

XX. *On the Probabilities of Survivorships between Two Persons of any given Ages, and the Method of determining the Values of Reversions depending on those Survivorships.* By Mr. William Morgan; communicated by the Rev. Richard Price, D. D. F. R. S.

Read May 8, 1788.

THE hypothesis of an equal decrement of life, adopted by M. DE MOIVRE, for the purpose of facilitating the computations of life annuities, has not only been rendered unnecessary by the late publication of many excellent tables deduced from real observations\*, but has likewise been found so very incorrect in some cases, that probably little or no recourse will ever be had to it in future. But though the direct application of this hypothesis may be laid aside, there is danger of its not being entirely abandoned; and mathematicians may still be led to reason from this principle, by deriving their rules from the *expectations* rather than from the *real probabilities* of life. The ingenious Mr. THOMAS SIMPSON has contented himself with this inaccurate method in his *Select Exercices*, and he has been followed in it by most other writers on the subject†. Even in those cases which involve only two lives, the errors are often

\* See Dr. PRICE's *Treatise on Reversionary Payments*, 4th edit.; and Mr. BARON MASERES on Life Annuities.

† Dr. PRICE's solutions of his 15th and 16th questions are exceptions to this remark.

considerable, especially if the expectations are derived from the *London*, the *Sweden*, or any other tables, in which the decrements of life are unequal. But when three lives are involved in the question, these errors are generally enormous; nor is it ever safe, when the ages of those lives differ very much, to have recourse to rules which are founded upon this principle. The three following problems, though the most common in the doctrine of survivorships, have never hitherto been solved in a manner strictly true. The second of them is of particular importance, and I have taken much pains to examine how far Mr. SIMPSON's solution of it may be depended upon. It has, indeed, been solved by M. DE MOIVRE \*, and Mr. DODSON †: but the first of these writers has erred most egregiously in the solution itself, and the other having derived his rule from a wrong hypothesis, has rendered it of no use. It is much to be wished that the solutions of all cases in reversions and survivorships were deduced, like the three following ones, from the real probabilities of life. Most of those which are now in use are at best but approximations, and can never be relied on with any tolerable degree of satisfaction.

#### PROBLEM I.

Supposing the ages of two persons, A and B, to be given; to determine the probabilities of survivorship between them from any table of observations.

#### SOLUTION.

Let  $a$  represent the number of persons living in the table at the age of A the younger of the two lives. Let  $a'$ ,  $a''$ ,  $a'''$ ,

\* See Mr. DE MOIVRE's 17th problem, and Dr. PRICE's remarks upon it in his *Treatise on Reversionary Payments*, Essay 3. Vol. I.

† See DODSON's *Mathematical Repository*, Prob. 23. Vol. III.

$a''''$ , &c. represent the decrements of life at the end of the 1st, 2d, 3d, 4th, &c. years from the age of A. Let  $b$  represent the number of persons living at the age of B the older of the two lives, and  $c, d, e, f$ , &c. the number of persons living at the end of the 1st, 2d, 3d, 4th, &c. years from the age of B. Supposing now it were required to determine the probability of B's surviving A in the first year. It is manifest that this event may take place either by A's dying before the end of the year and B's surviving that period, or by the extinction of both the lives, restrained however to the contingency of B's having died *last*. The probability that A dies in the first year, and that B survives it, is expressed by the fraction  $\frac{a'c}{ab}$ . The probability that both the lives die in this year is expressed by the fraction  $\frac{a' \cdot \overline{b-c}}{ab}$ ; and as it is very nearly an equal chance that A dies *first*, this fraction should be reduced *one-half*, and then it will become  $= \frac{a' \cdot \overline{b-c}}{2ab}$ . Hence the whole probability of B's surviving A in the first year will be  $= \frac{a'c}{ab} + \frac{a' \cdot \overline{b-c}}{2ab} = \frac{a' \cdot \overline{b+c}}{2ab}$ . In the same manner the probability of B's surviving A in the 2d, 3d, 4th, &c. years may be found  $= \frac{a'' \cdot \overline{c+d}}{2ab} \dots \frac{a''' \cdot \overline{d+e}}{2ab} \dots \frac{a'''' \cdot \overline{e+f}}{2ab}$ , &c. respectively; therefore, the whole probability of B's surviving A will be  $= \frac{1}{ab} \times \frac{\overline{b+c}}{2} a' + \frac{\overline{c+d}}{2} a'' + \frac{\overline{d+e}}{2} a''' + \frac{\overline{e+f}}{2} a''''$ , &c.

Having found by the preceding series the probability of B, the elder life's, surviving A the younger; the other expression,

which denotes the probability of A's surviving B, is well known to be the difference between the foregoing series and *unity*.

The sum of this series might easily be determined from tables of the expectations of single and joint lives\*. But no such table as the latter having ever been computed, it will by no means be found a laborious undertaking to compute a table of the probabilities of survivorship between two persons of all ages immediately from this series, without having recourse to the expectations of life. For if the probability of survivorship between any two persons be found, the probability between two persons one year younger is obtained with little difficulty; and by proceeding in this manner a whole table may be formed in less time than would be necessary for computing one of the expectations of two joint lives. To exemplify what I have said, I shall just set down a few operations for determining the probability of survivorship, according to the Northampton Table of Observations†, between two persons, whose common difference of age is 10 years.

\* The above series, expressing the probability of survivorship, may be easily found, from the solution of the next problem,  $= \frac{\beta \cdot \overline{K} - \overline{AK} - c \cdot \overline{C} - \overline{AC}}{2b}$ ; K

denoting the expectation of one life one year younger than B; C the expectation of a life one year older than B; AK and AC the expectations of the joint lives of A and K, and of A and C, respectively; and  $\beta$ , as in that problem, the number of persons living in the table of observations at the age of K.

† See Dr. PRICE's Treatise on Reversionary Payments, Tab. 6. Vol. II.

Age of B.	Age of A.	Probability of B's surviving A.	Probability of A's surviving B.
96	86	$\frac{1}{1 \times 145} \times \frac{0+1}{2} \times 34 = .1173$	$1 - .1173 = .8887$
95	85	$\frac{1}{4 \times 186} \times \frac{4+1}{2} \times 41 + 17 = .1606$	$1 - .1606 = .8394$
94	84	$\frac{1}{9 \times 234} \times \frac{9 \times 4}{2} \times 48 + 119.5 = .2049$	$1 - .2049 = .7951$
93	83	$\frac{1}{16 \times 289} \times \frac{16+9}{2} \times 55 + 431.5 = .2420$	$1 - .2420 = .7580$
92	82	$\frac{1}{24 \times 346} \times \frac{24+16}{2} \times 57 + 1119 = .2720$	$1 - .2720 = .7280$
91	81	$\frac{1}{34 \times 406} \times \frac{34+24}{2} \times 60 + 2259 = .2897$	$1 - .2897 = .7103$
90	80	$\frac{1}{46 \times 469} \times \frac{46+34}{2} \times 63 + 3999 = .3022$	$1 - .3022 = .6978$

It may easily be seen from these specimens in what manner the probabilities of survivorship between two younger lives are deduced from the probabilities between two older lives, provided their common difference of age be the same; for the numbers 17 . . 119.5 . . . 431.5, &c. in the 2d, 3d, 4th, &c. series are the sums of the series next preceding. Thus  $17 \text{ is } = 34 \times \frac{1}{2} \dots 119.5 \text{ is } = 41 \times \frac{1}{2} + 17 \dots 431.5 \text{ is } = 48 \times \frac{1}{2} + 119.5$ , &c. It may be necessary to observe further, that if the ages of the two persons be equal, the probability of survivorship between them being likewise equal, is expressed by the fraction  $\frac{1}{2}$ ; and that this affords an instance of the accuracy of the foregoing investigation; for the series expressing the probability in this case is the same with this fraction, the chance of survivorship becoming then (since  $a=b$ ;  $a'=b-c$ ;  $a''=c-d$ , &c.; and  $\overline{b+c} \times a' = \overline{b+c} \times \overline{b-c} = b^2 - c^2$ , &c.)  $= \frac{bb-cc}{2bb} + \frac{cc-dd}{2bb}$ , &c.  $= \frac{1}{2}$ .

Mr. SIMPSON, in his *Treatise on Annuities and Reversions* \* has given a curve whose area determines the probability of survivorship between two persons according to any table of observations. If one of the lives be not very young so that the equidistant ordinates may not be too few, this area is sufficiently correct. But if the eldest of the two lives is under 20 years of age, it becomes necessary to assume so many equidistant ordinates to render the solution accurate, when the decrements of life are unequal, that the operation is rendered much too laborious for use; nor do I know that it can be necessary to have recourse to this area in any case, especially as the true probabilities of survivorship are so easily computed from the preceding series.

The following table has been formed in the manner described above; and as no such table has ever been attempted before, I have been the more desirous to render it complete, by computing the probabilities of survivorship between two persons of all ages, whose common difference is not less than ten years.

Instead also of supposing certainty to be denoted by unity, I have assumed 100 for this purpose; so that the sums in the adjoining columns express the number of chances in 100 which are in favour of B or A's surviving the other.

\* Lemma 2, p. 100.

TABLE, shewing the probabilities of survivorship between two persons of all ages, whose common difference of age is not less than ten years, computed from the Northampton Table of Observations in Dr. PRICE's Treatise on Reverſionary Payments.

Ten years difference.			Twenty years difference.			Thirty years difference.		
Ages.	Probabilities.		Ages.	Probabilities.		Ages.	Probabilities.	
11-1	58.59	41.41	21-1	52.46	47.54	31-1	48.23	51.77
12-2	51.30	48.64	22-2	44.33	55.67	32-2	39.36	60.64
13-3	48.23	51.77	23-3	40.88	59.12	33-3	35.57	64.43
14-4	45.98	54.02	24-4	39.00	61.00	34-4	32.86	67.14
15-5	44.70	55.30	25-5	37.69	62.31	35-5	31.35	68.65
16-6	43.43	56.57	26-6	36.40	63.60	36-6	29.85	70.15
17-7	42.53	57.47	27-7	35.48	64.52	37-7	28.83	71.17
18-8	41.91	58.09	28-8	34.84	65.16	38-8	27.98	72.02
19-9	41.60	58.40	29-9	34.52	65.48	39-9	27.53	72.47
20-10	41.53	58.47	30-10	34.41	65.59	40-10	27.33	72.67
21-11	41.58	58.42	31-11	34.40	65.60	41-11	27.24	72.76
22-12	41.68	58.32	32-12	34.42	65.58	42-12	27.18	72.82
23-13	41.78	58.22	33-13	34.44	65.56	43-13	27.14	72.86
24-14	41.90	58.10	34-14	34.47	65.53	44-14	27.11	72.89
25-15	42.02	57.98	35-15	34.51	65.49	45-15	27.08	72.92
26-16	42.16	57.84	36-16	34.56	65.44	46-16	27.06	72.94
27-17	42.26	57.74	37-17	34.58	65.42	47-17	27.01	72.99
28-18	42.32	57.68	38-18	34.54	65.46	48-18	26.90	73.10
29-19	42.34	57.66	39-19	34.44	65.56	49-19	26.72	73.28
30-20	42.31	57.69	40-20	34.29	65.71	50-20	26.50	73.50
31-21	42.22	57.78	41-21	34.08	65.92	51-21	26.21	73.79
32-22	42.09	57.91	42-22	33.84	66.16	52-22	25.89	74.11
33-23	41.97	58.03	43-23	33.59	66.41	53-23	25.56	74.44
34-24	41.83	58.17	44-24	33.35	66.65	54-24	25.22	74.78
35-25	41.70	58.30	45-25	33.09	66.91	55-25	24.87	75.13
36-26	41.56	58.44	46-26	32.83	67.17	56-26	24.52	75.48
37-27	41.41	58.59	47-27	32.56	67.44	57-27	24.16	75.84
38-28	41.26	58.74	48-28	32.28	67.72	58-28	23.79	76.21
39-29	41.10	58.90	49-29	31.99	68.01	59-29	23.41	76.59
40-30	40.94	59.06	50-30	31.70	68.30	60-30	23.02	76.98
41-31	40.78	59.22	51-31	31.43	68.57	61-31	22.63	77.37
42-32	40.63	59.37	52-32	31.16	68.84	62-32	22.23	77.77
43-33	40.49	59.51	53-33	30.88	69.12	63-33	21.81	78.19

Ten years difference.			Twenty years difference.			Thirty years difference.		
Ages.		Probabilities.	Ages.		Probabilities.	Ages.		Probabilities.
44-34	40.34	59.66	54-34	30.60	69.40	64-34	21.38	78.62
45-35	40.19	59.81	55-35	30.32	69.68	65-35	20.93	79.07
46-36	40.04	59.96	56-36	30.03	69.97	66-36	20.47	79.53
47-37	39.88	60.12	57-37	29.73	70.27	67-37	20.01	79.99
48-38	39.71	60.29	58-38	29.43	70.57	68-38	19.54	80.46
49-39	39.54	60.46	59-39	29.13	70.87	69-39	19.06	80.94
50-40	39.38	60.62	60-40	28.82	71.18	70-40	18.59	81.41
51-41	39.23	60.77	61-41	28.49	71.51	71-41	18.10	81.90
52-42	39.07	60.93	62-42	28.14	71.86	72-42	17.58	82.42
53-43	38.90	61.10	63-43	27.74	72.26	73-43	17.05	82.95
54-44	38.81	61.19	64-44	27.33	72.67	74-44	16.53	83.47
55-45	38.53	61.47	65-45	26.90	73.10	75-45	16.04	83.96
56-46	38.35	61.65	66-46	26.46	73.54	76-46	15.60	84.40
57-47	38.16	61.84	67-47	26.01	73.99	77-47	15.15	84.85
58-48	37.97	62.03	68-48	25.55	74.45	78-48	14.68	85.32
59-49	37.77	62.23	69-49	25.09	74.91	79-49	14.16	85.84
60-50	37.56	62.44	70-50	24.61	75.39	80-50	13.61	86.39
61-51	37.30	62.70	71-51	24.06	75.94	81-51	13.04	86.96
62-52	37.00	63.00	72-52	23.49	76.51	82-52	12.46	87.54
63-53	36.68	63.32	73-53	22.91	77.09	83-53	11.94	88.06
64-54	36.34	63.66	74-54	22.35	77.65	84-54	11.60	88.40
65-55	35.97	64.03	75-55	21.81	78.19	85-55	11.30	88.70
66-56	35.58	64.42	76-56	21.31	78.69	86-56	11.04	88.96
67-57	35.17	64.83	77-57	20.80	79.20	87-57	10.80	89.20
68-58	34.74	65.26	78-58	20.24	79.76	88-58	10.63	89.37
69-59	34.30	65.70	79-59	19.59	80.41	89-59	10.25	89.75
70-60	33.85	66.15	80-60	18.90	81.10	90-60	9.65	90.35
71-61	33.38	66.62	81-61	18.23	81.77	91-61	8.68	91.32
72-62	32.90	67.10	82-62	17.57	82.43	92-62	7.54	92.46
73-63	32.46	67.54	83-63	17.03	82.97	93-63	6.18	93.82
74-64	32.04	67.96	84-64	16.73	83.27	94-64	4.93	95.07
75-65	31.70	68.30	85-65	16.55	83.45	95-65	3.68	96.32
76-66	31.45	68.55	86-66	16.29	83.71	96-66	2.58	97.42
77-67	31.20	68.80	87-67	16.33	83.62			
78-68	30.90	69.10	88-68	16.44	83.56			
79-69	30.47	69.53	89-69	16.23	83.77			
80-70	30.00	70.00	90-70	15.67	84.33			
81-71	29.58	70.42	91-71	14.50	85.50			
82-72	29.19	70.81	92-72	13.05	86.95			
83-73	28.90	71.10	93-73	11.08	88.92			
84-74	29.02	70.98	94-74	9.24	90.76			
85-75	29.17	70.83	95-75	7.17	92.83			
86-76	29.25	70.75	96-76	5.12	94.88			



Ten years difference.		
Ages.	Probabilities.	
87-77	29.42	70.58
88-78	29.94	70.06
89-79	30.29	69.71
90-80	30.22	69.78
91-81	28.97	71.03
92-82	27.20	72.80
93-83	24.20	75.80
94-84	20.49	79.51
95-85	16.06	83.94
96-86	11.73	88.27

Forty years difference.			Fifty years difference.			Sixty years difference.		
Ages.	Probabilities.		Ages.	Probabilities.		Ages.	Probabilities.	
41-1	43.57	56.43	51-1	39.16	60.84	61-1	34.94	65.06
42-2	33.98	66.02	52-2	28.87	71.13	62-2	24.13	75.87
43-3	29.85	70.15	53-3	24.46	75.54	63-3	19.52	80.48
44-4	26.88	73.12	54-4	21.28	78.72	64-4	16.19	83.81
45-5	25.20	74.80	55-5	19.48	80.52	65-5	14.28	85.72
46-6	23.53	76.47	56-6	17.68	82.32	66-6	12.37	87.63
47-7	22.30	77.70	57-7	16.35	83.65	67-7	10.92	89.08
48-8	21.40	78.60	58-8	15.35	84.65	68-8	9.82	90.18
49-9	20.86	79.14	59-9	14.74	85.26	69-9	9.12	90.88
50-10	20.59	79.41	60-10	14.42	85.58	70-10	8.73	91.27
51-11	20.45	79.55	61-11	14.22	85.78	71-11	8.46	91.54
52-12	20.35	79.65	62-12	14.06	85.94	72-12	8.25	91.75
53-13	20.26	79.74	63-13	13.90	86.10	73-13	8.05	91.95
54-14	20.18	79.82	64-14	13.76	86.24	74-14	7.88	92.12
55-15	20.11	79.89	65-15	13.62	86.38	75-15	7.75	92.25
56-16	20.05	79.95	66-16	13.50	86.50	76-16	7.68	92.32
57-17	19.96	80.04	67-17	13.35	86.65	77-17	7.59	92.41
58-18	19.81	80.19	68-18	13.14	86.86	78-18	7.43	92.57
59-19	19.59	80.41	69-19	12.86	87.14	79-19	7.19	92.81
60-20	19.31	80.69	70-20	12.53	87.47	80-20	6.90	93.10
61-21	18.96	81.04	71-21	12.11	87.89	81-21	6.55	93.45
62-22	18.55	81.45	72-22	11.65	88.35	82-22	6.16	93.84
63-23	18.12	81.88	73-23	11.20	88.80	83-23	5.81	94.19
64-24	17.68	82.32	74-24	10.75	89.25	84-24	5.56	94.44

Forty years difference.			Fifty years difference.			Sixty years difference.		
Ages.	Probabilities.		Ages.	Probabilities.		Ages.	Probabilities.	
65-25	17.22	82.78	75-25	10.32	89.68	85-25	5.32	94.68
66-26	16.76	83.24	76-26	9.91	90.09	86-26	5.11	94.89
67-27	16.29	83.71	77-27	9.50	90.50	87-27	4.90	95.10
68-28	15.81	84.19	78-28	9.07	90.93	88-28	4.73	95.27
69-29	15.32	84.68	79-29	8.61	91.39	89-29	4.48	95.52
70-30	14.83	85.17	80-30	8.15	91.85	90-30	4.13	95.87
71-31	14.34	85.66	81-31	7.70	92.30	91-31	3.63	96.37
72-32	13.84	86.16	82-32	7.27	92.73	92-32	3.10	96.90
73-33	13.35	86.65	83-33	6.88	93.12	93-33	2.48	97.52
74-34	12.87	87.13	84-34	6.60	93.40	94-34	1.94	98.06
75-35	12.42	87.58	85-35	6.35	93.65	95-35	1.40	98.60
76-36	11.99	88.01	86-36	6.12	93.88	96-36	0.95	99.05
77-37	11.56	88.44	87-37	5.91	94.09			
78-38	11.12	88.88	88-38	5.74	94.26			
79-39	10.64	89.36	89-39	5.48	94.52			
80-40	10.21	89.79	90-40	5.11	94.89			
81-41	9.68	90.32	91-41	4.55	95.45			
82-42	9.19	90.81	92-42	3.92	96.08			
83-43	8.76	91.24	93-43	3.15	96.85			
84-44	8.46	91.54	94-44	2.47	97.53			
85-45	8.18	91.82	95-45	1.80	98.20			
86-46	7.95	92.05	96-46	1.23	98.77			
87-47	7.74	92.26						
88-48	7.59	92.41						
89-49	7.32	92.68						
90-50	6.90	93.10						
91-51	6.17	93.83						
92-52	5.33	94.67						
93-53	4.32	95.68						
94-54	3.42	96.58						
95-55	2.51	97.49						
96-56	1.73	98.27						

Seventy years difference.			Eighty years difference.			Ninety years difference.		
Ages.	Probabilities.		Ages.	Probabilities.		Ages.	Probabilities.	
71-1	30.49	69.51	81-1	24.95	75.05	91-1	18.93	81.07
72-2	19.45	80.55	82-2	14.37	85.63	92-2	9.18	90.82
73-3	14.87	85.13	83-3	10.34	89.66	93-3	5.53	94.46
74-4	11.59	88.41	84-4	7.63	92.37	94-4	3.12	96.88
75-5	9.80	90.20	85-5	6.34	93.66	95-5	2.12	97.88
76-6	7.98	92.02	86-6	4.92	95.08	96-6	1.15	98.85
77-7	6.60	93.40	87-7	3.85	96.15			
78-8	5.53	94.47	88-8	3.04	96.96			
79-9	4.84	95.16	89-9	2.50	97.50			
80-10	4.44	95.56	90-10	2.16	97.84			
81-11	4.18	95.82	91-11	1.86	98.14			
82-12	3.97	96.03	92-12	1.57	98.43			
83-13	3.80	96.20	93-13	1.25	98.75			
84-14	3.72	96.28	94-14	0.97	99.03			
85-15	3.68	96.32	95-15	0.70	99.30			
86-16	3.70	96.30	96-16	0.49	99.51			
87-17	3.73	96.27						
88-18	3.75	96.25						
89-19	3.67	96.33						
90-20	3.45	96.55						
91-21	3.09	96.91						
92-22	2.63	97.37						
93-23	2.10	97.90						
94-24	1.64	98.36						
95-25	1.18	98.82						
96-26	0.80	99.20						

PROBLEM II.

Supposing the ages of A and B to be given; to determine, from any table of observations, the present value of the sum S payable on the contingency of one life's surviving the other.

## SOLUTION.

Let  $r$  denote  $\int 1$ . increased by its interest for a year, and let all the other symbols be the same as in the preceding problem. Let the life of B also be supposed to be the older of the two lives; and then it will follow, by reasoning as in the solution of that problem, that the present value of S to be received on death of A, should that happen in the life-time of B, will be expressed by the series

$$S \times \frac{b+c \cdot a'}{2abr} + \frac{c+d \cdot a''}{2abr^2} + \frac{d+e \cdot a'''}{2abr^3} + \frac{e+f \cdot a'''}{2abr^4}$$

&c. This series may be resolved into the two following;

$$\frac{S}{2} \times \frac{ca'}{abr} + \frac{da''}{abr^2} + \frac{ea'''}{abr^3} + \frac{fa'''}{abr^4} \quad \&c. + \frac{S}{2} + \frac{ba'}{abr} + \frac{ca''}{abr^2} + \frac{da'''}{abr^3} + \frac{fa'''}{abr^4} \quad \&c.$$

The first of these two series may be again resolved into

$$\frac{S}{2} \times \frac{c}{br} - \frac{ca-ca'}{abr} + \frac{d}{br^2} - \frac{da-da'-da''}{abr^2} + \frac{e}{br^3} - \frac{ea-ea'-ea''-ea'''}{abr^3} \quad \&c.$$

$$\left( -\frac{S}{2} \times \frac{da'}{abr^2} + \frac{ea'+ea''}{abr^3} \quad \&c. = \right) - S \times \frac{c}{2br} \times \frac{d}{cr} - \frac{da-da'}{acr} + \frac{e}{cr^2} -$$

$$\frac{ea-ea'-ea''}{acr^2} \quad \&c. \quad \text{Let B denote the value of an annuity on the}$$

life of B, C the value of an annuity on a life one year older than B, AB and AC the values of annuities on the joint lives of A and B and of A and C, and these series will be =

$$\frac{S \times B - AB}{2} - \frac{S \times c \times C - AC}{2br}. \quad \text{Again, the second series above men-}$$

$$\text{tioned, or } \frac{S}{2} \times \frac{ba'}{abr} + \frac{ca''}{abr^2} + \frac{da'''}{abr^3} \quad \&c., \text{ by pursuing the same steps}$$

$$\text{may be found} = \frac{\beta \times S}{2b} \times K - AK - \frac{S \cdot B - AB}{2r} \quad \text{where } \beta \text{ denotes the}$$

number of persons living at the age of a person one year younger than B, K the value of an annuity on that life,  
and

and AK the value of an annuity on the joint lives of A and K. The whole value of the survivorship is therefore =

$$S \times \frac{r-1 \cdot B-AB}{2r} + \frac{\beta \cdot K-AK}{2b} - \frac{c \cdot A-AC}{2br}. \quad Q. E. D.$$

Having now the value of the sum S payable on the contingency of B's surviving A, the value of the same sum, payable on the contingency of A's surviving B, is easily obtained by the well known method of subtracting the value found above from the whole value of the reversion after the extinction of the joint lives of A and B. It is evident, that the exactness of the above rule must depend on the accuracy with which the values of the single and joint lives are computed. Being possessed of such tables for all ages, even with respect to the joint lives, I have computed the following values, in order that it may be seen how far Mr. SIMPSON's approximation \* (the only rule now in use) may be depended on.

\* It must be remembered, that the correction explained by Dr. PRICE, in Vol. I. p. 39, &c. of his Treatise on Reversionary Payments, must be applied to Mr. SIMPSON's rule; that is, when the reversion is a *sum* and not an estate, the value found by the rule must be divided by £. 1, increased by its interest for a year.

		* Value of £. 100 payable on the death of A if B survives him.				Value of £. 100 payable on the death of A if B survives him.				Value of £. 100 payable on the death of A if B survives him.	
Age of B.	Age of A.	True value.	SIMP-SON'S value.	Age of B.	Age of A.	True value.	SIMP-SON'S value.	Age of B.	Age of A.	True value.	SIMP-SON'S value.
10	2	32.67	26.05	40	10	17.10	19.07	60	60	38.88	38.92
10	10	24.74	24.75	40	40	32.86	32.87	70	2	18.36	9.81
20	2	29.99	24.73	50	2	23.36	17.06	70	10	7.07	9.15
20	10	22.11	23.50	50	10	14.01	16.21	70	40	15.35	15.78
20	20	27.95	27.96	50	20	18.65	19.29	70	70	42.34	42.33
30	2	28.79	22.60	50	50	35.80	35.85	80	2	14.46	5.71
30	10	19.84	21.47	60	2	21.52	13.61	80	10	3.93	5.43
30	30	30.22	30.21	60	10	10.65	12.93	80	50	12.05	12.00
40	2	26.65	20.07	60	30	17.51	18.19	80	80	45.47	45.45

From this table it appears, that Mr. SIMPSON'S approximation in the middle stages of life is sufficiently accurate; but that it is exceedingly defective when the life of A is very young. It should also be remembered, that these values have been computed at a low rate of interest, and from the *Northampton* Table of Observations, in which the decrements of life come nearer to M. DE MOIVRE'S hypothesis than in any other table. But if the computations be made at a higher rate of interest even from this table, the approximation does not always agree so well, as will appear from the following specimens calculated at 5 per cent.

\* These values have been computed at 3 per cent. and from the *Northampton* Table of Observations.

Age of B Age of A		Value of £. 100 payable on the death of A if B survives him, by the Northampton table.		Age of B. Age of A.		Value of £. 100 payable on the death of A if B survives him, according to the Sweden Table of Observations, and at 4 per cent.		Age of B. Age of A.		True value.		SIMP-SON's value.	
		True value.	SIMP-SON's value.			True value.	SIMP-SON's value.			True value.	SIMP-SON's value.		
20	2	25.09	18.46	20	2	21.47	16.80	60	36	12.29	16.81		
20	10	15.49	17.54	20	14	15.42	17.82	60	42	16.11	19.58		
40	2	23.57	15.97	20	20	20.01	19.84	60	60	36.96	36.34		
60	2	19.83	11.75	40	4	23.53	14.22	76	40	9.21	9.81		
60	40	18.73	19.61*	40	16	13.71	16.23	76	52	12.58	14.00		
				40	28	17.60	20.44	76	64	23.81	22.81		
				40	40	27.62	27.00	76	76	42.90	43.29		
				60	24	9.39	13.01						

In order further to compare Mr. SIMPSON's approximation with the true value, I have inserted in the foregoing table a few computations deduced from the *Sweden* Table of Observations, in which the decrements of life are unequal. From these instances the approximation appears to be more defective in proportion as the probabilities of life differ from the hypothesis. It may be proper, however, to observe, that this table has been computed from the values of two joint-lives given by Dr. PRICE in his *Treatise on Reversionary Payments*, which values cannot be found with perfect correctness at every age, because given in that *Treatise* (Vol. II. p. 57, &c. and p. 144, &c.) only for ages that are either equal, or whose difference is five years or some multiple of five as in the *Northampton* Table, or six years or some multiple of six as in the *Sweden* Table. This circumstance may perhaps be supposed to affect the values of the survivorships computed from them, when the

\* These values at 3 per cent. are 21.92 and 22.78.

joint lives differ from each other in a greater or less degree than they have been given in those Tables. But by comparing the values of the reversions when the ages are equal (in which case Mr. SIMPSON gives the true rule) it will be seen, that the values of the joint lives are sufficiently correct for the purpose; and in order to put this matter out of all doubt, I have computed the values by the foregoing rule from the *exact* values of the joint lives by the *Northampton* Table, and also from those values which have been deduced from Dr. PRICE's Tables, and I have found them to agree exceedingly well\*.

It may be perceived, that in all the cases of *equal* lives I have computed the values of the survivorships by the preceding rule. This has only been done to prove its accuracy by comparing it with Mr. SIMPSON's rule, which is in this particular case, as I have already observed, perfectly right; for when the ages are equal, the chance of survivorship must also be equal, and therefore *half* the value of the reversion after the extinction of the joint lives will be the true value of the given sum payable on the death of either A or B, subject to the contingency of his being survived by the other life. The truth of this rule is self-evident; nor does it at all seem to depend on Mr. SIMPSON's solution in his *Select Exercises*. That it is capable, however, of being deduced from the foregoing series, may be demonstrated in the following manner. By

\* I shall just mention the following instances. When the age of B is 20 and that of A is 2 years, the value, by taking the joint lives from Dr. PRICE's Table, and approximating to the real values of those lives in the manner directed by him, Vol. II. p. 75. of his *Treatise*, is 29.59. When the ages of B and A are 40 and 2, 50 and 20, 60 and 2, and 70 and 10, the respective values are 26.93, 18.78, 21.57, and 7.08, which being compared with the values in p. 344. will be found to agree almost exactly with them.



reasoning as in p. 335. the value of the sum  $S$ , when the ages are equal, will be  $= S \times \frac{c}{br} \times \frac{b-c}{b} + \frac{b-c^2}{2bbr} + \frac{d}{br^2} \times \frac{c-d}{b} + \frac{c-d^2}{2bbr^2} + \&c. =$   
 $\frac{S}{2} \times \frac{bb}{bbr} + \frac{cc}{bbr^2} + \frac{dd}{bbr^3} + \&c. - \frac{S}{2} \times \frac{cc}{bbr} + \frac{dd}{bbr^2} + \frac{de}{bbr^3} + \&c. = \frac{S}{2r} + \frac{S}{2r} - \frac{S}{2}$   
 $\times BB$  (putting  $BB$  for the value of the two equal joint lives)  $=$   
 $\frac{S}{2} \times \frac{1}{r} - \frac{r-1}{r} \times BB =$  (since the perpetuity  $P$  is  $\frac{1}{r-1}$ , and the  
 interest of any sum is that sum multiplied by  $r-1$ )  
 $\frac{S}{2} \times \frac{P \cdot r - 1}{r} - \frac{BB \cdot r - 1}{r}$ , which is Mr. SIMPSON'S rule with Dr.  
 PRICE'S correction; that is, "half the sum multiplied by the  
 "difference between the perpetuity and the value of the equal  
 "joint lives, and divided by  $\text{£. 1}$  increased by its interest for a  
 "year."

### PROBLEM III.

The ages of  $A$  and  $B$  being given; to determine the value of the sum  $S$ , payable on the extinction of one life in particular, should that happen after the extinction of the other life.

### SOLUTION.

Supposing  $B$  to be the oldest of the two lives and the sum  $S$  to become payable on his decease, it is evident that this payment at the end of the first year must depend on the contingency of both lives being extinct before this period and of  $B$ 's dying *last*. Retaining the same symbols, and reasoning as I have done in the solution of the first problem, this value

will be expressed by the fraction  $\frac{S \cdot a' \cdot \overline{b-c}}{2abr}$ . The payment of the sum S at the end of the second year will depend on either of two events happening. First, that A and B both die in the second year after having survived the first, refrained, as above, to the contingency of B's having died last; secondly, that B dies in the second year and A in the first year. The value therefore of S for this year will be expressed by the two fractions  $\frac{S \cdot a'' \cdot \overline{c-d}}{2abr^2} + \frac{S \cdot \overline{c-d} \cdot a'}{abr^2}$ . Again, the payment of S in the third year will depend either on A and B's both dying in that year, and B having died *last*; or on B's dying in that year, and A's dying in the first or second years. The value therefore of S for this year will be  $= \frac{S \cdot a''' \cdot \overline{d-e}}{2abr^3} + \frac{S \cdot \overline{d-e} \cdot a' + a''}{abr^3}$ .

By proceeding in this manner for the other years the whole value of the reversion will be found  $= \frac{S}{2} \times \frac{a' \cdot \overline{b-c}}{abr} + \frac{a'' \cdot \overline{c-d}}{abr^2} + \frac{a''' \cdot \overline{d-e}}{abr^3} + \frac{a'''' \cdot \overline{e-f}}{abr^4} + \&c. + S \times \frac{a' \cdot \overline{c-d}}{abr^2} + \frac{a' + a'' \cdot \overline{d-e}}{abr^3} + \frac{a' + a'' + a''' \cdot \overline{e-f}}{abr^4} + \&c.$

The first of these series by proceeding in the same manner as in the solution of the second problem may be found  $=$

$$\frac{\beta}{2b} \times \overline{K-AB} - \frac{\overline{B-AB}}{2r} - \frac{1}{2} \cdot \overline{B-AB} + \frac{c}{2br} \times \overline{C-AC}; \text{ and the second series may be found } = -\frac{c}{br} \times \overline{C-AC} + \frac{\overline{B-AB}}{r}.$$

Hence the whole value of the reversion will be  $= S \times \frac{\beta r \cdot \overline{K-AB} - c \cdot \overline{C-AC}}{2br}$

$$- \frac{r-1 \cdot \overline{B-AB}}{2r}. \quad \text{Q. E. D.}$$

Having now the value of the sum S depending on the order of the two lives dying last, the value of the same sum depending

depending on the younger of the two lives dying last is easily obtained, by subtracting the value first found from the whole value of the reversion after the extinction of both lives.

The answers computed by this rule differ rather more from those computed by Mr. SIMPSON'S approximation than they do in the preceding problem. But I am fearful of becoming tedious, and therefore shall desist from inserting the comparative values in this case. It may not, however, be improper to exemplify the truth of this demonstration, by shewing in what manner the series may be resolved into the plain simple rule for computing the value of the reversion, when the ages are equal. The series in this case become

$$\frac{S \cdot \overline{b-c}^2}{2bbr} + \frac{S \cdot \overline{c-d}^2}{2bb^2r^2}, \&c. + \frac{S \cdot \overline{b-c} \cdot \overline{c-d}}{bb^2r^2} + \frac{S \cdot \overline{b-d} \cdot \overline{d-e}}{bb^3r^3} + \&c. =$$

$$\frac{S}{2} \times \frac{bb}{bbr} + \frac{cc-2bc}{bbr} + \frac{ad-2bd}{bb^2r^2} + \&c. + \frac{S}{2} \times \frac{2bc-cc}{bb^2r^2} + \frac{2bd-dd}{bb^3r^3}, \&c. =$$

$$S \times \frac{1}{2r} + \frac{1}{r} \times \frac{2B-BB}{2} - \frac{2B-BB}{2} = (\text{putting } L \text{ for } 2B-BB, \text{ the value of an annuity on the longest of the two equal lives})$$

$$\frac{S}{2} \times \frac{1}{r} + \frac{1-r \cdot L}{r} = \frac{S}{2} \times \frac{P \cdot r-1}{r} - \frac{L \cdot r-1}{r}. \quad Q.E.D.*$$

The *exact* value of all reversions depending on survivorships between two lives might be found in the same manner as the values in the preceding problems. With regard to the values of reversions depending on survivorships between three lives, I am sensible that the solutions of those cases would be rather difficult when deduced from the real probabilities of life. But they certainly might be effected; and *these* are the more necessary, inasmuch as the solutions derived from the expectations of life are often so very defective as not to deserve the name of approximations.

Chatham Place, Feb. 2, 1788.

\* See the latter part of the solution of the second problem, p. 347.