

V. *On the Method of determining, from the real Probabilities of Life, the Value of a contingent Reversion in which Three Lives are involved in the Survivorship.* By Mr. William Morgan; communicated by the Rev. Richard Price, D. D. F. R. S.

Read January 29, 1789.

IN a Paper which I had lately the honour of communicating to the Royal Society, respecting the method of determining the values of reversions depending on survivorships between two persons from the real probabilities of life, I observed, that the investigation of those cases in which three lives were involved in the survivorship (though attended with much more difficulty) might, however, be effected in a similar manner. The further pursuit of this subject has now convinced me that, as it is never safe, so likewise it can never be necessary to have recourse to the *expectations* of life in any case; and that the solution even of those problems which include three lives is far from being so formidable as at first sight it appears to be. I am sensible of the impropriety of entering minutely in this place into the vast variety of propositions which refer to the different orders of survivorship between three lives; but as the following problem seems to be of considerable importance on account of its being applied to the solution of many other problems, the demonstration of it, perhaps, may not be thought an improper addition to my former Paper.

PROBLEM.

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Supposing the ages of A, B, and C, to be given; to determine, from any table of observations, the value of the sum S payable on the contingency of C's surviving B, provided the life of A shall be then extinct.

SOLUTION.

Let  $a$  represent the number of persons living in the table at the age of A. Let  $a'$ ,  $a''$ ,  $a'''$ ,  $a''''$ , &c. represent the decrements of life at the end of the 1st, 2d, 3d, 4th, &c. years from the age of A. Let  $b$  represent the number of persons living at the age of B, and  $m$ ,  $n$ ,  $o$ ,  $p$ , &c. the number of persons living at the end of the 1st, 2d, 3d, 4th, &c. years from the age of B. In like manner let  $c$  represent the number of persons living at the age of C, and  $d$ ,  $e$ ,  $f$ ,  $g$ , &c. the number of persons living at the end of the 1st, 2d, 3d, 4th, &c. years from the age of C. Let  $r$  also denote the value of £. 1 increased by its interest for a year. In order to receive the sum S in the first year, it is necessary either that all the three lives shall have died in that year, A having died first, B next, and C last; or that only the two lives A and B shall have died (A having died first), and that C shall have lived to the end of that year. The probability that the three lives shall die in the first year is

$\frac{a' \cdot \overline{b-m} \cdot \overline{c-d}}{abc}$ . The probability that they shall die in the order above mentioned is  $\frac{a' \cdot \overline{b-m} \cdot \overline{c-d}}{6 \cdot abc}$ . The probability that both

A and B shall die in the first year is  $\frac{a' \cdot \overline{b-m}}{ab}$ . Half this frac-

tion, or  $\frac{a' \cdot \overline{b-m}}{2ab}$ , is the probability that the death of A shall happen before the death of B in this year. The probability that C shall survive A and B, restrained to the contingency of A's having died first, is  $\frac{a' \cdot \overline{b-m} \cdot d}{2abc}$ . The value therefore of the

sum S for the first year is  $S \times \frac{a' \cdot \overline{b-m} \cdot \overline{c-d}}{6 \cdot abcr} + \frac{a' \cdot \overline{b-m} \cdot d}{2abcr} = \frac{S}{abcr} \times$

$\frac{a'bc}{6} - \frac{a'mc}{6} + \frac{a'db}{3} - \frac{a'md}{3} \dots \dots \dots$ . In the second year the pay-

ment of the given sum will depend on either of four events happening. First, on the contingency of all the three lives dying in that year, A having died first, B next, and C last. 2dly, On the contingency of B's dying in that year, C's living to the end of it, and A's dying in the first year. 3dly, On the contingency of B's dying after A in the second year (both of them having survived the first year) and of C's living to the end of that year. 4thly, On the contingency of A's dying in the first year, and of B and C's both dying in the second year, B having died first. The probability of the first contingency is expressed by the fraction  $\frac{a'' \cdot \overline{m-n} \cdot \overline{d-e}}{6 \cdot abc}$ . The probability of

the second by the fraction  $\frac{a' \cdot \overline{m-n} \cdot e}{abc}$ . The probability of the

third by the fraction  $\frac{a'' \cdot \overline{m-n} \cdot e}{2 \cdot abc}$ . And the probability of the

fourth contingency by the fraction  $\frac{a' \cdot \overline{m-n} \cdot \overline{d-e}}{2abc}$ . These several

fractions, therefore, multiplied into  $\frac{S}{r^2}$  will be the value of the given sum for the second year, and may be easily found =

$\frac{S}{abcr^2} \times \frac{a'dm}{6} - \frac{a'dn}{6} + \frac{a'em}{3} - \frac{a'en}{3} + \frac{a'em}{2} - \frac{a'en}{2} + \frac{a'dm}{2} - \frac{a'dn}{2}$ . In like

manner the payment of the given sum in the third year will depend on the contingency of the same number of events as in the second year; that is, it will, first, depend on the contingency of all the three lives dying in that year, A having died first, B next, and C last; 2dly, on the contingency of B's dying in that particular year, C's living to the end of it, and A's dying in the first or second years; 3dly, on the contingency of B's dying after A in the third year (both of them having survived the two preceding years), and of C's living to the end of that year; and, 4thly, on the contingency of A's dying in the first or second year, and of B and C's both dying in the third year, C having died last. These several contingencies are expressed by the respective fractions

$$\frac{a''' \cdot \overline{n-o} \cdot \overline{e-f}}{6 \cdot abc} \dots \frac{\overline{a'+a''} \cdot \overline{n-o} \cdot f}{abc} \dots \frac{a'' \cdot \overline{n-o} \cdot f}{2abc} \dots \text{and}$$

$$\frac{\overline{a'+a''} \cdot \overline{n-o} \cdot \overline{e-f}}{2abc}.$$

Consequently the value of the sum S for

$$\text{the third year will be} = \frac{S}{abc r^3} \times \frac{a''' \cdot en}{6} - \frac{a''' \cdot eo}{6} + \frac{a''' \cdot fn}{3} - \frac{a''' \cdot fo}{3} +$$

$$\frac{a'+a'' \cdot fn}{2} - \frac{a'+a'' \cdot fo}{2} + \frac{a'+a'' \cdot en}{2} - \frac{a'+a'' \cdot eo}{2}.$$

And by reasoning in the same manner the value of the sum S for the fourth

$$\text{year may be found} = \frac{S}{abcr^4} + \frac{a'''' \cdot fo}{6} - \frac{a'''' \cdot fp}{6} + \frac{a'''' \cdot go}{3} - \frac{a'''' \cdot gp}{3} +$$

$$\frac{a'+a''+a''' \cdot go}{2} - \frac{a'+a''+a''' \cdot gp}{2} + \frac{a'+a''+a''' \cdot of}{2} - \frac{a'+a''+a''' \cdot fp}{2}.$$

If either B or C be the oldest of the three lives, these series continued to the extremity of that life will express the whole value of the reversion, which will be =  $\frac{S}{6} \times$

$$\frac{a'bc}{abcr} + \frac{a'dm}{abcr^2} + \frac{a''en}{abcr^3} + \frac{a'''' \cdot fo}{abcr^4} + \&c. + \frac{S}{2r} \times \frac{a'dm}{abcr} + \frac{a'+a'' \cdot en}{abcr^2} +$$

$$\begin{aligned}
& \frac{a' + a'' + a''' \cdot fo}{abcr^2} + \&c. - \frac{S}{6} \times \frac{a'mc}{abcr} + \frac{a''nd}{abcr^2} + \frac{a'''oe}{abcr^3} + \frac{a''''pf}{abcr^4} + \&c. \\
& \frac{S}{2r} \times \frac{a'dn}{abcr} + \frac{a' + a'' \cdot eo}{abcr^2} + \frac{a' + a'' + a''' \cdot fp}{abcr^3} + \&c. + \frac{S}{3} \times \frac{a'db}{abcr} + \frac{a'' \cdot em}{abcr^2} + \\
& \frac{a''' \cdot fn}{abcr^3} + \frac{a'''' \cdot go}{abcr^4} + \&c. + \frac{S}{2r} \times \frac{a'em}{abcr} + \frac{a' + a'' \cdot fn}{abcr^2} + \frac{a' + a'' + a''' \cdot go}{abcr^3} + \&c. \\
& - \frac{S}{3} \times \frac{a'md}{abcr} + \frac{a''en}{abcr^2} + \frac{a''' \cdot fo}{abcr^3} + \frac{a'''' \cdot gp}{abcr^4} + \&c. - \frac{S}{2r} \times \frac{a'en}{abcr} + \frac{a' + a'' \cdot fo}{abcr^2} + \\
& \frac{a' + a'' + a''' \cdot gp}{abcr^3} + \&c.
\end{aligned}$$

In order to sum up the first and second of these series let  $\beta$  represent the number of persons living at the age of F, a person one year younger than B, and  $\kappa$  the number of persons living at the age of K, a person one year younger than C. Let FK, BC, AFK, and ABC, represent the value of an annuity on the two and three joint lives of F and K, of B and C, of A, F and K, and of A, B and C respectively; then

$$\begin{aligned}
& \text{will the series } \frac{S}{6} \times \frac{a'bc}{abcr} + \frac{a'dm}{abcr^2} + \frac{a''en}{abcr^3} + \&c. \text{ be } = \frac{S \cdot \beta \cdot \kappa}{6 \cdot bc} \times \\
& \frac{bc}{\beta \kappa r} - \frac{a - a' \cdot bc}{a\beta \kappa r} + \frac{dm}{\beta \kappa r^2} - \frac{a - a' - a'' \cdot dm}{a\beta \kappa r^2} - \frac{a'dm}{a\beta \kappa r^2} + \frac{en}{\beta \kappa r^3} - \frac{a - a' - a'' - a''' \cdot en}{a\beta \kappa r^3} \\
& - \frac{a' + a'' \cdot en}{a\beta \kappa r^3}, \&c. = \frac{S \cdot \beta \cdot \kappa}{bc} \times \frac{FK - AFK}{6} \left( - \frac{S}{6r} \times \frac{a'dm}{abcr} + \frac{a' + a'' \cdot en}{abcr^2}, \&c. \right. \\
& \left. = - \frac{S}{6r} \times \frac{dm}{bcr} - \frac{a - a' \cdot dm}{abcr} + \frac{en}{bcr^2} - \frac{a - a' - a'' \cdot en}{abcr^2}, \&c. = \right) - \frac{S}{6r} \times \\
& BC - ABC. \text{ The sum, therefore, of the two first series, or of}
\end{aligned}$$

$$\frac{S}{6} \times \frac{a'bc}{abcr} + \frac{a'' \cdot dm}{abcr^2} + \&c. + \frac{S}{2r} \times \frac{a'dm}{abcr} + \frac{a' + a'' \cdot en}{abcr^2} + \&c. \text{ is } = \frac{S}{6} \times \frac{\beta \cdot \kappa \cdot FK - AFK}{bc} + \frac{S}{3r} \times BC - ABC.$$

Again, let P represent a life one year older than B, and let BK, PC, ABK, and APC, represent the values of annuities on the two and three joint lives of B and

B and K, P and C, A, B and K and of A, P and C: then the sum of the third and fourth series, or of  $-\frac{S}{6} \times$

$$\frac{a'mc}{abcr} + \frac{a''nd}{abcr^2} + \&c. - \frac{S}{2r} \times \frac{a'dn}{abcr} + \frac{a' + a'' \cdot eo}{abcr^2} + \&c. \text{ being } = -\frac{S \cdot z}{6c} \times$$

$$\frac{mc}{abr} - \frac{a - a' \cdot mc}{abr} + \frac{dn}{br^2} - \frac{a - a' - a'' \cdot dn}{abr^2} + \frac{eo}{br^3} - \frac{a - a' - a'' - a''' \cdot eo}{abr^3}, \&c.$$

$$\left( -\frac{S}{3r} \times \frac{a'dn}{abcr} + \frac{a' + a'' \cdot eo}{abcr^2}, \&c. = \right) - \frac{S \cdot m}{3br} \times \frac{a'dn}{acmr} + \frac{a' + a'' \cdot eo}{acmr^2} + \&c.,$$

will be  $= -\frac{S}{6} \times \frac{z \cdot BK - AbK}{c} - \frac{S}{3r} \times \frac{m \cdot FC - APC}{b}$ . The fifth fe-

ries,  $\frac{S}{3} \times \frac{a'db}{abcr} + \frac{a''em}{abcr^2} + \&c.$ , is  $= \frac{S}{3} \times \frac{\beta}{b} \times \frac{db}{\beta cr} - \frac{a - a' \cdot db}{a\beta cr} + \frac{em}{\beta cr^2} -$

$$\frac{a - a' - a'' \cdot em}{a\beta cr^2} + \frac{fn}{\beta cr^3} - \frac{a - a' - a'' - a''' \cdot fn}{a\beta cr^3}, \&c. - \frac{S}{3r} \times \frac{a'em}{abcr} + \frac{a' + a'' \cdot fn}{abcr^2}, \&c.$$

$=$  (putting FC and AFC for the values of the two and three joint lives of F and C and of A, F, and C)  $\frac{S}{3} \times \frac{\beta \cdot FC - AFC}{b} -$

$\frac{S}{3r} \times \frac{a'em}{abcr} + \frac{a' + a'' \cdot fn}{abcr^2}, \&c.$  The sum, therefore, of the fifth and

fixth series, or of  $\frac{S}{3} \times \frac{a'db}{abcr} + \frac{a''em}{abcr^2}, \&c. + \frac{S}{2r} \times \frac{a'em}{abcr} + \frac{a' + a'' \cdot fn}{abcr^2} + \&c.$

is  $= \frac{S}{3} \times \frac{\beta \cdot FC - AFC}{b} + \left( \frac{S}{6r} \times \frac{a'em}{abcr} + \frac{a' + a'' \cdot fn}{abcr^2}, \&c. = \right) \frac{S \times d}{6cr} \times$

$\frac{em}{bdr} - \frac{a - a' \cdot em}{abdr} + \frac{fn}{bdr^2} - \frac{a - a' - a'' \cdot fn}{abdr^2} + \&c.$  If T denote a life one

year older than C, and BT, and ABT denote the values of the two and three joint lives of B and T and of A, B, and T, this

last series will be  $= \frac{S}{6r} \times \frac{d \cdot BT - ABT}{c}$ , and consequently the sum

of the fifth and fixth series will be  $= \frac{S}{3} \times \frac{\beta \cdot FC - AFC}{b} + \frac{S}{6r} \times$

$\frac{d \cdot BT - ABT}{c}$ . Lastly, the seventh and eighth series, or -

$$\frac{S}{3} \times \frac{a'md}{abc} + \frac{a''en}{abc^2} + \&c. - \frac{S}{2r} \times \frac{a'en}{abc} + \frac{a' + a'' \cdot fo}{abc^2} + \&c. \text{ are } = - \frac{S}{3} \times$$

$$\frac{dm}{bc} - \frac{a - a' \cdot dm}{abc} + \frac{en}{bc^2} - \frac{a - a' - a'' \cdot en}{abc^2} + \&c. - \frac{S}{6r} \times \frac{a'en}{abc} + \frac{a' + a'' \cdot fo}{abc^2} + \&c.$$

$$= - \frac{S}{3} \times \overline{BC - ABC} - \left( \frac{S}{6r} \times \frac{md}{bc} \times \frac{en}{mdr} - \frac{a - a' \cdot en}{amdr} + \frac{of}{mar^2} - \right.$$

$$\left. \frac{a - a' - a'' \cdot of}{amdr^2} + \&c. \right) \frac{S}{6r} \times \frac{md \cdot \overline{PT - APT}}{bc}, \text{ where PT and APT}$$

represent the values of the two and three joint lives of P and T, and of A, P, and T. If these several expressions be added together, &c. we shall at last have  $\frac{S \cdot \alpha}{6c} \times \frac{\beta \cdot \overline{FK - AFK}}{b} - \overline{BK - ABK}$

$$+ \frac{S \cdot \beta}{3b} \times \overline{FC - AFC} - \frac{S \cdot r - 1}{3r} \times \overline{BC - ABC} - \frac{S \cdot m}{3br} \times \overline{PC - APC} +$$

$$\frac{S \cdot d}{6cr} \times \overline{BT - ABT} - \frac{m \cdot \overline{PT - APT}}{b}, \text{ for the value of the sum } S,$$

when either B or C are the oldest of the three lives.

In order to determine the value of the reversion when the life of A is the oldest of the three lives, let  $s, t, u, v, \&c.$  be the number of persons living at the end of the 1st, 2d, 3d, 4th, &c. years from the age of A, and let  $b', b'', b''', b''', \&c.$  be the decrements of life at the end of 1, 2, 3, 4, &c. years from the age of B; then, by reasoning as above, the value of the sum S for the first year will be expressed by the series  $\frac{S \times b' \cdot \overline{a-s} \cdot \overline{c-d}}{6abc} +$

$$\frac{S \times b' \cdot \overline{a-s} \cdot d}{2abc}, \text{ for the second year by the series } \frac{S \cdot b'' \cdot \overline{s-t} \cdot \overline{d-e}}{6abc^2} +$$

$$\frac{S \cdot b'' \cdot \overline{s-t} \cdot e}{2abc^2} + \frac{S \cdot b'' \cdot \overline{a-s} \cdot e}{abc^2} + \frac{S \cdot b'' \cdot \overline{a-s} \cdot \overline{d-e}}{2abc^2}, \text{ for the third year}$$

by the series  $\frac{S \cdot b''' \cdot \overline{t-u} \cdot \overline{e-f}}{6abc^3} + \frac{S \cdot b''' \cdot \overline{t-u} \cdot f}{2abc^3} + \frac{S \cdot b''' \cdot \overline{a-t} \cdot f}{abc^3} +$

$$\frac{S \cdot b''' \cdot \overline{a-t} \cdot \overline{e-f}}{2abc^3}, \text{ and so on for the remaining years of A's life.}$$

These several series may be found  $= \frac{S}{abc} \times -\frac{ab'c}{3} - \frac{b'cs}{6} - \frac{ab'd}{6} - \frac{b'ds}{3} +$

$ab'd$

$$\frac{ab'd}{2} + \frac{ab'e}{2} + \frac{S}{abc^2} \times -\frac{sb''d}{3} - \frac{b''dt}{6} - \frac{sb''e}{6} - \frac{be''t}{3} + \frac{ab''e}{2} + \frac{av''d}{2} + \frac{S}{abc^3} \times$$

$$-\frac{tb'''e}{3} - \frac{b'''eu}{6} - \frac{tb'''f}{6} - \frac{b'''fu}{3} + \frac{ab'''f}{2} + \frac{ab'''e}{2}, \&c. \&c. \text{ Let } \alpha \text{ represent}$$

the number of persons living at the age of H, a person one year younger than A; let N denote a person one year older than A, and let the several combinations BN, BNC, AB, &c. denote, as in the former case, the values of annuities on the joint lives of B and N, of B, N and C, of A and B, &c.; then by proceeding in the same manner as in the fore-

going demonstration the series  $\frac{ab'e}{3abc} + \frac{sb''d}{3abc^2} + \frac{tb'''e}{3abc^3} + \&c.$  may be found  $= \frac{\alpha x}{3ac} \times \overline{HK} - \overline{HBK} - \frac{AC - ABC}{3r}$ ; the series  $\frac{b'cs}{6abc} + \frac{b''dt}{6abc^2}$

$+ \frac{b'''eu}{6abc^3} + \&c. = \frac{x}{6c} \times \overline{AK} - \overline{ABK} - \frac{s}{6ar} \times \overline{NC} - \overline{NBC}$ ; the series

$\frac{ab'd}{6abc} + \frac{sb''e}{6abc^2} + \frac{tb'''f}{6abc^3} + \&c. = \frac{a}{6a} \times \overline{HC} - \overline{HBC} - \frac{d}{6cr} \times \overline{AT} - \overline{ABT}$ ;

and the series  $\frac{b'ds}{3abc} + \frac{b''et}{3abc^2} + \frac{b'''fu}{3abc^3} + \&c. = \frac{AC - ABC}{3} - \frac{sd}{3acr} \times$

$\overline{NT} - \overline{NTB}$ . These four series, therefore, supposing them all

to be positive quantities are  $= \frac{x}{3c} \times \frac{\alpha \cdot \overline{HK} - \overline{HBK}}{a} + \frac{\overline{AK} - \overline{ABK}}{2} +$

$\frac{\alpha}{6a} \times \overline{HC} - \overline{HBC} + \frac{r-1}{3r} \times \overline{AC} - \overline{ABC} - \frac{s}{6ar} \times \overline{NC} - \overline{NBC} - \frac{d}{3cr} \times$

$\frac{\overline{AT} - \overline{ABT}}{2} + \frac{s \cdot \overline{NT} - \overline{NBT}}{a}$ . With respect to the two remaining

series  $\frac{b'd}{2bcr} + \frac{b''e}{2bcr^2} + \frac{b'''f}{2bcr^3} + \&c. \dots$  and  $\frac{b'e}{2bcr} + \frac{b''d}{2bcr^2} + \frac{b'''e}{2bcr^3} + \&c.$ ,

these, it is evident, are to be continued after the decease of A till the extinction of the joint lives of B and C, and have been already proved in the solution of the second problem in my former Paper, to denote the value of the given sum payable if C should survive B. Let this value be represented

fented by R and the sum of the foregoing expressions (or  $\frac{x}{3c} \times \frac{a \cdot \overline{HK} - \overline{HBK}}{a} + \frac{\overline{AK} - \overline{ABK}}{2} + \frac{a}{6a} \times \overline{HC} - \overline{HBC}$ , &c.) by M, then will the value of the sum S (when A is the oldest of the three lives) be =  $S \times R - M$ . Q. E. D.

If the three lives be equal, the value of the given sum for the first year will be =  $\frac{S \cdot \overline{c-d}}{6 \cdot c^3 \cdot r} + \frac{S \cdot \overline{c-d}^2 \cdot d}{2c^3 \cdot r} = S \times \frac{1}{6r} + \frac{2a^3}{6c^3r} - \frac{3dd}{6c^2r}$ ; the value of the same sum for the second year will be =  $\frac{S \cdot \overline{a-d}^3}{6c^3r^2} + \frac{S \cdot \overline{d-e}^2 \cdot e}{2c^3r^2} + \frac{S \cdot \overline{d-e} \cdot \overline{c-d} \cdot e}{c^3r^2} + \frac{\overline{d-e}^3 \cdot \overline{c-d}}{2c^3r^2} = S \times \frac{2e^3}{6c^3r^2} - \frac{3ee}{6c^2r^2} - \frac{2d^3}{6c^3r^2} + \frac{3dd}{6c^2r^2}$ ; the value for the third year will be =  $S \times \frac{2f^3}{6c^3r^3} - \frac{3ff}{6c^2r^3} - \frac{2e^3}{6c^3r^3} + \frac{3ee}{6c^2r^3}$ , and so on for the other years to the extremity of life. Let CC and CCC denote the values of the two equal and three equal joint lives, the sum of these series may then be found =  $\frac{S}{6} \times \frac{1}{r} + \frac{2CCC}{r} - 3CC + \frac{3 \cdot CC - 2CCC}{r}$   
= (supposing the perpetuity, or  $\frac{1}{r-1}$ , to be denoted by V)  
 $\frac{S}{6} \times \frac{r-1}{r} \times V - 3 \cdot CC - 2CCC$ .

It must be here remembered, that from other principles it is well known, that the number of years purchase expressing the value of an *estate* or *perpetual annuity* to be entered upon at the failure of two out of any three equal lives is, "the difference  
" between three times the values of two equal joint lives, and  
" twice the values of three equal joint lives subtracted from  
" the perpetuity," or  $V - 3CC - 2CCC$ . The value, therefore, of such a reversion, supposing it to depend on the failure of the three equal lives in any one particular order, is (since there

there are six such orders equally probable)  $\frac{1}{6} \times V - \frac{3CC - 2CCC}{r}$ . But it appears, from the correction explained in Dr. PRICE's Treatise on Reversionary Payments, Vol. I. p. 34. that the value of a reversionary *sum* is always less than the value of an equivalent reversionary *estate* in the proportion of 1 to  $r$ . The sum being  $S$  the equivalent estate or perpetual annuity is always  $S \times \frac{r-1}{r}$ ; and consequently the value of the sum  $S$  depending on the ceasing of three equal lives in any one particular order and thus determined, is the same with that determined by the foregoing investigation, that is,  $\frac{S}{6} \times \frac{r-1}{r} \times V - \frac{3CC - 2CCC}{r}$ . The investigation, therefore, is right, and the correction and investigation demonstrate one another.

But the foregoing expression for determining the value of the reversion in this particular case is not only obtained immediately from the series, but also from the two different rules which have been given for determining the value when the lives are unequal; and hence a proof arises of the truth of these rules, as well as of the reasoning upon which they are founded. Thus the first rule, supposing the lives all equal,

becomes  $\frac{x^2}{c^2} \times \frac{KK - CKK}{6} - \frac{dd}{cc \cdot r} \times \frac{TT - CTT}{6} - \frac{r-1}{r} \times \frac{CC - CCC}{3} + \frac{x}{c} \times \frac{CK - CCK}{6} - \frac{d}{cr} \times \frac{CT - CCT}{6}$ , and the second rule becomes  $\frac{V - CC \cdot \frac{r-1}{r}}{2r} - \frac{x^2}{c^2} \times \frac{KK - CKK}{3} + \frac{dd}{cc \cdot r} \times \frac{TT - CTT}{3} - \frac{r-1}{r} \times \frac{CC - CCC}{3} - \frac{x}{c} \times \frac{CK - CCK}{3} + \frac{d}{cr} \times \frac{CT - CCT}{3}$ . Let the value according to the first rule be denoted by  $L$ , and the second rule will be =  $\frac{r-1 \cdot V - CC}{2r} - 2L - \frac{r-1 \cdot CC - CCC}{r} (= I)$ . Hence  $3I =$

$$\frac{r-1.V - CC - 2.r - 1.CC - CCC}{2r} \text{ and } L = \frac{r-1}{6r} \times V - 3CC - 2CCC.$$

Q. E. D.

Were we possessed of complete tables of the values of annuities on two and three joint lives, the preceding rules would give an easy and exact solution of this problem in all cases. But as such tables, computed for every age, would be a work of immense difficulty, especially in regard to the values of three joint lives, Mr. SIMPSON's rule for approximating to these from the given values of the two joint lives, has hitherto been adopted, and it seems upon the whole to answer the purpose very well. In the present problem it is attended with no other inconvenience than increasing the labour of the computations; for the values of the reversions derived from it appear in general to be perfectly correct. This is more fully ascertained by a table which Dr. PRICE has given in his *Treatise on Reversionary Payments* (Vol. II. Table 37.), of the values of three equal joint lives computed at 4 *per cent.* from the probabilities of life at NORTHAMPTON. By the assistance of this table, when the lives are of the same age, it is evident, from what has been already observed, that the exact value of the reversion may be easily obtained. The few following specimens computed from it, and compared with the values of the reversions deduced from the first and second of the preceding rules, demonstrate the accuracy of those rules: for, notwithstanding the approximated values of the three joint lives have been used in every instance in which the rules have been employed, yet the results approach so near the truth, even in the last stages of life, when the decrements are most irregular, that, though derived from these approximations, there can be little doubt of their correctness in almost every other period of life.

Common age.	Exact value * of £. 100. computed from Dr. PRICE's Tables of the values of two and three equal joint lives.	Value of £. 100. computed from the first of the foregoing rules, and from Mr. SIMPSON's approximation to the values of three joint lives.	Value of £. 100. computed from the second of the foregoing rules, and from Mr. SIMPSON's approximation to the values of three joint lives.
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70	-	12.000	-	12.005	-	12.000
75	-	12.944	-	12.943	-	12.943
80	-	13.840	-	13.810	-	13.880
85	-	14.450 †	-	14.670	-	14.340

Mr. DODSON ‡, and Mr. SIMPSON §, are the only writers who have solved, or rather who have approximated to the solution of this problem. But the former, by deducing his rules immediately from a wrong hypothesis, having rendered

\* That is, *one-sixth* part of the whole reversion.

† The several reversionions in this column, when computed from SIMPSON's approximation to the values of the three joint lives, are 12.012, 12.933, 13.847, and 14.803 respectively; which upon the whole differing nearly as much from the real values as those in the two other columns afford a convincing proof, that the very small deviation from the truth in these latter values proceeds not from any inaccuracy in the rules themselves, but solely from having used the *approximated* instead of the *real* values of the three joint lives. And this also will account for the difference in the values by the first and second rules. Were those values computed from tables which give the correct values of two and three joint lives at all ages, they would come out exactly the same. In the two first examples, where the values by one rule are true, it appears, that the values by the other rule are equally so. In the two last examples, where the values are not quite so accurate, it may be observed, that they differ as much in excess by one rule as they do in defect by the other; which must in general be the case from the very nature of those rules; for if L (or the value by the first rule) be greater than the truth, the difference between  $\frac{r-1 \cdot \bar{V}-CC}{2r}$  and 2L (or the value by the second rule) must be less than the truth; and, on the contrary, if L be less, this difference will be greater than the truth.

‡ See DODSON's Mathematical Repository, Vol. III. Questions 42, 43, &c.

§ See SIMPSON's Select Exercises, Prob. 38.

most of them (especially those in which three lives are concerned) of no use, it will be unnecessary to take notice of what he has done on the subject. With regard to the latter, whose rule is not only the sole guide for determining the value of this reversion, but also the source from which a great variety of other problems are solved, perhaps it may not be improper to examine how far his solution is to be depended upon; and the following examples have therefore been computed for this purpose.

T A B L E I.

Ages of			Value, by SIMPSON's rule, of £. 100. payable on the contingent reversion specified in this pro- blem, when either C or B are el- dest, according to the <i>Northamp-</i> <i>ton Table</i> and at 4 per cent.			True value of the same reversion computed from the first rule in the fore- going solution.	
C.	B.	A.					
80	70	40	-	1.926	-	-	1.179
75	65	25	-	1.873	-	-	1.032
65	50	15	-	2.090	-	-	1.690
70	80	40	-	6.615	-	-	6.117
50	65	15	-	5.580	-	-	3.879
78	78	20	-	2.583	-	-	1.982
45	60	12	-	5.571	-	-	4.133
60	45	12	-	2.292	-	-	1.686

T A B L E II.

Ages.			Value of the same reversion by SIMPSON's rule, when A is the oldest of the three lives.			True value of the same reversion by the second rule in the foregoing solution.	
C.	B.	A.					
24	65	75	-	34.636	-	-	31.792
65	24	75	-	6.305	-	-	7.895
49	9	69	-	7.351	-	-	5.960
18	78	78	-	37.554	-	-	33.019

T A B L E

T A B L E III.

Common Age.	Value of the same reversion by SIMPSON'S rule, when the ages of the three lives are equal.				True value of the same reversion.
70	-	-	13.20	-	12.00
75	-	-	14.98	-	12.94
80	-	-	16.58	-	13.84
85	-	-	17.86	-	14.45

By comparing the values in the preceding tables, Mr. SIMPSON'S rule appears in almost every instance to be exceedingly incorrect. Even when the lives are equal (in which case it might have been expected to be sufficiently accurate) it seems to deviate, in old age at least, so widely from the truth as to be unfit for use. When C or B are eldest (which, however, is a case that does not often occur), the results sometimes exceed the truth *one-half*, and generally by more than *one-third* of the real value. When A is the oldest of the three lives (which is the most common case) these results are erroneous in nearly an equal degree. Nay, in some cases, Mr. SIMPSON'S rule is not only wrong but absurd. Thus, in the last example in the second table, the value of £. 100. payable on the contingency of C aged 18 surviving B aged 78 after A aged 78, is by this rule = £. 37.554. The value, therefore, of the same sum on the contingency of C's surviving A after B is also £. 37.554. Hence the value of £. 100. on the contingency of C's surviving A and B (without the restriction of one dying before the other) is  $2 \times 37.554 =$  £. 75.108\*. By another rule of Mr. SIMPSON †, the value

\* See SIMPSON'S Select Exercises, Prob. 39. † Ibid. Prob. 32.

of £. 100, on the contingency of C's surviving B only, is no more than £. 74\*. Now it is self-evident, that this latter value, instead of being *less*, ought to have been *greater* than the former, inasmuch as the probability of C's surviving only one life must be greater than that of his surviving two lives.

Many additional instances might be produced in which this rule, being made the basis upon which the solutions of other problems are founded, leads to conclusions equally erroneous. But these enquiries would be improper here; and I shall only observe, that had the foregoing examples been computed from the SWEDEN or LONDON, instead of the NORTHAMPTON Table, this rule would have appeared to be still more incorrect than it does from those computations.

When Mr. SIMPSON wrote his Select Exercises, he was in a great measure obliged to have recourse to DE MOIVRE's hypothesis, for want of those excellent tables of the real probabilities of life, and also of the values of single and joint lives which have been since published. Had he been possessed of these, it is most likely that his superior abilities would have directed him to a more accurate method of investigation. At present there can be no just reason for ever recurring to this wretched hypothesis. The solutions of all cases of two and even of three lives may be effected without much difficulty from principles strictly true. But I must here take my leave of this subject, hoping that its importance may engage other mathematicians to the further prosecution of it.

\* The true values are £. 66.038, and £. 74.884, respectively.

