XV. *Of the Temperament of those musical Instruments, in which the Tones, Keys, or Frets, are fixed, as in the Harpsichord, Organ, Guitar,* &c. *By Mr.* Tiberius Cavallo, *F. R. S.*

Read April 3, 1788.

THE scale of music, which is used at present, consists of seven principal notes or sounds, which musicians denote by the letters of the alphabet A, B, C, D, E, F, and G; which, together with some intermediate ones, commonly called flats and sharps, and the octave of the first, make 13 sounds.

When those sounds are considered with respect to the first, they are called by the following names, *viz.* the prime or key-note, the second minor, second, third minor, third major, fourth, fourth major, fifth, sixth minor, sixth major, seventh minor, seventh major, and octave.

Musical sounds are produced by the vibrations of the sonorous bodies, and they are acuter or graver as the vibrations performed in a given time are more or less in number; so that if a string vibrating 100 times in a second produces a certain sound, and another string vibrating 120 times in a second produces another sound, the latter is said to be acuter, higher, or sharper than the former.

The number of vibrations performed in a certain time principally depends on the thickness, length, and elasticity of the sonorous bodies; but as the simplest sonorous bodies, and the fittest for examination, are those strings which are equal in every

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other reſpect, excepting in their lengths, becauſe the number of vibrations, which they perform in a given time, is ſimply in the proportion of their lengths, we ſhall conſider only thoſe in the preſent inveſtigation, the number of vibrations performed by other ſorts of ſonorous bodies being eaſily deduced from them.

As the above-mentioned 13 ſounds are all different from each other, the ſtrings which produce them differ in length, and of courſe in the number of the vibrations, which, when ſtruck, they perform in a certain time. Here follow the proportions which the numbers of vibrations performed in a given time, or the length of the ſtrings which expreſs thoſe 13 ſounds, bear to the firſt, prime, or key-note.

Firſt	1	Fourth	$\frac{3}{4}$	Seventh minor	$\frac{5}{9}$
Second minor	$\frac{15}{16}$	Fourth major	$\frac{32}{45}$	Seventh major	$\frac{8}{15}$
Second	$\frac{8}{9}$	Fifth	$\frac{2}{3}$	Octave	$\frac{1}{2}$
Third minor	$\frac{5}{6}$	Sixth minor	$\frac{5}{8}$		
Third major	$\frac{4}{5}$	Sixth major	$\frac{3}{5}$		

If, inſtead of many ſtrings having thoſe lengths in order to expreſs the 13 ſounds, or notes of an octave, one ſtring be divided according to thoſe proportions, and this ſtring be ſtopped conſecutively in the different points or diviſions; on being ſtruck, it will expreſs the correſponding ſounds. Thus, if a ſtring ſtretched between two fixed points, as CZ fig. 1. Tab. III. be ſtruck, it will produce a ſound called the prime, firſt or key-note; if it be ſtopped in the middle, one-half of the ſtring will ſound the octave, its length, compared to that of the whole ſtring, being in the proportion of 2 to 1; if two-thirds of the ſtring be cauſed to vibrate, the ſound produced will be the fifth, its length,

length, compared to that of the whole ſtring, being as 2 to 3, and ſo of the reſt.

The higheſt ſound of the octave is expreſſed by the half of the ſtring; and if this half be divided again in the ſame manner or proportion, a higher octave will be obtained, the higheſt note of which will be expreſſed by a quarter of the original ſtring. This quarter may be divided again into a higher octave, and ſo on; therefore, a ſtring ſo divided may expreſs the ſounds of all the keys of an harpſichord or organ.

In regard to thoſe diviſions it muſt be obſerved, that as the notes of the ſecond octave bear the ſame proportion to the firſt note of that octave as the notes of the firſt octave reſpectively bear to the firſt note of that octave, or to the whole ſtring; and as the length of the ſtring expreſſing the firſt note of the ſecond octave, *viz.* *cZ*, fig. 1. is half the length of CZ, the firſt note of the firſt octave, it follows, that the length of the ſtring of every note in the ſecond octave is half the length of the correſponding note in the firſt octave; thus *g* in the ſecond octave is 120 inches long, and G in the firſt octave is 240 inches long, *viz.* twice 120. Hence, when the diviſions of the firſt octave are aſcertained, in order to find the diviſions of the notes of the ſecond octave, we need only take the half of the lengths expreſſing the notes in the firſt octave. By the very ſame reaſoning it is evident, that to find the diviſions for the third octave, we need only take the halves of the lengths which expreſs the notes of the ſecond octave, or the quarters of thoſe of the firſt octave, and ſo of the reſt.

The ſtring or line CZ, fig. 1. is divided in the above-mentioned manner, and in order to avoid confuſion, the diviſions of the principal notes only of the firſt and ſecond octave are annexed to it. The numbers under the line expreſs the lengths from

from Z to the divisions to which they stand near. The letters just over the lines are the names of the notes or sounds expressed by the corresponding lengths of the string. The fractional numbers express the proportion which each particular division bears to the whole string; and the Roman numbers denote the numerical names of each note with respect to its distance from the first, which is always included. For example, suppose the whole string to be called C, and to be 360 inches long: then, if this string be stopped in G, the part GZ will be 240 inches, *viz.* two-thirds of the whole string CZ; the sound expressed by it when struck is called G, and it is the fifth note from C, which is the first or key-note. Again, if this string be stopped in A, the part AZ will be equal to 216 inches, *viz.* three-fifths of the whole; the sound produced by it is called A, and is a sixth to the key-note C, &c.

It is evident, that if any of those divisions be considered as the first or key-note, then the other notes, though they retain their alphabetical names, must have their numerical names altered accordingly: for example, if we take D for the key-note, then A will be the fifth of it, whereas A was the sixth when C was considered as the key-note; thus also B is the third of G, and the seventh of C, and so on.

Thus much having been premised, we may proceed to shew the meaning of what is called *the temperament* in a system of musical sounds, and likewise the necessity of it. For this purpose it is necessary to recollect, first, that the string, divided in the above-mentioned manner, exhibits the various notes or sounds of the keys of an harpsichord, the pipes of an organ, &c. Secondly, that those divisions remain unalterable, so that the harpsichord, when tuned, cannot be altered in the course of performing on it; and, thirdly, that when any one of

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thoſe notes or diviſions is conſidered as the key-note, its ſecond, third, fourth, fifth, &c. muſt bear their reſpective proportions, according to what has been ſaid in the preceding pages.

Now, if amongſt the diviſions of the ſtring CZ, fig. 1. we take D for the firſt or key-note, its length being 320 inches, the length of its fifth muſt be $213\frac{1}{3}$ inches, *viz.* two-thirds of 320, that being the proportion which the fifth muſt bear to the key-note; but amongſt the diviſions of the ſtring, there is none equal to $213\frac{1}{3}$ inches; therefore, there is not a note among them which may ſerve for a fifth to D: however, as the length of AZ, *viz.* 216, is the neareſt to $213\frac{1}{3}$, this A muſt be taken for the fifth of D. It is evident, that this is an imperfect fifth of D; but if, in order to render it perfect, we make AZ equal to $213\frac{1}{3}$ inches inſtead of 216, then it will be a redundant ſixth to C, when C is conſidered as the key-note; the beſt expedient, therefore, is to divide the imperfection between the two lengths, *viz.* to make AZ neither ſo long as 216, nor ſo ſhort as $213\frac{1}{3}$, which will render the diſagreeable ſenſation, ariſing from the improper length, the leaſt poſſible. This alteration of the juſt lengths of ſtrings, neceſſary for adapting them to ſeveral key-notes, is called *the temperament:* and the beſt temperament in a ſet of muſical ſounds is evidently ſuch a partition of the natural imperfections, as will render all the chords equally and the leaſt diſagreeable poſſible.

What has been exemplified in D and A may be ſaid of all the other notes; ſo that if any one of them be a perfect third, fifth, &c. with reſpect to one key-note, it will be found to be imperfect with reſpect to others. Hence it is manifeſt, firſt, that in a ſet of muſical keys, pipes, or frets, a temperament is abſolutely neceſſary; and, ſecondly, that the harpſichord, organ,

organ, guitar, or any other instrument in which the notes are fixed, so as not to be alterable by the performer's hands, must be imperfect even when tuned in the best manner possible; for by the temperament we can divide, but not annihilate, the imperfection.

Other instruments, in which the notes are not fixed, as the violin, violoncello, &c. are perfect, because the performer stops the strings upon them in different places, even for sounding the notes of the same name. Thus a skilful performer, in order to sound A, will stop the string a little farther from the bridge when he plays in the key of C, *viz.* when C is considered as the key-note, than when he plays in the key of D.

Most people imagine, that the scale of music is capable of many different temperaments; and, agreeable to this supposition, the writers on harmonics have proposed different temperaments; but it will be shewn in the sequel, that the nature of the scale admits of only one temperament capable of rendering the imperfection and the harmony equal throughout; and that it is impossible to form a different and more advantageous scale.

Before we begin with the investigation of this subject, it will be necessary to explain certain principles, the want of which may possibly raise some doubts in the minds of those persons who are not much acquainted with the theory of musical sounds. In the first place it must be observed, that the proportion of 2 to 3 for the fifth, the proportion of 1 to 2 for the octave, and in short the proportions of all the notes, are not assumed at pleasure; but they have been determined from constant experience, *viz.* from the agreeable or disagreeable effects produced when two different notes are sounded at the same time.

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To render this more evident, let two ſtrings equal in every reſpect be ſtruck at the ſame time, and they will expreſs the ſame ſound preciſely, ſo that no ear can perceive any difference between them, and it is almoſt impoſſible to diſtinguiſh whether the ſound ariſes from two ſtrings, or from one only, excepting from the loudneſs. But if one of thoſe ſtrings be ſucceſſively ſtopped in different parts of its length, whilſt the other remains open as before, and if at every time they be both ſtruck together, their combined ſounds will be found to produce different effects, *viz.* ſometimes more or leſs pleaſing, and at other times more or leſs diſagreeable. When the combinations of the two ſounds are agreeable, they are called *concords*; and when diſagreeable, they are called *diſcords*.

Experience evinces, that the beſt concord is when the length of one ſtring is to the length of the other as 1 to 2, every other circumſtance being the ſame in both. This proportion forms the octave. The next beſt concord is the fifth, *viz.* when the lengths of the two ſtrings are as 2 to 3, after which come the proportions of 3 to 4, 4 to 5, 3 to 5, 5 to 6, and 5 to 8, for the other concords. The other proportions beſides theſe are diſagreeable in a greater or leſs degree, unleſs they are greater than the proportion of 1 to 2; but in that caſe it will be found, that the proportions which produce agreeable combinations are the double, quadruple, octuple, &c. of thoſe mentioned above, *viz.* are their octaves, double octaves, &c.: thus the proportion of 1 to 4 produces a very agreeable concord, becauſe 1 to 4 is the double of 1 to 2, *viz.* it expreſſes a double octave.

Secondly, it appears, from the foregoing obſervations, that if we have the length of a ſtring, or the proportion of a note in any part of the ſtring, we may eaſily find its octaves by taking

taking its double, or its half, or the double of the double, &c.: for instance, in fig. 1. if *cZ* be given equal to 90 inches, we may find its octave below by taking twice 90, *viz.* 180, or the octave of this octave, which is 360, *viz.* equal to twice 180, or to four times 90; and, on the other side, we may find the octave above of the given note by taking its half, which is 45, &c.

It is now necessary to shew why within the octave there are admitted only thirteen different notes, *viz.* eight principal ones, and five others, called sharps and flats.

The line XY, fig. 2. represents a musical string, the length of which is supposed to be divided into 13286025 equal parts *. On one side of this line there are the divisions of seven successive octaves, *viz.* the half of XZ, a quarter of it, &c.; and on the other side are the divisions of a series of fifths, *viz.* the fifth of the whole string, the fifth of this fifth, and so on, which are found by taking two-thirds of the whole string, then two-thirds of those two thirds, and so on.

Here we take notice only of the octaves and fifths, because they are the principal and the best concords; so that a temperament being required, it is necessary first to take care, that these concords be not rendered insufferable to the ear, the rest admitting of a greater latitude in the temperament or deviation from the perfect state. Besides, it will appear in the sequel, that all the other notes are derived from the series of successive fifths.

In whatever key a piece of music is performed, its fifth is the most predominant of its concords; and as the notes of

* This number has been chosen, because the line may in that case be divided into the necessary number of successive octaves and fifths, without any common fractions, which renders the operation more easy and more perspicuous; otherwise there might have been assumed any other number.

music must be so ordered as that, for the sake of modulation, any note may be considered as the key-note; therefore having found the fifth of the whole string by taking two-thirds of its length, which gives a note called G, we must suppose, that this G may be considered as the key-note, consequently must find its fifth, which gives D, as shewn in the figure, and so on, until we find one of those successive fifths, which coincides with one of the successive octaves; for after that, to find more successive fifths would be only repeating the same thing over again.

Indeed, if we carry the succession of octaves and of fifths indefinitely far, we shall find, that no one of the fifths ever coincides perfectly with one of the octaves, and therefore the division would have no end. However, as the length of the seventh octave comes so very near to the twelfth fifth, we must be contented with taking this seventh octave for the fifth of F, the difference between this note and the perfect fifth of F being about the hundredth part of its length; whereas, if we carry on the succession of fifths and of octaves, we shall find, that amongst thirty and more fifths none comes nearer to one of the octaves than the above-mentioned one, as may be seen in the following table, which contains a series of successive octaves, and another series of successive fifths, as in fig. 2.; but in the table the divisions are carried farther on than in the figure:

Octaves.	Fifths.
13286025	13286025.
6643012,5	8857350.
3321506,25	5904900.
1660753,125	3936600.
830376,5625	2624400.

Octaves

Octaves.	Fifths.
415188,28125	1749600.
207594,140625	1166400.
☞ 103797,0703125	777600.
51898,5351 +	518400.
25949,2675 +	345600.
12974,6387 +	230400.
6487,3193 +	153600.
3243,654 +	☞ 102400.
1621,827 +	68266,6 +
810,913 +	45511 +
405,456 +	30340,7 +
202,728 +	20227 +
101,364 +	6742,3 +
50,662 +	4494,8 +
25,331 +	2996,6 +
12,665 +	1997,7 +
	1331,8 +
	887,8 +
	592 +
	394,6 +
	263 +
	175,4 +
	117 +
	78 +
	52 +
	34,6 +
	23 +
	15,3 +

The number of fifths then in this series is twelve, and as, from what has been said above, when the division expressing a certain note has been assigned in any part of a string, we may

easily find all its octaves above and below, it follows, that by finding all the octaves of those twelve divisions we shall have twelve distinct notes within half the string, *viz.* within the first octave of the whole string; to which, if the sound of the whole string be added, we shall have thirteen different sounds, which shews why an octave comprehends neither more nor less than thirteen notes.

Without dwelling any longer upon the names or number of those notes, I shall immediately proceed to find out the temperament.

It has been shewn above, and it is expressed in fig. 2. that the length of the string for the last fifth is shorter than the length of the last octave, and also that one of them must be necessarily taken for both purposes; but here we must consult nature, examining by the ear which of the two is least disagreeable. This, however, is soon decided; for imperfect octaves are quite insufferable, whereas a certain degree of imperfection in the fifths is tolerable; therefore we are necessitated to leave the octaves perfect, and to let the seventh octave serve for the fifth of F. In this case it is evident, that each of the notes in the succession of fifths is a perfect fifth to its preceding note, excepting the last, which would be by much too flat, and therefore it is necessary to divide the imperfection equally among them all.

For this purpose it must be considered, that as the twelve successive fifths, together with the whole string or first note, are each two-thirds of its preceding note; they form a geometrical series, the ratio of which is $\frac{2}{3}$, its extremes are 13286025 and 102400, and the number of terms is 13. But because instead of 102400, which is the last fifth, we must take the number 103797,0703125 (*viz.* the length of the last octave) for

for the laſt term of the ſeries; therefore the problem is reduced to the finding out of eleven mean proportionals between the two numbers 13286025 and 103797,0703125.

It is demonſtrated in almoſt every treatiſe on algebra and arithmetic, that in a geometrical progreſſion, as the above-mentioned one, the firſt or ſmalleſt extreme is to the laſt or greateſt extreme as unity is to a power of the ratio, the index of which is equal to the number of terms leſs one. Hence, in our caſe, in which the number of terms, including the two extremes, is 13, we ſhall have 103797,0703125 : 13286025 :: 1 : R^{12}, from which the ratio is found by dividing the ſecond number by the firſt, and extracting the twelfth root from the quotient, *viz.* $\frac{13286025}{103797,0703125} = 128$; and $\overline{128}\rceil^{\frac{1}{12}} = 1,4983069$, which is the ratio ſought.

The ratio having been aſcertained, the ſucceſſion of tempered fifths is thus eaſily determined; *viz.* divide the length of the whole ſtring by this ratio, and the quotient gives the firſt tempered fifth; divide this fifth by the ſame ratio, and the quotient gives the ſecond tempered fifth; divide this ſecond fifth by the ſame ratio, and ſo on till the laſt fifth, which comes out equal to 103797, 21735 (ſee fig. 3.) which is ſo nearly equal to the length of the ſeventh octave, that the difference is truly inſignificant; but, if greater accuracy were required, we need only extract the proper root of 128 to a greater number of decimals.

Fig. 3. ſhews the diviſions of the ſtring XZ tempered in the above-mentioned manner; *viz.* the ſucceſſive fifths have been aſcertained firſt, and then, by taking their octaves, the whole ſet of diviſions has been completed. By comparing this figure with fig. 2. one may eaſily perceive, how ſmall is

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the difference between the perfect fifths of the latter, and the tempered ones of the former.

The divisions, thus ascertained, form a series of notes, in which the octaves only are perfect; but all the fifths, all the thirds, and in short all the chords of the same denomination, are equally tempered throughout: so that whichever of them is taken for the key-note, its fifth, sixth, &c. will have always the same proportion to it, and consequently will always produce the same harmony when sounded with it.

It is evident, that, besides this, there can be no other temperament capable of producing equal harmony; for when the extremes of a geometrical series and number of mean proportionals are given, there can be but one set of those means: thus, if we are to find two mean proportionals between the numbers 2 and 16, these are necessarily 4 and 8; nor is it possible to assign any others.

If, on the other hand, we endeavour to find a better temperament by introducing more than thirteen notes within the limits of an octave, we shall find it impracticable, because it has been shewn, in the preceding pages, that after the number thirteen, if the succession of fifths be carried farther on, they will recede more from a coincidence with any one of the octaves.

This explanation of the nature, origin, and necessity of the temperament has been thought necessary for the sake of perspicuity; but the same end may be obtained by the following easier method. As the thirteen notes of an octave must be arranged so, that whichever of them be taken for the first or key-note, the second, third, fourth, &c. may bear the same constant proportion to it; therefore it follows, that they must be in a geometrical proportion one of the other, so as to form a

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ſeries of thirteen numbers, the extremes of which are the whole ſtring and its half, *viz.* any number and its half. The ratio of this ſeries is found in the ſame manner as in the other ſeries, *viz.* the greateſt extreme is divided by the leaſt, and the twelfth root of the quotient is the ratio ſought. But the extremes are any aſſumed number and its half: and as the quotient of a number divided by the half of the ſame number is always equal to two; therefore, whatever be the length of the ſtring, the ratio is always $\overline{2}|^{\frac{1}{12}} = 1{,}0594+$, and if the length of the whole ſtring be divided by this ratio, *viz.* 1,0594+, the quotient will be the length of the ſtring expreſſing the ſecond note, which, divided by the ſame ratio, gives the third note, and ſo on; or elſe, inſtead of dividing the length of the whole ſtring by the ratio, you may multiply the half of it by the ratio, the product of which will give the ſeventh note, which multiplied by the ſame ratio gives the ſixth, and ſo on in a retrograde order, which will give the tempered notes of the octaves as well as the former method. By this means the following diviſions for the notes of an octave have been calculated, the length of the whole ſtring having been ſuppoſed equal to 100000.

I.	100000.
✻ b	94387.
II.	89090.
✻ b	84090.
III.	79370.
IV.	74915.
✻ b	70710.
V.	66743.
✻ b	62997.
VI.	59462.
✻ b	56123.

VII. 52973.
VIII. 50000.

If a monochord be divided in this manner, and a harpsichord tuned by it, this instrument will then be tuned so, that whichever note be taken for the first or key-note, its fifth, sixth, &c. will produce the same effect respectively.

Thus far I have endeavoured to explain this subject in the most familiar manner, avoiding as much as possible the mathematical language and symbols; having found, by experience, that intricate mathematical disquisitions, especially on this subject, are understood only by a few able mathematicians, but that they are neither comprehended, nor even read, by those who might wish to understand, or to use them. It is now necessary to consider this subject with respect to the practice.

At present, the harpsichords and organs are commonly tuned so, that some concords are very agreeable to the ear, whilst others are quite intolerable; or, in other words, when the performer plays in certain keys, the harmony is very pleasing, in others the harmony is just tolerable, and in some other keys the harmony is quite disagreeable.

The best keys to be played in are the keys of C, of F, of E flat, of B flat, of G and of D in the major mood, and the keys of C, of D, of A, and of B, in the minor mood. Next to those come the less agreeable keys of A, A flat, and E in the major mood; besides those, the rest are disagreeable in a greater or less degree, so that out of twelve keys*, which, on account of the two moods, *viz.* the major and the minor, become twenty-four, there are hardly fourteen that can be

* The octave must be excepted, in this respect it being the same as the first note.

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used; and for this reason most of the modern compositions in music are written in those keys.

So far the common method of tuning answers some purpose; for as long as the performer is to play in certain keys only, it is much better to have them tuned in the most advantageous manner, than to let those be tuned in a less perfect manner for the sake of others, which he does not intend to use. Hence the great harpsichord players generally have their instruments tuned in a peculiar manner, *viz.* so as to give the most advantageous effect to those concords which they more frequently use in their compositions. And hence also, the harpsichords and organs are always tuned different from each other, unless they be tuned by the same person with equal attention, and without any particular instructions.

This practice cannot conveniently be laid aside, *viz.* when the instrument is to be tuned for solo playing; and for a certain style of music, it is very proper to tune it so as to give the greatest effect to those combinations of sounds, which are mostly used in those compositions. But the case is far different when the instrument is to serve for accompanying other instruments in every sort of music, or the voices of good singers; for then the disagreement becomes very audible; and for this purpose the harpsichord or organ ought to be tuned according to the above demonstrated temperament of equal harmony, which is the only one that can possibly take place.

When the compositions of old masters are performed in concert, and with the organ or harpsichord tuned in the common manner, the effect is frequently very disagreeable. This is particularly the case with the songs of HANDEL, GALLUPPI, LEO, PERGOLESE, and others, who wrote in a great variety

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of keys, and very often in thoſe, for which the common way of tuning is not at all calculated.

In order to hear the effect of the above-mentioned temperament of equal harmony, I had a monochord made in a very accurate manner, and upon it I laid down the diviſions for the thirteen notes of an octave properly tempered in the manner explained above. After a great deal of trouble in adjuſting the moveable fret, correcting the diviſions, &c. I at laſt ſucceeded ſo well as to render the diviſions exact within at leaſt the 300th part of an inch, and every part of the inſtrument was rendered ſufficiently ſteady and unalterable.

This being done, I had a large harpſichord, with a ſingle uniſon (in order to judge the better of the effect), tuned very accurately by the help of the monochord. With this inſtrument, in whatever key the performer played, the harmony was perfectly equal throughout, and the effect was the ſame as if one played in the key of E natural on a harpſichord tuned in the uſual manner.

I ſhall, therefore, conclude with ſaying, that when the harpſichord, organ, &c. is to ſerve for ſolo playing, and for a particular ſort of muſic, it is proper to tune in the uſual manner, *viz.* ſo as to give the greateſt effect to thoſe concords which occur more frequently in that ſort of muſic; but that when the inſtrument is to ſerve for accompanying other inſtruments or human voices, and eſpecially when modulations and tranſpoſitions are to be practiſed, then it muſt be tuned according to the temperament of equal harmony, which has been explained in the preceding pages.