XL. *Methodus inveniendi Lineas Curvas ex proprietatibus Variationis Curvaturæ.* *Auctore* Nicolao Landerbeck, *Mathef. Profeſſ. in Acad. Upſalienſi Adjuncto.* *Communicated by* Nevil Maſkelyne, *D. D. F. R. S. and Aſtronomer Royal.*

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PARS SECUNDA*.

CURVAS, ex proprietate variationis curvaturæ invenire, indice per functionem coordinatarum cujuſdam expreſſo, problema etſi indeterminatum eſt; juvat tamen ad curvas cognoſcendas, quum facile et ſponte ſeſe offerunt conditiones determinantes qui rei conveniunt et quæ in caſu quovis examini ſubjecto locum habent. Quo conſilio et qua arte calculum inire oporteat, ut et hæc et his affinia peragenda ſint, quæ ad curvas ex curvaturæ variatione cognoſcendas pertineant, per theoremata quæ ſequuntur, exponere conabor.

THEOREMA I. (Vide tab. XXI. fig. 2.)

Si curvæ cujuſdam LC index variationis curvaturæ ſit T, radius curvedinis R, ſinus anguli BCD p, poſito ſinu toto 1, arcus curvæ LC z coordinatæ perpendiculares x et y earumque fluxiones dp, dz, dx, et dy dicantur, erit $\frac{dz}{\int T dz} = -\frac{dp}{\sqrt{1-p^2}}$.

Quoniam $dx = -R dp$ et $dz = -\frac{dx}{\sqrt{1-p^2}}$ habetur $\frac{dz}{R} = -\frac{dp}{\sqrt{1-p^2}}$

* See Vol. LXXIII. p. 456.

 et

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et quum $dR = Tdz$ erit $R = \int Tdz$ et ſubſtitutione $\frac{dz}{\int Tdz} = -\frac{dp}{\sqrt{1-p^2}}$.

Cor. 1. Hinc obtinetur $\frac{dx}{R} = -dp$, $\frac{dy}{R} = -\frac{pdp}{\sqrt{1-p^2}}$ et $\frac{dz}{R} = -\frac{dp}{\sqrt{1-p^2}}$.

Cor. 2. Si Tangens anguli BCD per r, Secans per s deſignentur habetur $\frac{dz}{\int Tdz} = -\frac{dr}{1+r^2}$ et $\frac{dz}{\int Tdz} = -\frac{ds}{s\sqrt{s^2-1}}$.

Schol. 1. Ex hoc theoremate facilis deducitur methodus generaliter calculandi variationem curvaturæ curvæ cujuſcumque. Nam $\int Tdz = -\frac{dz\sqrt{1-p^2}}{dp}$, quantitas vero $\frac{dz\sqrt{1-p^2}}{dp}$ datur, data inter x et y relatione. Sit valor quantitatis $-\frac{dz\sqrt{1-p^2}}{dp} = Z$ functioni curvæ z, $\int Tdz = Z$ et ſumtis fluxionibus $Tdz = \dot{Z}dz$ qua $T = \dot{Z}$ functioni ipſius z. Si valor quantitatis $-\frac{dz\sqrt{1-p^2}}{dp} = P$ per p expreſſus, erit $\int Tdz = P$ ſumtiſque fluxionibus $Tdz = \dot{P}dp$ et $T = \frac{\dot{P}dp}{dz}$, quæ functio eſt quantitatis p, in poteſtate ſemper eſt $\frac{dp}{dz}$ per p exprimere.

Schol. 2. Hujus etiam theorematis ſubſidio inveniri poſſunt curvæ ex data relatione inter T et z, R et z, R et y, et R et p. Si enim ſit $T = Z$ functioni quantitatis z, erit $\int Tdz = \int Zdz + A$, vi theorematis $\frac{dz}{\int Zdz + A}$ $\left(= \frac{dz}{\int Tdz}\right) = -\frac{dp}{\sqrt{1-p^2}}$ et integratione $\int \frac{dz}{\int Zdz + A} + C = -\frac{dp}{\sqrt{1-p^2}}$. Poſita $\int \frac{dz}{\int Zdz + A} + C = b$ et N numerus

merus cujus logarithmus hyperbolicus 1 habetur $\sqrt{1-p^2} = \frac{N^{b\sqrt{-1}} - N^{-b\sqrt{-1}}}{2\sqrt{-1}}$ et $p = \frac{N^{b\sqrt{-1}} + N^{-b\sqrt{-1}}}{2}$, quæ functiones sunt quantitatis z, quibus positis $\overset{|}{Z}$ et $\sqrt{1-\overset{|}{Z}^2}$ respective proveniunt $x\left(=\int dz\sqrt{1-p^2}\right) = \int Zdz$ et $y\left(=\int pdz\right) = \int dz\sqrt{1-\overset{|}{Z}^2}$ quarum alterutra curvarum indoles innotescit.

Si $R = X$ functioni abscissæ x provenit $\frac{dx}{X}\left(=\frac{dx}{R}\right) = -dp$ et integratione $\overset{|}{X}\left(=C-\int\frac{dx}{R}\right) = p$ unde $\sqrt{1-p^2} = \sqrt{1-\overset{|}{X}^2}$ et $y\left(=\int\frac{pdx}{\sqrt{1-p^2}}\right) = \int\frac{\overset{|}{X}dx}{\sqrt{1-\overset{|}{X}^2}}$ æquatio curvæ indolem exprimens.

Et si $R = Y$ functioni ordinatæ y, habetur $\frac{dy}{Y}\left(=\frac{dy}{R}\right) = -\frac{pdp}{\sqrt{1-p^2}}$ et integratione $\overset{|}{Y}\left(=\int\frac{dy}{Y}+C\right) = \sqrt{1-p^2}$, unde $p = \sqrt{1-\overset{|}{Y}^2}$ et $x\left(=\int\frac{dy\sqrt{1-p^2}}{p}\right) = \int\frac{\overset{|}{Y}dy}{\sqrt{1-\overset{|}{Y}^2}}$ quæ exprimit naturam curvæ.

Hinc colligitur quod quoties Tdz perfecte integretur et $\int\frac{dz}{\int Zdz+A}$ obtineatur per arcus circulares dum aut $\int\overset{|}{Z}dz$ aut $\int dz\sqrt{1-\overset{|}{Z}^2}$ absolutam admittat integrationem, curvæ erunt rectificabiles, et algebraicæ, si relatio inter x et z vel inter y et z in relationem algebraicam inter x et y permutari possit.

Evidens etiam est quod si X functio est algebraica quantitatis x vel Y quantitatis y, et non solum $\frac{dx}{X}$ vel $\frac{dy}{Y}$ sed etiam $\frac{\overset{|}{X}dx}{\sqrt{1-\overset{|}{X}^2}}$ vel $\frac{\overset{|}{Y}dy}{\sqrt{1-\overset{|}{Y}^2}}$ quantitates perfecte integrabiles, curvæ evadunt algebraicæ, alias transcendentes.

Exempl.

Exempl. 1. Invenienda ſit curva ubi variatio curvaturæ $T = \frac{3 \cdot \overline{8a+27z}|^{\frac{2}{3}} - za^{\frac{2}{3}}}{a^{\frac{1}{3}}\sqrt{\overline{8a+27z}|^{\frac{2}{3}} - 4a^{\frac{2}{3}}}}$. Ut ſimplicior reddatur calculus ponatur $\overline{8a+27z}|^{\frac{2}{3}} = u$ et $a^{\frac{2}{3}} = b$ erit $z = \frac{u^{\frac{3}{2}} - 8b^{\frac{3}{2}}}{27}$, $dz = \frac{du\sqrt{u}}{18}$, $T = \frac{3\overline{u-2b}}{\sqrt{b}\sqrt{u-4b}}$ et $\int T dz = \frac{u\sqrt{u}\sqrt{u-4b}}{18\sqrt{b}} + A$; ſit conſtans hæc $A = a$, quod accidit evaneſcente $\int T dz u = b$, habetur per theorema $\frac{du\sqrt{b}}{u\sqrt{u-4b}} \left(= \frac{dz}{\int T dz}\right) = -\frac{dp}{\sqrt{1-p^2}}$ et integratione $\int \frac{du\sqrt{b}}{u\sqrt{u-4b}} + C = -\int \frac{dp}{\sqrt{1-p^2}}$, cujus æquationis termini quum ſint arcus circulares quorum ſinus $\sqrt{1-p^2} = \frac{\sqrt{u-4b}}{\sqrt{u}}$ et coſinus $p = \frac{2\sqrt{b}}{\sqrt{u}}$, poſito arcu conſtanti $C = 0$, obtinetur $y \left(= \int p dz\right) = \int \frac{du\sqrt{b}}{9} + B) = \frac{\sqrt{u-4b}\sqrt{b}}{9}$ nam $B = \frac{4b\sqrt{b}}{9}$, poſita $y = 0$ et $u = 4b$, atque $x \left(= \int dz\sqrt{1-p^2}\right) = \int \frac{du\sqrt{u-4b}}{18} = \frac{\overline{u-4b}|^{\frac{3}{2}}}{27}$ quibus æquationibus exterminata u et ſubſtituta a habetur $y^3 = ax^2$ æquatio pro parabola cubica.

Exempl. 2. Si ſit variatio curvaturæ $T = \frac{2z}{a}$ erit $\int T dz \left(= \int \frac{2zdz}{a}\right) = \frac{z^2}{a} + A$ et ſi $Z = 0$ $\int T dz = a$ erit conſtans $A = a$, atque vi theorematis $\frac{adz}{a^2+z^2} \left(= \frac{dz}{\int T dz}\right) = -\frac{dp}{\sqrt{1-p^2}}$ et integratione $\int \frac{adz}{a^2+z^2} + C = -\int \frac{dp}{\sqrt{1-p^2}}$; poſito arcu conſtanti $C = 0$ cæteri ſunt æquales eorumque ſinus et coſinus, unde $\sqrt{1-p^2} = \frac{z}{\sqrt{a^2+z^2}}$, $p = \frac{a}{\sqrt{a^2+z^2}}$ et $dx \left(= dz\sqrt{1-p^2}\right) = \frac{zdz}{\sqrt{a^2+z^2}}$ et dy (=

$pdz) = \frac{adz}{\sqrt{a^2+z^2}}$, quibus conſtat curvam eſſe catenariam.

Exempl. 3. Sit variatio curvaturæ $T = \frac{a-z}{\sqrt{2az-z^2}}$, evadit $\int Tdz = \sqrt{2az-z^2}$, per theorema $\frac{dz}{\sqrt{2az-z^2}} \left(= \frac{dz}{\int Tdz}\right) = -\frac{dp}{\sqrt{1-p^2}}$ et per integrationem $\int \frac{dz}{\sqrt{2az-z^2}} + C = -\int \frac{dp}{\sqrt{1-p^2}}$, ſi arcus ille conſtans $C = 0$, cæteri ſunt æquales eorumque ſinus et coſinus, quo $\sqrt{1-p^2} = \frac{\sqrt{2az-z^2}}{a}$, $p = \frac{a-z}{a}$ et $y \left(= \int pdz\right) = \int \frac{a-z}{a} dz\Big) = \frac{2az-z^2}{a}$ æquatio pro cycloide ordinaria.

THEOREMA II.

Manentibus antea adhibitis denominationibus erit $\frac{dx}{y+\int Tdx} = -\frac{dp}{\sqrt{1-p^2}}$.

Quoniam $\frac{dx}{R} = -dp$, erit dividendo per $\sqrt{1-p^2}$, $\frac{dx}{R\sqrt{1-p^2}} = -\frac{dp}{\sqrt{1-p^2}}$. Propter $1 : \sqrt{1-p^2} :: CD(R) : CF = R\sqrt{1-p^2}$, ſed $dz : dx :: Tdz : Tdx$, quæ fluxio eſt ipſius DE, quare $DE = \int Tdx$, unde $CF = y + \int Tdx$ qua pro $R\sqrt{1-p^2}$ ſubſtituta, prodit $\frac{dx}{y+\int Tdx} = -\frac{dp}{\sqrt{1-p^2}}$.

Cor. 1. Quantitas $dy + Tdx$ ſemper eſt perfecte integrabilis. Nam $Tdx = -\frac{ddx\sqrt{1-p^2}}{dp}$ et $dy = \frac{pdx}{\sqrt{1-p^2}}$ unde $dy + Tdx = \frac{pdx}{\sqrt{1-p^2}} - \frac{ddx\sqrt{1-p^2}}{dp}$ et integratione $y + \int Tdx = -\frac{dx\sqrt{1-p^2}}{dp}$.

Cor.

Cor. 2. Dicatur ſemichorda curvaturæ CF F, obtinetur $\frac{dx}{F} = -\frac{dp}{\sqrt{1-p^2}}$, $\frac{dy}{F} = -\frac{pdp}{1-p^2}$ et $\frac{dz}{F} = -\frac{dp}{1-p^2}$.

Cor. 3. Si Tangens anguli BCD per r, Secans per s deſignentur habetur $\frac{dx}{y+\int Tdx} = -\frac{dr}{1+r^2}$ et $\frac{dx}{y+\int Tdx} = -\frac{ds}{s\sqrt{s^2-1}}$.

Schol. 1. Per hoc theorema via etiam patet calculandi generaliter variationem curvaturæ. Eſt enim $y+\int Tdx = -\frac{dx\sqrt{1-p^2}}{dp}$, quantitas vero $\frac{dx\sqrt{1-p^2}}{dp}$ datur data inter x et p relatione. Sit valor quantitatis $-\frac{dx\sqrt{1-p^2}}{dp} = X$ functioni abſciſſæ x æquatione ad curvam inventus, erit $\int Tdx = X - y$ et ſumtis fluxionibus $Tdx = \dot{X}dx - dy$, qua $T = \dot{X} - \frac{dy}{dx}$ ubi tam $\dot{X}$ quam $\frac{dy}{dx}$ ſunt functiones abſciſſæ x. Si valor quantitatis $-\frac{dx\sqrt{1-p^2}}{dp} = P$ per p expreſſus, erit $\int Tdx = P - y$ ſumtiſque fluxionibus $Tdx = \dot{P}dp - dy$, qua $T = \frac{\dot{P}dp}{dx} - \frac{p}{\sqrt{1-p^2}}$ ubi $\frac{\dot{P}dp}{dx}$ functio eſt quantitatis p, nam $\frac{dp}{dx}$ per p exprimi poteſt.

Schol. 2. Hoc adhibito theoremate inveniri etiam poſſunt curvæ, ex data relatione inter T et x, F et x, F et y, F et z, et F et p. Poſita enim T functione quantitatis x, patet per curvarum quadraturas, aut perfectam aut imperfectam quantitatis Tdx obtineri integrationem. Sit $\int Tdx = \dot{X} + \int \dot{X}dx$ functioni vel algebraicæ vel ex parte tranſcendenti ipſius x, cujus terminis homogeneus valor ipſius $y = \int \ddot{X}dx$ capiatur, iſque ejus indolis ut $\int \overline{\dot{X} + \ddot{X}}dx$, vel quod idem eſt $y + \int Tdx =$

$X + \int \overline{\overset{\text{I}}{X} + \overset{\text{II}}{X}} dx$ integratione absoluta habeatur, permanente $Tdz = Tdx\sqrt{1 - \overset{\text{II}}{X}^2}$ perfecte integrabili. Per theorema deinde habetur $\frac{dx}{X + \int \overline{\overset{\text{I}}{X} + \overset{\text{II}}{X}} dx} \left(= \frac{dx}{y + \int Tdx}\right) = -\frac{dp}{\sqrt{1-p^2}}$, et per integrationem $\int \frac{dx}{X + \int \overline{\overset{\text{I}}{X} + \overset{\text{II}}{X}} dx} + C = -\int \frac{dp}{\sqrt{1-p^2}}$, si ponatur $\int \frac{dx}{X + \int \overline{\overset{\text{I}}{X} + \overset{\text{II}}{X}} dx} + C = k$ et N basi logarithmorum hyperbolicorum, erit $\sqrt{1-p^2} = \frac{N^{k\sqrt{-1}} - N^{-k\sqrt{-1}}}{2\sqrt{-1}}$ et $p = \frac{N^{k\sqrt{-1}} + N^{-k\sqrt{-1}}}{2}$, $\sqrt{1-p^2}$ et p igitur sunt functiones ipsius x, quæ si ponantur $\sqrt{1 - \overset{\text{III}}{X}^2}$ et $\overset{\text{III}}{X}$, habetur $y \left(= \int \frac{pdx}{\sqrt{1-p^2}}\right) = \int \frac{\overset{\text{III}}{X} dx}{\sqrt{1 - \overset{\text{I}}{X}^2}}$, æquatio qua curvæ internoscuntur.

Si sit $F = Y$ functioni quantitatis y erit per Cor. 2. $\frac{dy}{Y} \left(= \frac{dy}{F}\right) = -\frac{dp}{1-p^2}$ et integratione $\int \frac{dy}{Y} + \log. C = \log. \sqrt{1-p^2}$, ponatur $\int \frac{dy}{Y} = k$ et N logarithmorum basi, erit facto ad quantitates absolutas transitu $CN^k = \sqrt{1-p^2}$, $p = \sqrt{1 - C^2N^{2k}}$ et $x \left(= \int \frac{dy\sqrt{1-p^2}}{p}\right) = \int \frac{CN^k dy}{\sqrt{1 - C^2N^{2k}}}$, æquatio quæ indolem curvæ indigitat.

Si $F = Z$ functioni ipsius z erit $\frac{dz}{Z} \left(= \frac{dz}{F}\right) = -\frac{dp}{1-p^2}$ et integratione $\int \frac{dz}{Z} + \log. C = \log. \sqrt{\frac{1-p}{1+p}}$, et si $\int \frac{dz}{Z} = k$ et N basi logarithmica habetur $p = \frac{1 - C^2N^{2k}}{1 + C^2N^{2k}}$ et $y = \int pdz) = \int \frac{\overline{1 - C^2N^{2k}}\, dz}{1 + C^2N^{2k}}$ qua curvæ cognoscuntur.

Conſtat hinc quod quoties $X + \int \overset{\text{I}}{X} dx$ perfecta integratione habeatur $\int \frac{dx}{X + \int \overset{\text{I}}{X} + \overset{\text{II}}{X} dx}$ per arcus circulares dum $\frac{\overset{\text{III}}{X} dx}{\sqrt{1 - \overset{\text{III}}{X}^2}}$ abſolutam admittat integrationem curva ſit algebraica, ſi vero aliter evenerit tranſcendens.

Quoties $\frac{dy}{Y}$ ſit integrale logarithmicum et $\frac{CN^k dy}{\sqrt{1 - C^2 N^{2k}}}$ abſolutam admittat integrationem curva eſt algebraica, in aliis caſibus tranſcendens.

Et quoties $\int \frac{dz}{Z}$ per logarithmos inveniatur, $\frac{1 - C^2 N^{2k} dz}{1 + C^2 N^{2k}}$ abſolute ſit integrabilis pariter ac $\frac{2CN^k dz}{1 + C^2 N^{2k}}$ curva eſt algebraica, alias tranſcendens.

Exempl. 1. Si ſit variatio curvaturæ $T = \frac{3 \cdot \overline{b^2 - a^2} x \sqrt{a^2 - x^2}}{a^3 b}$ erit $\int T dx \left(= \frac{\overline{a^2 - b^2} \cdot \overline{a^2 - x^2} \sqrt{a^2 - x^2}}{a^3 b} \right) = \frac{a \sqrt{a^2 - x^2}}{} - \frac{x^2 \sqrt{a^2 - x^2}}{ab} - \frac{b \sqrt{a^2 - x^2}}{a} + \frac{bx^2 \sqrt{a^2 - x^2}}{a^3}$, ſi ponatur $y = \frac{b \sqrt{a^2 - x^2}}{a}$ habetur $y + \int T dx = \frac{a^4 + \overline{b^2 - a^2} x^2 \sqrt{a^2 - x^2}}{a^3 b}$, adhibendo theorema $\frac{a^3 b dx}{a^4 + \overline{b^2 - a^2} x^2 \sqrt{a^2 - x^2}} \left(= \frac{dx}{y + \int T dx} \right) = - \frac{dp}{\sqrt{1 - p^2}}$ et integrando $\int \frac{a^3 b dx}{a^4 + \overline{b^2 - a^2} x^2 \sqrt{a^2 - x^2}} + C = - \int \frac{dp}{\sqrt{1 - p^2}}$, cujus termini ſunt arcus circulares quorum ſinus $\sqrt{1 - p^2} = \frac{a \sqrt{a^2 - x^2}}{\sqrt{a^4 + \overline{b^2 - a^2} x^2}}$ et coſinus $p = \frac{bx}{\sqrt{a^4 + \overline{b^2 - a^2} x^2}}$ evaneſcente arcu conſtanti C, quare $y \left(= \int \frac{p dx}{\sqrt{1 - p^2}} \right) = \int \frac{b dx}{a \sqrt{a^2 - x^2}} \Big) = \frac{b \sqrt{a^2 - x^2}}{a}$ et in hoc caſu curva eſt ellipſis.

Exempl. 2. Sit jam variatio curvaturæ $T = \frac{2\sqrt{2ax+x^2}}{a}$ erit $\int T dx = \frac{x\sqrt{2a+x^2}}{a} + \int \frac{x dx}{\sqrt{2ax+x^2}}$ et posita $y = \int \frac{a dx}{\sqrt{2ax+x^2}}$ perfecta integratione habetur $y + \int T dx = \frac{\overline{a+x}\sqrt{2ax+x^2}}{a}$. Theorematis itaque auxilio erit $\frac{a dx}{\overline{a+x}\sqrt{2ax+x^2}}$ $\left(= \frac{dx}{y + \int T dx} = -\frac{dp}{\sqrt{1-p^2}}\right.$, et integratione $\int \frac{a dx}{\overline{a+x}\sqrt{2ax+x^2}} = C = -\int \frac{dp}{\sqrt{1-p^2}}$, si vero arcus ille constans $C = 0$ cæteri sunt æquales eorumque sinus et cosinus, unde $\sqrt{1-p^2} = \frac{\sqrt{2ax+x^2}}{a+x}$, $p = \frac{a}{a+x}$ et $y\left(= \int \frac{p dx}{\sqrt{1-p^2}}\right) = \int \frac{a dx}{\sqrt{2ax+x^2}}$, æquatio indicans curvam esse catenariam.

THEOREMA III.

Dicatur cosinus anguli BCD q, posito radio 1, cæterisque manentibus denominationibus erit $\frac{dy}{\int T dy - x} = \frac{dq}{\sqrt{1-q^2}}$.

Est enim $\frac{ay}{R} = aq$, qua per $\sqrt{1-q^2}$ divisa, dat $\frac{dy}{R\sqrt{1-q^2}} = \frac{dq}{\sqrt{1-q^2}}$; et ob $1 : \sqrt{1-q^2} :: CD\ (R) : CG = R\sqrt{1-q^2}$, sed $dz : dy :: T dz : T dy$ cujus integrale est $AE = \int T dy$, unde $CG\ (= AE - AB) = \int T dy - x$, qua pro $R\sqrt{1-q^2}$ substituta, prodit $\frac{dy}{\int T dy - x} = \frac{dq}{\sqrt{1-q^2}}$.

Cor. 1. Semper $T dy - dx$ admittit perfectam integrationem. Etenim $T dy = \frac{ddy\sqrt{1-q^2}}{dq}$ et $dx = \frac{q dy}{\sqrt{1-q^2}}$, quibus $T dy - dx = \frac{ddy\sqrt{1-q^2}}{dq} - \frac{q dy}{\sqrt{1-q^2}}$ et integratione $\int T dy - x = \frac{dy\sqrt{1-q^2}}{aq}$.

Cor. 2. Dicatur ſemichorda curvaturæ CG G, habetur $\frac{dy}{G} = \frac{dq}{\sqrt{1-q^2}}$, $\frac{dx}{G} = \frac{qdq}{1-q^2}$ et $\frac{dz}{G} = \frac{dq}{1-q^2}$.

Cor. 3. Dicatur cotangens anguli BCD t, et coſecans v erit $\frac{dy}{\int Tdy - x} = \frac{dt}{1+t^2}$ et $\frac{dy}{\int Tdy - x} = \frac{dv}{v\sqrt{v^2-1}}$.

Schol. 1. Quoniam $\int Tdy - x = \frac{dy\sqrt{1-q^2}}{dq}$ datur ex data relatione inter y et q, ſit $\frac{dy\sqrt{1-q^2}}{dq} = Y$ functioni ordinatæ y erit $\int Tdy = Y - x$ ſumtiſque fluxionibus $Tdy = \dot{Y}dy - dx$ qua $T = \dot{Y} - \frac{dx}{dy}$ functioni ipſius y. Si autem $\frac{dy\sqrt{1-q^2}}{dq} = Q$ functioni ipſius q erit $\int Tdy = Q - x$ et ſumtis fluxionibus $Tdy = \dot{Q}dq - dx$, qua habetur $T = \frac{\dot{Q}dq}{dy} - \frac{q}{\sqrt{1-q^2}}$ per q.

Schol. 2. Hujus theorematis auxilio elicere licet curvas data relatione inter T et y, G et y, G et x, G et z, et G et q. Si enim ſit T functio ipſius y generaliter $\int Tdy = Y + \int \dot{Y}dy + A$, quæ functio eſt algebraica ipſius y quoties $\int \dot{Y}dy$ abſolute ſumi poſſit. Aſſumatur $x = \int \ddot{Y}dy$, tali ipſius y functioni ut non ſolum $\int Tdy - x = Y + \int \overline{\dot{Y} + \ddot{Y}}dy$ ſed etiam $\int Tdz = \int Tdy \sqrt{1 - \ddot{Y}^2}$ abſoluta integratione habeantur, provenit vi theorematis $\frac{dy}{Y + \int \overline{\dot{Y} + \ddot{Y}}dy + A} \left(= \frac{dy}{\int Tdy - x}\right) = \frac{dq}{\sqrt{1-q^2}}$ et integratione $\int \frac{dy}{Y + \int \overline{\dot{Y} + \ddot{Y}}dy + A} + C = \int \frac{dq}{\sqrt{1-q^2}}$. Poſita $\frac{dy}{Y + \int \overline{\dot{Y} + \ddot{Y}}dy + A} + C = l$ et N baſi logarithmica erit $q = \frac{N^{l\sqrt{-1}} - N^{-l\sqrt{-1}}}{2\sqrt{-1}}$ et $\sqrt{1-q^2}$

$= \frac{N^{l\sqrt{-1}} + N^{-l\sqrt{-1}}}{2}$ quæ functiones ſunt quantitatis y, quibus poſitis $\overset{\text{III}}{Y}$ et $\sqrt{1 - \overset{\text{III}}{Y}{}^2}$ prodit x $\left(= \int \frac{qdy}{\sqrt{1-q^2}}\right) = \int \frac{\overset{\text{III}}{Y}dy}{\sqrt{1 - \overset{\text{III}}{Y}{}^2}}$ æquatio quæ indolem curvarum indicat.

Si $G = X$ functioni ipſius x erit per Cor. 2. $\frac{dx}{X}$ $\left(= \frac{dx}{G}\right) = \frac{qdq}{1-q^2}$, et integratione log. CN^l $\left(= \int \frac{dx}{X} + \log. C\right) = \log. \frac{1}{\sqrt{1-q^2}}$ ſi $\int \frac{dx}{X} = l$, exinde $\sqrt{1-q^2} = \frac{1}{CN^l}$, $q = \frac{\sqrt{C^2N^{2l} - 1}}{CN^l}$ et y $\left(= \int \frac{dx\sqrt{1-q^2}}{q}\right)$ $= \int \frac{dx}{\sqrt{C^2N^{2l} - 1}}$, quæ curvæ naturam indigitat.

Si $G = Z$ functioni ipſius z erit $\frac{dz}{Z}$ $\left(= \frac{dz}{G}\right) = \frac{dq}{1-q^2}$, et integratione log. CN^l $\left(= \frac{dz}{Z} + C\right) = \log. \sqrt{\frac{1+q}{1-q}}$ ſi $\int \frac{dz}{Z} = l$, unde $q = \frac{C^2N^{2l}}{1 + C^2N^{2l}}\sqrt{1-q^2} = \frac{2CN^l}{1 + C^2N^{2l}}$ x $\left(= \int qdz\right) = \int \frac{\sqrt{C^2N^{2l} - 1}\,dz}{1 + C^2N^{2l}}$ et y $\left(= \int dz\sqrt{-1q^2}\right) = \int \frac{2CN^l dz}{1 + C^2N^{2l}}$ quibus curvæ cognoſcuntur.

Patet hinc quod quando $Y + \int \overset{\text{I}}{Y}dy$ algebraice habeatur $\int \frac{dy}{Y + \int \overset{\text{I}}{Y} + \overset{\text{II}}{Y}dy + A}$ per quadraturam circuli, et $\int \frac{\overset{\text{III}}{Y}dy}{\sqrt{1 - \overset{\text{III}}{Y}{}^2}}$ etiam obtineatur algebraice, curvæ evadunt algebraicæ, ſecus vero tranſcendentes.

Quando $\int \frac{dx}{X}$ vel $\int \frac{dz}{Z}$ obtineatur per logarithmos, et $\int \frac{dx}{\sqrt{C^2N^{2l} - 1}}$, vel tam $\int \frac{\sqrt{C^2N^{2l} - 1}\,dz}{1 + C^2N^{2l}}$ quam $\int \frac{2CN^l dz}{1 + C^2N^{2l}}$ abſoluta integratione, curvæ erunt algebraicæ.

Exempl.

Exempl. 1. Sit index variationis curvaturæ $T = \frac{6y}{a}$ erit $\int T dy = \frac{3y^2}{a} + A$, si quantitas illa constans $A = \frac{a}{2}$ quod evenit quum $\int T dy = \frac{a}{2}$ et $y = 0$; sumatur $x = \frac{y^2}{a}$ erit vi theorematis $\frac{2ady}{a^2+4y^2}$ $\left(= \frac{dy}{\int T dy - x}\right) = \frac{dq}{\sqrt{1-q^2}}$ et integratione $\int \frac{2ady}{a^2+4y^2} + C = \int \frac{dq}{\sqrt{1-q^2}}$, cujus æquationis termini quoniam sint arcus circulares quorum sinus $q = \frac{2y}{\sqrt{a^2+4y^2}}$ et cosinus $\sqrt{1-q^2} = \frac{a}{\sqrt{a^2+4y^2}}$, arcu constanti $C = 0$, obtinetur $x \left(= \int \frac{qdy}{\sqrt{1-q^2}}\right) = \frac{y^2}{a}$ æquatio pro parabola Apolloniana.

Exempl. 2. Si sit $T = \frac{a^2}{y\sqrt{a^2-y^2}}$ habetur $\int T dy = \int \frac{dy\sqrt{a^2-y^2}}{y} - \sqrt{a^2-y^2} + A$, si quantitas illa constans $A = 0$ quod evenit quum $\int T dy = 0$ et $y = a$, et assumatur $x = \int \frac{dy\sqrt{a^2-y^2}}{y}$, evadit per theorema $-\frac{dy}{\sqrt{a^2-y^2}} \left(= \frac{dy}{\int T dy - x}\right) = \frac{dq}{\sqrt{1-q^2}}$, et per integrationem $-\int \frac{dy}{\sqrt{a^2-y^2}} + C = \int \frac{dq}{\sqrt{1-q^2}}$, quorum arcuum sinus $q = \frac{\sqrt{a^2-y^2}}{a}$ et cosinus $\sqrt{1-q^2} = \frac{y}{a}$ si constans ille $C = 0$, atque inde $dx \left(\frac{qdy}{\sqrt{1-q^2}}\right) = \frac{dy\sqrt{1-y^2}}{y}$ qua patet curvam esse tractoriam.

THEOREMA IV.

Dicatur summa tangentium angulorum HCD et BCD H, et differentia tangentium angulorum HCD et CKB K, retentis reliquis denominationibus erit $\frac{dx}{\int H dx} = -\frac{dp}{\sqrt{1-p^2}}$ et $\frac{dy}{\int K dy} = \frac{dq}{\sqrt{1-q^2}}$.

Quoniam

Quoniam $dy = \frac{pdx}{\sqrt{1-p^2}}$ erit $dy + Tdx = \overline{T + \frac{p}{\sqrt{1-p^2}}}\, dx$ et quum $H = T + \frac{p}{\sqrt{1-p^2}}$ habetur $dy + Tdx = Hdx$. Eodem modo quum $dx = \frac{qdy}{\sqrt{1-q^2}}$ erit $\int Tdy - dx = T - \frac{q}{\sqrt{1-q^2}}\, dy$, sed $K = T - \frac{q}{\sqrt{1-q^2}}$, unde $\int Tdy - x = Kdy$. Per theorema igitur 2 et 3 provenit $\frac{dx}{\int Hdx} = -\frac{dp}{\sqrt{1-p^2}}$ et $\frac{dy}{\int Kdy} = \frac{dq}{\sqrt{1-q^2}}$.

Cor. Si sit ut antea tangens anguli BCD r, cotangens t, secans s, et cosecans v, erit $\frac{dx}{\int Hdx} = -\frac{dr}{r+r^2}$ et $\frac{dx}{\int Hdx} = -\frac{ds}{s\sqrt{s^2-1}}$, $\frac{dy}{\int Kdy} = \frac{dt}{1+t^2}$ et $\frac{dy}{\int Kdy} = \frac{dv}{v\sqrt{v^2-1}}$.

Schol. Ope hujus theorematis invenire licet curvas, data relatione inter H et x atque K et y. Itaque sit $H = X$ functioni ipsius x erit $\int Hdx = \int Xdx + A$, vi theorematis $\frac{dx}{\int Xdx + A}$ $\left(= \frac{dx}{\int Hdx}\right) = -\frac{dp}{\sqrt{1-p^2}}$, et integratione $\int \frac{dx}{\int Xdx + A} + C = -\int \frac{dp}{\sqrt{1-p^2}}$. Posita $\int \frac{dx}{\int Xdx + A} + C = m$, et N logarithmorum basi prodit $\sqrt{1-p^2} = \frac{N^{m\sqrt{-1}} - N^{-m\sqrt{-1}}}{2\sqrt{-1}}$ et $p = \frac{N^{m\sqrt{-1}} + N^{-m\sqrt{-1}}}{2}$, quibus functionibus quantitatis x positis $\sqrt{1-X^2}$ et X provenit æquatio $y \left(= \int \frac{pdx}{\sqrt{1-p^2}}\right) = \frac{Xdx}{\sqrt{1-X^2}}$ naturam curvarum exprimens.

Si $K = Y$ functioni quantitatis y, eadem calculandi ratione habetur $x \left(= \int \frac{qdy}{\sqrt{1-q^2}}\right) = \frac{Ydy}{\sqrt{1-Y^2}}$ æquatio qua curvæ cognoscuntur.

Quando

Quando $\int Xdx$ vel $\int Ydy$ abſoluta integratione, $\int \frac{dx}{\int Xdx + A}$ vel $\int \frac{dy}{\int Ydy + A}$ per rectificationem circuli, et $\int \frac{Xdx}{\sqrt{1 - X^2}}$ vel $\int \frac{Ydy}{\sqrt{1 - Y^2}}$ integratione perfecta obtineantur, curva eſt algebraica.

Exempl. 1. Si ſit $H = \frac{a + 12x}{2\sqrt{a}\sqrt{x}}$ erit $\int Hdx = \frac{\overline{a + 4x}\sqrt{x}}{\sqrt{a}} + A$, et poſita $A = 0$ habetur per theorema $\frac{dx\sqrt{a}}{a + 4x\sqrt{x}} \left(= \frac{dx}{\int Hdx}\right) = -\frac{dp}{\sqrt{1 - p^2}}$ et per integrationem $\int \frac{dx\sqrt{a}}{a + 4x\sqrt{x}} + C = \int \frac{dp}{\sqrt{1 - p^2}}$, cujus termini quum ſint arcus circulares quorum ſinus $\sqrt{1 - p^2} = \frac{2\sqrt{x}}{\sqrt{a + 4x}}$ et coſinus $p = \frac{\sqrt{a}}{\sqrt{a + 4x}}$, poſita $C = 0$, obtinetur $y \left(= \int \frac{pdx}{\sqrt{1 - p^2}}\right) = \sqrt{ax}$, quæ parabolam Apolloniam exprimit.

Exempl. 2. Sit $H = \frac{2a^4 - x^4}{ax^2\sqrt{a^2 - x^2}}$ erit $\int Hdx = \frac{\overline{x^2 - 2a^2}\sqrt{a^2 - x^2}}{ax} + A$, et ſi $A = 0$, per theorema $\frac{axdx}{x^2 + 2a^2\sqrt{a^2 - x^2}} \left(= \frac{dx}{\int Hdx}\right) = -\frac{dp}{\sqrt{1 - p^2}}$ et per integrationem $\int \frac{axdx}{x^2 - 2a^2\sqrt{a^2 - x^2}} + C = -\int \frac{dp}{\sqrt{1 - p^2}}$, et ſi $C = 0$, $\sqrt{1 - p^2} = \frac{\sqrt{a^2 - x^2}}{\sqrt{2a^2 - x^2}}$ $p = \frac{a}{\sqrt{2a^2 - x^2}}$ et $y \left(= \int \frac{pdx}{\sqrt{1 - p^2}}\right) = \int \frac{adx}{\sqrt{a^2 - x^2}}$ æquatio pro curva ſinuum.

Exempl. 3. Si ſit $K = \frac{\overline{5a^2 + 6y^2}.y}{a\sqrt{a^2 + y^2}}$ erit $\int Kdy = \frac{\overline{a^2 + 2y^2}\sqrt{a^2 + y^2}}{a^2} + A$, ſi $A = 0$ habetur per theorema $\frac{a^2dy}{a^2 + 2y^2\sqrt{a^2 + y^2}} \left(= \frac{dy}{\int Kdy}\right) = \frac{dq}{\sqrt{1 - q^2}}$ et integratione $\int \frac{a^2dy}{a^2 + 2y^2\sqrt{a^2 + y^2}} + C = -\int \frac{dp}{\sqrt{1 - p^2}}$, qua $q = \frac{y}{\sqrt{a^2 + 2y^2}}$, $\sqrt{1 - q^2} = \frac{\sqrt{a^2 + y^2}}{\sqrt{a^2 + 2y^2}}$, ſi $C = 0$, unde $x \left(= \int \frac{qdy}{\sqrt{1 - q^2}}\right) = \sqrt{a^2 + y^2}$ æquatio pro hyperbola æquilatera.

Exempl.

Exempl. 4. Sit $K = \frac{y}{\sqrt{a^2 - y^2}}$ erit $\int K dy = A - \sqrt{a^2 - y^2}$ et si $A = 0$, per theorema $-\frac{dy}{\sqrt{a^2 - y^2}} \left(= \frac{dy}{\int K dy}\right) = \frac{dq}{\sqrt{1 - q^2}}$ et per integrationem $-\int \frac{dy}{\sqrt{a^2 - y^2}} + C = \int \frac{dq}{\sqrt{1 - q^2}}$ qua $q = \frac{\sqrt{a^2 - y^2}}{a}$, $\sqrt{1 - q^2} = \frac{y}{a}$ et $dy \left(= \frac{q dy}{\sqrt{1 - q^2}}\right) = \frac{dy \sqrt{a^2 - y^2}}{y}$ quæ Tractoriam exprimit.

THEOREMA V.

Designetur productum tangentium angulorum HCD et BCD per U, et angulorum HCD et CKB per V cæteris manentibus erit $\frac{dx}{\int U dx - x} = -\frac{dp}{p}$ et $\frac{dy}{Y + \int V dy} = \frac{dq}{q}$.

Quoniam $dy = \frac{p dx}{\sqrt{1 - p^2}}$ et $U = \frac{Tp}{\sqrt{1 - p^2}}$ erit $T dy \left(= \frac{Tp dx}{\sqrt{1 - p^2}}\right) = U dx$, et integratione $\int T dy = \int U dx$ qua $\int T dy - x = \int U dx - x$. Et quoniam $dx = \frac{q dy}{\sqrt{1 - q^2}}$ et $V = \frac{Tq}{\sqrt{1 - q^2}}$ erit $T dx \left(= \frac{Tq dy}{\sqrt{1 - q^2}}\right) = V dy$, $\int T dx = \int V dy$ et $y + \int T dx = y + \int V dy$. Theoremate 2. et 3. prodit $\frac{dx}{\int U dx - x} = -\frac{dp}{p}$ et $\frac{dy}{y + \int V dy} = \frac{dq}{q}$.

Cor. Si anguli BCD tangens, cotangens, &c. designentur ut antea, habetur $\frac{dx}{\int U dx - x} = -\frac{dr}{r \cdot 1 + r^2}$, $\frac{dy}{y + \int V dy} = -\frac{dt}{t \cdot 1 + t^2}$, &c.

Schol. Per hoc theorema curvæ inveniuntur ex data relatione inter U et x, atque inter V et y. Si enim sit $U = X$ functioni ipsius x erit $\int U dx = \int X dx + A$, per theorema $\frac{dx}{\int X dx - x + A} \left(= \frac{dx}{\int U dx - x}\right) = -\frac{dp}{p}$, et per integrationem $\int \frac{dx}{\int X dx - x + A} + \log. C =$

log. $\frac{1}{p}$. Ponatur $\int \frac{dx}{\int Xdx - x + A} = n$ et N basi logarithmica, erit $\frac{1}{p} = CN^n$, $p = \frac{1}{CN^n}$, $\sqrt{1-p^2} = \frac{\sqrt{C^2N^{2n}-1}}{CN^n}$ et y $\left(= \frac{pdx}{\sqrt{1-p^2}}\right) = \int \frac{dx}{\sqrt{C^2N^{2n}-1}}$ qua æquatione curvarum indoles innotescit.

Si $V = Y$ functioni ipsius y, eadem calculandi ratione provenit x $\left(= \int \frac{qdy}{\sqrt{1-q^2}}\right) = \int \frac{CN^n dy}{\sqrt{1-C^2N^{2n}}}$ qua curvæ cognoscuntur.

Evidens hinc est quod quoties $\int Xdx$ vel $\int Ydy$ algebraice $\int \frac{dx}{\int Xdx - x + A}$ vel $\int \frac{dy}{y + \int Ydy + A}$ per logarithmos, atque $\int \frac{dx}{\sqrt{C^2N^{2n}-1}}$ vel $\int \frac{CN^n dy}{\sqrt{1-C^2N^{2n}}}$ integratione absoluta, obtineantur, curva est algebraica.

Exempl. 1. Si sit $U = 3$ erit $\int Udx = 3x + A$, si vero $\int Udx = \frac{a}{2}$ quando $x = 0$ erit $A = \frac{a}{2}$ et $\int Udx - x = \frac{a+4x}{2}$. Per theorema igitur $\frac{2dx}{a+4x}$ $\left(= \frac{dx}{\int Udx - x}\right) = -\frac{dp}{p}$ et per integrationem log. $\sqrt{a+4x} + \log. C = \log. \frac{1}{p}$, posita $p = 1$ dum $x = 0$ log. $C = -$ log. $\sqrt{a}$, unde facto a logarithmis transitu $\frac{\sqrt{a+4x}}{\sqrt{a}} = \frac{1}{p}$, qua $p = \frac{\sqrt{a}}{\sqrt{a+4x}}$, $\sqrt{1-p^2} = \frac{2\sqrt{x}}{\sqrt{a+4x}}$ et y $\left(= \int \frac{pdx}{\sqrt{1-p^2}}\right) = \int \frac{dx\sqrt{a}}{2\sqrt{x}}\Big) = \sqrt{ax}$ æquatio pro Parabola Apolloniana.

Exempl. 2. Sit $U = \frac{x^3 - 4a^3}{x\sqrt{x}}$ erit $\int Udx = \frac{x^3 - 2a^3}{3x^2} + A$, si autem $\int Udx = 0$ et $x = a\sqrt[3]{2}$, erit $A = 0$ et $\int Udx - x = \frac{2 \cdot a^3 - x^3}{3x^2}$. Vi igitur theorematis erit $\frac{3x^2dx}{2a^3 - x^3}\left(= \frac{dx}{\int Udx - x}\right) = -\frac{dp}{p}$, et integratione

$\log. \frac{a\sqrt{a}}{\sqrt{a^3-x^3}} + \log. C = \log. \frac{1}{p}$ qua $p = \frac{\sqrt{a^3-x^3}}{a\sqrt{a}}$; $\sqrt{1-p^2} = \frac{x\sqrt{x}}{a\sqrt{a}}$ et $y \left(= \int \frac{pdx}{\sqrt{1-p^2}}\right) = \frac{dx\sqrt{a^3-x^3}}{x\sqrt{x}}$ æquatio ad curvam quæsitam.

Exempl. 3. Si $V = -\frac{1}{2}$ erit $\int Vdy = A - \frac{y}{2}$, posita $\int Vdy = 0$ et $y = 0$ erit $A = 0$ et $y + \int Vdy = \frac{y}{2}$. Per theorema obtinetur $\frac{2dy}{y}\left(= \frac{dy}{y + \int Vdy}\right) = \frac{dq}{q}$ et per integrationem $\log. y^2 + \log. C = \log. q$, si $q = 1$ et $y = a$ erit $\log. C = -\log. a^2$, unde $q = \frac{y^2}{a^2}$, $\sqrt{1-q^2} = \frac{\sqrt{a^4-y^4}}{a^2}$ atque $dx \left(= \frac{qdy}{\sqrt{1-q^2}}\right) = \frac{y^2dy}{\sqrt{a^4-y^4}}$, curva ergo est Elastica,

Exempl. 4. Sit $V = \frac{a^2-2y^2}{y^2}$ erit $\int Vdy = A - \frac{a^2+2y^2}{y}$, si $\int Vdy = -3a$ et $y = a$ erit $A = 0$, indeque $y + \int Vdy = -\frac{a^2+y^2}{y}$. Theorematis ope habetur $-\frac{ydy}{a^2+y^2}\left(= \frac{dy}{y + \int Vdy}\right) = \frac{dq}{q}$ et integratione $\log. \frac{1}{\sqrt{a^2+y^2}} + \log. C = \log. q$, si $q = 1$ et $y = 0$ erit $\log. C = \log. a$ et exinde $q = \frac{a}{\sqrt{a^2+y^2}}$, $\sqrt{1-q^2} = \frac{y}{\sqrt{a^2+y^2}}$ et $dx \left(= \frac{qdy}{\sqrt{1-q^2}}\right) = \frac{ady}{y}$ æquatio pro Logarithmica.

THEOREMA VI.

Dicatur ED L, et AE M, retentis præterea adhibitis denominationibus erit $\frac{dL}{T} = dx$ et $\frac{dM}{T} = dy$.

Quoniam $dz : dx :: Tdz\ (dR) : Tdx$ habetur $dL = Tdx$ et

$\frac{dL}{T} = dx$. Et quoniam $dz : dy :: Tdz\ (dR) : Tdy$ obtinetur $dM = Tdy$ et $\frac{dM}{T} = dy$.

Cor. Quum $Tdy = Udx$ et $Tdx = Vdy$, erit ſubſtitutione $\frac{dM}{U} = dx$ et $\frac{dL}{V} = dy$.

Schol. Hoc adhibito theoremate inveniri poſſunt curvæ data relatione inter T et L, T et M, atque inter U et M et V et L. Ponatur $L = \overset{|}{T}$ functioni quantitatis T habetur per theorema $\frac{d\overset{|}{T}}{T}\left(= \frac{dL}{T}\right) = dx$ et integratione $\int \frac{d\overset{|}{T}}{T} + C = x$ qua T per x datur. Curvæ deinde per theorema 2. elici poſſunt.

Si $M = \overset{|}{T}$ ipſius T functioni, habetur eodem modo T per y. Si $M = \overset{|}{U}$ functioni ipſius U, obtinetur U per x, et ſi $L = \overset{|}{V}$ functioni quantitatis V, datur V per y. Per theorema deinde 3. et 5. curvæ inveniuntur.

Evidens quidem eſt quod curvæ eſſe non poſſunt algebraicæ niſi $\int \frac{dL}{T}$, $\int \frac{dM}{T}$, $\int \frac{dM}{U}$ vel $\int \frac{dL}{V}$, obtineantur integratione abſoluta.

Exempl. 1. Si ſit $L = \frac{aT^3}{54}$ erit $dL = \frac{aT^2 dT}{18}$, et per hoc theorema $\frac{aTdT}{18}\left(= \frac{dL}{T}\right) = dx$ et integratione $\frac{aT^2}{36} + C = x$ qua $T = \frac{6\sqrt{x}}{\sqrt{a}}$, ſi $C = 0$. Per theorema 2. reperitur $y = \sqrt{ax}$, æquatio pro Parabola Apolloniana.

Exempl. 2. Si ſit $M = -\int \frac{aT^2 dT}{2 \cdot \overline{1+T^2}|^2}$ erit $dM = -\frac{aT^2 dT}{2 \cdot \overline{1+T^2}|^2}$ et ope theorematis $-\frac{aTdT}{2 \cdot \overline{1+T^2}|^2}\left(= \frac{dM}{T}\right) = dy$, et integratione $\frac{a}{4 \cdot \overline{1+T^2}} + C = y$, qua ſi $C = 0$, $T = \frac{\sqrt{a-4y}}{2\sqrt{y}}$. Per theorema 3. habetur

$dx =$

$dx = \frac{2dy\sqrt{y}}{\sqrt{a-4y}}$, æquatio pro Cycloide ordinaria.

Exempl. 3. Sit $L = -a\sqrt{V}$ erit $dL = -\frac{adV}{2\sqrt{V}}$ et per theorema $-\frac{adV}{2V\sqrt{V}}\left(=\frac{dL}{V}\right) = dy$ et integratione $\frac{a}{\sqrt{V}} + C = y$, et si $C = 0$, habetur $V = \frac{a^2}{y^2}$ et deinde per theorema 5. $dx = \frac{dy\sqrt{a^2-y^2}}{y}$, qua constat curvam esse Tractoriam.

THEOREMA VII.

Dicatur ut antea CF F et CG G, et summa tangentium angulorum HCD et BCD, H, et differentia tangentium angulorum HCD et CKB, K, erit $\frac{dF}{H} = dx$ et $\frac{dG}{K} = dy$.

Quoniam $dF\ (= dy + Tdx) = Hdx$ et $dG\ (= \int T dy - x) = Kdy$ provenit $\frac{dF}{H} = dx$ et $\frac{dG}{K} = dy$.

Cor. Quum $F = -\frac{dx\sqrt{1-p^2}}{dp}$ et $G = \frac{dy\sqrt{1-q^2}}{dp}$ provenit divisione $\frac{dF}{FH} = -\frac{dp}{\sqrt{1-p^2}}$ atque $\frac{dG}{GK} = \frac{dq}{\sqrt{1-q^2}}$.

Schol. Auxilio hujus theorematis inveniuntur curvæ ex data relatione inter F et H, G et K, H et p atque K et q. Nam si sit $F = \overset{\prime}{H}$ functioni ipsius H, vel $G = \overset{\prime}{K}$ functioni ipsius K, habetur per theorema $\frac{d\overset{\prime}{H}}{H}\left(=\frac{dF}{H}\right) = dx$ et integratione $\int\frac{d\overset{\prime}{H}}{H} + C = x$ qua H per x datur. Eodem modo $\frac{d\overset{\prime}{K}}{K}\left(=\frac{dG}{K}\right) = dy$ et integratione $\int\frac{d\overset{\prime}{K}}{K} + C = y$ qua K per y obtinetur. Theorema 4. ulterius progredienti viam monstrat ad curvas inveniendas.

Patet

Patet quod curva non ſit algebraica niſi $\int \frac{dH}{H}$ vel $\int \frac{dK}{K}$ obtineantur perfecta integratione.

Exempl. 1. Si ſit $F = \frac{a}{\sqrt{1+H^2}}$ habetur per theorema $-\frac{adH}{\overline{1-H^2}|^{\frac{3}{2}}} \left(= \frac{dF}{H}\right) = dx$, et integratione $\frac{aH}{\sqrt{1-H^2}} + C = -x$ qua $H = -\frac{x}{\sqrt{a^2-x^2}}$, poſita $C = 0$. Per theorema deinde 4. provenit $y = \sqrt{a^2-x^2}$ æquatio pro circulo.

Exempl. 2. Sit $F = \frac{a \cdot \overline{H^3+H^2+6\sqrt{H^2-12}}}{108}$, erit per theorema $\frac{a \cdot \overline{H^2-6+H\sqrt{H^2-12}} \cdot dH}{36\sqrt{H^2-12}} \left(= \frac{dF}{H}\right) = dx$ et integratione facta $\frac{a \cdot \overline{H^2-6+H\sqrt{H^2-12}}}{72} + C = x$, et poſita $C = 0$ habetur $H = \frac{a+12x}{2\sqrt{a}\sqrt{x}}$, unde per theorema 4. prodit $y = \sqrt{ax}$ æquatio pro Parabola Apolloniana.

Exempl. 3. Sit $G = -\frac{a \cdot \overline{4+K^2}}{4}$ erit per theorema $\frac{adK}{2} \left(= \frac{dG}{K}\right) = dy$, et integratione $\frac{aK}{2} + C = y$, et ſi $C = 0$ $K = \frac{2y}{a}$ unde per theorema 4. $dx = \frac{ady}{y}$, qua conſtat curvam eſſe Logarithmicam.

THEOREMA VIII.

Dicatur ut antea productum tangentium angulorum HCD et BCD U, et productum tangentium angulorum HCD et CKB V manentibus reliquis denominationibus erit $\frac{dG}{U-1} = dx$ et $\frac{dF}{1+V} = dy$.

Quoniam

Quoniam $G = \int T dy - x$ erit $dG = T dy - dx$, ſed $T dy = U dx$, unde $dG = \overline{U-1}\, dx$ et $\frac{dG}{U-1} = dx$. Eeodem modo quum $F = y + \int T dx$ erit $dF = dy + T dx$, ſed $T dx = V dy$ quare $dF = \overline{1+V}\, dy$ et $\frac{dF}{1+V} = dy$.

Cor. Quoniam $G = \frac{dy\sqrt{1-q^2}}{dq}$ et $F = -\frac{dx\sqrt{1-p^2}}{dp}$, habetur ſubſtitutione debita $\frac{dG}{G \,.\, U-1} = -\frac{dp}{p}$ et $\frac{dF}{F^2\, 1+V} = \frac{dq}{q}$.

Schol. Ope hujus theorematis indagantur curvæ data relatione inter G et U vel inter F et V. Nam ſi ſit $G = \overset{1}{U}$ functioni quantitatis U vel $F = \overset{1}{V}$ functioni quantitatis V obtinetur per theorema in caſu priori $\frac{d\overset{1}{U}}{U-1}\left(= \frac{dG}{U-1}\right) = dx$ et integratione $\int \frac{d\overset{1}{U}}{U-1} + C = x$, qua U per x habetur; in poſteriori $\frac{d\overset{1}{V}}{1+V}\left(= \frac{dF}{1+V}\right) = dy$ et integratione $\int \frac{d\overset{1}{V}}{1+V} + C = y$, qua V habetur per y. Per theorema deinde 5. curvæ cognoſcuntur.

Datur etiam per Cor. U in p, et V in q, et conſequenter T in p vel q, nam $U = \frac{Tp}{\sqrt{1-p^2}}$ et $V = \frac{Tq}{\sqrt{1-q^2}}$.

Conſtat hinc quod curvæ non ſint algebraicæ niſi $\int \frac{d\overset{1}{U}}{U-1}$ vel $\int \frac{d\overset{1}{V}}{1+V}$ obtineantur integratione abſoluta.

Exempl. 1. Si ſit $G = \frac{a \,.\, \overline{U-3}}{2}$ erit per theorema $\frac{a dU}{2\,\overline{U-1}}\left(= \frac{dG}{U-1}\right) = dx$ et integratione $\log. 1 - U + \log. C = \frac{2x}{a}$ et ſi $C = \frac{a^2}{2}$

log.

log. $\frac{a^2 \cdot \overline{1-U}}{2} = \frac{2x}{a}$ et $\frac{a \cdot \overline{1-U}}{2} = N^{\frac{2x}{a}}$ qua $U = \frac{a^2 - 2N^{\frac{2x}{a}}}{a^2}$. Per theorema deinde 5. habetur $dy = \frac{dx N^{\frac{x}{a}}}{a}$ qua conſtat curvam eſt Logarithmicam.

Exempl. 2. Si ſit $T = \frac{a \cdot \overline{V-1}\sqrt{V+2}}{3\sqrt{3}}$ erit per theorema $\frac{adV}{2\sqrt{3}\sqrt{V+2}} \left(= \frac{dF}{1+V}\right) = dy$ et per integrationem $\frac{a\sqrt{V+2}}{\sqrt{3}} = y$ qua $V = \frac{3y^2 - 2a^2}{a^2}$; et per theorema 5. $dx = \frac{dy\sqrt{y^2-a^2}}{a}$, æquatio ad curvam cujus conſtructio a quadratura hyperbolæ dependet.

THEOREMA IX.

Sint LC et $l\,c$ duæ curvæ eandem habentes Evolutam QD, dicatur radiorum oſculi CD cD conſtans differentia cC b, curvæ $l\,c$ variatio curvaturæ S, ceteriſque ut antea manentibus erit $\frac{dR}{\overline{R-bS}} = -\frac{dp}{\sqrt{1-p^2}}$.

Quoniam radius curvaturæ DH evolutæ ſit $RT = R - bS$, erit $\frac{1}{\overline{R-bS}} = \frac{1}{RT}$, quæ per dR $(= Tdz) = -\frac{RTdp}{\sqrt{1-p^2}}$ multiplicata, moſtrat eſſe $\frac{dR}{\overline{R-bS}} = -\frac{dp}{\sqrt{1-p^2}}$.

Cor. Si ſint ut antea tangens anguli BCD r et ſecans s, habetur $\frac{dR}{\overline{R-bS}} = -\frac{dr}{1+r^2}$ et $\frac{dR}{\overline{R-bS}} = -\frac{ds}{S\sqrt{S^2-1}}$.

Schol. Subſidio hujus theorematis invenire licet curvas, data relatione inter S et R vel inter S et T nam $\frac{S}{T} = \frac{R}{\overline{R-b}}$. Itaque ſi

ponatur

ponatur $S = \overset{1}{R}$ functioni radii curvedinis R, erit $\frac{dR}{R - b\overset{1}{R}} \left(= \frac{dR}{R - bS}\right)$ $= -\frac{dp}{\sqrt{1-p^2}}$, et integratione $\int \frac{dR}{R - b\overset{1}{R}} + C = -\int \frac{dp}{\sqrt{1-p^2}}$. Sit $\int \frac{dR}{R - b\overset{1}{R}} + C = f$ et N logarithmorum basi habetur $\sqrt{1-p^2} = \frac{N^{f\sqrt{-1}} - N^{-f\sqrt{-1}}}{2\sqrt{-1}}$ et $p = \frac{N^{f\sqrt{-1}} + N^{-f\sqrt{-1}}}{2}$ functionibus quantitatis R, quibus R per p exprimi potest. Per theorema igitur 1. curvas internoscere valemus.

Si $R = \overset{1}{S}$ functioni quantitatis S habetur $\frac{d\overset{1}{S}}{\overset{1}{S} - bS} \left(= \frac{dR}{R - bS}\right)$ $= -\frac{dp}{\sqrt{1-p^2}}$, et integratione $\int \frac{d\overset{1}{S}}{\overset{1}{S} - bS} + C = -\int \frac{dp}{\sqrt{1-p^2}}$, posita $\int \frac{d\overset{1}{S}}{\overset{1}{S} - bS} + C = g$, erit $\sqrt{1-p^2} = \frac{N^{g\sqrt{-1}} - N^{-g\sqrt{-1}}}{2 - \sqrt{1}}$ et $p = \frac{N^{g\sqrt{-1}} + N^{-g\sqrt{-1}}}{2}$ quibus S per p datur. Per theoremata Partis I. invenire licet curvas omnes eandem evolutam habentes.

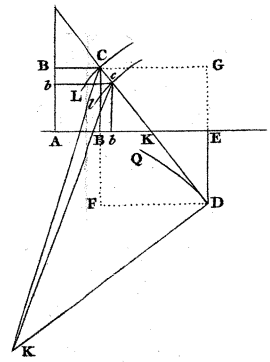
Hinc videtur, quod curvæ non sint algebraicæ nisi $\int \frac{dR}{R - b\overset{1}{R}}$ vel $\int \frac{d\overset{1}{S}}{\overset{1}{S} - bS}$ per circuli rectificationem obtineatur.

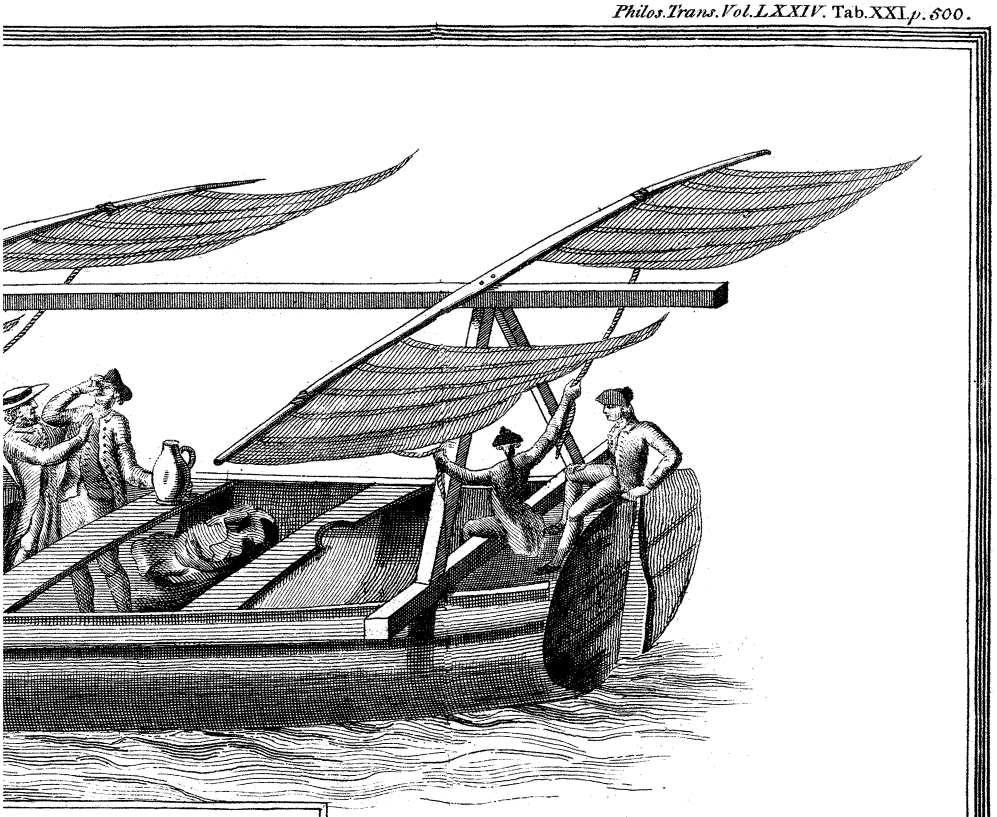
Exempl. 1. Si sit $S = \frac{2R}{\sqrt{a} \cdot \sqrt{R-a}}$ supposita $b = a$, erit per theorema $\frac{dR\sqrt{a}}{2R\sqrt{R-a}} \left(= \frac{dR}{R - bS}\right) = -\frac{dp}{\sqrt{1-p^2}}$ et integratione $\int \frac{dR\sqrt{a}}{2R\sqrt{R-a}} + C = -\int \frac{dp}{\sqrt{1-p^2}}$, si vero arcus ille constans $C = 0$ erit $\sqrt{1-p^2} = \frac{\sqrt{R-a}}{\sqrt{R}}$ qua $R = ap^2$, et per Cor. 1. Theor. 1. habetur $dy = \frac{adx}{\sqrt{x^2 - a^2}}$, æquatio pro Catenaria.

Exempl. 2. Sit $S = \frac{5a^2 + R^2}{a \cdot a - 5R}$, posita $b = \frac{a}{5}$ erit per theorema $-\frac{adR}{a^2+R^2} \left(= \frac{dR}{R - bS}\right) = -\frac{dp}{\sqrt{1-p^2}}$ et facta integratione $-\int \frac{adR}{a^2+R^2} + C = -\int \frac{dp}{\sqrt{1-p^2}}$, qua si $C = 0$, habetur $\sqrt{1-p^2} = \frac{R}{\sqrt{a^2+R^2}}$ et $R = \frac{a\sqrt{1-p^2}}{p}$. Per theorema 1. $dx = \frac{dy\sqrt{a^2-y^2}}{y}$ qua constat curvam esse Tractoriam.

Fig. 1.

Fig. 2.





Basire S.

Fig. 1.

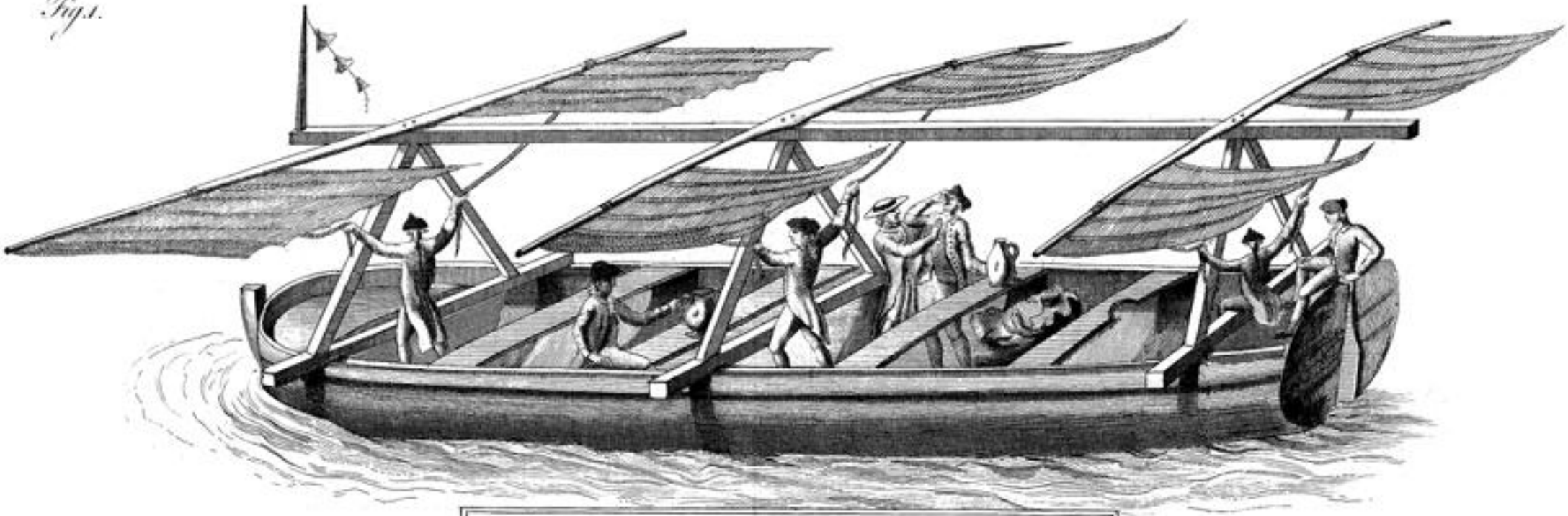
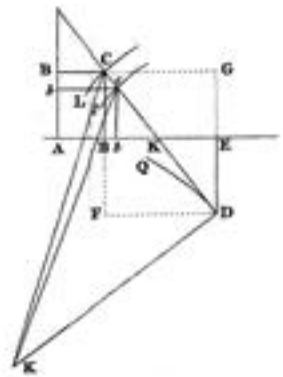


Fig. 2.



Basire S.