

XXVIII. *On the Summation of Series, whose general Term is a determinate Function of z the Distance from the first Term of the Series.* By Edward Waring, M. D. Lucasian Professor of the Mathematics at Cambridge, and Fellow of the Societies of London and Bononia.

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P R O B L E M.

THE sum S being given, to find a series of which it is the sum.

1. Reduce the sum S into a converging series, proceeding according to the dimensions of any small quantities, and it is done. For example: let any algebraical function S of an unknown or small quantity x be assumed, reduce it into a converging series proceeding according to the dimensions of x , and there results a series whose sum is S . 2. Let $A, B, C, \&c.$ be algebraical functions of x ; reduce the $\int Ax, \int Bx, \int Cx, \&c.$ into a converging series, proceeding according to the dimensions of x , and the problem is done.

It is always necessary to find the values of x , between which the abovementioned serieses converge. Reduce the algebraical function S in the first example, and the algebraical functions $A, B, C, \&c.$ in the second into their lowest terms; and in such a manner, that the quantities contained in the numerator and denominator may have no denominator: make the deno-

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minator in the first example, and the denominator in the second, and every distinct irrational quantity contained in them respectively $=0$; and also every distinct irrational quantity contained in the numerators $=0$. Suppose α the least root affirmative or negative (but not $=0$) of the abovementioned resulting equations; then a series ascending according to the dimensions of x will always converge, if the value of x is contained between α and $-\alpha$; but if x be greater than α or $-\alpha$, the abovementioned series will diverge. Let π be the greatest root of the abovementioned resulting equations; then a series descending according to the reciprocal dimensions of x will converge, if x be greater than $\pm \pi$; but, if less, not. When impossible roots $a \pm b\sqrt{-1}$ are contained in the equations, an ascending series will converge, if x be less than the least root $\pm \alpha$, and $\pm (a-b)$ and $\pm (a+b)$; or more generally, if x be less than the least root $\pm \alpha$, and x^{n+1} at an infinite distance n , be infinitely less than

$$\frac{2a^n - 2 \cdot n \cdot \frac{n-1}{2} a^{n-2} b^2 + 2 \cdot n \cdot \frac{n-1}{2} \cdot \frac{n-2}{3} \cdot \frac{n-3}{4} a^{n-4} b^4 - \&c.}{(a^2 + b^2)^n} :$$

a descending series will always converge, when x is greater than the greatest root of the resulting equations; and x^{n-1} , when n is infinite, is infinitely greater than $(a+b)^n$ and $(a-b)^n$; or more generally than $2a^n - 2n \cdot \frac{n-1}{2} a^{n-2} b^2 + 2n \cdot \frac{n-1}{2} \cdot \frac{n-2}{3}$

$$\frac{n-3}{4} a^{n-4} b^4 - \&c.$$

This follows from Caput 3. of the *Meditationes Algebraicæ*.

Cor. It appears from hence, that, if the ascending series converges, the descending will diverge; and, *vice versa*, if the descending converges, the ascending will diverge, unless all the roots of the above-mentioned resulting equations may be deemed

of

of equal magnitude, as $+a$ and $-a$, $a\sqrt{-1}$, &c. and $x=a$; in which case sometimes both series may become the same converging series, &c.

When x , in the preceding cases, is equal to the least or greatest root, the series will sometimes converge, and sometimes not, as is shewn in the above-mentioned chapter. Whether the sum of a series, whose general term is given, can be found or not, will in many cases appear from the law of the multinomial and other more general serieses.

2. There are serieses which always converge, whatever may be the value of x ; as, for example, the series $1 + \frac{1}{2 \cdot 3} x + \frac{1}{2 \cdot 3 \cdot 4 \cdot 5} x^2 + \&c.$ or $1 + \frac{x}{2^2} + \frac{x^2}{3^3} + \frac{x^3}{4^4} + \&c.$ &c. always converge, whatever may be the value of x ; but it may be observed, that these serieses never arise from the expansion of algebraical functions of x , or the before-mentioned fluents; but, in a few cases, they may from fluxional equations. There are also serieses which never converge as $1 + 1 \cdot 2x + 1 \cdot 2 \cdot 3x^2 + 1 \cdot 2 \cdot 3 \cdot 4x^3 + \&c.$ to which the preceding remark may be applied.

3. In the year 1757 some papers, which contained the first edition of my *Meditationes Algebraicæ*, were sent to the Royal Society, in which was contained the following rule, viz. let S be a given function of the quantity x , which expand into a series $(a + bx^m + cx^{2m} + \&c.)$ proceeding according to the dimensions of x ; in the quantity S , for x^m write αx^m , βx^m , γx^m , &c. where α , β , γ , &c. are roots of the equation $x^n - 1 = 0$; and let the resulting quantities be A , B , C , D , &c. then will $\frac{A+B+C+D+\&c.}{n}$ be equal to the sum of the first, $2n+1$, $3n+1$, &c. terms in infinitum. This method, in the preface to the

last edition of the *Meditationes Algebraicæ*, is rendered more correct and general.

4. Let the sum of a series required be expressed by a function of a quantity z , the distance from the first term of the series, then will the general term be the difference between the two successive sums generally expressed.

5. Let the general term be an algebraical function of z :

if, let it be $\frac{az^m + bz^{m-1} + cz^{m-2} + \&c.}{z+e \cdot z+e+1 \cdot z+e+2 \dots z+e+n-1} = T$, where m and n are whole numbers; and m (if the sum of an infinite series of terms is required) less than n by two or more:

then the general term $\frac{az^m + bz^{m-1} + \&c.}{z+e \cdot z+e+1 \cdot z+e+2 \dots z+e+n-1} =$

$$\frac{\gamma}{z+e \cdot z+e+1} + \frac{\delta}{z+e \cdot z+e+1 \cdot z+e+2} + \frac{\epsilon}{z+e \cdot z+e+1 \cdot z+e+2 \cdot z+e+3} + \&c. \dots \frac{\theta}{z+e \cdot z+e+1 \dots z+e+n-1}$$

whence if

$z+e+2 \cdot z+e+3 \cdot z+e+4 \dots z+e+n-1 = z^{n-2} + Az^{n-3} + Bz^{n-4} + \&c.$;
 $z+e+3 \cdot z+e+4 \cdot z+e+5 \dots z+e+n-1 = z^{n-3} + A'z^{n-4} + B'z^{n-5} + \&c.$;
 $z+e+4 \cdot z+e+5 \dots z+e+n-1 = z^{n-4} + A''z^{n-5} + B''z^{n-6} + \&c.$ and so on;
 then, if $m = n-2$, will $\gamma = a$, $\delta = b - \gamma A$, $\epsilon = c - \delta A' - \gamma B$, $\zeta = d - \epsilon A'' - \delta B' - \gamma C$, &c.; whence the integral *in infinitum*, or sum of the infinite series, will be $\frac{\gamma}{z+e} + \frac{\delta}{2 \cdot z+e \cdot z+e+1} +$

$$\frac{\epsilon}{3 \cdot z+e \cdot z+e+1 \cdot z+e+2} + \&c.$$

The reduction of the general term T into quantities of the before given formulæ was published in the *Meditationes*, printed in the year 1774. It was before reduced into formulæ of the same kind nearly by Mr. NICHOLE in the *Paris Acts*.

2d, Let the general term be $T' =$

$$\frac{az^b + bz^{b-1} + cz^{b-2} + \&c.}{z+e \cdot z+e+1 \cdot z+e+2 \dots z+e+n-1 \times z+f \cdot z+f+1 \cdot z+f+2 \dots z+f+m-1 \times z+g \cdot z+g+1 \cdot z+g+2 \dots z+g+l-1 \times \&c.}$$

where

where b is a whole number less than $n+m+l+\&c.$ (if it be greater, then the fraction can easily be reduced into a rational quantity $ax^{b-n-m-l-\&c.} + \&c.$ and a fraction of the before-mentioned kind); then will $T' = \left(\frac{a}{z+e} + \frac{a'}{z+f} + \frac{a''}{z+g} + \&c. \right) +$

$$\left(\frac{\beta}{z+e.z+e+1} + \frac{\beta'}{z+f.z+f+1} + \frac{\beta''}{z+g.z+g+1} + \&c. \right) +$$

$$\left(\frac{\gamma}{z+e.z+e+1.z+e+2} + \frac{\gamma'}{z+f.z+f+1.z+f+2} + \frac{\gamma''}{z+g.z+g+1.z+g+2} \right.$$

$$\left. + \&c. \right) \dots \left(\frac{X}{z+e.z+e+1 \dots z+e+n-1} + \frac{X'}{z+f.z+f+1 \dots z+f+m-1} \right.$$

$$\left. + \frac{X''}{z+g.z+g+1 \dots z+g+l-1} + \&c. \right); \text{ whence its integral in in-}$$

finitum, that is, the sum of the infinite series can be found when $\alpha=0$, $\alpha'=0$, $\alpha''=0$, &c.; and consequently b not greater than $n+m+l+\&c.-2$; otherwise not. If b is not greater than $n+m+l+\&c.-2$, then will $\alpha+\alpha'+\alpha''+\&c.=0$, for else the sum would be infinite.

Let the number of quantities ($e, f, g, \&c.$) be r , then from r independent integrals of a series, whose term is T' ; or from $(r - 1)$ independent sums of infinite serieses, whose term is T' ; that is, where b is not greater than $n + m + l + \&c. - 2$; can be deduced the sum of all infinite serieses of the before-mentioned formulæ, whose general term is T' .

If any factors are deficient in the denominator, as suppose the term to be $z + e \times z + e + 3 \times z + e + n - 1$; multiply the numerator and denominator by the deficient factors, viz. by $z + e + 1 \cdot z + e + 2 \times z + e + 4 \cdot z + e + 5 \dots z + e + n - 2$, and it acquires the preceding formula; and so in the following examples.

3d, Let the denominator be $\frac{x+e^\pi}{x+e+2} \times \frac{x+e^\pi}{x+e+1} \times \frac{x+e^\pi}{x+e+n-1} \times \frac{x+e^{\pi'}}{x+e+\mu} \times \frac{x+e^{\pi'}}{x+e+\mu+1} \times$

$\overline{x+e+\mu+2}^{\pi'} \times \&c. \times \overline{x+f}^{\rho} \times \overline{x+f+1}^{\rho} \times \overline{x+f+2}^{\rho} \dots \times$
 $\overline{x+f+m-1}^{\rho} \times \&c. = D$, where $\pi, \pi', \rho, \&c.$; $\mu, \&c.$ are
 whole numbers; and the general term is $\frac{az^b + bz^{b-1} + cz^{b-2} + \&c.}{D}$

$= T''$; then, if the dimensions of z in the numerator be less than its dimensions in the denominator, will $T'' =$

$$\left(\frac{\alpha}{z+e} + \frac{\alpha'}{(z+e)^2} + \frac{\alpha''}{(z+e)^3} \dots \frac{\alpha^{\pi+\pi'-1}}{(z+e)^{\pi+\pi'}} + \frac{\beta}{z+f} + \frac{\beta'}{(z+f)^2} + \frac{\beta''}{(z+f)^3} \dots \right.$$

$$\left. \frac{\beta^{\rho-1}}{(z+f)^{\rho}} + \&c. \right) + \left(\frac{\gamma}{z+e \cdot z+e+1} + \frac{\theta}{z+f \cdot z+f+1} + \&c. \right) + \&c.$$

and in general there will be included all terms of the formulæ,

$$\frac{A(\overline{z+e+i}^p - \overline{z+e}^p)}{(z+e)^p \cdot (z+e+1)^p \dots (z+e+i)^p},$$

$$\frac{B((z+f+i')^{p'} - (z+f)^{p'})}{(z+f)^{p'} \cdot (z+f+1)^{p'} \dots (z+f+i')^{p'}}, \&c.$$

$$\frac{C((z+e+\mu+i'')^{p''} - (z+e+\mu)^{p''})}{(z+e+\mu)^{p''} \cdot (z+e+\mu+1)^{p''} \dots (z+e+\mu+i'')^{p''}}, \&c.$$

where $A, B, C, \&c.$ $\alpha, \alpha', \&c.$ $\beta, \beta', \&c.$ $\gamma, \theta, \&c.$ denote invariable quantities; and $p, p', p'', \&c.$ are whole numbers not greater than $\pi, \rho, \pi', \&c.$ respectively; and $i, i', i'', \&c.$ are whole numbers not greater than $n-1, m-1, \&c.$

If all the quantities $\alpha, \alpha', \alpha'', \&c.$ $\beta, \beta', \beta'', \&c.$ $\&c.$ are $=0$, the sum of the series can be expressed in finite terms of the quantity z , otherwise not; and also if b be less than the dimensions of z in the denominator by two or more, then will $\alpha + \beta + \&c. = 0$, otherwise the sum would be infinite.

From $\pi + \pi' + \rho + \&c. - 1$ independent sums of infinite series of this kind can be deduced the sums of all infinite series of the same kind.

This method may be extended to infinite series, in which exponentials as e^z are contained, which will easily be seen from some subsequent propositions; but in my opinion the subsequent method of finding the sum of serieses is to be preferred to the preceding one, both for its generality and facility.

6. 1. Let the general term be $(az^b + bz^{b-1} + cz^{b-2} + \&c.) \times (z+e)^{-1} \cdot (z+e+1)^{-1} \cdot (z+e+2)^{-1} \dots (z+e+n-1)^{-1}$; where b is a whole number less than n by two or more, when the sum of an infinite series is required.

Assume for the sum the quantity $(z+e)^{-1} \cdot (z+e+1)^{-1} \cdot (z+e+2)^{-1} \dots (z+e+n-2)^{-1} \times (az^{b'} + \beta z^{b'-1} + \gamma z^{b'-2} + \&c.)$; find the difference between this sum and its successive one $(z+e+1)^{-1} \cdot (z+e+2)^{-1} \cdot (z+e+3)^{-1} \dots (z+e+n-1)^{-1} \times (\alpha z^{b'} + \beta z^{b'-1} + \gamma z^{b'-2} + \&c.)$, which will be $-(z+e)^{-1} \cdot (z+e+1)^{-1} \cdot (z+e+2)^{-1} \dots (z+e+n-1)^{-1} \times (\alpha z^{b'} + \beta z^{b'-1} + \gamma z^{b'-2} + \&c.) - z+e+n-1 \times (az^{b'} + \beta z^{b'-1} + \gamma z^{b'-2} + \&c.) = \overline{b'-n+1} \alpha z^{b'} + \&c.$; then make the terms of this difference equal to the correspondent terms of the given quantity $az^b + bz^{b-1} + \&c.$ and there result $b' = b$, $-\overline{b-n+1} \times \alpha = a$, and consequently $\alpha = \frac{-a}{b-n+1}$, &c.

2. Let the general term be $(z+e)^{-1} \cdot (z+e+1)^{-1} \cdot (z+e+2)^{-1} \dots (z+e+n-1)^{-1} \times (z+f)^{-1} \cdot (z+f+1)^{-1} \cdot (z+f+2)^{-1} \dots (z+f+m-1)^{-1} \times (az^b + bz^{b-1} + cz^{b-2} + \&c.)$. Assume the quantity $(z+e)^{-1} \cdot (z+e+1)^{-1} \dots (z+e+n-2)^{-1} \times (z+f)^{-1} \cdot (z+f+1)^{-1} \cdot (z+f+2)^{-1} \dots (z+f+m-2)^{-1} \times (\alpha z^{b'} + \beta z^{b'-1} + \gamma z^{b'-2} + \&c.)$ for the sum of the series sought; and thence deduce the general term, which suppose equal to the given general term, and from equating their corresponding parts easily can be deduced the index b' and co-efficients α, β, γ , &c. and consequently the sum of the series sought.

3. Let

3. Let the general term reduced to its lowest dimensions be

$$\frac{z+e^{-\pi}}{z+e+1^{-\pi}} \dots \frac{z+e+n-1^{-\pi}}{rz+f^{-e}} \times \frac{rz+f+r^{-e}}{rz+f+2r^{-e}} \dots \frac{rz+f+m-1r^{-e}}{z+g^{-\sigma}} \times \frac{z+g+1^{-\sigma}}{z+g+l-1^{-\sigma}} \times \&c. \times (az^{b'} + bz^{b'-1} + cz^{b'-2} + \&c.).$$

If it be required to reduce the term $\frac{rz+f}{z+g+l-1^{-\sigma}}$, &c. to a conformity with the rest, for $\frac{rz+f}{z+g+l-1^{-\sigma}}$, &c. substitute $z + \frac{f}{r} \times r^{-e}$, &c. and it is done. Assume for the integral or sum the quantity

$$S = \frac{z+e^{-\pi}}{z+e+1^{-\pi}} \dots \frac{z+e+n-2^{-\pi}}{rz+f^{-e}} \cdot \frac{rz+f+r^{-e}}{rz+f+2r^{-e}} \dots \frac{rz+f+m-2r^{-e}}{z+g^{-\sigma}} \times \frac{z+g+1^{-\sigma}}{z+g+l-2^{-\sigma}} \times \&c. \times (az^{b'} + \beta z^{b'-1} + \&c.) = S,$$

find its successive sum by writing $z+1$ for z in the sum S , and let the quantity resulting be S' ; then will the general term be $S - S'$, which equate to the given general term, that is, their correspondent quantities; and thence may be deduced the index b' and co-efficients α , β , &c.; and consequently the sum sought. If the series does not terminate, then the sum will be expressed by a series proceeding *in infinitum*, according to the reciprocal dimensions of z .

From $\pi + e + \sigma + \&c. - 1$ independent integrals of the above-mentioned kind can be deduced the integrals of all quantities of the same kind; that is, where b is any whole affirmative number whatever, and the co-efficients a , b , c , &c. are any how varied.

If any factor $z+g$ in the denominator, &c. has no other $z+g+l-1$, which differs from it by a whole number $l-1$; or the factor $rz+f$ has no correspondent factor $rz+f+mr$, where m is a whole number; then the integral of the above-mentioned series cannot be expressed in finite terms of the quantity z . In like manner, if the dimensions of z in the numerator are less than

than its dimensions in the denominator by unity, then the integral of the general term cannot be expressed by a finite algebraical function of z . If the number of terms to be added be infinite, it is well known that the sum in this case will be infinite.

It may be observed, that in finding the sum of a series, whose general term is given, all common divisors of the numerator and denominator must be rejected, otherwise serieses may appear difficult to be summed, which are very easy: for example, let the series be

$$\frac{1}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} + \frac{4}{4 \cdot 5 \cdot 6 \cdot 7 \cdot 8} + \frac{9}{7 \cdot 8 \cdot 9 \cdot 10 \cdot 11} + \&c.$$

$$= \frac{1}{3} \left(\frac{1}{1 \cdot 2 \cdot 4 \cdot 5} + \frac{2}{4 \cdot 5 \cdot 7 \cdot 8} + \frac{3}{7 \cdot 8 \cdot 10 \cdot 11} + \&c. \right),$$

whose general term is $\frac{z+1}{3z+1 \cdot 3z+4 \times 3z+2 \cdot 3z+5}$; and by assuming, as

is before taught, $\frac{1}{3z+1} \times \frac{1}{3z+2} \times \alpha$ for the sum sought;

and finding its general term $\frac{1}{3z+1} \times \frac{1}{3z+4} \times \frac{1}{3z+2} \times$

$\frac{1}{3z+5} \times 18z+1 \times \alpha$, which equating to the general term

given, there results $18\alpha=1$, and the sum sought $= \frac{1}{18} \times$

$$\frac{1}{3z+1 \cdot 3z+2}.$$

Ex. 2. Let the series be $\frac{1}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} + \frac{14}{5 \cdot 6 \cdot 7 \cdot 8 \cdot 9} +$

$$\frac{55}{9 \cdot 10 \cdot 11 \cdot 12 \cdot 13} + \frac{140}{13 \cdot 14 \cdot 15 \cdot 16 \cdot 17} + \&c. = \frac{1}{24} \left(\frac{1}{1 \cdot 5} + \frac{1}{5 \cdot 9} + \right.$$

$$\left. \frac{1}{9 \cdot 13} + \frac{1}{13 \cdot 17} + \&c. \right),$$

of which the general term is $\frac{1}{24} \times \frac{1}{4z+1 \cdot 4z+5}$; and consequently the sum deduced is $\frac{1}{24} \times$

$$\frac{1}{4} \times \frac{1}{4z+1}.$$

These are serieses given by Mr. DE MOIVRE, and esteemed by Dr. TAYLOR *altioris indaginis*.

Some other writers have made some serieses to appear more difficult to be summed, by not reducing them to their lowest terms.

7. Having given the principles of a general method of finding the sum of a series, when its general term can be expressed by algebraical, and not exponential, functions of z , the distance from the first term of the series; it remains to perform the same when exponentials are included.

1. Let S the sum be any algebraical function of z multiplied into $e^z = x^z$; then will the general term be $Se^z - eS'e^z = (S - eS')e^z$; whence, from the general term Te^z being given, assume quantities in the same manner (with the same denominator, &c.) as when no exponential was involved, which multiplied into e^z , suppose to be the sum; from the sum find its general term, and equate it to the given one by equating their correspondent co-efficients, and it is done.

Ex. Let the general term be $\frac{z+2}{2z+1 \cdot 2z+3} \times e^{z+1}$: assume for the sum sought $\frac{\alpha}{2z+1} \times e^{z+1}$, whence the general term is $\left(\frac{\alpha}{2z+1} - \frac{\alpha e}{2z+3}\right) e^{z+1} = \frac{2\alpha(1-e)z+3\alpha-\alpha e}{2z+1 \cdot 2z+3} \times e^{z+1}$; equate it to the given term, and there results $2\alpha(1-e) = 1$ and $3\alpha - \alpha e = 2$, and consequently $e = \frac{1}{3}$ and $\alpha = \frac{3}{4}$, if the series can be summed.

The same observation, *viz.* that if any factor in the denominator or irrational quantity have no other correspondent to it; for example, if the factor be $z+g$, and there is no correspondent one $x+g+n$, where n is a whole number, then its integral cannot be expressed by a finite algebraical function of z .

In the same manner may the sums be found, when the terms are exponentials of superior orders; for the exponential, irra-

tional, &c. quantities in the denominators of the sums may be easily deduced from the preceding principles; and thence, by proceeding as is before taught, the sum required.

The principles of all these cases have been given in the *Meditationes*.

8. Mr. JAMES BERNOULLI found summable serieses by assuming a series V, whose terms at an infinite distance are infinitely little, and subtracting the series diminished by any number (*l*) of terms from the series itself, &c.

It is observed in the *Meditationes*, that if $T(m)$, $T(m+n)$, $T(m+n+n')$, $T(m+n+n'+n'')$, &c. be the terms at m , $m+n$, $m+n+n'$, $m+n+n'+n''$, &c. distances from the first, and $aT(m) + bT(m+n) + cT(m+n+n') + dT(m+n+n'+n'') + \&c.$ be the general term, it will be summable, when $a+b+c+d+\&c.=0$; the sum of the series will be $a(T(m) + T(m+1) + T(m+2) + \dots + T(m+n+n'+n''+\&c.-1)) + b(T(m+n) + T(m+n+1) + T(m+n+2) + \dots + T(m+n+n'+n''+\&c.-1)) + c(T(m+n+n') + T(m+n+n'+1) + \dots + T(m+n+n'+n''+\&c.-1)) + \&c.=H$. If the sum $a+b+c+d+\&c.$ be not $=0$, and the series $T(m) + T(m+1) + T(m+2) + \&c.$ in *infinitum* be a converging one $=S$, then will the sum of the resulting series be $(a+b+c+d+\&c.)S - (b+c+d+\&c.) (T^m \dots + T^{m+n-1}) - (c+d+\&c.) (T^{m+n} \dots + T^{m+n+n'-1}) - (d+\&c.) (T^{m+n+n'} \dots + T^{m+n+n'+n''-1}) - \&c.$

8. 2. Let the series V consist of terms, which have only one factor in the denominator, and its numerator $=1$; that is, let the general term be $\frac{1}{rz+e}$, and the series consequently

$\frac{1}{e} + \frac{1}{r+e} + \frac{1}{2r+e} + \&c.=V$; from the before-mentioned addition

or subtraction there follows $\frac{a}{rz+e} + \frac{b}{rz+r+e} + \frac{c}{rz+2r+e} + \&c.=$

$\frac{\alpha z^m + \beta z^{m-1} + \gamma z^{m-2} + \&c.}{rz + e . rz + r + e . rz + 2r + e . \&c.}$; where m is not greater than the number (N) of factors in the denominator diminished by unity. From $\alpha, \beta, \gamma, \&c.$ r and e being given, easily can be acquired by simple equations, or known theorems, the required co-efficients $a, b, c, \&c.$ If $m = N - 1$ and α and $a + b + c + d + \&c. = 0$, then the sum of the series resulting will be finite.

8. 3. If the terms of the series assumed $\frac{1}{e} - \frac{1}{r+e} + \frac{1}{2r+e} - \frac{1}{3r+e} + \&c.$ be alternately affirmative and negative; then by the preceding case find $\frac{\alpha z^m + \beta z^{m-1} + \gamma z^{m-2} + \&c.}{rz + e . rz + r + e . rz + 2r + e + \&c.} = \frac{a}{rz + e} + \frac{b}{rz + r + e} + \frac{c}{rz + 2r + e} + \&c.$ Where the terms of the resulting series are alternately affirmative and negative, let the two subsequent terms be supposed $\frac{\alpha z^m + \beta z^{m-1} + \gamma z^{m-2} + \&c.}{rz + e . rz + r + e \dots rz + n - 1r + e}$
 $= \frac{a}{rz + e} + \frac{b}{rz + r + e} + \&c.$ and $\frac{\alpha z^{n-m} + \beta z^{n-m-1} + \gamma z^{n-m-2} + \&c.}{rz + r + e . rz + 2r + e \dots rz + nr + e} = \frac{a}{rz + r + e} + \frac{b}{rz + 2r + e} + \&c.$ of which the one is affirmative and the other negative: reduce the resulting series to an affirmative one by subtracting the subsequent term from its preceding, and it becomes $\frac{(rz + nr + e)(\alpha z^m + \beta z^{m-1} + \&c.) - (rz + e)(\alpha z^{n-m} + \beta z^{n-m-1} + \&c.)}{rz + e . rz + r + e . rz + 2r + e \dots rz + nr + e}$
 $= \frac{n - mr \alpha z^m + \&c.}{rz + e . rz + r + e \dots rz + nr + e} = \frac{a}{rz + e} + \frac{b - a}{rz + r + e} + \&c.$ In this case, since two terms are added into one, the distance from the first term of the series will be $\frac{z}{2}$, which suppose $= w$; and write $2w$ for z in the above-mentioned term, and there results $\frac{n - mr \alpha z^m + \&c.}{rz + e . rz + r + e \dots rz + nr + e} = \frac{n - mr \alpha \times 2^m w^m + \&c.}{2rw + e . 2rw + r + e \dots 2rw + nr + e} = \frac{a}{2rw + e}$

$\frac{a}{2rw+e} + \frac{b-a}{2rw+r+e} + \&c.$; whence the sum of any series, whose general term is $\frac{a'w^m + b'w^{m-1} + \&c.}{2rw+e \cdot 2rw+r+e \dots 2rw+nr+e}$, where m is a whole number less than n by two or more, and w the distance from the first term of the series can be found from the sum of the series $\frac{1}{e} - \frac{1}{r+e} + \frac{1}{2r+e} - \frac{1}{3r+e} + \&c.$

9. Let there be two serieses $\frac{1}{e} + \frac{1}{e+r} + \frac{1}{e+2r} + \&c. = S$ and $\frac{1}{f} + \frac{1}{f+r} + \frac{1}{f+2r} + \frac{1}{f+3r} + \&c. = S'$, whose general terms are respectively $+\frac{1}{e+rz}$ and $+\frac{1}{f+rz}$; then from the sum of these two serieses can be collected the sum of any series, whose general term is

$$\frac{\alpha z^m + \beta z^{m-1} + \&c.}{rz+e \cdot rz+e+r \cdot rz+e+2r \dots rz+n-1r+e \times rz+f \cdot rz+r+f \dots rz+f+m-1r}$$

$$= \frac{a}{rz+e} + \frac{b}{rz+e+r} + \frac{c}{rz+e+2r} \dots + \frac{\lambda}{rz+n-1r+e} + \frac{a'}{rz+f} + \frac{b'}{rz+r+f}$$

$$+ \frac{c'}{rz+2r+f} \dots + \frac{\mu'}{rz+m-1r+f};$$

where $e-f$ is not a whole number. Let $a+b+c \dots +\lambda=0$, and $a'+b'+c' \dots +\mu'=0$, then the sum will be $a \left(\frac{1}{rz+e} + \frac{1}{rz+r+e} \dots + \frac{1}{rz+n-2r+e} \right) + b \left(\frac{1}{rz+e+r} + \frac{1}{rz+e+2r} \dots + \frac{1}{rz+n-2r+e} \right) + c \left(\frac{1}{rz+e+2r} + \frac{1}{rz+e+3r} + \dots + \frac{1}{rz+n-2r+e} \right) + \&c.$

$$+ a' \left(\frac{1}{rz+f} + \frac{1}{rz+r+f} \dots + \frac{1}{rz+m-2r+f} \right) + b' \left(\frac{1}{rz+r+f} + \dots + \frac{1}{rz+m-2r+f} \right) + \&c.$$

2. If the serieses are $\frac{1}{e} - \frac{1}{e+r} + \frac{1}{e+2r} - \&c.$ and $\frac{1}{f} - \frac{1}{f+r} + \frac{1}{f+2r} - \&c.$; then from the sum of these two serieses can be collected by the principles given above the sum of any series, whose general term is

$$\alpha z^m + \beta z^{m-1} + \gamma z^{m-2} + \&c.$$

$$2rz + e. 2rz + r + e. 2rz + 2r + e \dots 2rz + n-1 + e \times 2rz + f. 2rz + r + f. 2rz + 2r + f \dots 2rz + m-1 + f$$

The same principle may be applied to find the sum of any series of the abovementioned sort, in whose denominator are contained other factors, $rz + g$, $rz + g + r$, &c. &c.; or $2rz + g$, $2rz + g + r$, $2rz + g + 2r$, &c. Like propositions may be deduced from serieses, in which r and r' , &c. and the factors $rz + e$ and $r'z + g$, &c. denote different quantities.

10. An apparently more general method may be given from assuming a series or serieses as before; and adding every two, three, four, &c. (n) successive terms together for terms of a new series beginning from the first, second, third, &c. n^{th} term; and in general adding together two, three, &c. n successive general terms; and in their sum writing for z the distance from the first term of the series $2z + a$, $3z + a$, &c. $nz + a$; there will result the general term of a series not to be found from the above-mentioned addition.

Ex. Let the series assumed be $\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \&c.$ in infinitum, of which the general term beginning from the first is $\frac{1}{z+1}$; add three successive general terms $\frac{1}{z+1} + \frac{1}{z+2} + \frac{1}{z+3} = \frac{3z^2 + 12z + 11}{z+1 \cdot z+2 \cdot z+3}$; in this term for z write $3z$, and there results $\frac{27z^2 + 36z + 11}{3z+1 \cdot 3z+2 \cdot 3z+3}$. In the same manner, if the beginning is instituted from the second or third term of the given series, the terms resulting will be $\frac{3z^2 + 18z + 26}{z+2 \cdot z+3 \cdot z+4}$ and $\frac{3z^2 + 24z + 47}{z+3 \cdot z+4 \cdot z+5}$. In these terms for z write $3z$, and there result $\frac{27z^2 + 54z + 26}{3z+2 \cdot 3z+3 \cdot 3z+4}$ and $\frac{27z^2 + 72z + 47}{3z+3 \cdot 3z+4 \cdot 3z+5}$.

If

If the terms of the given series are alternately affirmative and negative, the terms of the resulting series will be alternately affirmative and negative, if n be an odd number; otherwise its terms will be all affirmative. The sum of this series will be finite or infinite, as the sum of the series $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \&c.$ is finite or infinite; but from it, by the preceding method of addition or subtraction of Mr. BERNOULLI's, or a like method applied to more serieses, may be found the sums of different finite serieses.

It may be observed, that from Mr. BERNOULLI's addition or subtraction can never be deduced the serieses which arise from this method; for, by his method, the denominator can never have any factors but what are contained in the denominators of the given series, viz. (in the series $\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \&c.$), $z + l$, where l is a whole number; but by this method are introduced into the denominator the factors $2z + l$, $3z + l$, $\&c.$ and $nz + l$, or which may be reduced to the same $(z + \frac{l}{n}) \times n$.

If n successive general terms of the serieses arising from Mr. BERNOULLI's addition or subtraction be added together, and in the quantity thence arising for z the distance from the first term of the series be substituted nz , there will be produced serieses of the above-mentioned formula.

II. Multiply two converging serieses $a + bx + cx^2 + dx^3 + \&c.$ $= S$ and $\alpha + \beta x + \gamma x^2 + \&c. = V$, or find any rational and integral function of them, and the series resulting will be finite and $= S \times V$, $\&c.$ Let $\alpha + \beta x + \gamma x^2 + \&c. x^m = V$ be finite, and the resulting series will be finite and $= S \times V$, $\&c.$ If S be a series converging or not, whose ultimate terms are less than any finite quantity, then will the series $(a + bx + cx^2 + \&c.) \times (\alpha + \beta x + \gamma x^2 + \&c. x^m) = V \times S$ be a converging one, if $\alpha + \beta x + \gamma x^2 + \dots \&c. x^m = 0$; which case was given by Mr. DE MOIVRE.

Mr.

Mr. BERNOULLI's addition, &c. can be applied to serieses of this kind. For example, let the given series be $\frac{1}{e} + \frac{1}{e+1}x + \frac{1}{e+2}x^2 + \&c. = S$. From this series subtract the same series diminished by m terms, viz. $\frac{1}{e+m}x^m + \frac{1}{e+m+1}x^{m+1} + \frac{1}{e+m+2}x^{m+2} + \&c.$ and there remains $\frac{e+m-ex^m}{e \cdot m+e} + \frac{e+m+1-e+1x^m}{e+1 \cdot e+m+1}x + \frac{e+m+2-e+2x^m}{e+2 \cdot e+m+2}x^2 + \frac{e+m+3-e+3x^m}{e+3 \cdot e+m+3}x^3 + \&c.$; for x^m write A , then will the series become $\frac{m-eA}{e \cdot m+e} + \frac{e+m+1-e+1A}{e+1 \cdot e+m+1}x + \frac{e+m+2-e+2A}{e+2 \cdot e+m+2}x^2 + \frac{e+m+3-e+3A}{e+3 \cdot e+m+3}x^3 + \&c. = \frac{1}{e} + \frac{1}{e+1}x + \frac{1}{e+2}x^2 \dots \dots \frac{1}{e+m-1}x^{m-1}.$

Let the general term be $\frac{ax^m + bx^{m-1} + cx^{m-2} + \&c.}{z+e \cdot z+e+1 \cdot z+e+2 \dots z+e+n-1} \times x^z$
 $= \left(\frac{\alpha}{z+e} + \frac{\beta}{z+e+1} + \frac{\gamma}{z+e+2} \dots \frac{X}{z+e+n-1} \right) x^z$. Suppose $\beta = \beta'x$, $\gamma = \gamma'x^2$, $\delta = \delta'x^3$, $\dots X = X'x^{n-1}$; then will the sum of the above-mentioned series be $(\alpha + \beta' + \gamma' + \delta' + \&c.) \times S - \frac{1}{e}(\beta' + \gamma' + \delta' + \&c.) - \frac{1}{e+1}(\gamma' + \delta' + \&c.) - \frac{1}{e+2}(\delta' + \&c.) - \&c.$

From the sum of the series $\frac{1}{e} - \frac{x}{e+1} + \frac{x^2}{e+2} - \&c.$ by these and the principles before delivered can be deduced the sum of any series, whose general term is

$$\frac{ax^m + bx^{m-1} + \&c.}{2z+e \cdot 2z+e+1 \cdot 2z+e+2 \cdot 2z+e+3 \times \&c.} x^z.$$

In like manner from the sum of the serieses $\frac{x}{e} + \frac{x^2}{e+1} + \frac{x^3}{e+2} + \&c.$ $\frac{x}{f} + \frac{x^2}{f+1} + \frac{x^3}{f+2} + \&c.$ $\frac{x}{g} + \frac{x^2}{g+1} + \frac{x^3}{g+2} + \&c.$ &c. can be deduced

deduced the sum of any series, whose general term is

$$\frac{az^m + bz^{m-1} + \&c.}{2z + e. 2z + e + 1. \&c. \times 2z + f. 2z + f + 1. \&c. \times 2z + g. 2z + g + 1. \&c.} \times x^z.$$

And also from the sum of the serieses $\frac{1}{e} - \frac{1}{e+1}x + \frac{x^2}{e+2} - \&c.$
 $\frac{1}{f} - \frac{x}{f+1} + \frac{x^2}{f+2} - \&c.$ $\frac{1}{g} - \frac{x}{g+1} + \frac{x^2}{g+2} - \&c.$ $\&c.$ can be deduced
 the sum of any series, whose general term is

$$\frac{az^m + bz^{m-1} + \&c.}{2z + e. 2z + e + 1. \&c. \times 2z + f. 2z + f + 1. \&c. \times 2z + g. 2z + g + 1. \&c.} \times x^z.$$

The method of adding more terms of a given series together, as before taught, may be applied to these and all other serieses. For example: let the given series be $1 + \frac{1}{2}x + \frac{1}{3}x^2 + \frac{1}{4}x^3 + \&c.$; add two terms constantly together, and it becomes

$$1 + \frac{1}{2}x + \&c. = \frac{2+x}{2} + \frac{4+3x}{3 \cdot 4} x^2 + \frac{6+5x}{5 \cdot 6} x^3 + \&c. = \frac{2+A}{2} + \frac{4+3A}{3 \cdot 4} x^2 + \frac{6+5A}{5 \cdot 6} x^3 + \&c. \text{ whence the general term is } \frac{2z+2+(2z+1)}{2z+2}$$

$\frac{x}{2z+1} x^{2z}$. From the methods before given of addition, subtraction, and multiplication; and the serieses found by this method, can be derived serieses, whose sums are known.

12. Suppose a given series $ax^n + bx^{n\pm 1} + cx^{n\pm 2} + dx^{n\pm 3} + \&c.$ whose sum p is either an algebraical, exponential, or fluent fluxion of x ; multiply the equation $p = ax^n + bx^{n\pm 1} + cx^{n\pm 2} + dx^{n\pm 3} + \&c.$ into $x^{\pm r-n}$, and there results $ax^{\pm r} + bx^{\pm r\pm 1} + cx^{\pm r\pm 2} + \&c.$; find the fluxion of this equation, and there follows $\frac{1}{x}$ multiplied into the fluxion of the quantity $(x^{\pm r-n}p)$

$= \pm r ax^{\pm r-1} + (\pm r \pm 1) bx^{\pm r\pm 1-1} + (\pm r \pm 2) cx^{\pm r\pm 2-1} + \&c.$
 of which the general term is $(\pm r \pm z) \times t$, where z denotes the distance from the first term of the series, and t

is the term in the given series, whose distance from the first is z . In the same manner may be deduced the sum of a series whose general term is $t' \times \overline{\pm r \pm zs} \times \overline{\pm r' \pm z \pm ns}$, or by repeated operations $t' \times \overline{ez^2 + fz + g}$, where t' is a term of the given equation, whose distance from the first term is z . And in general, from the sum of a given series, whose fluxion can be found, and whose general term is t' , can be deduced by continued multiplication, and finding the fluxion, the sum of a series or quantity, of which the general term is $A't'$, where A is any function of the following kind $a'z^m + b'z^{m-1} + c'z^{m-2} + \&c.$ in which z denotes the distance from the first term of the series, and m a whole number. It is to be observed, that if the given series converges in a ratio, which is at least equal to the ratio of the convergency of some geometrical series, the resulting equation will always converge. But if in a less ratio, then it will sometimes converge, sometimes not, according to the ratio which the successive terms of the resulting series have to each other at an infinite distance.

Corollary. $\frac{p \cdot p + 1 \cdot p + 2 \cdot p + 3 \cdot \dots \cdot p + z}{r \cdot r + 1 \cdot r + 2 \cdot r + 3 \cdot \dots \cdot r + z} =$
 $\frac{p + z \cdot p + z - 1 \cdot p + z - 2 \cdot p + z - 3 \cdot \dots \cdot z + r + 1}{r \cdot r + 1 \cdot r + 2 \cdot r + 3 \cdot \dots \cdot p - 1}$, if $p - r$ be a whole affirmative number; but this latter quantity has the formula above-mentioned $az^m + bz^{m-2} + cz^{m-3} + \&c.$; and consequently, if the sum of the series $a + bx^r + cx^{2r} + dx^{3r} + \&c. = p$ be known, by this method can be deduced the sum of the series $a + \frac{p}{r}bx^r + \frac{p \cdot p + 1}{r \cdot r + 1}cx^{2r} + \frac{p \cdot p + 1 \cdot p + 2}{r \cdot r + 1 \cdot r + 2}dx^{3r} + \&c.$

Ex. I. Since $a + x^n = a^n \left(1 + \frac{m}{n} \times \frac{x}{a} + \frac{m}{n} \times \frac{m-1}{2n} a^{-2} x^2 + \frac{m}{n} \cdot \frac{m-2}{2n} \cdot \frac{m-3}{3n} a^{-3} x^3 + \&c. \right)$; multiply the successive terms of this series into

into the terms of the series $1, \frac{p}{r}, \frac{p \cdot p+1}{r \cdot r+1}, \&c.$ and a series is deduced $a^{\frac{m}{n}} + \frac{p \cdot m}{r \cdot n} a^{+\frac{m}{n}-1} x + \frac{p \cdot p+1 \times m \cdot m-n}{r \cdot r+1 \cdot n \cdot 2n} x^{+\frac{m}{n}-2} + \&c.$ whose sum is known, if the sum of the series $= a + x^{+\frac{m}{n}}$ is known.

Ex. 2. If the series begins from the $l+1^{\text{th}}$ term of the above-mentioned binomial theorem $a^{\frac{m}{n}} + \frac{m}{n} a^{+\frac{m}{n}-1} x + \&c.$ viz. the series be $H \times 1 + \frac{m-l+1n}{l+2n} \frac{x}{a} + \frac{m-l+2n}{l+3n} \frac{x^2}{a^2} + \frac{m-l+3n}{l+4n} \frac{x^3}{a^3} + \&c.$ of which let the respective terms be multiplied into $1, \frac{p}{r}, \frac{p \cdot p+1}{r \cdot r+1}, \&c.$ there will result a series whose sum is known.

Ex. 3. From the rule first given by me for finding the sum of the terms at b distances from each other, the sum of the series $1 + \frac{m-l+1n}{l+2 \cdot n} \times \frac{m-l+2n}{l+3n} \cdot \frac{m-l+bn}{l+b+1n} \times \frac{x^b}{a^b} + P \times \frac{m-l+b+1n}{l+b+2n} \times \frac{m-l+b+2n}{l+b+3n} \dots \frac{m-l+2bn}{l+2b+1n} \frac{x^{2b}}{a^{2b}} + \&c.$ where P denotes the coefficient of the preceding term, can be deduced; and consequently the sum of the series deduced from multiplying the successive terms of this series into the quantities $1, \frac{p}{r}, \frac{p \cdot p+1}{r \cdot r+1}, \&c.$ respectively.

The general principles of this case were first delivered by Mr. BERNOULLI, Mr. DE MOIVRE, Mr. EULER, &c.

12. Assume the series $a + bx^n + cx^{2n} + \&c. = p$, multiply it into $x^{n-1}x$, and find the fluent, then will $\frac{1}{a} x^n p - \frac{1}{a} \int x^n p = \frac{1}{a} a x^n +$

$\frac{1}{\alpha+n} bx^{\alpha+n} + \frac{1}{\alpha+2n} cx^{\alpha+2n} + \&c.$; multiply this equation into $x^{\beta-\alpha-1}x$, and find the fluent of the equation resulting, which will be $\frac{1}{\beta} \times \frac{1}{\alpha} x^{\beta}p - \frac{1}{\alpha} \cdot \frac{1}{\beta} \int x^{\beta}p - \frac{1}{\alpha} \times \frac{1}{\beta-\alpha} x^{\beta-\alpha} \int x^{\alpha}p + \frac{1}{\alpha} \cdot \frac{1}{\beta-\alpha} \int x^{\beta}p = \frac{1}{\alpha} \cdot \frac{1}{\beta} ax^{\beta} + \frac{1}{\alpha+n} \cdot \frac{1}{\beta+n} bx^{\beta+n} + \frac{1}{\alpha+2n} \cdot \frac{1}{\beta+2n} cx^{\beta+2n} + \&c.$; divide by x^{β} , and there results $\frac{1}{\beta} \cdot \frac{1}{\alpha} p + \frac{1}{\alpha} \cdot \frac{1}{\alpha-\beta} x^{-\alpha} \int x^{\alpha}p + \frac{1}{\beta} \cdot \frac{1}{\beta-\alpha} x^{-\beta} \int x^{\beta}p = \frac{1}{\alpha} \cdot \frac{1}{\beta} a + \frac{1}{\alpha+n} \cdot \frac{1}{\beta+n} bx^n + \&c.$; and in general $\frac{1}{\alpha} \cdot \frac{1}{\beta} \cdot \frac{1}{\gamma} \cdot \&c. p + \frac{1}{\alpha} \cdot \frac{1}{\alpha-\beta} \cdot \frac{1}{\alpha-\gamma} x^{-\alpha} \int x^{\alpha}p + \frac{1}{\beta} \cdot \frac{1}{\beta-\alpha} \cdot \frac{1}{\beta-\gamma} \cdot \&c. x^{-\beta} \int x^{\beta}p + \frac{1}{\gamma} \cdot \frac{1}{\gamma-\alpha} \cdot \frac{1}{\gamma-\beta} \cdot \&c. x^{-\gamma} \int x^{\gamma}p = \frac{1}{\alpha} \cdot \frac{1}{\beta} \cdot \frac{1}{\gamma} \cdot \&c. a + \frac{1}{\alpha+n} \cdot \frac{1}{\beta+n} \cdot \frac{1}{\gamma+n} \cdot \&c. bx^n + \frac{1}{\alpha+2n} \cdot \frac{1}{\beta+2n} \cdot \frac{1}{\gamma+2n} \cdot \&c. cx^{2n} + \&c.$ whence the law of continuation is immediately manifest.

Hence, if no two quantities $\alpha, \beta, \gamma, \delta, \&c.$ be equal to each other; and the successive terms $a, b, c, d, \&c.$ of any series $a + bx^n + cx^{2n} + \&c. = p$ be divided by $\alpha, \beta, \gamma, \delta, \&c.$; $\overline{\alpha+n} \cdot \overline{\beta+n} \cdot \overline{\gamma+n} \cdot \overline{\delta+n} \cdot \&c.$; $\alpha+2n \cdot \beta+2n \cdot \gamma+2n \cdot \delta+2n \cdot \&c. \cdot \&c.$; and in general by $\alpha+nz \cdot \beta+nz \cdot \gamma+nz \cdot \delta+nz \cdot \&c. \cdot \&c.$; then can the sum of the series be found from the fluents of the fluxions $x^{\alpha}p, x^{\beta}p, x^{\gamma}p, x^{\delta}p, \&c.$ as has been observed in the Meditations. If two are equal, *viz.* $\alpha=\beta$, then also the fluent of the fluxion $\frac{x}{x} \int x^{\alpha}p$ is required. If three are equal *viz.* $\alpha=\beta=\gamma$; then it is necessary to find the fluent of the fluxion $\frac{x}{x} \int \frac{x}{x} \int x^{\alpha}p$; and so on.

1. Let $p = \frac{x}{1 \pm x^n}$; and if the differences of the quantities $\alpha, \beta, \gamma, \delta, \&c.$ are divisible by n , from the fluent of the fluxion

fluxion $x^a \dot{p}$ can be deduced the fluents of all the other fluxions $x^b \dot{p}$, $x^c \dot{p}$, &c.; and in general, if $\alpha - \beta$ is divisible by n , then from the fluent of the fluxion $x^a \dot{p}$ can be deduced the fluent of the fluxion $x^b \dot{p}$.

2. Suppose p = the terms of the binomial theorem expanded according to the dimensions of x , viz. $(a + bx^n)^{\frac{r}{s}}$ =

$a^{\frac{r}{s}} + \frac{r}{s} a^{\frac{r}{s}-1} bx^n + \&c.$ beginning from the first or any other terms; then, if α , β , &c. divided by n give whole affirmative numbers, will all the fluxions $x^a \dot{p}$, $x^b \dot{p}$, $x^c \dot{p}$, &c. be integrable; and if the differences of the quantities α , β , γ , δ , &c. are divisible by n , from the fluent of the fluxion $x^a \dot{p}$ can be deduced the fluents of the fluxions $x^b \dot{p}$, $x^c \dot{p}$, &c.

If p denotes the sum of the alternate or terms whose distance from each other are m , of the binomial theorem, the same may be applied.

3. If $p = \overline{a + bx^n + cx^{2n} + \&c.}$; and α , β , γ , δ , &c. divided by n give whole affirmative numbers, then from $\int x^a \dot{p}$ can be deduced all the remainder $\int x^b \dot{p}$, $\int x^c \dot{p}$, &c.: and in general from two can be deduced all the remainder.

To find when the sum of any series of this kind can be found, add together each of the fluents, which can be found from each other, and not otherwise, and suppose their sum = 0; and so of any other similar fluent, and from the resulting equations can be discovered when the series can be integrated.

13. If the general term of a series contains in it more variable quantities, z , v , w , &c.; then find the sum of the series, first, from the hypothesis that one of them (z) is only variable,

ble, which, properly corrected, let be A ; in the quantity A suppose all the quantities invariable but some other v , and find the sum of the series thence resulting, which let be B , and so on; and the sum of the series will be deduced.

Ex. Let the term be $\frac{1}{z \cdot z+1 \times v \cdot v+1 \cdot v+2}$; the dimensions of z and v , &c. in the denominator must be at least greater than its dimensions in the numerator by a quantity greater than the number of the quantities z , v , &c. which proceed *in infinitum* increased by unity. First, suppose z only variable, and the sum of the infinite series resulting will be $\frac{1}{z \cdot v \cdot v+1 \cdot v+2} = A$; then suppose v only variable, and the sum resulting will be $\frac{1}{2z \cdot v \cdot v+1} = B$, which is the sum required.

If it be supposed, that the quantities z and v , &c. in the same term shall never have the same values, then suppose z and v always to have the same values, and the general term $\frac{1}{z \cdot z+1 \cdot v \cdot v+1 \cdot v+2}$ becomes $\frac{1}{z^2 \cdot z+1^2 \cdot z+2}$, of which let the sum be V , then will $B - V$ be the sum required.

On this and some other subjects more have been given in the *Meditationes*.

14. If the sum of the series cannot be found in finite terms, and it is necessary to recur to infinite series; it is observed in the *Meditationes* to be generally necessary to add so many terms together, that the distance from the first term of the series may considerably exceed the greatest root of an equation resulting from the general term made $= 0$; and afterwards a series more converging may commonly be deduced from the fluents of fluxions resulting from neglecting all but the greatest quantities in the general terms resulting; and by other

different methods. Mr. NICHOLAS BERNOULLI and Mr. MOMMORT investigated the sum of the series (P) $A + Br + Cr^2 + \&c.$ by a series (Q) $\frac{A}{1-r} + \frac{d'r}{(1-r)^2} + \frac{d''r^2}{(1-r)^3} + \frac{d'''r^3}{(1-r)^4} + \&c.$; where d' , d'' , d''' , &c. denote the successive differences of the terms A, B, C, D, &c. If r be negative, the denominators become $1+r$, $(1+r)^2$, $(1+r)^3$, &c.

It has been observed, in the *Meditationes*, that in swift converging series the series P will converge more swiftly than the series Q; in series converging according to a geometrical ratio, sometimes the one will converge more swift, and sometimes the other. In other series, which converge more slow, where most commonly r nearly = 1, it cannot in general be said, which of the serieses will converge the swiftest. The preceding remark, *viz.* the addition of the first terms of the series, is necessary in most cases of finding the sums by serieses of this kind.

It is not unworthy of observation, that in almost all cases of infinite series, the convergency depends on the roots of the given equations, which remark was first published in the *Meditationes*. For example: in finding approximates to the roots of given equations the convergency depends on how much the approximates given are more near to one root than to any other; and consequently, when two or more roots or values of an unknown quantity are nearly equal, different rules are to be applied, which are improvements of the rule of false. This rule, and the above-mentioned observations were first given in the *Meditationes Algebraicæ et Analyticæ*, with several other additions on similar subjects.

Many more things concerning the summation of series, which depend on fluxional, &c. equations, might be added; but I shall conclude this paper with congratulating myself, that some algebraical inventions published by me have been since thought not unworthy of being published by some of the greatest mathematicians of this or any other age.

1st, In the year 1757, I sent to the Royal Society the first edition of my *Meditationes Algebraicæ*: they were printed and published in the years 1760 and 1762, with *Properties of Curve Lines*, under the title of *Miscellanea Analytica*, and a copy of them sent to Mr. EULER in the beginning of the year 1763, in which was contained a resolution of algebraical equations, not inferior, on account of its generality and facility, to any yet published (*viz.* $y = a \sqrt[n]{p} + b \sqrt[n]{p^2} + c \sqrt[n]{p^3} + \dots \sqrt[n]{p^{n-1}}$). This resolution was published by Mr. EULER in the *Petersburg Acts* for the year 1764. Whether Mr. EULER ever received my book, I cannot pretend to say; nor is it material: for the fact is, that it was published by me in the year 1760 and 1762, and first by Mr. EULER in the year 1764. Mr. DE LA GRANGE and Mr. BEZOUT have ascribed this resolution to Mr. EULER, as first published in the year 1764, not having seen (I suppose) my *Miscell. Analyt.* Mr. BEZOUT found from it some new equations, of which the resolution is known, and applied it to the reduction of equations: more new equations are given, and the resolution rendered more easy by me in the *Philosophical Transactions*. 2d, In the above-mentioned *Miscell. Analyt.* an equation is transformed into another, of which the roots are the squares of the differences of the roots of the given equation; and it is asserted in that book, that if the coefficients

co-efficients of the terms of the resulting equations change continually from + to - and - to +, the roots of the given equation are all possible, otherwise not; and in a paper, inserted by me in the Philosophical Transactions for the year 1764, in which is found from this transformation, when there are none, two or four impossible roots contained in an algebraical equation of four or five dimensions; it is observed, that there will be none or four, &c. impossible roots contained in the given equation, if the last term be + or -; and two, &c. on the contrary, if the last term be - or +. These observations and transformation have been since published and explained in the Berlin Acts for the years 1767 and 1768, by Mr. DE LA GRANGE. 3d. In the Miscell. Anal. an equation is transformed into another, whose roots are the squares, &c. of the roots of a given equation; and it is asserted, that there are at least so many impossible roots contained in the given equation, as there are continual progresses in the resulting equation from + to + and - to -. It is afterwards remarked, that these rules sometimes find impossible roots when Sir ISAAC NEWTON's, and such like rules, fail; and that Sir ISAAC NEWTON's, &c. will find them, when this rule fails. This rule may somewhat further be promoted by first changing the given equation, whose root is x , into another whose root is $\sqrt{-1}x$; but, in my opinion, the rule of HARRIOT's, which only finds whether there are impossible roots contained in a cubic equation or not, is to be preferred to these rules, which, in equations of any dimensions, of which the impossible roots cannot generally be found from the rules, seldom find the true number. 4th, It is remarked, that rules which discover the true number of impossible roots require immense calculations, since they must necessarily find, when

the roots become equal. In order to this, in the Miscell. Anal. there is found an equation, whose roots are the reciprocals of the differences of any two roots of the given equation; and from finding a quantity (π) greater than the greatest root of the given, and $\left(\frac{1}{A}\right)$ greater than the greatest root of the resulting equation, and substituting π , $\pi - A$, $\pi - 2A$, &c. for x in the given equation; will always be found the true number of impossible roots. 5th, In the same book are assumed two equations $(nx^{n-1} - \overline{n-1}px^{n-2} + \overline{n-2}qx^{n-3} - \&c. = 0$ and $x^n - px^{n-1} + \&c. = w)$, and thence deduced an equation, whose root is w , from which, in some cases, can be found the number of impossible roots.

6. In the Miscell. Anal. is given the law of a series, and its demonstration, which finds the sum of the powers of the roots of a given equation from its co-efficients. Mr. EULER has since published the same in the Petersburg Acts. Mr. DE LA GRANGE printed a property of this series, also printed by me about the same time; viz. that if the series was continued *in infinitum*, the powers would observe the same law as the roots, which indeed immediately follows from the series itself; but from thence with the greatest sagacity he deduces the law of the reversion of the series ($y = a + bx + cx^2 + dx^3 + \&c.$): it has since been given in a different manner from similar principles in the Medit. Analyt. 7. In the Miscell. Analyt. the law of a series is given for finding the sum of all quantities of this kind ($\alpha^n \times \beta^n \times \gamma^n \times \delta^n \times \&c. + \&c.$) where α , β , γ , δ , &c. denote the roots of a given equation, from the powers of the roots of the given equation. This law, with a different notation, has been since published in the Paris Acts by Mr. VANDERMONDE; who indeed mentions that he

had heard, that a series for that purpose was contained in my book, but had not seen it. In the same book is given a method of finding the aggregates of any algebraical functions of each of the roots of given equations, which is somewhat improved in the latter editions. 8. In the same book are assumed

$\frac{az^n + bz^{n-1} + \&c.}{pz^m + qz^{m-1} + \&c.}$ and $\frac{Az^{n'} + Bz^{n'-1} + \&c.}{pz^{m'} + qz^{m'-1} + \&c.}$, where z is any rational

quantity whatever for x and y , the unknown quantities of a given equation of two or more dimensions. 9. In the Miscell. Analyt. a biquadratic ($x^4 + 2px^3 = qx^2 + rx + s$, of which no term is destroyed) is reduced to a quadratic ($x^2 + px + n = \sqrt{p^2 + 2n + qx} + \sqrt{s + n^2}$;) and in the second edition of it, printed in the years 1767, 1768, 1769, and published in the beginning of the year 1770, the values of n are found

$\frac{\alpha\beta + \gamma\delta}{2}$, $\frac{\alpha\gamma + \beta\delta}{2}$, and $\frac{\alpha\delta + \beta\gamma}{2}$; and the fix values of $\sqrt{y^2 + 2n + q}$ respectively $\frac{\alpha + \beta - \gamma - \delta}{2}$, $\frac{\alpha + \gamma - \beta - \delta}{2}$, $\frac{\alpha + \delta - \beta - \gamma}{2}$, and their nega-

tives; and the fix values of $\sqrt{s + n^2}$ respectively $\frac{\alpha\beta - \gamma\delta}{2}$, $\frac{\alpha\gamma - \beta\delta}{2}$, $\frac{\alpha\delta - \beta\gamma}{2}$, and their negatives. 10. From a given biquadratic

($y^4 + qy^3 + ry + s = 0$) by assuming $y^2 + ay + b = v$ and a and b such quantities as to make the second and fourth terms of the resulting equations to vanish, there results an equation ($v^4 + Av^2 + B = 0$) of the formula of a quadratic. Mr. DE LA GRANGE has ascribed this resolution to Mr. TSCHIRNHAUSEN; but in the Leipzig Acts the resolution of a cubic is given by Mr. TSCHIRNHAUSEN, but not of a biquadratic: his general design seems to be the extermination of all the terms.

11. Mr. EULER or Mr. DE LA GRANGE found, that if α be a root of the equation $x^n - 1 = 0$, where n is a prime number, $\alpha, \alpha^2, \alpha^3, \dots, \alpha^{n-1}, 1$ will be (n) roots of it. More on a similar subject has been added in the last edition of the *Medit. Algebr.* 12. It is observed in the *Miscell. Analyt.* that CARDAN's or SCIPIO FERREUS's resolution of a cubic is a resolution of three different cubic equations; and in the *Medit. Algeb.* 1770, the three cubics are given, and the rationale of the resolution (for example: if α, β , and γ , be the roots of the cubic equation $x^3 + qx - r = 0$, then is given the function of the above roots, which are the roots of the reducing equation $z^6 - rz^3 = q^3$); and also the rationale of the common resolution of biquadratics. 13. It is asserted in the *Miscell.* that if the terms ($My^n + by^{n-1}x + cy^{n-2}x^2 + \&c.$ and $Ny^m + By^{m-1}x + Cy^{m-2}x^2 + \&c.$) of two equations of n and m dimensions, which contain the greatest dimensions of x and y have a common divisor, the equation whose root is x or y , will not ascend to $n \times m$ dimensions; and if the equation, whose root is x or y , ascends to $n \times m$ dimensions, the sum of its roots depends on the terms of n and $n - 1$ dimensions in the one, and m and $m - 1$ dimensions in the other equation, &c. It is also asserted, in the *Miscell.* that if three algebraical equations of n, m , and r dimensions contain three unknown quantities x, y , and z , the equation, whose root is x or y or z , cannot ascend to more than $n . m . r$ dimensions. 14. Mr. BEZOUT has given two very elegant propositions for finding the dimensions of the equation whose root is x or y , &c; where x, y , &c. are unknown quantities contained in two or more (b) algebraical equations of π, ρ, σ , &c. dimensions, and in which some of the unknown quantities do not ascend to the above π, ρ, σ , &c. dimensions

dimensions respectively. In demonstrating these propositions he uses one (amongst others) before given by me (*viz.* if an equation of n dimensions contains m unknown quantities, the number of different terms which may be contained in it will be $\overline{n+1} \cdot \frac{n+2}{2} \cdot \frac{n+3}{3} \cdot \cdot \frac{n+m}{m}$). In the *Medit.* 1770 there is given a method of finding in many cases the dimensions of the equation, whose root is x or y , &c.; from which one, if not both, of the above-mentioned cases may more easily be deduced, and others added. 15. In the *Medit.* 1770 is observed, that if there be n equations containing m unknown quantities, where n is greater than m , there will be $n-m$ equations of conditions, &c. 16. In the *Miscell.* is given and demonstrated the subsequent proposition; *viz.* if two equations contain two unknown quantities x and y , in which x and y are similarly involved; the equation, whose root is x or y will have twice the number of roots which the equation, whose root is $x+y$, x^2+y^2 , &c. has. In the *Medit.* 1770 the same reasoning is applied to equations, which have two, three, four, &c. quantities similarly involved. 17. Mr. DE LA GRANGE has done me the honour to demonstrate my method of finding the number of affirmative and negative roots contained in a biquadratic equation. A demonstration of my rule for finding the number of affirmative, negative, and impossible roots contained in the equation $x^n + Ax^m + B = 0$ is also omitted, on account of its ease and length. From the *Medit.* the investigation of finding the true number of affirmative and negative roots appears to be as difficult a problem as the finding the true number of impossible roots; and it further appears, that the common methods in both cases can seldom be depended on. But their faults lie on different sides,

the one generally finds too many, the other too few. 18. In the *Medit.* 1770, from the number of impossible roots in a given equation ($x^n - px^{n-1} + \&c. = 0$) is found the number of impossible roots in an equation, whose roots (v) have any assignable relation to the roots of a given equation; and examples are given in the relation ($nx^{n-1} - n - 1px^{n-2} + \&c. = v$); and in an equation, whose roots are the squares of the differences of the roots of the given equation. 19. It is observed in the *Medit.* 1770, that in two or more equations, having two or more unknown quantities, the same irrationality will be contained in the correspondent values of each of the unknown quantities, unless two or more values of one of them are equal, &c. The same observation is also applied to the coefficients of an equation deduced from a given equation. 20. In the *Miscell.* was published a new method of exterminating, from a given equation, irrational quantities, by finding the multipliers, which, multiplied into it, give a rational product. 21. In the *Medit.* 1770, are given the different resolutions of a certain quantity $(a^2 + rb^2)^{2m+1}$ and $(a^2 + rb^2)^{2m+2}$ into quantities of the same kind. 22. Mr. DE LA GRANGE has very elegantly demonstrated Mr. WILSON's celebrated property of prime numbers contained in my book. In the last edition of the *Medit.* the same property is demonstrated, and some similar ones added. 23. In the *Miscell.* is given a method of finding all the integral correspondent values of the unknown quantities of a given simple equation, having two or more unknown quantities; and, in the *Medit.* 1770, are given methods of reducing simple and other algebraical equations into one, so that some unknown quantities may be exterminated; and if the unknown quantities of the resulting equations be integral or rational,

rational, the unknown quantities exterminated may also be integral or rational. 24. In the *Medit.* are given rules for finding the different and correspondent roots of an equation, whose resolution is given. 25. Mr. DE LA GRANGE has recommended my new transformation of equations, published in the *Miscell.* which perhaps is not less general nor elegant than any yet published; and in the *Meditat.* 1770 is given a method very useful in finding the co-efficients.

If either here, or in the preface to the *Medit. Algebraicæ*, I have ascribed to myself any algebraical, or in the properties of curve lines any geometrical, or in the *Medit. Analyt.* any analytical invention, which has been before published by any other person, I can only plead ignorance of it, and shall on the very first conviction acknowledge it.

I must further add, that I have been able to carry my algebraical improvements into geometry; for from them, with some geometrical principles added, I have (unless I am deceived) deduced as many new properties of conic sections and curve lines as have been published by any one since the great geometrician APOLLONIUS.

