}{dy} \\ b&=\frac{dN}{dx}-\frac{dM}{dz} \\ c&=\frac{dL}{dy}-\frac{dM}{dx} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1')$$

equations the same in form as equations (1).

As before we obtain equations (3), (4), and (5), while instead of (6) we have the simple equations $P=\frac{dL}{dt}$ and the two others.

Substituting for $\frac{dL}{dt}$ we obtain an equation of the same form as (7), which may also be put into the form (8). Equations (9) and (10) will also follow.

Just as we have obtained equations by considering the growth of the magnetic induction to its present state so we may obtain corresponding equations by considering the growth of the electric induction.

Let $\frac{Adx}{4\pi}$, $\frac{Bdy}{4\pi}$, $\frac{Cdz}{4\pi}$ be the algebraic sum of the number of electric induction tubes

which have cut dx, dy, dz drawn from a point in such a way as to create magnetic intensities in the positive direction along dx, dy, dz.

The excess of the number of tubes which have passed in over those which have passed out through the boundary of any area will be equal to the time integral of the total current through the area.

The components of the total current are

$$u=p+\frac{df}{dt} \qquad v=q+\frac{dg}{dt} \qquad w=r+\frac{dh}{dt}$$

p, q, and r being the components of the conduction current or the number of tubes dissipated per second, and f, g, h the components of the induction actually existing.

As in the last case, if we put $f'=\int udt$, &c., we at once obtain the equations

$$\left.\begin{aligned} 4\pi f'&=\frac{dC}{dy}-\frac{dB}{dz} \\ 4\pi g'&=\frac{dA}{dz}-\frac{dC}{dx} \\ 4\pi h'&=\frac{dB}{dx}-\frac{dA}{dy} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

Corresponding to the current equations (2) we have three equations obtained from the condition that the rate of increase of magnetic induction through an area is equal to the integral of the electric intensity round it in the negative direction. These are

$$\left.\begin{aligned} \frac{da}{dt}&=\frac{dQ}{dz}-\frac{dR}{dy} \\ \frac{db}{dt}&=\frac{dR}{dx}-\frac{dP}{dz} \\ \frac{dc}{dt}&=\frac{dP}{dy}-\frac{dQ}{dx} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

If $C_{\prime}$ is the specific conductivity we may by OHM's law put the current equations after integrating in the form

$$f'=C_{\prime}\int Pdt+\frac{KP}{4\pi}$$

$$g'=C_{\prime}\int Qdt+\frac{KQ}{4\pi}$$

$$h'=C_{\prime}\int Rdt+\frac{KR}{4\pi}$$

whence in media where K is constant

$$\frac{dg'}{dz}-\frac{dh'}{dy}=C_{\prime}\int\left(\frac{dQ}{dz}-\frac{dR}{dy}\right)dt+\frac{K}{4\pi}\left(\frac{dQ}{dz}-\frac{dR}{dy}\right)$$

$$=C_{\prime}\int\frac{da}{dt}dt+\frac{K}{4\pi}\frac{da}{dt}\text{ from (13)}$$

$$=C_{\prime}a+\frac{K}{4\pi}\frac{da}{dt}$$

with two similar equations.

Finding the values of the left hand from (12) we obtain

$$\left.\begin{aligned}4\pi C_{\prime}a+K\frac{da}{dt}&=-\nabla^2A-\frac{d}{dx}\left(\frac{dA}{dx}+\frac{dB}{dy}+\frac{dC}{dz}\right)\\4\pi C_{\prime}b+K\frac{db}{dt}&=-\nabla^2B-\frac{d}{dy}\left(\frac{dA}{dx}+\frac{dB}{dy}+\frac{dC}{dz}\right)\\4\pi C_{\prime}c+K\frac{dc}{dt}&=-\nabla^2C-\frac{d}{dz}\left(\frac{dA}{dx}+\frac{dB}{dy}+\frac{dC}{dz}\right)\end{aligned}\right\}\quad\ldots\ldots\quad(14)$$

If we assume

$$A'=-\frac{1}{4\pi}\iiint\left(4\pi C_{\prime}a+K\frac{da}{dt}\right)\frac{1}{r}dxdydz$$

with corresponding values for B′ and C′ and

$$M=\frac{1}{4\pi}\iiint\left(\frac{dA}{dx}+\frac{dB}{dy}+\frac{dC}{dz}\right)\frac{1}{r}dxdydz$$

then

$$\left.\begin{aligned}A&=A'-\frac{dM}{dx}\\B&=B'-\frac{dM}{dy}\\C&=C'-\frac{dN}{dy}\end{aligned}\right\}\quad\ldots\ldots\ldots\ldots\quad(15)$$

are solutions of (13).

We may obtain by substitution from (15) in (12) values for f', g', h' corresponding to the values of the magnetic induction in (9), viz.:

$$4\pi f'=\frac{dC'}{dy}-\frac{dB'}{dz}$$

and two others; where A′, B′, C′ are given in terms of the magnetic induction as above.

It is only in special cases, such as that of a straight wire with a steady current, that the magnetic intensity will be equal to 4π times the number of electric induction tubes passing through unit length per second. In all cases the line integral of the magnetic intensity round a closed curve is equal to 4π times the number of electric tubes passing through the boundary, but the electric tubes may be more crowded in

some parts than in others, while the magnetic intensity is not altered in a corresponding manner. For instance, the magnetic tubes will be continued through an insulated conductor in the field, while in the steady state no electric tubes pass through it. But each element adds to the line integral the quantity which, after Mr. BOSANQUET, I have called the magnetomotive force, this being equal to 4π times the number of electric tubes passing through the element. But it only adds it on integrating round the whole of the closed curve.

The intensity at any point will therefore be the resultant of the intensities produced by the magnetomotive forces in the various elements. Perhaps the simplest mode of finding it is as follows.

The components of the magnetomotive force produced in a cube dx, dy, dz parallel to the three edges will be

$$\frac{dA}{dt}dx,\ \frac{dB}{dt}dy,\ \frac{dC}{dt}dz.$$

for $\frac{1}{4\pi}\frac{dA}{dt}, \frac{1}{4\pi}\frac{dB}{dt}, \frac{1}{4\pi}\frac{dC}{dt}$, are by definition the rates at which electric tubes are cutting unit lengths parallel to the axes.

But these magnetomotive forces would be produced by currents round the cube in planes perpendicular to the axes respectively, and equal to

$$\frac{1}{4\pi}\frac{dA}{dt}dx,\ \frac{1}{4\pi}\frac{dB}{dt}dy,\ \frac{1}{4\pi}\frac{dC}{dt}dz.$$

for the line integral of the intensity round a curve threading a current is $4\pi\times$ current. But the magnetic intensity at any point due to a current is equal to that of a magnetic shell of strength (*i.e.*, intensity $\times$ thickness), equal numerically to the current bounding the shell.

If we suppose the thickness of the shell equal to that of the cube, the effect is the same as if the cube were magnetised with intensity having components

$$\frac{1}{4\pi}\frac{dA}{dt},\ \frac{1}{4\pi}\frac{dB}{dt},\ \frac{1}{4\pi}\frac{dC}{dt}.$$

The potential of such a distribution of magnetisation is (MAXWELL, vol. ii., p. 29, equation (23)).

$$V=\frac{1}{4\pi}\iiint\left(\frac{dA}{dt}\frac{dp}{dx}+\frac{dB}{dt}\frac{dp}{dy}+\frac{dC}{dt}\frac{dp}{dz}\right)dxdydz$$

where $p=\frac{1}{r}$, and the magnetic intensity is given by

$$\alpha=-\frac{dV}{dx},\quad \beta=-\frac{dV}{dy},\quad \gamma=-\frac{dV}{dz}.$$

It may be noticed that in a steady field $\frac{dA'}{dt}, \frac{dB'}{dt}, \frac{dC'}{dt}$ are all zero, so that

$$V=-\frac{1}{4\pi}\iiint\left(\frac{d}{dx}\frac{dM}{dt}\frac{dp}{dx}+\frac{d}{dy}\frac{dM}{dt}\frac{dp}{dy}+\frac{d}{dz}\frac{dM}{dt}\frac{dp}{dz}\right)dxdydz$$

where

$$M=\frac{1}{4\pi}\iiint\left(\frac{dA}{dx}+\frac{dB}{dy}+\frac{dC}{dz}\right)\frac{1}{r}dxdydz$$

We may obtain equations of the same form as those given in (14) without any hypothesis as to the movement of electric induction tubes, merely assuming that the total current through a curve is equal to $4\pi\times$ line integral of magnetic intensity round the curve.

We start with the following definitions. Let A, B, C be the time integrals of the components of magnetic intensity since the origin of the system.

Then

$$A=\int\alpha dt, \qquad B=\int\beta dt, \qquad C=\int\gamma dt$$

and

$$\alpha=\frac{dA}{dt}, \qquad \beta=\frac{dB}{dt}, \qquad \gamma=\frac{dC}{dt}$$

We have the equations

$$4\pi u=\frac{d\gamma}{dy}-\frac{d\beta}{dz}$$

and two others.

Integrating with respect to t we have

also

$$\left.\begin{aligned}4\pi\int udt=4\pi f'&=\frac{dC}{dy}-\frac{dB}{dz}\\ 4\pi g'&=\frac{dA}{dz}-\frac{dC}{dx}\\ 4\pi h'&=\frac{dB}{dx}-\frac{dA}{dy}\end{aligned}\right\}$$

which are of the same form as (12).

Hence exactly as before we obtain equations (14) and their solutions (15).

The equations for the magnetic intensity are now

$$\alpha=\frac{dA}{dt}, \qquad \beta=\frac{dB}{dt}, \qquad \gamma=\frac{dC}{dt}$$

If we differentiate (14) with respect to t, and substitute from these equations for magnetic intensity, we obtain

$$4\pi\mu C_{\prime}\frac{d\alpha}{dt}+K\mu\frac{d^2\alpha}{dt^2}=-\nabla^2\alpha-\frac{d}{dx}\left(\frac{d\alpha}{dx}+\frac{d\beta}{dy}+\frac{d\gamma}{dz}\right)$$

with corresponding equations for β and γ.

Differentiating the second of these with respect to z, and the third with respect to y, and subtracting, we obtain

$$4\pi\mu C_{\prime}\frac{du}{dt}+K\mu\frac{d^2u}{dt^2}=-\nabla^2 u$$

with corresponding equations for v and w.

These correspond to MAXWELL'S equations (7), p. 395.

In conclusion it may be remarked that the equations found in this paper give the same expression for the rate of Transfer of Energy as that in my previous paper derived from MAXWELL'S equations involving F, G, and H.