PHILOSOPHICAL TRANSACTIONS.

I. *Calculations to determine at what Point in the Side of a Hill its Attraction will be the greatest, &c.* By Charles Hutton, *LL.D. and F. R. S.* *In a Letter to* Nevil Maſkelyne, *D. D. F. R. S. and Aſtronomer Royal.*

Read Nov. 11, 1779.

Royal Military Acad.
Sept. 21, 1779.

DEAR SIR,

AS the experiment of determining the univerſal attraction of matter, which you lately conducted with ſo much accuracy and ſucceſs, is of ſo great im-

portance and curiofity, that it may probably be hereafter repeated by the learned in other countries, and in other fituations; and as the utmoft precifion is defirable in fo nice an experiment; I have compofed the inclofed paper to determine the beft part of a hill for making the obfervations, fo as to obtain the greateft quantity of attraction.

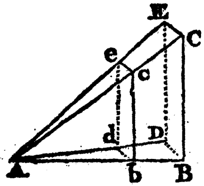
I have no doubt, Sir, that the determination of this point will appear of fome confequence in the opinion of one who has the improvement of ufeful knowledge fo much at heart; and if the manner in which it is here made, meet your approbation, I fhall defire the favour of your prefenting it to the Royal Society.

I have the honour to be, &c.

1. The great ſucceſs of the experiment, lately made by the Royal Society, on the hill Schehallien, to determine the univerſal attraction of matter, and the important conſequences that have reſulted from it, may probably give occaſion to other experiments of the ſame kind to be made elſewhere: and as all poſſible means of accuracy and facility are to be deſired in ſo delicate and laborious an undertaking; it has occurred to me that it might not be unuſeful to add, by way of ſupplement to my paper of calculations relative to the above-mentioned experiment, an inveſtigation of the height above the bottom of a hill, at which its horizontal attraction ſhall be the greateſt; ſince that is the height at which commonly the obſervations ought to be made, and ſince this beſt point of obſervation has never been any where determined that I know of, but has been variouſly ſpoken of or gueſſed at, it being ſometimes accounted at $\frac{1}{3}$, and ſometimes at $\frac{1}{2}$ of the height of the hill; whereas from theſe inveſtigations it is found to be generally at about only $\frac{1}{4}$ of the altitude from the bottom.

2. Let ABCEDA be part of a cuneus of matter, its ſides or faces being the two ſimilar right-angled triangles ABC, ADE meeting in the point A, and forming the indefinitely ſmall angle BAD. Then of any ſection

 bced,



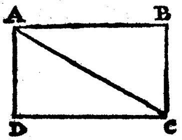
bced, perpendicular to the planes ABD and ADE, the attraction on a body at A in the direction AB, is equal to the conſtant quantity Ss; where S = ſin. ∠BAC and s = ſin. ∠BAD, to the radius 1.

For, firſt, ſince the magnitude of the flowing ſection is every where as Ab^2, and the attraction of the particles of matter inverſely as the ſame, or as $\frac{1}{Ab^2}$; therefore their product or $\frac{Ab^2}{Ab^2}$ or 1 (a conſtant quantity) is as the force of attraction of bced.

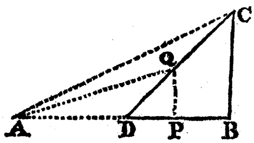
Then to find what that quantity is. Put AB $= a$, and BC $= x$; then BD or CE (the diſtance between the two planes at the diſtance AB) is $= as$. Now the force of a particle in the line CE is as $\frac{1}{AC^2}$ in the direction AC, and therefore it is as $\frac{AB}{AC^3}$ in the direction AB; conſequently the force of the whole lineola CE in the direction AB is $\frac{AB.CE}{AC^3}$; and therefore the fluxion of the force of the ſection BCED or $\dot{f}$ is $= \frac{AB.CE}{AC^3}.\dot{BC} = \frac{a.as.\dot{x}}{\overline{a^2+x}|^{\frac{3}{2}}} = \frac{a^2 s\dot{x}}{\overline{a^2+x^2}|^{\frac{3}{2}}}$; and the fluent gives $f = \frac{sx}{\sqrt{a^2+x^2}} = s \times \frac{BC}{AC} = s$S for the attraction itſelf.

3. To

3. To find now the attraction of the whole right-angled cuneus on a body at A in the direction AB.—Since the force of each ſection is Ss by the laſt article, therefore the force of all the ſections, the number of them being AB or a, is $a s$S $= s \,.\, \text{AB} \,.\, \frac{\text{BC}}{\text{AC}}$ the force of the whole cuneus ABCEDA.

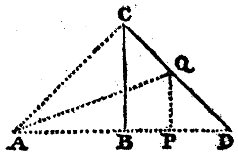


4. To find the attraction of the rectangular part ABCD on A in the direction AB; ABCD being one ſide of the cuneus, and AD its edge.—Put $\text{AD} = \text{BC} = b$, and $\text{AB} = x$. Then, the force of any ſection as BC being always as Ss by Art. 2, the fluxion of the force or $\dot{f}$ will be $= \text{S}s\dot{x} = s\dot{x} \times \frac{\text{BC}}{\text{AC}} = s\dot{x} \times \frac{b}{\sqrt{b^2+x^2}} = \frac{bs\dot{x}}{\sqrt{b^2+x^2}}$; and the fluent is $f = bs \times$ hyp. log. of $\frac{x+\sqrt{b^2+x^2}}{b} = s \,.\, \text{BC} \times$ hyp. log. $\frac{\text{AB}+\text{AC}}{\text{BC}}$ = the attraction of ABCD.



5. To find the attraction of the right-angled part BCD of a cuneus whoſe edge paſſes through A the place of the body attracted.—Put $\text{AB} = a$, $\text{BC} = b$, $\text{BD} = c$, $\text{DA} = d = a - c$, $\text{DC} = e$, $\text{AC} = g$, and $\text{DP} = x$. Then, the force of any ſection PQ being ſtill as Ss, the fluxion of the force of the part DPQ is $\dot{f} = \text{S}s\dot{x} = s\dot{x} \times \frac{\text{PQ}}{\text{AQ}} = \frac{bsx\dot{x}}{\sqrt{c^2d^2+2c^2dx+e^2x^2}}$; and the correct fluent when

when $x = c$ is $f = \frac{bcs}{ee} \times \overline{g - d - \frac{dc}{e} \times \text{hyp. log.}\ \frac{ce + eg + dc}{de + dc}}$ = the force of a body at A in the direction AB.



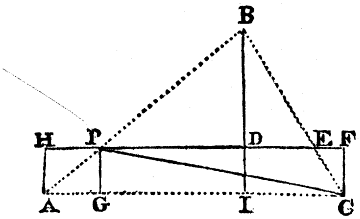
6. To find the attraction of the right-angled part BCD on the point A.—Using here again the same notation as in the last article, we have $\dot{f} = ss\dot{x} = s\dot{x} \times \frac{PQ}{AQ} = \frac{bsx\dot{x}}{\sqrt{c^2d^2 - 2c^2dx + e^2x^2}}$. The correct fluent of which, when $x = c$, is

$$f = \frac{bcs}{ee} \times \overline{g - d + \frac{dc}{e} \times \text{hyp. log.}\ \frac{ce + eg - dc}{de - dc}}.$$

7. To apply now these premises to the finding of the place where the attraction of a hill is greatest, it will be necessary to suppose the hill to have some certain figure. That position is most convenient for observing the attraction, in which the hill is most extended in the east and west direction. Supposing then such a position of a hill, and that it is also of a uniform height and meridional section throughout; the point of observation must evidently be equally distant from the two ends. But instead of being only considerably extended, I shall suppose the hill to be indefinitely extended to the east and to the west of the point of observation, in order that the investigation may be mathematically true, and yet at the same time sufficiently exact for the before-said limited extent also. It will also come nearest to the

I practical

practical experiment, to suppose the hill to be a long triangular prism, so that all its meridional sections may be similar triangles. Let therefore the triangle ABC represent its section by a vertical plane passing through the meridian, or one side of an indefinitely thin cuneus whose edge is in PG; or rather PBCGP the side of one cuneus, and PAG the side of another, their common edge being the line PG perpendicular to the base AC; P being the required point in the side AB where the attraction of the section ABC, or indefinitely thin cuneus, shall be greatest in a direction parallel to the horizon AC. And then from the foregoing suppositions, it is evident that in whatever point of AB the attraction of ABC is greatest, there also will the attraction of the whole hill be the greatest.



8. Now draw HPDEF parallel to AC; and AH, PG, BI, CF, perpendicular to the same. Then it is evident that at the point P, in the direction PF, the attraction of PBCGP is affirmative, and that of PAG negative. But PBCGP is = PBD + BDE + PFCG − EFC; and PAG = PHAG − PHA. Therefore the attractions of PBD, BDE, PFCG, PHA, are affirmative; and those of EFC, PHAG, negative.

Put

Put now BI $= a$, AI $= b$, IC $= c$, AB $= d$, BC $= e$, AC $= g = b + c$, and PG $= x$, the altitude of the point P above the bottom. Alſo let $s =$ the ſine of the indefinitely ſmall angle of the cuneus to rad. 1; and $q^2 = \sqrt{a^2g^2 - 2abgx + d^2x^2}$.

Then by Art. 3, the attraction

$$\text{of} \begin{cases} \text{PBD is } s \,.\, \text{PD} \,.\, \dfrac{\text{BD}}{\text{BP}} = sb \times \dfrac{a-x}{d}, \\ \text{PHA is } s \,.\, \text{PH} \,.\, \dfrac{\text{PG}}{\text{PA}} = sb \times \dfrac{x}{d}. \end{cases}$$

By Art. 4, the attraction

$$\text{of} \begin{cases} \text{PFCG is } s \,.\, \text{PG} \times \text{h. l. } \dfrac{\text{PF}+\text{PC}}{\text{PG}} = sx \times \text{h. l. } \dfrac{ag - bx + qq}{ax}, \\ \text{PGAH is } s \,.\, \text{PG} \times \text{h. l. } \dfrac{\text{PH}+\text{PA}}{\text{PG}} = sx \times \text{h. l. } \dfrac{b+d}{a}. \end{cases}$$

By Art. 5, the attraction of EFC is

$$s \,.\, \frac{\text{EF}\,.\,\text{FC}}{\text{EC}^2} \times \overline{\text{PC} - \text{PE} - \frac{\text{PE}\,.\,\text{EF}}{\text{EC}} \times \text{h. l. } \frac{\text{EC}^2 + \text{EC}\,.\,\text{PC} + \text{PE}\,.\,\text{EF}}{\text{PE}\,.\,\text{EC} + \text{PE}\,.\,\text{EF}}}$$

$$= \frac{sc}{ee} \times \overline{qq - g \,.\, \overline{a - x} - gc \,.\, \frac{a-x}{e} \times \text{h. l. } \frac{acg + eqq + aax - bcx}{g \,.\, \overline{c+e} \,.\, \overline{a-x}}}.$$

Laſtly by Art. 6, the attraction of BDE is

$$s \,.\, \frac{\text{BD}\,.\,\text{DE}}{\text{BE}^2} \times \overline{\text{PB} - \text{PE} + \frac{\text{PE}\,.\,\text{DE}}{\text{BE}} \times \text{h. l. } \frac{\text{BE}^2 + \text{BE}\,.\,\text{BP} - \text{PE}\,.\,\text{DE}}{\text{PE}\,.\,\text{BE} - \text{PE}\,.\,\text{DE}}}$$

$$= \frac{sc}{ee} \,.\, \overline{a - x} \times \overline{d - g + \frac{cg}{e} \times \text{h. l. } \frac{ee + de - cg}{eg - cg}}.$$

Theſe quantities being collected together with their proper ſigns, and contracted, we have

$$s \times \begin{Bmatrix} \dfrac{ab}{d} + c \,.\, \dfrac{ad - qq - dx}{ee} + x \times \text{hyp. log. } \dfrac{ag + qq - bx}{\overline{b+d} \,.\, x} + \\ \dfrac{ccg \,.\, \overline{a-x}}{e^3} \times \text{h. l. } \dfrac{\overline{ee + de - cg} \,.\, \overline{acg + eqq + a^2x - bcx}}{gg \,.\, \overline{ee - cc} \,.\, \overline{a-x}} \end{Bmatrix}$$

for the whole attraction in the direction PE.

9. Having now obtained a general formula for the meaſure of the attraction in any ſort of triangle, if the particular values of the letters be ſubſtituted which any practical caſe may require, and the fluxion of this attraction be put $= 0$, the root of the reſulting equation will be the required height from the bottom of the hill.

10. But for a more particular ſolution in ſimpler terms, let us ſuppoſe the triangle ABC to be iſoſceles, in which caſe we ſhall have $d = e$, and $g = 2b = 2c$, and then the above general formula will become

$$s \times \left\{ \begin{array}{l} \frac{2ad - qq - dx}{dd} b + x \times \text{h. l. } \frac{2ab + qq - bx}{\overline{b + d} \,.\, x} \\ + \frac{a - x}{d^3} \,.\, 2b^3 \times \text{h. l. } \frac{2ab^2 + dq^2 - \overline{b^2 - a^2} \,.\, x}{2b^2 \,.\, \overline{a - x}} \end{array} \right\}$$

for the value of the attraction in the caſe of the iſoſceles triangle, where q^2 is $= \sqrt{4a^2b^2 - 4ab^2x + d^2x^2}$. And the fluxion of this expreſſion being equated to 0, the equation will give the relation between a and x for any values of b and d, by a proceſs not very troubleſome.

11. Now it is probable that the relation between a and x, when the attraction is greateſt, will vary with the various relations between b and d, or between b and a. Let us therefore find the limits of that relation, between which it may always be taken, by uſing two particular extreme caſes, the one in which the hill is

very fteep, and the other in which it is very flat, or a very fmall in refpect of b or d.

12. And firft let us fuppofe the triangular fection to be equilateral; in which cafe the angle of elevation is 60°, which being a degree of fteepnefs that can fcarcely ever happen, this may be accounted the firft extreme cafe. Here then we fhall have $d = 2b = \frac{2}{3}a\sqrt{3}$, and the formula in Art. 10, will become $s \times : \frac{2a - r - x}{2} + x \times$ h. l. $\frac{2a + 2r - x}{3x} + \frac{a - x}{4} \times$ h. l. $\frac{a + 2r + x}{a - x}$ for the value of the attraction in the cafe of the equilateral triangle, in which r is $= \sqrt{a^2 - ax + x^2}$.

13. Or if we take $x = na$, where n expreffes what part of a is denoted by x, the laft formula will become $sa \times : 1 - \frac{1}{2}n - \frac{1}{2}\sqrt{1 - n + n^2} + n \times$ h. l. $\frac{2 - n + 2\sqrt{1 - n + n^2}}{3n} +$ $\frac{1 - n}{4} \times$ h. l. $\frac{1 + n + 2\sqrt{1 - n + n^2}}{1 - n}$ for the cafe of the equilateral triangle.

14. To find the maximum of the expreffion in the laft article, put its fluxion $= 0$, and there will refult this equation $1 + \frac{1 + n}{\sqrt{1 - n + n^2}} = 2$ h. l. $\frac{2 - n + 2\sqrt{1 - n + n^2}}{3n} -$ $\frac{1}{2}$h. l. $\frac{1 + n + 2\sqrt{1 - n + n^2}}{1 - n}$; the root of which is $n = \cdot 251999$. Which fhews that, in the equilateral triangle, the height from the bottom to the point of greateft attraction, is only

only $\frac{1}{500}$th part more than $\frac{1}{4}$ of the whole altitude of the triangle. And this is the limit for the ſteepeſt kind of hills.

15. Let us find now the particular values of the meaſure of attraction ariſing by taking certain values of n varying by ſome ſmall difference, in order to diſcover what part of the greateſt attraction is wanting by obſerving at different altitudes.

16. And firſt uſing the value of n (·251999) as found in the 14th article, the general formula in Art. 13, gives $sa \times 1{\cdot}0763700$ for the meaſure of the greateſt attraction.

17. If $n = \frac{3}{10}$, or $x = \frac{3}{10}a$; the ſame formula gives $\frac{sa}{20} \times : 17 - \sqrt{79} + 6\,\text{h. l.}\,\frac{17 + 2\sqrt{79}}{9} + \frac{7}{2}\,\text{h. l.}\,\frac{13 + 2\sqrt{79}}{7} =$ $sa \times 1{\cdot}0702512$ for the attraction at $\frac{3}{10}$ of the altitude, which is ſomething leſs than the other.

18. If $n = \frac{4}{10} = \frac{2}{5}$; the formula gives $\frac{sa}{20} \times : 16 - \sqrt{76} + 8\,\text{h. l.}\,\frac{8 + \sqrt{76}}{6} + 3\,\text{h. l.}\,\frac{7 + \sqrt{76}}{3} = sa \times 1{\cdot}0224232$ for the attraction at $\frac{4}{10}$ or $\frac{2}{5}$ of the altitude; leſs again than the laſt was.

19. If $n = \frac{5}{10} = \frac{1}{2}$; the formula gives $\frac{1}{4}sa \times : 3 - \sqrt{3} - 2\,\text{h. l.}\,3 + \frac{5}{2}\,\text{h. l.}\,\overline{3 + 2\sqrt{3}} = sa \times {\cdot}9340963$ for the attraction at half way up the hill; ſtill leſs again than the laſt.

20. If $n = \frac{6}{10} = \frac{3}{5}$; the formula gives $\frac{sa}{20} \times : 14 - \sqrt{76} + 12 \text{h. l.} \frac{7 + \sqrt{76}}{9} + 2 \text{h. l.} \frac{8 + \sqrt{76}}{2} = sa \times {\cdot}8109843$ for the attraction at $\frac{6}{10}$ or $\frac{3}{5}$ of the altitude from the bottom; being still less than the last was. And thus the quantity of attraction is continually less and less the higher we ascend up the hill above the ·251999 part, or in round numbers ·252 part of the altitude. Let us now descend, by trying the numbers below ·252; and first,

21. If $n = {\cdot}25 = \frac{1}{4}$; the same formula in Art. 13, gives $\frac{1}{8} sa \times : 7 - \sqrt{13} + 2 \text{h. l.} \frac{7 + 2\sqrt{13}}{3} + \frac{3}{2} \text{h. l.} \frac{5 + 2\sqrt{13}}{3} = sa \times 1{\cdot}0763589$ for the attraction at $\frac{1}{4}$ of the altitude; and is very little less than the maximum.

22. If $n = \frac{2}{10} = \frac{1}{5}$; the formula gives $\frac{1}{10} sa \times : 9 - \sqrt{21} + 2 \text{h. l.} \frac{9 + 2\sqrt{21}}{3} + 2 \text{h. l.} \frac{3 + \sqrt{21}}{2} = \frac{1}{10} sa \times : 9 - \sqrt{21} + 2 \text{h. l.} \frac{23 + 5\sqrt{21}}{2} = sa \times 1{\cdot}0684622$ for the attraction at $\frac{2}{10}$ or $\frac{1}{5}$ of the altitude; and is something less than at $\frac{1}{4}$ of the altitude.

23. If $n = \frac{1}{10}$; the formula gives $\frac{sa}{20} \times : 19 - \sqrt{91} + 2 \text{h. l.} \frac{19 + 2\sqrt{91}}{3} + \frac{9}{2} \text{h. l.} \frac{11 + 2\sqrt{91}}{9} = sa \times {\cdot}9986188$ for the attraction at $\frac{1}{10}$ of the altitude; still less than the last was. And, lastly,

24. If $n = 0$, or the point be at the bottom of the hill;

hill; the formula gives $\frac{1}{4}sa \times \overline{2 + \text{h. l. } 3} = sa \times {\cdot}7746531$ for the attraction at the bottom of the hill; which is between $\frac{2}{3}$ and $\frac{3}{4}$ of the greateſt attraction, being ſomething greater than $\frac{2}{3}$ but leſs than $\frac{3}{4}$ of it.

25. The annexed table exhibits a ſummary of the calculations made in the preceding articles; where the firſt column ſhews at what part of the altitude of the hill the obſervation is made; the ſecond column contains the correſponding numbers which are proportional to the attraction; and the third column ſhews what part of the greateſt attraction is loſt at each reſpective place of obſervation, or how much each is leſs than the greateſt.

$\frac{6}{10}$	8109843	$\frac{1}{4}$
$\frac{5}{10}$	$934096\dot{3}$	$\frac{2}{13}$
$\frac{4}{10}$	10224232	$\frac{1}{20}$
$\frac{3}{10}$	10702512	$\frac{1}{180}$
$\frac{252}{1000}$	10763700	0
$\frac{1}{4}$	10763589	$\frac{1}{97852}$
$\frac{2}{10}$	10684622	$\frac{1}{134}$
$\frac{1}{10}$	9986188	$\frac{1}{14}$
0	7746531	$\frac{2}{7}$

26. Having now ſo fully illuſtrated the caſe of the firſt extreme, or limit, let us ſearch what is the limit for the other extreme, that is, when the hill is very low or flat. In this caſe b is nearly equal to d, and they are both very great in reſpect of a; conſequently the formula for the attraction in Art. 10, will become barely $s \times : x \times \text{h. l. } \frac{2a-x}{x} + 2\,.\,\overline{a-x} \times \text{h. l. } \frac{2a-x}{a-x}$; the fluxion of which being put $= 0$, we obtain $0 = \text{h. l. } \frac{2a-x}{x} - 2\,\text{h. l. } \frac{2a-x}{a-x} = \text{h. l. } \frac{2a-x}{x} - \text{h. l. } \overline{\frac{2a-x}{a-x}}\Big|^2 = \text{h. l. } \frac{\overline{a-x}|^2}{x\,.\,\overline{2a-x}}$; hence therefore $\overline{a-x}|^2 = x\,.\,\overline{2a-x}$, and $x = a \times \overline{1 - \sqrt{\tfrac{1}{2}}} = {\cdot}2929a$.

$\cdot 2929a$. Which ſhews that the other limit is $\frac{29}{100}$; that is, when the hill is extremely low, the point of greateſt attraction is at $\frac{29}{100}$ of the altitude, like as it is at $\frac{25}{100}$ when the hill is very ſteep. And between theſe limits it is always found, it being nearer to the one or the other of them, as the hill is flatter or ſteeper.

27. Thus then we find that at $\frac{1}{4}$ of the altitude, or very little more, is the beſt place for obſervation, to have the greateſt attraction from a hill in the form of a triangular priſm of an indefinite length. But when its length is limited, the point of greateſt attraction will deſcend a little lower; and the ſhorter the hill is, the lower will that point deſcend. For the ſame reaſon, all pyramidal hills have their place of greateſt attraction a little below that above determined. But if the hill have a conſiderable ſpace flat at the top, after the manner of a fruſtum, then the ſaid point will be a little higher than as above found. Commonly, however, $\frac{1}{4}$ of the altitude may be uſed for the beſt place of obſervation, as the point of greateſt attraction will ſeldom differ ſenſibly from that place. And when uncommon circumſtances may produce a difference too great to be intirely neglected, the obſerver muſt exerciſe his judgment in gueſſing at the neceſſary change he ought to make in the place of obſervation, ſo as to obtain the beſt effect which the concomitant circumſtances will admit of.