XXV. *A new Method of finding Fluents by Continuation. By the Rev.* Samuel Vince, *A. M. F. R. S.*

Read July 6, 1786.

ART. I. Put $\dot{F} = \frac{x^m \dot{x}}{a^n + x^n} = x^{m-n}\dot{x} - a^n x^{m-2n}\dot{x} + a^{2n}x^{m-3n}\dot{x} -$ &c. $\pm \frac{a^{vn}x^{m-vn}\dot{x}}{a^n+x^n}$, then $F = \frac{x^{m-n+1}}{m-n+1} - \frac{a^n x^{m-2n+1}}{m-2n+1} + \frac{a^{2n}x^{m-3n+1}}{m-3n+1} -$ &c. $\pm W$, where W reprefents the fluent of the laft term. Now $\frac{x^m\dot{x}}{\overline{a^n+x^n}|^r} = \frac{x^m\dot{x}\times\overline{a^n+x^n}|^{1-r}}{a^n+x^n}$; hence $\int\frac{x^m\dot{x}}{\overline{a^n+x^n}|^r} = \int\frac{x^m\dot{x}\times\overline{a^n+x^n}|^{1-r}}{a^n+x^n} = F \times \overline{a^n+x^n}|^{1-r} - \int F \times \frac{\overline{1-r}\,.\,nx^{n-1}\dot{x}}{\overline{a^n+x^n}|^r} =$ (by fubftituting for F its value in the latter quantity) $F\times\overline{a^n+x^n}|^{1-r} - \frac{\overline{1-r}\,.\,n}{m-n+1} \times \int\frac{x^m\dot{x}}{\overline{a^n+x^n}|^r} + \frac{\overline{1-r}\,.\,na^n}{m-2n+1} \times \int\frac{x^{m-n}\dot{x}}{\overline{a^n+x^n}|^r} - \frac{\overline{1-r}\,.\,na^{2n}}{m-3n+1} \times \int\frac{x^{m-2n}\dot{x}}{\overline{a^n+x^n}|^r} +$ &c. $\pm \int W \times \frac{\overline{1-r}\,.\,nx^{n-1}\dot{x}}{\overline{a^n+x^n}|^r}$; tranfpofe $-\frac{\overline{1-r}\,.\,n}{m-n+1} \times \int\frac{x^m\dot{x}}{\overline{a^n+x^n}|^r}$ and divide both fides of the equation by $\frac{m-rn+1}{m-n+1}$ and we have $\int\frac{x^m\dot{x}}{\overline{a^n+x^n}|^r} = \frac{m-n+1}{m-rn+1} \times F \times \overline{a^n+x^n}|^{1-r} + \frac{m-n+1}{m-rn+1} \times \frac{\overline{1-r}\,.\,na^n}{m-2n+1} \times \int\frac{x^{m-n}\dot{x}}{\overline{a^n+x^n}|^r} - \frac{m-n+1}{m-rn+1} \times \frac{\overline{1-r}\,.\,na^{2n}}{m-3n+1} \times \int\frac{x^{m-2n}\dot{x}}{\overline{a^n+x^n}|^r} +$ &c. $\pm \frac{m-n+1}{m-rn+1} \times$

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$\int W \times \frac{\overline{1-r} \cdot nx^{n-1}\dot{x}}{\overline{a^n+x^n}|^r}$; now the fluent of the laſt term is $\mp \frac{m-n+1}{m-rn+1} \times W \times \overline{a^n+x^n}|^{1-r} \pm \frac{m-n+1}{m-rn+1} \times \int \frac{a^{vn}x^{m-vn}\dot{x}}{\overline{a^n+x^n}|^r}$; hence by ſubſtituting this quantity for the laſt term, it is manifeſt, that the firſt part $\mp \frac{m-n+1}{m-rn+1} \times W \times \overline{a^n+x^n}|^{1-r}$ will be deſtroyed by the laſt term of $\frac{m-n+1}{m-rn+1} \times F \times \overline{a^n+x^n}|^{1-r}$, when we ſubſtitute for F its value; hence if we put $M = \frac{x^{m-n+1}}{m-n+1} - \frac{a^n x^{m-2n-1}}{m-2n+1} + \frac{a^{2n}x^{m-3n+1}}{m-3n+1}$ – &c. omitting the laſt term $\pm W$, we have $\int \frac{x^m\dot{x}}{\overline{x^n+x^n}|^r} = \frac{m-n+1}{m-rn+1} \times M \times \overline{a^n+x^n}|^{1-r} + \frac{m-n+1}{m-rn+1} \times \frac{\overline{1-r} \cdot na^n}{m-2n+1} \times \int \frac{x^{m-n}\dot{x}}{\overline{a^n+x^n}|^r} - \frac{m-n+1}{m-rn+1} \times \frac{\overline{1-r} \cdot na^{2n}}{m-3n+1} \times \int \frac{x^{m-2n}\dot{x}}{\overline{a^n+x^n}|^r} + \text{\&c.} \pm \frac{m-n+1}{m-rn+1} \times \times a^{vn} \times \int \frac{x^{m-vn}\dot{x}}{\overline{a^n+x^n}|^r}$; hence, if the fluent of the laſt term be given, we have the general law of continuation by which we may find the fluent of $\frac{x^m\dot{x}}{\overline{a^n+x^n}|^r}$. If the fluxion be $\frac{x^m\dot{x}}{\overline{a^n+x^n}|^r}$ all the terms after the firſt will be negative, and the laſt always poſitive.

Ex. 1. Given the fluent of $\frac{\dot{x}}{\sqrt{1+x^2}}$ to find the fluent of $\frac{x^{2s}\dot{x}}{\sqrt{1+x^2}}$.

Here $n=2$, $a=1$, $m=2s$, $r=\frac{1}{2}$, $M = \frac{x^{2n-1}}{2n-1} - \frac{x^{2n-3}}{2n-3}$ + &c. to $\pm x$,

$\pm x$, and the fluent of $\frac{\dot{x}}{\sqrt{1+x^2}}$ is the hyp. log. $x+\sqrt{1+x^2}$ which call Q; hence $\int\frac{x^{2s}\dot{x}}{\sqrt{1+x^2}}=\frac{2s-1}{2s}\times M\times\sqrt{1+x^2}+\frac{2s-1}{2s\cdot 2s-3}\times\int\frac{x^{2s-2}\dot{x}}{\sqrt{1+x^2}}-\frac{2s-1}{2s\cdot 2s-5}\times\int\frac{x^{2s-4}\dot{x}}{\sqrt{1+x^2}}+\&c.\pm\frac{2s-1}{2s}\times Q.$

If $s=1, \int\frac{x^2\dot{x}}{\sqrt{1+x^2}}=\frac{1}{2}\sqrt{1+x^2}\times x-\frac{1}{2}Q=\alpha.$

$s=2, \int\frac{x^4\dot{x}}{\sqrt{1+x^2}}=\frac{3}{4}\sqrt{1+x^2}\times\overline{\frac{x^3}{3}-x}+\frac{3}{4}\alpha+\frac{3}{4}Q=\beta.$

$s=3, \int\frac{x^6\dot{x}}{\sqrt{1+x^2}}=\frac{5}{6}\sqrt{1+x^2}\times\overline{\frac{x^5}{5}-\frac{x^3}{3}+x}+\frac{5}{6\cdot 3}\beta-\frac{5}{6}\alpha-\frac{5}{6}Q=\gamma.$

$s=4, \int\frac{x^8\dot{x}}{\sqrt{1+x^2}}=\frac{7}{8}\sqrt{1+x^2}\times\overline{\frac{x^7}{7}-\frac{x^5}{5}+\frac{x^3}{3}-x}+\frac{7}{8\cdot 5}\gamma-\frac{7}{8\cdot 3}\beta+\frac{7}{8}\alpha+\frac{7}{8}Q.$

&c. &c. &c.

Ex. 2. To find the fluent of $x^{\frac{s}{2}}\dot{x}\sqrt{2+x}$, given the fluent of $x^{-\frac{1}{2}}\dot{x}\sqrt{2+x}$, and s an odd number.

Here $a=2$, $n=1$, $r=-\frac{1}{2}$, $\frac{s}{2}=m$, $m-vn=-\frac{1}{2}$, or $\frac{s}{2}-v=-\frac{1}{2}$; $\therefore v=\frac{s+1}{2}$, $M=\frac{2x^{\frac{s}{2}}}{s}-\frac{4x^{\frac{s}{2}-1}}{s-2}+\frac{8x^{\frac{s}{2}-2}}{s-4}-$ &c. and the fluent (Q) of $x^{-\frac{1}{2}}\dot{x}\sqrt{2+x}$ is $\pi+\sqrt{2x+x^2}$, where $\pi=$ hyp. log. $1+x+\sqrt{2x+x^2}$; hence $\int x^{\frac{s}{2}}\dot{x}\sqrt{2+x}=\frac{s}{s+3}\times\overline{2+x}|^{\frac{3}{2}}\times M+\frac{s}{s+3}\times\frac{b}{s-2}\times\int x^{\frac{s}{2}-1}\dot{x}\sqrt{2+x}-\frac{s}{s+3}\times\frac{12}{s-4}\times\int x^{\frac{s}{2}-2}\dot{x}\sqrt{2+x}+\frac{s}{s+3}\times\frac{24}{s-6}\times\int x^{\frac{s}{2}-3}\dot{x}\sqrt{2+x}$ &c. $\pm\frac{s}{s+3}\times 2^{\frac{s+1}{2}}\times\int x^{-\frac{1}{2}}\dot{x}\sqrt{2+x}.$

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If $s=1, \int x^{\frac{1}{2}}\dot{x}\sqrt{2+x} = \frac{1}{4}\times\overline{2+x}|^{\frac{3}{2}}\times 2x^{\frac{1}{2}} - \frac{1}{2}Q = \alpha.$

$s=3, \int x^{\frac{3}{2}}\dot{x}\sqrt{2+x} = \frac{1}{2}\times\overline{2+x}|^{\frac{3}{2}}\times\overline{\frac{2x^{\frac{3}{2}}}{3} - 4x^{\frac{1}{2}}} + 3\alpha + 2Q = \beta.$

$s=5, \int x^{\frac{5}{2}}\dot{x}\sqrt{2+x} = \frac{5}{8}\times\overline{2+x}|^{\frac{3}{2}}\times\overline{\frac{2x^{\frac{5}{2}}}{5} - \frac{4x^{\frac{3}{2}}}{3} + 8x^{\frac{1}{2}}} + \frac{5}{4}\beta -$
$\frac{15}{2}\alpha - 5Q = \gamma.$

$s=7, \int x^{\frac{7}{2}}\dot{x}\sqrt{2+x} = \frac{7}{10}\times\overline{2+x}|^{\frac{3}{2}}\times\overline{\frac{2x^{\frac{7}{2}}}{7} - \frac{4x^{\frac{5}{2}}}{5} + \frac{8x^{\frac{3}{2}}}{3} - 16x^{\frac{1}{2}}} +$
$\frac{21}{25}\gamma - \frac{14}{5}\beta + \frac{84}{5}\alpha + \frac{56}{5}Q.$

&c. &c. &c.

II. Let $\dot{F}\dfrac{x^n\dot{x}}{a+bx^m+x^{2m}} = x^{n-2m}\dot{x} - Px^{n-3m}\dot{x} + Qx^{n-4m}\dot{x} - \&c. \pm$ $\dfrac{Vx^{n-sm}\dot{x}}{a+bx^m+x^{2m}} \pm \dfrac{Wx^{\overline{n-s+1}\cdot m}\dot{x}}{a+bx^m+x^{2m}}$, then $F = \dfrac{x^{n-2m+1}}{n-2m+1} - \dfrac{Px^{n-3m-1}}{n-3m+1} + \dfrac{Qx^{n-4m+1}}{n-4m+1}$ $- \&c. \pm T \pm U$, where T and U are put for the fluents of the two laſt terms, and P, Q, &c. for the co-efficients ariſing from the diviſion. Now, $\int\dfrac{x^n\dot{x}}{\sqrt{a+bx^m+x^{2m}}} = \int\dfrac{x^n\dot{x}\times\overline{a+bx^m+x^{2m}}|^{1-r}}{a+bx^m+x^{2m}} =$ $F\times\overline{a+bx^m+x^{2m}}|^{1-r} - \int F + \dfrac{\overline{1-r}\cdot mbx^{m-1}\dot{x} + \overline{1-r}\cdot 2mx^{2m-1}\dot{x}}{\overline{a+bx^m+x^{2m}}|^r} =$ (by ſubſtituting for F its value in the latter quantity, and putting A, B, C, &c for the co-efficients which ariſe in conſequence thereof) $F\times\overline{a+bx^m+x^{2m}}|^{1-r} - A\times\int\dfrac{x^n\dot{x}}{\overline{a+bx^m+x^{2m}}|^r} + B\times$

$$\int\frac{x^{n-m}\dot{x}}{\overline{a+bx^m+x^{2m}}|^r} - C\times\int\frac{x^{n-2m}\dot{x}}{\overline{a+bx^m+x^{2m}}|^r} + \&c.$$

$$\mp \int \mathrm{T} \times \frac{\overline{1-r} \,.\, mbx^{m-1}\dot{x} + \overline{1-r} \,.\, 2mx^{2m-1}\dot{x}}{\overline{a+bx^m+x^{2m}}|^r}$$

$\mp \int \mathrm{U} \times \frac{\overline{1-r} \,.\, mbx^{m-1}\dot{x} + \overline{1-r} \,.\, 2mx^{2m-1}\dot{x}}{\overline{a+bx^m+x^{2m}}|^r}$; hence by transposition and division we have $\int \frac{x^n \dot{x}}{\overline{a+bx^m+x^{2m}}|^r} = \frac{1}{1+\mathrm{A}} \times \mathrm{F} \times \overline{a+bx^m+x^{2m}}|^{1-r}$

$$+ \frac{\mathrm{B}}{1+\mathrm{A}} \times \int \frac{x^{n-m}\dot{x}}{\overline{a+bx^m+x^{2m}}|^r} - \frac{\mathrm{C}}{1+\mathrm{A}} \times \int \frac{x^{n-2m}\dot{x}}{\overline{a+bx^m+x^{2m}}|^r} + \&c.$$

$$\mp \int \frac{\mathrm{T}}{1+\mathrm{A}} \times \frac{\overline{1-r} \,.\, mbx^{m-1}\dot{x} + \overline{1-r} \,.\, 2mx^{2m-1}\dot{x}}{\overline{a+bx^m+x^{2m}}|^r}$$

$\mp \int \frac{\mathrm{U}}{1+\mathrm{A}} \times \frac{\overline{1-r} \,.\, mbx^{m-1}\dot{x} + \overline{1-r} \,.\, 2mx^{2m-1}\dot{x}}{\overline{a+bx^m+x^{2m}}|^r}$. Now the fluents of these two last terms are $\mp \frac{\mathrm{T}}{1+\mathrm{A}} \times \overline{a+bx^m+x^{2m}}|^{1-r} \pm \frac{\mathrm{V}}{1+\mathrm{A}}$ $\int \frac{x^{n-sm}\dot{x}}{\overline{a+bx^m+x^{2m}}|^r}$ and $\mp \frac{\mathrm{U}}{1+\mathrm{A}} \times \overline{a+bx^m+x^{2m}}|^{1-r} \pm \frac{\mathrm{W}}{1+\mathrm{A}} \int \frac{x^{n-\overline{s+1}\,.\,m}\dot{x}}{\overline{a+bx^m+x^{2m}}|^r}$ respectively; hence, by substituting these for the last term, it is manifest that $\mp \frac{\mathrm{T}}{1+\mathrm{A}} \times \overline{a+bx^m+x^{2m}}|^{1-r}$ and $\mp \frac{\mathrm{U}}{1+\mathrm{A}}$ $\times \overline{a+bx^m+x^{2m}}|^{1-r}$ will be destroyed by the two last terms of $\frac{1}{1+\mathrm{A}} \times \mathrm{F} \times \overline{a+bx^m+x^{2m}}|^{1-r}$ when we substitute for F its value; hence, if we put $\mathrm{M} = \frac{x^{n-2m-1}}{n-2m+1} - \frac{\mathrm{P}x^{n-3m+1}}{n-3m+1} + \frac{\mathrm{Q}x^{n-4m+1}}{n-4m+1} - \&c.$ omitting the two last terms $\pm$ T and $\pm$ U, we shall have $\int \frac{x^n \dot{x}}{\overline{a+bx^m+x^{2m}}|^r} = \frac{1}{1+\mathrm{A}} \times \mathrm{M} \times \overline{a+bx^m+x^{2m}}|^{1-r} + \frac{\mathrm{B}}{1+\mathrm{A}} \times$ $\int \frac{x^{n-m}\dot{x}}{\overline{a+bx^m+x^{2m}}|^r} - \frac{\mathrm{C}}{1+\mathrm{A}} \times \int \frac{x^{n-2m}\dot{x}}{\overline{a+bx^m+x^{2m}}|^r} + \&c. \pm \frac{\mathrm{V}}{1+\mathrm{A}} \times$

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$\int \frac{x^{n-sm}\dot{x}}{\overline{a+bx^m+x^{2m}}|^r} \pm \frac{W}{1+A} \times \frac{x^{n-\overline{s+1}.m}\dot{x}}{\overline{a+bx^m+x^{2m}}|^r}$; hence if the two last fluents be given, we have the general law of continuation up to $\frac{x^n\dot{x}}{\overline{a+bx^m+x^{2m}}|^r}$ in the same manner as before.

III. In general, if we proceed as in the two last articles, we shall find $\int \frac{x^n\dot{x}}{\overline{a+bx^m+\&c.\ x^{tm}}|^r} = \frac{M}{P} \times \overline{a+bx^m+\&c.\ x^{tm}}|^{1-r} + \frac{A}{P} \times \int \frac{x^{n-m}\dot{x}}{\overline{a+bx^m+\&c.\ x^{tm}}|^r} + \frac{B}{P} \times \int \frac{x^{n-2m}\dot{x}}{\overline{a+bx^m+\&c.\ x^{tm}}|^r} + \&c. \pm \frac{T}{P} \times \int \frac{x^{n-sm}\dot{x}}{\overline{a+bx^m+\&c.\ x^{tm}}|^r} \pm \frac{V}{P} \times \int \frac{x^{n-\overline{s+1}.m}\dot{x}}{\overline{a+bx^m+\&c.\ x^{tn}}|^r} \pm$ &c. where the number of these last terms is t, and $M = \frac{x^{n-tm+1}}{n-tm+1} - \frac{Qx^{n-\overline{t+1}.m-1}}{n-\overline{t+1}.m+1}$ + &c. omitting, as before, the terms at the end arising from the remainders. Hence if the last t fluents be given, we can by continuation find the required fluent.

Because the division of $\frac{x^n\dot{x}}{a+bx^m+\&c.\ x^{tm}}$ may be expressed by an *ascending* series $x^n\dot{x} - Qx^{n+m}\dot{x} + Rx^{n+2m}\dot{x} -$ &c. it is manifest, that by the same method we may continue the fluents downwards as well as upwards.

IV. Let $\dot{F} = \frac{x^n\dot{x}}{1-x} = - x^{n-1}\dot{x} - x^{n-2}\dot{x} - x^{n-3}\dot{x} - \&c. - x^{n-r}\dot{x} + \frac{x^{n-r}\dot{x}}{1-x}$, then $F = -\frac{x^n}{n} - \frac{x^{n-1}}{n-1} - \frac{x^{n-2}}{n-2} - \&c. - \frac{x^{n-r+1}}{n-r+1} + W$, where W is the fluent of the last term. Now $\frac{x^n\dot{x}}{\sqrt{1-x^2}} = \frac{x^n\dot{x}}{1-x} \times \sqrt{\frac{1-x}{1+x}}$,

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hence $\int \frac{x^n \dot{x}}{\sqrt{1-x^2}} = \int \frac{x^n \dot{x}}{1-x} \times \sqrt{\frac{1-x}{1+x}} = F \times \sqrt{\frac{1-x}{1+x}} + \int F \times \frac{\dot{x}}{\sqrt{1+x^2} \times \overline{1+x}} = F \times \sqrt{\frac{1-x}{1+x}} - \int \frac{x^n \dot{x}}{n\sqrt{1-x^2} \times \overline{1+x}} - \int \frac{x^{n-1}\dot{x}}{\overline{n-1} \,.\, \sqrt{1-x^2} \times \overline{1+x}}$ $- \int \frac{x^{n-2}\dot{x}}{\overline{n-2} \,.\, \sqrt{1-x^2} \times \overline{1+x}} - \&c. - \int \frac{x^{n-r+1}\dot{x}}{\overline{n-r+1} \,.\, \sqrt{1-x^2} \times \overline{1+x}} +$ $\int W \times \frac{\dot{x}}{\sqrt{1-x^2} \times \overline{1+x}}$. But

$$\frac{x^n \dot{x}}{\sqrt{1-x^2} \times \overline{1+x}} = \frac{x^{n-1}\dot{x}}{n\sqrt{1-x^2}} - \frac{x^{n-2}\dot{x}}{n\sqrt{1-x^2}} + \frac{x^{n-3}\dot{x}}{n\sqrt{1-x^2}} - \&c. \mp \frac{x^{n-r}\dot{x}}{n\sqrt{1-x^2}} \pm \frac{x^{n-r}\dot{x}}{n\sqrt{1-x^2} \times \overline{1+x}}$$

$$\frac{x^{n-1}\dot{x}}{\sqrt{1-x^2} \times \overline{1+x}} = \frac{x^{n-2}\dot{x}}{\overline{n-1} \,.\, \sqrt{1-x^2}} - \frac{x^{n-3}\dot{x}}{\overline{n-1} \,.\, \sqrt{1-x^2}} + \&c. \pm \frac{x^{n-r}\dot{x}}{\overline{n-1} \,.\, \sqrt{1-x^2}} \mp \frac{x^{n-r}\dot{x}}{\overline{n-1} \,.\, \sqrt{1-x^2} \times \overline{1+x}}$$

$$\frac{x^{n-2}\dot{x}}{\sqrt{1-x^2} \times \overline{1+x}} = \frac{x^{n-3}\dot{x}}{\overline{n-2} \,.\, \sqrt{1-x^2}} - \&c. \mp \frac{x^{n-r}\dot{x}}{\overline{n-2} \,.\, \sqrt{1-x^2}} \pm \frac{x^{n-r}\dot{x}}{\overline{n-2} \,.\, \sqrt{1-x^2} \times \overline{1+x}}$$

&c. &c. &c.

$$\frac{x^{n-r+1}\dot{x}}{\overline{n-r+1} \,.\, \sqrt{1-x^2} \times \overline{1+x}} = \frac{x^{n-r}\dot{x}}{\overline{n-r+1} \,.\, \sqrt{1+x^2}} - \frac{x^{n-r}\dot{x}}{\overline{n-r+1} \,.\, \sqrt{1-x^2} \times \overline{1+x}}.$$

Hence $\int \frac{x^n \dot{x}}{\sqrt{1-x^2}} = F \times \sqrt{\frac{1-x}{1+x}} - \frac{1}{n} \times \int \frac{x^{n-1}\dot{x}}{\sqrt{1-x^2}} + \overline{\frac{1}{n} - \frac{1}{n-1}} \times$ $\int \frac{x^{n-2}\dot{x}}{\sqrt{1-x^2}} - \overline{\frac{1}{n} - \frac{1}{n-1} + \frac{1}{n-2}} \times \int \frac{x^{n-3}\dot{x}}{\sqrt{1-x^2}} + \&c. \pm \overline{\frac{1}{n} \mp \frac{1}{n-1} \pm \&c.}$ $\overline{- \frac{1}{n-r+1}} \times \int \frac{x^{n-r}\dot{x}}{\sqrt{1-x^2}} \mp \overline{\frac{1}{n} \pm \frac{1}{n-1} \mp \&c. + \frac{1}{n-r+1}} \times \int \frac{x^{n-r}\dot{x}}{\sqrt{1-x^2} \times \overline{1+x}}$ $+ \int W \times \frac{\dot{x}}{\sqrt{1-x^2} \times \overline{1+x}}$. Now $\int W \times \frac{\dot{x}}{\sqrt{1-x^2} \times \overline{1+x}} = -W \times$ $\overline{\frac{1-x}{1+x}}\Big|^{\frac{1}{2}} + \int \frac{x^{n-r}\dot{x}}{\sqrt{1-x^2}}$; also $\int \frac{x^{n-r}\dot{x}}{\sqrt{1-x^2} \times \overline{1+x}} = -x^{n-r} \times \sqrt{\frac{1-x}{1+x}} +$ $\overline{n-r} \times \int \frac{x^{n-r-1}\dot{x}}{\sqrt{1-x^2}} - \overline{n-r} \times \int \frac{x^{n-r}\dot{x}}{\sqrt{1-x^2}}$; hence, by substituting these quantities in the two last terms, it is manifest that $- W \times \overline{\frac{1-x}{1-x}}\Big|^{\frac{1}{2}}$ will be destroyed by the last term of

F ×

$F \times \overline{\frac{1-x}{1+x}}\Big|^{\frac{1}{2}}$ when we ſubſtitute for F its value; therefore, if we put $M = -\frac{x^n}{n} - \frac{x^{n-1}}{n-1} - \frac{x^{n-2}}{n-2} - \&c. - \frac{x^{n-r+1}}{n-r+1}$, we ſhall have $\int \frac{x^n \dot{x}}{\sqrt{1+2}} = \overline{M \pm \frac{1}{n} \mp \frac{1}{n-1} \pm \&c. - \frac{1}{n-r+1}} \times \overline{\frac{1-x}{1+x}}\Big|^{\frac{1}{2}} - \frac{1}{n} \times \int \frac{x^{n-1}\dot{x}}{\sqrt{1-x^2}} + \overline{\frac{1}{n} - \frac{1}{n-1}} \times \int \frac{x^{n-2}\dot{x}}{\sqrt{1-x^2}} - \overline{\frac{1}{n} - \frac{1}{n-1} + \frac{1}{n-2}} \times \int \frac{x^{n-3}\dot{x}}{\sqrt{1-x^2}} + \&c.$ $\overline{\pm \frac{1}{n} \mp \frac{1}{n-1} \pm \&c. - \frac{1}{n-r+1} \times \overline{n-r+1} + 1} \times \int \frac{x^{n-r}\dot{x}}{\sqrt{1-x^2}} \mp \overline{\frac{1}{n} \pm \frac{1}{n-1} \mp \&c. + \frac{1}{n-r+1}} \times \overline{n-r} \times \int \frac{x^{n-r-1}\dot{x}}{\sqrt{1-x^2}}$. Hence, if the two laſt fluents be given, we have the general law of continuation up to the fluent of $\frac{x^n \dot{x}}{\sqrt{1-x^2}}$, where the index of x without the vinculum increaſes by unity each time. And in the ſame manner we may (by increaſing the index of x without by m) find the fluent of $\frac{x^n \dot{x}}{\sqrt{a^{2m}-x^{2m}}}$ if we have given the fluents of $\frac{x^{n-rm}\dot{x}}{\sqrt{a^{2m}-x^{2m}}}$ and $\frac{x^{n-\overline{r+1}.m}\dot{x}}{\sqrt{a^{2m}-x^{2m}}}$. Thus we have a general law of continuation, where the index of x without is increaſed by half the index under the vinculum.

V. Aſſume $\dot{F} = \frac{x^n \dot{x}}{x^m - b} = x^{n-m}\dot{x} + bx^{n-2m}\dot{x} + b^2 x^{n-3m}\dot{x} + \&c. + \frac{b^r x^{n-rm}\dot{x}}{x^m - b}$, then $F = \frac{x^{n-m+1}}{n-m+1} + \frac{bx^{n-2m+1}}{n-2m+1} + \frac{b^2 x^{n-3m+1}}{n+3m+1} + \&c. + W$, where W is put for the fluent of the laſt term. Now $\int x^n \dot{x} \sqrt{\frac{x^m - a}{x^m - b}} = \int \frac{x^n \dot{x}}{x^m - b} \times \sqrt{\overline{x^m - a} \times \overline{x^m - b}} = F \times \sqrt{\overline{x^m - a} \times \overline{x^m + b}}$

$-\int F \times \frac{2mx^{2m-1}\dot{x} - \overline{a+b}\,.\,mx^{m-1}\dot{x}}{2\sqrt{x^m - a \times x^m - b}}$ = (by substituting for F its value in the latter quantity, and putting A, B, C, &c. for the co-efficients which arise therefrom) $F \times \sqrt{\overline{x^m - a} \times \overline{x^m - b}} -$

$$A\int \frac{x^n\dot{x}}{\sqrt{x^m - a \times x^m - b}} - B\int \frac{x^{n-m}\dot{x}}{\sqrt{x^m - a \times x^m - b}} - C\int \frac{x^{n-2m}\dot{x}}{\sqrt{x^m - a \times x^m - b}} -$$

&c. $-\int W \times \frac{2mx^{2m-1}\dot{x} - \overline{a+b}\,.\,mx^{m-1}\dot{x}}{2\sqrt{x^m - a \times x^m - b}}$. Now $\frac{x^n\dot{x}}{\sqrt{x^m - a \times x^m - b}} =$

$\frac{x^n\dot{x}}{x^m - a}\sqrt{\frac{x^m - a}{x^m - b}}$, $\frac{x^{n-m}\dot{x}}{\sqrt{x^m - a \times x^m - b}} = \frac{x^{n-m}\dot{x}}{x^m - a}\sqrt{\frac{x^m - a}{x^m - b}}$, $\frac{x^{n-2m}\dot{x}}{\sqrt{x^m - a \times x^m - b}} = \frac{x^{n-2m}\dot{x}}{x^m - a}$

$\sqrt{\frac{x^m - a}{x^m - b}}$, &c. But

$$\frac{x^n\dot{x}}{x^m - a} = x^{n-m}\dot{x} + ax^{n-2m}\dot{x} + a^2x^{n-3m}\dot{x} + \&c. + a^{r-1}x^{n-rm}\dot{x} + \frac{a^r x^{n-rm}\dot{x}}{x^m - a}$$

$$\frac{x^{n-m}\dot{x}}{x^m - a} = x^{n-2m}\dot{x} + ax^{n-3m}\dot{x} + \&c. + a^{r-2}x^{n-rm}\dot{x} + \frac{a^{r-1}x^{n-rm}\dot{x}}{x^m - a}$$

$$\frac{x^{n-2m}\dot{x}}{x^m - a} = x^{n-3m}\dot{x} + \&c. + a^{r-3}x^{n-rm}\dot{x} + \frac{a^{r-2}x^{n-rm}\dot{x}}{x^m - a}$$

&c. &c. &c.

Hence, by substitution, $\int x^n\dot{x}\sqrt{\frac{x^m - a}{x^m - b}} = F \times \sqrt{\overline{x^m - a} \times \overline{x^m - b}}$

$$- A\int x^{n-m}\dot{x}\sqrt{\frac{x^m - a}{x^m - b}} - \overline{Aa + B} \times \int x^{n-2m}\dot{x}\sqrt{\frac{x^m - a}{x^m - b}} - \overline{Aa^2 + Ba + C}$$

$$\times \int x^{n-3m}\dot{x}\sqrt{\frac{x^m - a}{x^m - b}} - \&c. - \overline{Aa^{r-1} + Ba^{r-2} + Ca^{r-3} + \&c.} \times$$

$$\times \int x^{n-rm}\dot{x}\sqrt{\frac{x^m - a}{x^m - b}} - \overline{Aa^r + Ba^{r-1} + Ca^{r-2} + \&c.} \times \int \frac{x^{n-rm}\dot{x}}{\sqrt{x^m - a + x^m - b}}$$

$$- \int W \times \frac{2mx^{2m-1}\dot{x} - \overline{a+b}\,.\,mx^{m-1}\dot{x}}{2\sqrt{x^m - a \times x^m - b}}.$$ But

$-\int W \times \frac{2mx^{2m-1}\dot{x} - \overline{a+b}.mx^{m-1}\dot{x}}{2\sqrt{\overline{x^m-a} \times \overline{x^m-b}}}$ is $-W \times \sqrt{\overline{x^m-a} \times \overline{x^m-b}}$ $+b^r \times \int x^{n-rm}\dot{x}\sqrt{\frac{x^m-a}{x^m-b}}$; hence, by ſubſtituting this for the laſt term, it is manifeſt, that $-W \times \sqrt{\overline{x^m-a} \times \overline{x^m-b}}$ will be deſtroyed by the laſt term of $F \times \sqrt{\overline{x^m-a} \times \overline{x^m-b}}$ when we ſubſtitute for F its value; therefore, if we put $M = \frac{x^{n-m+1}}{n-m+1} + \frac{bx^{n-2m+1}}{n-2m+1} + \&c. + \frac{b^{r-1}x^{n-rm+1}}{n-rm+1}$, we have $\int x^n\dot{x}\sqrt{\frac{x^m-a}{x^m-b}} = M \times \sqrt{\overline{x^m-a} \times \overline{x^m-b}} - A\int x^{n-m}\dot{x}\sqrt{\frac{x^m-a}{x^m-b}} - \overline{Aa+B} \times \int x^{n-2m}\dot{x}\sqrt{\frac{x^m-a}{x^m-b}} - \overline{Aa^2+Ba+C} \times \int x^{n-3m}\dot{x}\sqrt{\frac{x^m-a}{x^m-b}} - \&c. - \overline{Aa^{r-1}+Ba^{r-2}+\&c.-b^r} \times \int x^{n-rm}\dot{x}\sqrt{\frac{x^m-a}{x^m-b}} - \overline{Aa^r+Ba^{r-1}+Ca^{r-2}+\&c.} \times \int \frac{x^{n-rm}\dot{x}}{\sqrt{\overline{x^m-a} \times \overline{x^m-b}}}$. Hence if the laſt two fluents be given, we have the general law of continuation up to the fluent of $x^n\dot{x}\sqrt{\frac{x^m-a}{x^m-b}}$.

The utility of finding fluents by continuation was manifeſt to Sir ISAAC NEWTON, who firſt propoſed it; and ſince his time ſome of the moſt eminent mathematicians have employed much of their attention upon it. The method which I have inveſtigated and exemplified in this Paper I offer as being entirely new; and at the ſame time it not only exhibits, at once, the general law up to the required fluent, but alſo appears, from ſome of the inſtances here given, to be more extenſive and convenient in its application than any method hitherto offered.

The

The general resolution of the given fluxion into a series of fluxions of the same kind, where the index of the unknown quantity without the vinculum keeps decreasing or increasing either by the index under or by half the index, has not, that I know of, before been given; which furnishes us at once not only with a very easy method of *continuing* fluents, but also points out a very simple method of investigating the fluent of the given fluxion *without* continuation. For if $\int \dot{A} = p + b\int \dot{B} + c\int \dot{C} + d\int \dot{D} + \&c.$ $\int \dot{B} = p' + c'\int \dot{C} + d'\int \dot{D} + \&c.$ $\int \dot{C} = p'' + d''\int \dot{D} + \&c.$ &c. &c. then if for $\int \dot{B}$, $\int \dot{C}$, &c. &c. we substitute their respective values, we shall get a general series for $\int \dot{A}$ without continuation. The extent of any new method is, at first, seldom obvious; and how far that which is here proposed may be successfully employed in other cases will best appear from its application. Different methods will always be found to have their uses in particular cases; for where one becomes impracticable another will often be found to succeed; and I hope that which is here offered will contribute something towards facilitating the investigation of fluents.