# XVII. *On the Method of determining, from the real Probabilities of Life, the Values of contingent Reversions, in which Three Lives are involved in the Survivorship.* *By* William Morgan, *Esq. F. R. S.*

Read May 15, 1794.

In the last paper which I communicated to the Royal Society on the doctrine of survivorships, I concluded with observing that, as far as my own judgment could discover, I had then given rules for determining the values of reversions depending upon three lives in every case which admitted of an exact solution, and that the remaining cases, which were nearly equal in number to those I had already investigated, involved a contingency for which it appeared very difficult to find such a general expression as should not render the rules too complicated and laborious. Since that period I have bestowed much time and attention on this subject, and have at length so far succeeded as to give me reason now to hope that it is capable of being entirely exhausted. It is not my present design to enter into the investigation of all the problems which still remain to be solved. I shall here confine myself to a few of the most important, reserving the conclusion of the subject for some future opportunity.

The contingency to which I have alluded in this and my former paper, as opposing the great difficulty in those problems which I have not yet solved, is that of *one life's failing*

*after another in a given time.* It becomes necessary, therefore, previous to any other investigation, to deduce a general method of ascertaining such an event, and for this purpose I shall subjoin the following lemma.

## LEMMA.

To determine, from any table of observations, the probability that B the elder dies *after* A the younger of two lives, either in any given number of years, or during the whole continuance of the life of B.

## SOLUTION.

This event can take place in the first year only by the extinction of both lives, A having died first; the probability of which will be expressed by the fraction $\frac{\overline{b-m}\,.\,a'}{2ab}$.* In the second year the probability will be increased; for the event may have taken place, as above mentioned, in the first year, or the lives may have failed in the second year, A having died first; or B may have died in this year, and A in the first year. The expression, therefore, for the second year will be $\frac{\overline{b-m}\,.\,a'}{2ab} + \frac{\overline{m-n}\,.\,a''}{2ab} + \frac{\overline{m-n}\,.\,a'}{ab} = \frac{1}{ab} \times \overline{\frac{\overline{b-n}\,.\,a'}{2} + \frac{\overline{m-n}\,.\,\overline{a'+a''}}{2}}$. In the third year the probability will be still further increased; for, in addition to the foregoing contingencies, the event may have taken place by the extinction of the two lives in the third year, A having died first; or by the extinction of the life of A

* In order to avoid unnecessary repetitions, I have uniformly in this paper preserved the same symbols as in my last paper.—See Phil. Trans. Vol. LXXXI. page 247.

in the first or second years, and of the life of B in the third year. Therefore, the probability for the third year will be expressed by $\frac{\overline{b-m}.a'}{2ab} + \frac{\overline{m-n}.a''}{2ab} + \frac{\overline{m-n}.a'}{ab} + \frac{\overline{n-o}.a'''}{2ab} + \frac{\overline{n-o}.\overline{a'+a''}}{ab} = \frac{1}{ab} \times \overline{\frac{\overline{b-o}.a'}{2} + \frac{\overline{m-o}.\overline{a'+a''}}{2} + \frac{\overline{n-o}.\overline{a''+a'''}}{2}}$. By proceeding in the same manner for the fourth year, the probability will be found $= \frac{1}{ab} \times \overline{\frac{\overline{b-p}.a'}{2} + \frac{\overline{m-p}.\overline{a'+a''}}{2} + \frac{\overline{n-p}.\overline{a''+a'''}}{2} + \frac{\overline{o-p}.\overline{a'''+a''''}}{2}}$; and supposing $x$ to denote the difference between the ages of B, and of the oldest person in the table, and $y$ and $z$ respectively the number of persons living at the two last ages in the same table, the whole probability of the elder life's dying after the younger will be $= \frac{1}{2ab}$ into $\overline{b-z}.a' + \overline{m-z}.\overline{a'+a''} + \overline{n-z}.\overline{a''+a'''} + \overline{o-z}.\overline{a'''+a''''} \ldots + \overline{y-z}.\overline{a^{x-1}+a^{x}}$. Now, since it is well known that the probability of both lives failing in $x$ years, without any regard to the order of their extinction, is $= \frac{\overline{b-z} \times \overline{a'+a''+a'''+a'''' \ldots + a^{x}}}{ab}$ (or supposing $\pi$ to be the number of persons living at the end of $x$ years from the age of A) $= \frac{\overline{b-z}.\overline{a-\pi}}{ab}$, it is evident that, if the foregoing series be subtracted from this fraction, the probability will be obtained of the younger person's dying after the elder in $x$ years. In the first paper which I communicated to the Royal Society on this subject,* I not only described the most concise method of computing a table of the probabilities of survivorship between any two given lives, but computed a comprehensive one for persons of all ages, whose common difference was not less than ten years. As the contingency in this lemma is of considerable

* See Phil. Trans. Vol. LXXVIII. p. 335.

importance, and the solutions of a great number of problems require that it should be previously ascertained, I have computed a similar table on the present occasion; and it will appear from the following operations that both are formed in nearly the same manner.

| Ages of B. | A | Probability of B's dying after A. | | Probability of A's dying after B. |
|---|---|---|---|---|
| *95 | 85 | $\frac{1}{4\times186}\times\frac{41\times3}{2}=$ - - | .0827 | $\frac{3\times41}{4\times186}-.0827=.0827$ |
| 94 | 84 | $\frac{1}{9\times234}\times\frac{48\times8}{2}+\frac{\overline{48+41}\times3}{2}=$ - - | .1546 | $\frac{8\times89}{9\times234}-.1546=.1835$ |
| 93 | 83 | $\frac{1}{16\times289}\times\frac{55\times15}{2}+\frac{\overline{55+48}\times8}{2}+\frac{\overline{48+41}\times3}{2}=$ - | .2072 | $\frac{15\times144}{16\times289}-.2072=.2599$ |
| 92 | 82 | $\frac{1}{24\times346}\times\frac{57\times23}{2}+\frac{\overline{57+55}\times15}{2}+\frac{\overline{55+48}\times8}{2}+\frac{\overline{48+41}\times3}{2}=$ | .2422 | $\frac{23\times201}{24\times346}-.2422=.3145$ |
| 91 | 81 | $\frac{1}{34\times406}\times\frac{60\times33}{2}+\frac{\overline{60+57}\times23}{2}+\frac{\overline{57+55}\times15}{2}+$ &c. $=$ | .2696 | $\frac{33\times261}{34\times406}-.2696=.3544$ |
| 90 | 80 | $\frac{1}{46\times469}\times\frac{63\times45}{2}+\frac{\overline{63+60}\times33}{2}+\frac{\overline{60+57}\times23}{2}+$ &c. $=$ | .2864 | $\frac{45\times324}{46\times469}-.2864=.3894$ |
| 89 | 79 | $\frac{1}{62\times534}\times\frac{65\times61}{2}+\frac{\overline{65+63}\times45}{2}+\frac{\overline{63+60}\times33}{2}+$ &c. $=$ | .2907 | $\frac{61\times389}{62\times534}-.2907=.4260$ |

From these specimens it will be readily seen, that the probability between two younger lives is derived from that of the two preceding older ones, without any addition of labour; for the sum of all the terms of the series, excepting the two first, is constantly obtained from the foregoing operations. Thus, when the ages of B and A are 92 and 82, the two terms $\frac{\overline{55+48}\times8}{2}+\frac{\overline{48+41}\times3}{2}$ form a part of the preceding series, which expresses the probability between two persons aged 93 and 83.

* This, and all the other computations in this paper, are deduced from the Northampton Table, in Dr. PRICE's Treatise on Annuities, Vol. II. p. 36. edit. 5th.

And in like manner when their ages are 91 and 81, the three terms $\frac{\overline{57+55} \times 15}{2} +$ &c. form a part of the series which denotes the probability between two persons, aged 92 and 82. By proceeding with these operations, a table may be formed for all lives, whose common difference of age is the same, with little more trouble than in the single case of the two youngest lives. If a table of the probabilities of survivorship be already formed (such as that to which I have referred in my first paper), the operations in the present case may be exceedingly abridged; and it will not perhaps be improper here to explain the manner in which this is effected. By exchanging the symbols $c$, $d$, $e$, &c. in the solution in my first paper, for their equals $m$, $n$, $o$, &c. in the present solution, the series expressing the probability of B's surviving A will become $= \frac{1}{ab} \times \overline{\frac{b \cdot a'}{2} + \frac{m \cdot \overline{a'+a''}}{2} + \frac{n \cdot \overline{a''+a'''}}{2} \ldots\ldots + \frac{z \cdot a^x}{2}}$, which exceeds the series expressing the probability of B's dying after A by $\frac{z}{ab} \times \overline{a'+a''+a''' \ldots\ldots + \frac{a^x}{2}}$ (or supposing $z$, as in the Northampton Table, to be $= 1$) by $\frac{a-\pi}{ab}$ nearly. If, therefore, the given probability of B's surviving A be denoted by Y, the probability of B's dying after A will be $= \mathrm{Y} - \frac{a-\pi}{ab}$, and the probability of A's dying after B will be $= \frac{a-\pi}{a} - \mathrm{Y}$. The following table has been computed in this manner, excepting the first and the two last divisions, where the difference of age between the two lives is 10, 80, and 90 years. In these cases, the probabilities have been deduced from the series in this lemma, and chiefly with the view of proving the accuracy of the table in my first paper. It is

however necessary to observe, that in the abovementioned series $\frac{z}{ab} \times \overline{a' + a'' + a''' + \ldots\ldots \frac{a^x}{2}}$, the last term, $\left(\frac{a^x}{2}\right)$ by supposing the whole series $= \frac{a-\pi}{ab}$, is taken $= a^x$, and therefore the difference between Y and the probability of B's dying after A, is not *exactly* expressed above. Regard has been had to this circumstance in the following table, in all cases where the age of the eldest life exceeds 86 years. But under that age it is omitted, as the expression $\frac{a-\pi}{ab}$ becomes then true to four places of decimals, and of consequence sufficiently correct for any useful purpose.

# TABLE,

## Shewing the probability of one life's dying after another.*

| Ten years difference. | | | | Twenty years difference. | | | | Thirty years difference. | | | | Forty years difference. | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Ages. | | Youngest. | Eldest. | Ages. | | Youngest. | Eldest. | Ages. | | Youngest. | Eldest. | Ages. | | Youngest. | Eldest. |
| 1 | 11 | .3973 | .5858 | 1 | 21 | .3885 | .5244 | 1 | 31 | .3384 | .4821 | 1 | 41 | .2908 | .4355 |
| 2 | 12 | .4664 | .5136 | 2 | 22 | .4536 | .4431 | 2 | 32 | .3934 | .3934 | 2 | 42 | .3355 | .3396 |
| 3 | 13 | .4962 | .4822 | 3 | 23 | .4803 | .4086 | 3 | 33 | .4155 | .3555 | 3 | 43 | .3526 | .2983 |
| 4 | 14 | .5176 | .4597 | 4 | 24 | .4934 | .3898 | 4 | 34 | .4307 | .3284 | 4 | 44 | .3642 | .2686 |
| 5 | 15 | .5297 | .4469 | 5 | 25 | .5028 | .3767 | 5 | 35 | .4382 | .3133 | 5 | 45 | .3694 | .2518 |
| 6 | 16 | .5417 | .4342 | 6 | 26 | .5120 | .3638 | 6 | 36 | .4456 | .2983 | 6 | 46 | .3746 | .2351 |
| 7 | 17 | .5501 | .4252 | 7 | 27 | .5183 | .3546 | 7 | 37 | .4498 | .2881 | 7 | 47 | .3777 | .2228 |
| 8 | 18 | .5559 | .4190 | 8 | 28 | .5223 | .3482 | 8 | 38 | .4533 | .2796 | 8 | 48 | .3791 | .2138 |
| 9 | 19 | .5586 | .4159 | 9 | 29 | .5237 | .3450 | 9 | 39 | .4541 | .2751 | 9 | 49 | .3788 | .2084 |
| 10 | 20 | .5591 | .4152 | 10 | 30 | .5232 | .3439 | 10 | 40 | .4532 | .2731 | 10 | 50 | .3772 | .2057 |
| 11 | 21 | .5583 | .4157 | 11 | 31 | .5223 | .3438 | 11 | 41 | .4516 | .2722 | 11 | 51 | .3747 | .2043 |
| 12 | 22 | .5571 | .4167 | 12 | 32 | .5209 | .3440 | 12 | 42 | .4497 | .2716 | 12 | 52 | .3719 | .2033 |
| 13 | 23 | .5558 | .4178 | 13 | 33 | .5187 | .3449 | 13 | 43 | .4476 | .2712 | 13 | 53 | .3690 | .2024 |
| 14 | 24 | .5544 | .4189 | 14 | 34 | .5179 | .3445 | 14 | 44 | .4453 | .2709 | 14 | 54 | .3659 | .2016 |
| 15 | 25 | .5530 | .4201 | 15 | 35 | .5162 | .3449 | 15 | 45 | .4430 | .2706 | 15 | 55 | .3626 | .2009 |
| 16 | 26 | .5513 | .4215 | 16 | 36 | .5144 | .3454 | 16 | 46 | .4405 | .2704 | 16 | 56 | .3591 | .2003 |
| 17 | 27 | .5500 | .4225 | 17 | 37 | .5128 | .3456 | 17 | 47 | .4381 | .2699 | 17 | 57 | .3551 | .1994 |
| 18 | 28 | .5490 | .4232 | 18 | 38 | .5117 | .3452 | 18 | 48 | .4360 | .2688 | 18 | 58 | .3523 | .1978 |
| 19 | 29 | .5486 | .4233 | 19 | 39 | .5109 | .3442 | 19 | 49 | .4342 | .2670 | 19 | 59 | .3491 | .1956 |
| 20 | 30 | .5485 | .4230 | 20 | 40 | .5105 | .3427 | 20 | 50 | .4325 | .2648 | 20 | 60 | .3459 | .1928 |
| 21 | 31 | .5490 | .4221 | 21 | 41 | .5105 | .3406 | 21 | 51 | .4312 | .2618 | 21 | 61 | .3428 | .1893 |
| 22 | 32 | .5498 | .4209 | 22 | 42 | .5107 | .3382 | 22 | 52 | .4298 | .2586 | 22 | 62 | .3399 | .1852 |
| 23 | 33 | .5506 | .4196 | 23 | 43 | .5110 | .3356 | 23 | 53 | .4284 | .2553 | 23 | 63 | .3244 | .1809 |
| 24 | 34 | .5515 | .4183 | 24 | 44 | .5110 | .3332 | 24 | 54 | .4268 | .2519 | 24 | 64 | .3339 | .1765 |
| 25 | 35 | .5524 | .4169 | 25 | 45 | .5112 | .3306 | 25 | 55 | .4253 | .2484 | 25 | 65 | .3307 | .1719 |
| 26 | 36 | .5533 | .4155 | 26 | 46 | .5112 | .3280 | 26 | 56 | .4235 | .2449 | 26 | 66 | .3274 | .1673 |
| 27 | 37 | .5543 | .4140 | 27 | 47 | .5113 | .3253 | 27 | 57 | .4212 | .2413 | 27 | 67 | .3238 | .1626 |
| 28 | 38 | .5553 | .4125 | 28 | 48 | .5114 | .3225 | 28 | 58 | .4199 | .2376 | 28 | 68 | .3201 | .1578 |

* In the table in the LXXVIIIth Vol. of the Philosophical Transactions, the *certainty* of one life's surviving the other is denoted by 100. In this table the *certainty* of both lives becoming extinct is denoted by unity, this number being better suited to the solution in the following problems. It may not be improper to add, that both tables, though deduced from the decrements of life at Northampton, may be safely used, even when the values of the life annuities are derived from a different source, as the probabilities they express are very nearly the same, from whatever table of observations they are computed.

| Ten years difference. | | | | Twenty years difference. | | | | Thirty years difference. | | | | Forty years difference. | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Ages. | | Youngest. | Eldest. | Ages. | | Youngest. | Eldest. | Ages. | | Youngest. | Eldest. | Ages. | | Youngest. | Eldest. |
| 29 | 39 | .5562 | .4110 | 29 | 49 | .5115 | .3196 | 29 | 59 | .4179 | .2338 | 29 | 69 | .3164 | .1528 |
| 30 | 40 | .5573 | .4094 | 30 | 50 | .5115 | .3167 | 30 | 60 | .4159 | .2299 | 30 | 70 | .3122 | .1479 |
| 31 | 41 | .5583 | .4078 | 31 | 51 | .5112 | .3140 | 31 | 61 | .4136 | .2260 | 31 | 71 | .3077 | .1430 |
| 32 | 42 | .5592 | .4063 | 32 | 52 | .5108 | .3113 | 32 | 62 | .4112 | .2220 | 32 | 72 | .3029 | .1380 |
| 33 | 43 | .5601 | .4048 | 33 | 53 | .5104 | .3085 | 33 | 63 | .4089 | .2177 | 33 | 73 | .2977 | .1331 |
| 34 | 44 | .5608 | .4034 | 34 | 54 | .5099 | .3057 | 34 | 64 | .4066 | .2134 | 34 | 74 | .2921 | .1282 |
| 35 | 45 | .5613 | .4022 | 35 | 55 | .5092 | .3029 | 35 | 65 | .4037 | .2089 | 35 | 75 | .2858 | .1237 |
| 36 | 46 | .5625 | .4003 | 36 | 56 | .5086 | .3000 | 36 | 66 | .4009 | .2043 | 36 | 76 | .2788 | .1194 |
| 37 | 47 | .5635 | .3986 | 37 | 57 | .5072 | .2969 | 37 | 67 | .3978 | .1997 | 37 | 77 | .2715 | .1150 |
| 38 | 48 | .5646 | .3968 | 38 | 58 | .5071 | .2939 | 38 | 68 | .3945 | .1950 | 38 | 78 | .2637 | .1106 |
| 39 | 49 | .5655 | .3951 | 39 | 59 | .5051 | .2909 | 39 | 69 | .3910 | .1902 | 39 | 79 | .2559 | .1057 |
| 40 | 50 | .5664 | .3934 | 40 | 60 | .5049 | .2878 | 40 | 70 | .3872 | .1854 | 40 | 80 | .2470 | .1014 |
| 41 | 51 | .5669 | .3920 | 41 | 61 | .5038 | .2845 | 41 | 71 | .3829 | .1805 | 41 | 81 | .2384 | .0960 |
| 42 | 52 | .5676 | .3904 | 42 | 62 | .5026 | .2810 | 42 | 72 | .3785 | .1753 | 42 | 82 | .2286 | .0910 |
| 43 | 53 | .5684 | .3886 | 43 | 63 | .5017 | .2770 | 43 | 73 | .3736 | .1699 | 43 | 83 | .2174 | .0865 |
| 44 | 54 | .5692 | .3868 | 44 | 64 | .5007 | .2728 | 44 | 74 | .3681 | .1647 | 44 | 84 | .2040 | .0834 |
| 45 | 55 | .5701 | .3849 | 45 | 65 | .4995 | .2685 | 45 | 75 | .3617 | .1598 | 45 | 85 | .1898 | .0803 |
| 46 | 56 | .5709 | .3830 | 46 | 66 | .4982 | .2641 | 46 | 76 | .3544 | .1553 | 46 | 86 | .1743 | .0776 |
| 47 | 57 | .5708 | .3811 | 47 | 67 | .4967 | .2596 | 47 | 77 | .3465 | .1508 | 47 | 87 | .1576 | .0751 |
| 48 | 58 | .5723 | .3792 | 48 | 68 | .4950 | .2550 | 48 | 78 | .3383 | .1460 | 48 | 88 | .1393 | .0731 |
| 49 | 59 | .5729 | .3773 | 49 | 69 | .4945 | .2503 | 49 | 79 | .3308 | .1407 | 49 | 89 | .1214 | .0700 |
| 50 | 60 | .5737 | .3751 | 50 | 70 | .4907 | .2455 | 50 | 80 | .3207 | .1351 | 50 | 90 | .1032 | .0649 |
| 51 | 61 | .5748 | .3725 | 51 | 71 | .4885 | .2400 | 51 | 81 | .3105 | .1293 | 51 | 91 | .0865 | .0569 |
| 52 | 62 | .5762 | .3695 | 52 | 72 | .4860 | .2342 | 52 | 82 | .2993 | .1234 | 52 | 92 | .0691 | .0476 |
| 53 | 63 | .5778 | .3662 | 53 | 73 | .4830 | .2284 | 53 | 83 | .2864 | .1180 | 53 | 93 | .0520 | .0363 |
| 54 | 64 | .5792 | .3629 | 54 | 74 | .4793 | .2227 | 54 | 84 | .2706 | .1143 | 54 | 94 | .0324 | .0252 |
| 55 | 65 | .5811 | .3591 | 55 | 75 | .4747 | .2173 | 55 | 85 | .2530 | .1110 | 55 | 95 | .0126 | .0126 |
| 56 | 66 | .5830 | .3551 | 56 | 76 | .4691 | .2122 | 56 | 86 | .2338 | .1079 | | | | |
| 57 | 67 | .5848 | .3511 | 57 | 77 | .4628 | .2070 | 57 | 87 | .2127 | .1049 | | | | |
| 58 | 68 | .5868 | .3467 | 58 | 78 | .4561 | .2013 | 58 | 88 | .1891 | .1025 | | | | |
| 59 | 69 | .5886 | .3423 | 59 | 79 | .4494 | .1947 | 59 | 89 | .1657 | .0979 | | | | |

| Ten years difference. | | | | Twenty years difference. | | | | Thirty years difference. | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| Ages. | | Youngest. | Eldest. | Ages. | | Youngest. | Eldest. | Ages. | | Youngest. | Eldest. |
| 60 | 70 | .5904 | .3377 | 60 | 80 | .4421 | .1876 | 60 | 90 | .1424 | .0909 |
| 61 | 71 | .5921 | .3330 | 61 | 81 | .4332 | .1808 | 61 | 91 | .1204 | .0801 |
| 62 | 72 | .5937 | .3281 | 62 | 82 | .4175 | .1740 | 62 | 92 | .0973 | .0674 |
| 63 | 73 | .5946 | .3236 | 63 | 83 | .4103 | .1683 | 63 | 93 | .0740 | .0520 |
| 64 | 74 | .5950 | .3193 | 64 | 84 | .3935 | .1649 | 64 | 94 | .0467 | .0364 |
| 65 | 75 | .5942 | .3159 | 65 | 85 | .3737 | .1626 | 65 | 95 | .0184 | .0184 |
| 66 | 76 | .5921 | .3133 | 66 | 86 | .3512 | .1607 | | | | |
| 67 | 77 | .5896 | .3106 | 67 | 87 | .3256 | .1591 | | | | |
| 68 | 78 | .5868 | .3075 | 68 | 88 | .2956 | .1586 | | | | |
| 69 | 79 | .5848 | .3030 | 69 | 89 | .2651 | .1549 | | | | |
| 70 | 80 | .5822 | .2981 | 70 | 90 | .2338 | .1473 | | | | |
| 71 | 81 | .5784 | .2936 | 71 | 91 | .2032 | .1338 | | | | |
| 72 | 82 | .5729 | .2893 | 72 | 92 | .1695 | .1166 | | | | |
| 73 | 83 | .5649 | .2860 | 73 | 93 | .1335 | .0933 | | | | |
| 74 | 84 | .5508 | .2866 | 74 | 94 | .0878 | .0682 | | | | |
| 75 | 85 | .5342 | .2871 | 75 | 95 | .0361 | .0361 | | | | |
| 76 | 86 | .5148 | .2868 | | | | | | | | |
| 77 | 87 | .4912 | .2869 | | | | | | | | |
| 78 | 88 | .4601 | .2899 | | | | | | | | |
| 79 | 89 | .4260 | .2907 | | | | | | | | |
| 80 | 90 | .3894 | .2864 | | | | | | | | |
| 81 | 91 | .3544 | .2696 | | | | | | | | |
| 82 | 92 | .3145 | .2422 | | | | | | | | |
| 83 | 93 | .2599 | .2072 | | | | | | | | |
| 84 | 94 | .1835 | .1546 | | | | | | | | |
| 85 | 95 | .0827 | .0827 | | | | | | | | |

**Fifty years difference.**

| Ages. | | Youngest. | Eldest. |
|---|---|---|---|
| 1 | 51 | .2419 | .3914 |
| 2 | 52 | .2761 | .2885 |
| 3 | 53 | .2879 | .2444 |
| 4 | 54 | .2954 | .2126 |
| 5 | 55 | .2979 | .1946 |
| 6 | 56 | .3005 | .1766 |
| 7 | 57 | .3011 | .1633 |
| 8 | 58 | .3014 | .1533 |
| 9 | 59 | .2998 | .1472 |
| 10 | 60 | .2972 | .1440 |
| 11 | 61 | .2940 | .1420 |
| 12 | 62 | .2906 | .1404 |
| 13 | 63 | .2870 | .1388 |
| 14 | 64 | .2833 | .1373 |
| 15 | 65 | .2793 | .1359 |
| 16 | 66 | .2751 | .1347 |
| 17 | 67 | .2707 | .1332 |
| 18 | 68 | .2662 | .1311 |
| 19 | 69 | .2617 | .1283 |
| 20 | 70 | .2570 | .1250 |
| 21 | 71 | .2524 | .1208 |
| 22 | 72 | .2476 | .1162 |
| 23 | 73 | .2424 | .1116 |
| 24 | 74 | .2369 | .1071 |
| 25 | 75 | .2308 | .1028 |
| 26 | 76 | .2242 | .0987 |
| 27 | 77 | .2174 | .0945 |
| 28 | 78 | .2103 | .0902 |
| 29 | 79 | .2031 | .0856 |
| 30 | 80 | .1956 | .0809 |
| 31 | 81 | .1876 | .0763 |
| 32 | 82 | .1788 | .0720 |
| 33 | 83 | .1692 | .0680 |
| 34 | 84 | .1580 | .0650 |
| 35 | 85 | .1460 | .0624 |
| 36 | 86 | .1333 | .0598 |
| 37 | 87 | .1198 | .0574 |
| 38 | 88 | .1052 | .0553 |
| 39 | 89 | .0909 | .0523 |
| 40 | 90 | .0770 | .0481 |
| 41 | 91 | .0641 | .0420 |
| 42 | 92 | .0509 | .0350 |
| 43 | 93 | .0380 | .0265 |
| 44 | 94 | .0235 | .0182 |
| 45 | 95 | .0090 | .0090 |

**Sixty years difference.**

| Ages. | | Youngest. | Eldest. |
|---|---|---|---|
| 1 | 61 | .1957 | .3491 |
| 2 | 62 | .2184 | .2411 |
| 3 | 63 | .2245 | .1950 |
| 4 | 64 | .2276 | .1617 |
| 5 | 65 | .2275 | .1426 |
| 6 | 66 | .2275 | .1235 |
| 7 | 67 | .2266 | .1090 |
| 8 | 68 | .2251 | .0980 |
| 9 | 69 | .2226 | .0910 |
| 10 | 70 | .2194 | .0870 |
| 11 | 71 | .2156 | .0843 |
| 12 | 72 | .2114 | .0822 |
| 13 | 73 | .2070 | .0802 |
| 14 | 74 | .2022 | .0785 |
| 15 | 75 | .1973 | .0768 |
| 16 | 76 | .1909 | .0764 |
| 17 | 77 | .1845 | .0755 |
| 18 | 78 | .1780 | .0739 |
| 19 | 79 | .1713 | .0714 |
| 20 | 80 | .1643 | .0685 |
| 21 | 81 | .1569 | .0649 |
| 22 | 82 | .1490 | .0610 |
| 23 | 83 | .1405 | .0574 |
| 24 | 84 | .1305 | .0548 |
| 25 | 85 | .1201 | .0523 |
| 26 | 86 | .1091 | .0499 |
| 27 | 87 | .0975 | .0476 |
| 28 | 88 | .0851 | .0456 |
| 29 | 89 | .0731 | .0427 |
| 30 | 90 | .0615 | .0389 |
| 31 | 91 | .0510 | .0335 |
| 32 | 92 | .0402 | .0277 |
| 33 | 93 | .0298 | .0209 |
| 34 | 94 | .0183 | .0143 |
| 35 | 95 | .0059 | .0059 |

**Seventy years difference.**

| Ages. | | Youngest. | Eldest. |
|---|---|---|---|
| 1 | 71 | .1544 | .3045 |
| 2 | 72 | .1622 | .1942 |
| 3 | 73 | .1604 | .1484 |
| 4 | 74 | .1573 | .1156 |
| 5 | 75 | .1523 | .0977 |
| 6 | 76 | .1477 | .0795 |
| 7 | 77 | .1433 | .0657 |
| 8 | 78 | .1390 | .0550 |
| 9 | 79 | .1346 | .0481 |
| 10 | 80 | .1301 | .0440 |
| 11 | 81 | .1250 | .0414 |
| 12 | 82 | .1193 | .0392 |
| 13 | 83 | .1137 | .0375 |
| 14 | 84 | .1068 | .0366 |
| 15 | 85 | .0993 | .0361 |
| 16 | 86 | .0912 | .0360 |
| 17 | 87 | .0821 | .0362 |
| 18 | 88 | .0723 | .0360 |
| 19 | 89 | .0623 | .0350 |
| 20 | 90 | .0525 | .0327 |
| 21 | 91 | .0433 | .0286 |
| 22 | 92 | .0342 | .0235 |
| 23 | 93 | .0253 | .0177 |
| 24 | 94 | .0155 | .0121 |
| 25 | 95 | .0059 | .0059 |

**Eighty years difference.**

| Ages. | | Youngest. | Eldest. |
|---|---|---|---|
| 1 | 81 | .1294 | .2485 |
| 2 | 82 | .1189 | .1426 |
| 3 | 83 | .1042 | .1027 |
| 4 | 84 | .0901 | .0756 |
| 5 | 85 | .0768 | .0626 |
| 6 | 86 | .0649 | .0484 |
| 7 | 87 | .0546 | .0377 |
| 8 | 88 | .0457 | .0294 |
| 9 | 89 | .0382 | .0239 |
| 10 | 90 | .0318 | .0203 |
| 11 | 91 | .0274 | .0158 |
| 12 | 92 | .0204 | .0140 |
| 13 | 93 | .0149 | .0105 |
| 14 | 94 | .0091 | .0071 |
| 15 | 95 | .0035 | .0035 |

**Ninety years difference.**

| Ages. | | Youngest. | Eldest. |
|---|---|---|---|
| 1 | 91 | .1098 | .1803 |
| 2 | 92 | .0759 | .0844 |
| 3 | 93 | .0510 | .0480 |
| 4 | 94 | .0291 | .0234 |
| 5 | 95 | .0110 | .0110 |

In the solution of all the problems which involve the contingency in the foregoing table, the constant method of ascertaining it has hitherto been, by taking half the probability of the two lives becoming extinct in a given time, both in the case of the elder life's dying after the younger, and of the younger's dying after the elder. When the ages of the two lives are very different, this method (as I have observed in my former paper) must be incorrect. I have taken considerable pains to determine the extent of the inaccuracy, and for this purpose have computed the following table.

# TABLE,

Shewing the value, at 4 per cent. of £ 1, payable at the end of a given time, provided the life of B shall have failed after the life of A.

| Ages. | | 5 years. | | 10 years. | | 15 years. | | 20 years. | | 25 years. | | 30 years. | | 40 years. | | Life. | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| B * | A | True val. | Approx. | True val. | Approx. | True val. | Approx. | True val. | Approx. | True val. | Approx. | True val. | Approx. | True val. | Approx. | True val. | Approx. |
| 20 | 10 | .00134 | .00133 | .00448 | .00472 | .00889 | .00979 | .01382 | .01513 | .01879 | .02030 | .02304 | .02457 | .02978 | .03118 | .02099 | .02473 |
| 35 | 15 | .00194 | .00206 | .00732 | .00785 | .01452 | .01528 | .02245 | .02316 | .02977 | .03041 | .03588 | .03667 | .04392 | .04528 | .03153 | 03936 |
| 45 | 15 | 00250 | .00266 | .00883 | 01017 | .01882 | .01980 | .02835 | .02958 | .03710 | .03838 | .04456 | .04599 | | | .03663 | .04829 |
| 50 | 30 | .00504 | .00503 | .01654 | .01656 | .03057 | .03087 | .04454 | .04523 | .05723 | .05872 | .06639 | .06897 | | | .05197 | .06819 |
| 55 | 15 | .00347 | .00369 | .01173 | .01377 | .02464 | .02640 | .03710 | .03925 | .04683 | .05000 | | | | | .04022 | .05644 |
| 65 | 20 | .00725 | .00730 | .02402 | .02410 | .04203 | .04326 | .05460 | .05898 | .05525 | .06692 | | | | | .04696 | 06800 |
| 70 | 40 | .01413 | .01421 | .04303 | .04477 | .06948 | .07697 | | | | | | | | | .06688 | .10334 |
| 75 | 60 | .03357 | .03572 | .09545 | 10372 | .12293 | .15521 | | | | | | | | | .11324 | .17570 |
| 80 | 70 | .07653 | .08051 | .15576 | .18868 | | | | | | | | | | | .15915 | .23503 |
| 85 | 10 | .01136 | .01374 | | | | | | | | | | | | | .01987 | .03500 |

* In these specimens (which are sufficient to give an idea of the difference between the true values and the approximation in all cases) I have constantly supposed the life of B to be the eldest. But the differences would have been the same if his life had been the youngest; only that in this case the true values would have varied as much in *excess* as they here do in *defect*. This is obvious from the nature of the approximation.

From this table it appears, that the approximations and exact values do not differ much from each other till the last years of B's life, and that the principal inaccuracy in adopting the approximation will arise after the extinction of the life of B, when it becomes necessary to multiply the fraction expressing the probability of his dying after A into the remaining series of the solution. But this perhaps will be better understood from the following problems, and from the computations which are made to prove the correctness of the general rules.

PROBLEM I.

To find the value of an annuity on the life of C after A, on the particular condition that A's life when it fails shall fail before the life of B.

SOLUTION.

As the approximation appears from the preceding table to be always sufficiently correct, except in the two or three last years of B's life, it is evident, that if the fractions which express the probability of B's dying after A in those years, be either confined only to the value of the annuity during that short period, or be not involved at all in the computation, no great inaccuracy will arise from having recourse to the ordinary method of determining that probability, provided the solution be founded on real observations of life, and not on Mr. De Moivre's hypothesis. In the present problem, when C or A is the oldest of the three lives, the abovementioned fractions either never enter into the computation, or are confined to the last years of A's life; and in both cases they are combined

with another contingency, which necessarily renders them of less consequence. The solution, therefore, particularly in the former case, becomes very easy; and even in the latter, by the assistance of the table in my first paper,* it becomes equally simple and correct. But when B is the oldest of the three lives, the above fractions are combined with a series which is often of considerable importance, and consequently the common method of solution fails in this case. Yet even here, being possessed of the table deduced from the foregoing lemma, it is attended with little or no difficulty, and a general rule as short and accurate is obtained as in the other cases. This however will be more satisfactorily proved by the following operations.

1st. *Let C be the oldest of the three lives.* In the first year the payment of the annuity depends on one or other of two events; either that A and B both die (B having died last), and that C lives, the probability of which event is expressed by $\frac{a' . \overline{b-m} . d}{2abc}$, or that only A dies, and that B and C both live, which probability is expressed by $\frac{a'md}{abc}$. The value, therefore, of the annuity for the first year will be $= \frac{a'bd}{2abcr} + \frac{a'md}{2abcr}$. In the second year, the payment of the annuity depends nearly on the same events: 1st. that A and B both die in the first or second year (B having died last), and that C lives to the end of this term, which is $= \frac{e}{c} \times \frac{\overline{b-n} . \overline{a'+a''}}{2ab}$; or 2dly, that only A has died before the end of the second year, and that B and C have both lived, which is $= \frac{ne . \overline{a'+a''}}{abc}$. Hence the value of the annuity for the

* Phil. Trans. Vol. LXXVIII. p. 337.

second year will be $= \frac{be\,.\,\overline{a'+a''}}{2abcr^2} + \frac{ne\,.\,\overline{a'+a''}}{2abcr^2}$. In the third year, by following the same steps, the value of the annuity will be found $= \frac{bf\,.\,\overline{a'+a''+a'''}}{2abcr^3} + \frac{of\,.\,\overline{a'+a''+a'''}}{2abcr^3}$, and in the remaining years of C's life the value of the annuity may be determined in a similar manner. The whole value of the annuity therefore, when C is the oldest of three lives, will be expressed by the two series $\frac{a'd}{2acr} + \frac{\overline{a'+a''}\,.\,e}{2acr^2} + \frac{\overline{a'+a''+a'''}\,.\,f}{2acr^3} +$ &c. and $\frac{a'md}{2abcr} + \frac{\overline{a'+a''}\,.\,ne}{2abcr^2} + \frac{\overline{a'+a''+a'''}\,.\,of}{2abcr^3} +$ &c. The first of these series is $= \frac{\text{C}-\text{AC}}{2}$, and the second is $= \frac{\text{BC}-\text{ABC}}{2}$; hence the required value in this case is $= \frac{\text{C}-\text{AC}}{2} + \frac{\text{BC}-\text{ABC}}{2}$.

Secondly. *Let A be the oldest of the three lives,* and if $z$ denote the number of years between the ages of A and of the last person in the table, C′ the value of an annuity on the life of C for $z$ years, and B′C′ the same value on the two joint lives of B and C; the value of the annuity for the first $z$ years will evidently in this case be $= \frac{\text{C}'-\text{AC}}{2} + \frac{\text{B}'\text{C}'-\text{ABC}}{2}$. At the expiration of this term the life of A is necessarily extinct, and consequently the value of the annuity for the remaining years of C's life (supposing $\delta$, $\varepsilon$, $\zeta$, $\eta$, &c. to denote the number of persons living in the table at the end of $\overline{z+1}$, $\overline{z+2}$, $\overline{z+3}$, &c. years, and $\varphi$ to denote the probability of B's surviving A*) will be $= \varphi \times \overline{\frac{\delta}{cr^{z+1}} + \frac{\varepsilon}{cr^{z+2}} + \frac{\zeta}{cr^{z+3}} + \text{\&c.}} = \varphi\,.\,\overline{\text{C}-\text{C}'}$. The whole value of the annuity therefore, when A is the oldest

* See the table in the LXXVIIIth Vol. of the Phil. Trans. p. 337. N. B. When this table is used in the present and following problems, certainty must be denoted by *unity.*

of the three lives, will be $= \frac{C'+B'C'}{2} + \phi \,.\, \overline{C - C'} - \frac{AC+ABC}{2}$.

Thirdly. *If B be the oldest of the three lives,* let $x$ denote the number of years between the ages of B and of the last person in the table, C′ the value of an annuity on the life of C for $x$ years, A′C′ the same value on the joint lives of A and C, and $\pi$ the probability (found by the table in the foregoing lemma), that B dies after A. Then, by proceeding as above, the value of the annuity in this case will be found $= \frac{C'-A'C'}{2} + \pi \times \overline{C - C'} + \frac{BC-ABC}{2}$.. Q. E. D.

When the lives are all equal, the general rule deduced either from the series or the foregoing expressions becomes $= \frac{C-CCC}{2}$, which is known to be the exact value in this case from self-evident principles.

As this method of solution is applicable to a great number of problems, I have thought it necessary to make the following computations, with the view of determining how far it may be depended upon. It is to be observed, that the first series of fractions in the above solution, or $\frac{a' \,.\, \overline{b-m} \,.\, d}{2abcr} + \frac{\overline{a'+a''} \,.\, \overline{b-n} \,\, e}{2abcr^2} +$ &c. should have been (according to the lemma), in order to express the exact value, $\frac{a' \,.\, \overline{b-m} \,.\, d}{2abcr} + \frac{d' \,.\, \overline{b-n} + \overline{a'+a''} \,.\, \overline{m-n} \,.\, e}{2abcr^2} +$ &c. and that it is impossible to find a general expression which shall be equal to this latter series and at the same time fit for use. This has rendered it necessary to have recourse to the present approximation. But in the first column of the following examples, each term of this last series has been separately computed, so that by comparing the values in that and the second column an exact idea may be formed of the accuracy of the preceding rules.

| Ages of | | | Annuity of £ 1. | |
|---|---|---|---|---|
| C. | B. | A. | True value. | Approximation. |
| 78 | 15 | 75 | .138 | .146 |
| 70 | 20 | 65 | .406 | .408 |
| 81 | 70 | 80 | .530 | .550 |
| 15 | 85 | 10 | 13.833 | 13.782 |
| 15 | 75 | 15 | 11.385 | 11.230 |
| 35 | 75 | 15 | 8.968 | 8.834 |
| 15 | 65 | 20 | 8.485 | 8.379 |
| 15 | 80 | 70 | 9.893 | 9.529 |
| 15 | 10 | 85 | .647 | .698 |
| 15 | 15 | 75 | 1.038 | 1.193 |
| 35 | 15 | 75 | .786 | .920 |
| 15 | 20 | 65 | 1.769 | 1.876 |
| 15 | 70 | 80 | 4.307 | 4.671 |

From these examples it appears, that when C or A is the oldest of the three lives, the approximated and the true values agree sufficiently near for any useful purpose; and that even when B is the oldest, the difference is almost as inconsiderable. It should likewise be observed, that these examples are cases in which the difference is likely to be greatest, and therefore a nearer approximation need not be required. Both Mr. SIMPSON and myself have given solutions of this problem, and in most of the foregoing examples the values derived from them are more correct than could have been expected; but these solutions being founded on a wrong hypothesis, are not so correct as the present, except when C is the oldest of the

three lives, nor are they even more simple, so that it can now be seldom necessary to have recourse to them. Without the assistance of the preceding lemma, and the computations which have been just made, it would not have been possible to have ascertained the degree of accuracy of any approximation; and therefore were no other end answered by them, this of itself would be of sufficient consequence to deserve the time and labour which I have bestowed upon this subject. But it will appear, in the solution of some of the succeeding problems, that the use and application of this lemma, and especially of the table deduced from it, are much more extensive and important.

## PROBLEM II.

To find the value of an annuity during the life of C, after the decease of A, provided A should survive B.

## SOLUTION.

The payment of this annuity depends only on one contingency; and that is, the extinction of the two lives of A and B before the end of each year (B having died first), and the continuance of the life of C to the end of those respective years. The value therefore of the annuity for the first year will be $= \frac{d \cdot \overline{b - m} \cdot a'}{2abcr}$, for the second year $= \frac{e \cdot \overline{b-n} \cdot \overline{a' + a''}}{2abcr^2}$, for the third year $= \frac{f \cdot \overline{b-o} \cdot \overline{a' + a'' + a'''}}{2abcr^3}$, and so on for the remaining years. The value of the annuity (*when C is the oldest life*) will consequently be expressed by the two series $\frac{da'}{2acr} + \frac{e \cdot \overline{a' + a''}}{2acr^2} +$

$\frac{f \cdot \overline{a'+a''+a'''}}{2acr^3} +$ &c. . . . and $- \frac{mda'}{2abcr} - \frac{ne \cdot \overline{a'+a''}}{2abcr^2} - \frac{of \cdot \overline{a'+a''+a'''}}{2abcr^3} -$ &c. $= \frac{C-AC}{2} - \frac{BC-ABC}{2}$. *If A be the oldest of the three lives* this rule will be insufficient. Let $z$, $C'$, $B'C'$ and $\phi$ denote the the same quantities as in the second part of the preceding problem; then will the value of the annuity in this case, for the first $z$ years, be $= \frac{C'-AC}{2} - \frac{B'C'-ABC}{2}$, and its value for the remaining years of C's life $= \overline{1-\phi} \cdot \overline{C-C'}$; for the payment of it during this last term depends on the contingency of C's living so long, and of A's having survived B, which probability is $= 1-\phi$; therefore the whole value will be $= C - \frac{C'+B'C'}{2} - \phi \cdot \overline{C-C'} - \frac{AC-ABC}{2}$.

If B *be the oldest of the three lives*, let $x$, $\pi$, $C'$, $A'C'$, $\delta$, $\epsilon$, $\zeta$, &c. denote the same quantities as in the third part of the foregoing problem; also let $x'$ denote the sum of the decrements of the life of A for $x$ years, and $\alpha'$, $\alpha''$, $\alpha'''$, &c. the decrements of the same life in the $\overline{x+1}$st, $\overline{x+2}$d, $\overline{x+3}$d, &c. years. The value of the annuity for the first $x$ years will it is evident be $= \frac{C'-A'C'}{2} - \frac{BC-ABC}{2}$. In the $\overline{x+1}$st year the payment of it will depend on the contingency of A's having died after B in $\overline{x+1}$ years, and C's having lived to the end of this term. As the life of B becomes necessarily extinct in $x$ years, it is plain that the probability of A's dying after him in $\overline{x+1}$ years must be $= \frac{x'+\alpha'}{a} - \pi$, and therefore that the value of the annuity in this year will be $= \frac{x'+\alpha'}{a} \times \frac{\delta}{cr^{x+1}} - \frac{\pi \cdot \delta}{cr^{x+1}}$. In the same manner the value of the annuity in the $\overline{x+2}$d,

$\overline{x+3}$d, &c. years will be $= \frac{x'+\alpha'+\alpha''}{a} \times \frac{\varepsilon}{cr^x+^2} - \frac{\pi\,.\,\varepsilon}{cr^x+^2}$ . . . . . $\frac{x'+\alpha'+\alpha''+\alpha'''}{a} \times \frac{\zeta}{cr^x+^3} - \frac{\pi\,.\,\zeta}{cr^x+^3}$ . . . &c. But the series $\frac{x'+\alpha'}{a} \times \frac{\delta}{cr^x+^1} + \frac{x'+\alpha'+\alpha''}{a} \times \frac{\varepsilon}{cr^x+^2}$ + &c. is $= \overline{C-C'} - \overline{AC-A'C'}$, and the series $\frac{\pi\,.\,\delta}{cr^x+^1} + \frac{\pi\,.\,\varepsilon}{cr^x+^2}$ + &c. is $= \pi\,.\,\overline{C-C'}$, the whole value of the annuity therefore, when B is the eldest, will be $=$

$$C - AC - \frac{C'-A'C'}{2} - \frac{BC-ABC}{\;} - \pi\,.\,\overline{C-C'}.$$

COROLLARY.

If the solution of either of these two problems be given, the solution of the other problem may be immediately derived from it; for the value of the reversion in one is no more than the difference between the value of the reversion in the other, and the value of an annuity on the life of C after A. In other words, let the value found by either of these problems be called Q, and the required value of the reversion in the other problem, supposing the ages of A, B, and C to be the same in both, will be $= \overline{C-AC} - Q$. This deduction is self-evident, and if applied to any of the foregoing rules will be found to confirm the truth of the solution.

PROBLEM III.

To find the value of a given sum payable on the death of A and C, provided B should survive one life in particular (A).

SOLUTION.

In the first year the payment of the given sum will depend upon either of two events; 1st, that all the three lives shall drop (B having survived A) which is $= \frac{\overline{b-m}\,.\,\overline{c-d}\,.\,a'}{2abc}$. 2dly, That B shall live, and only A and C die, which is $= \frac{\overline{c-d}\,.\,ma'}{abc}$. The value therefore of the given sum in the first year will be $= \frac{S\,.\,a'}{abcr} \times \overline{\frac{bc}{2}+\frac{mc}{2}-\frac{bd}{2}-\frac{md}{2}}$. In the second and following years the payment of S will depend upon either of seven events: 1st. that all the three lives drop in the year, B having survived A. 2dly, That B lives, and only A and C die in the year. 3dly, that A dies in the year, C dies in any of the foregoing years, and B lives. 4thly, That B dies after A in the year, and C dies in any of the foregoing years. 5thly, That C dies in the year, and B dies after A in any of the foregoing years. 6thly, That B and C both die in the year, and A dies in any of the foregoing years. And 7thly, That C dies in the year, A in any of the foregoing years, and B lives. From the several fractions expressing these contingencies the value of the given sum will be found

$$= \frac{S}{2ab} \times \overline{\frac{a'b}{r}+\frac{a''m}{r^2}+\frac{a'''.n}{r^3}+\&c.} + \frac{S}{2ab} \times \overline{\frac{a'm}{r}+\frac{a''n}{r^2}+\frac{a'''.o}{r^3}+\&c.} - \frac{S}{2abc} \times \overline{\frac{a'bd}{r}+\frac{a''me}{r^2}+\frac{a'''nf}{r^3}+\&c.} -$$

$$\frac{S}{2abc} \times \overline{\frac{a'md}{r}+\frac{a''ne}{r^2}+\frac{a'''of}{r^3}+\&c.} + \frac{S}{2acr} \times \overline{\frac{a'd}{r}+\frac{\overline{a'+a''}\,.\,e}{r^2}+\&c.} +$$

$$\frac{S}{2abcr} \times \overline{\frac{a'md}{r}+\frac{\overline{a'+a''}\,.\,ne}{r^2}+\&c.} - \frac{S}{2acr} \times \overline{\frac{a'e}{r}+\frac{\overline{a'+a''}\,.f}{r^2}+\&c.} -$$

$$\frac{S}{2abcr} \times \overline{\frac{a'me}{r}+\frac{\overline{a'+a''}\,.\,nf}{r^2}+\&c.} = E + \frac{S}{2r} \times \overline{C-CA-\overline{r-1}}\,.$$

$$\overline{BC-ABC} - \frac{S\,.\,\beta}{2b} \times \overline{FC-AFC} + \frac{S\,.\,d}{2cr} \times \frac{m\,.\,\overline{PT-APT}}{b} -$$

$\overline{T-AT}$. If *A be the oldest of the three lives*, let $z$, $C'$, $B'C'$, and $\varphi$ denote the same quantities as in the second part of prob. I, and let $F'C'$ be the value of the joint lives of F and C for $z$ years, it will then be evident that the value of the reversion for the first $z$ years will be $= E + \frac{S}{2r} \times \overline{C'-A'C'-\overline{r-1}.\overline{B'C'-ABC}} - \frac{S.\beta}{2b} \times \overline{F'C'-AFC} + \frac{S.d}{2cr} \times \overline{\frac{m.\overline{P'T'-APT}}{b} - \overline{T'-AT}}$, and its value after this term $= \frac{S.\overline{r-1}}{r^{z+1}} \times \frac{\varphi.k}{c} \times \overline{V-C^z}$; $C^z$ being the value of an annuity on a life $z$ years older than C, and $k$ the number of persons living at the age of $C^z$... If *B be the oldest of the three lives*, the value, by proceeding as above, may be easily found $= E + \frac{S}{2r} \times \overline{C'-A'C'-\overline{r-1}.\overline{BC-ABC}} - \frac{S.\beta}{2b} \times \overline{FC-AFC} + \frac{S.d}{2cr} \times \overline{\frac{m.\overline{PT-APT}}{b} - \overline{T'-A'T'}} + \frac{S.\overline{r-1}}{r^{x+1}} \times \frac{\pi.q}{c} \times \overline{V-C^x}$; $C'$, $A'C'$, $\pi$ and $x$ denoting the same quantities as in the third part of prob. I, $C^x$ the value of an annuity on a life $x$ years older than C, $q$ the number of persons living at the age of $C^x$.. and $T'$ and $A'T'$ the values of annuities on the single life of T, and on the joint lives of A and T for $x$ years.

But the solution of this problem may be obtained rather more easily by the assistance of the first problem in this paper, and of the second problem which I communicated to the Royal Society in the year 1788.* For the value of a given sum payable on the death of A and C should B survive A, is evidently "the difference between the value of that sum depending on "the contingency of B's surviving A, and the value of an

* Phil. Trans. Vol. LXXVIII. p. 341.

"annuity equal to the interest of the given sum during the life "of C after A, provided A should die before B." The first of these is E, and if an annuity of £1. by prob. I, be denoted by Q, the second will be $= \frac{S.\overline{r-1}.Q}{r}$. The required value therefore will be $= E - \frac{S.\overline{r-1}.}{r} \times Q$ . . . . . If *the three lives be equal,* the general theorem will become $= \frac{S.\overline{r-1}.}{r} \times \overline{V - CC - C - CCC}$, which may be derived from either of the foregoing rules, or from the different series given above.

### PROBLEM IV.

To find the value of a given sum S, payable on the death of A and C, should B die before one life in particular (A).

### SOLUTION.

The payment of S in the first year depends on the contingency of the three lives having become extinct (A having survived B), which is expressed by $\frac{\overline{b-m}.\overline{c-d}.a'}{2abc}$, and therefore the value of S in this year will be $= \frac{S.a'}{2abcr} \times \overline{bc - mc - bd + ma'}$. In the second and following years the sum S will become payable if either of five events should take place. 1st, If the *three lives* should drop in the year (B having died before A). 2dly, If C should die in any of the preceding years, and A die after B in that particular year. 3dly, If B and C should die in any of the preceding years, and only A die in that year. 4thly, If B should die in any of the preceding

years, and A and C both die in that year: and 5thly, If A should die af er B in any of t e preceding years, and C die in that year. From the different fractions expressing those probabilities, the value of S may be found $= \frac{S}{a} \times \overline{\frac{a'}{r} + \frac{a''}{r^2} + \frac{a'''}{r^3} + \&c.}$ $+ \frac{S}{acr} \times \overline{\frac{da'}{r} + \frac{e \cdot \overline{a' + a''}}{r^2} + \&c.} - \frac{S}{2ab} \times \overline{\frac{a'm}{r} + \frac{a''n}{r^2} + \frac{a''' \cdot o}{r^3} + \&c.} +$ $\frac{S}{2abc} \times \overline{\frac{a'bd}{r} + \frac{a''me}{r^2} + \frac{a''' \cdot nf}{r^3} + \&c.} + \frac{S}{2abc} \times \overline{\frac{a'md}{r} + \frac{a''ne}{r^2} + \frac{a'''of}{r^3} + \&c.}$ $- \frac{S}{2acr} \times \overline{\frac{da'}{r} + \frac{e \cdot \overline{a' + a''}}{r^2} + \&c.} - \frac{S}{2abcr} \times \overline{\frac{a'md}{r} + \frac{\overline{a' + a''} \cdot ne}{r^2} + \&c.} +$ $\frac{S}{2acr} \times \overline{\frac{ea'}{r} + \frac{f \cdot \overline{a' + a''}}{r^2} + \&c.} + \frac{S}{2abcr} \times \overline{\frac{a'me}{r} + \frac{\overline{a' + a''} \cdot nf}{r^2} + \&c.}$ The three first of these series are $= \frac{S \cdot \overline{r - 1}}{r} \times \overline{V - A - C + AC}$, and the remaining eight denote the value of S by the third problem, with contrary signs. If this last value be called Y, and the value of an annuity of £ 1. on the longest of the two lives of A and C be called Z, the required value will be $= \frac{S \cdot \overline{r - 1}}{r} \times \overline{V - Z} - Y$; that is, the value of the given sum in this case is "the difference between its value after the extinc-"tion of the lives of A and C, on the contingency of B's sur-"viving A, and the whole value of the reversion after the "death of A and C, without any restriction." This rule is self-evident, and proves the truth of the foregoing investigations. The solution of this problem may also be derived from the second problem in this paper, and the third problem in my paper communicated in the year 1788.* In other words, "the "value of S in the present case is equal to the difference be-"tween its value after the death of A and B, provided B

* Phil. Trans. Vol. LXXVIII. p. 347.

"should die before A, and the value of an annuity equal to "the interest of S during the life of C after A, provided A "should survive B." Let the first of these values be denoted by W, and the second by X, and the required value will be $= W - \frac{S.\overline{r-1}}{r} \times X$. When the three lives are equal, the value of the reversion evidently becomes $= \frac{S.\overline{r-1}}{2r} \times \overline{V-L}$, which expression may be easily derived from either of the rules given above, or immediately from the series themselves.

Having given so many examples of the accuracy of the rules in the first and second problems, it becomes unnecessary to add any further examples in regard to the two foregoing problems, as the solutions of the latter are derived from those of the former, and consequently are equally correct in all cases.

### PROBLEM V.

To find the value of a given sum payable on the *decease* of B and C, should their lives be the *last* that shall fail of the three lives A, B, and C.

### SOLUTION.

In the first year the given sum can be received only provided the three lives shall have failed, and the life of A have been the first that became extinct. In the second and following years it may be received provided either of four events shall have happened: 1st, If all the three lives shall have failed in that year, A dying first. 2dly, If A shall have died in any of the foregoing years, and B and C both died in that

year. 3dly, If B and A shall have both died in the foregoing years (B dying last), and C died in that year. 4thly, If C and A shall have both died in the foregoing years (C dying last), and B died in that year. From the fractions expressing these several contingencies the value of the reversion will be found

$$= \frac{S}{3abc} \times \overline{\frac{a'bc}{r} + \frac{a''md}{r^2} + \frac{a'''ne}{r^3} + \&c.} - \frac{S}{3abc} \times \overline{\frac{a'mc}{r} + \frac{a''nd}{r^2} + \frac{a'''oe}{r^3} + \&c.}$$

$$- \frac{S}{2abcr} \times \overline{\frac{nd\,.\,a'}{r} + \frac{eo\,.\,\overline{a'+a''}}{r^2} + \&c.} - \frac{S}{3abc} \times \overline{\frac{a'bd}{r} + \frac{a''me}{r^2} + \frac{a'''nf}{r^3} + \&c.}$$

$$- \frac{S}{2abcr} \times \overline{\frac{me\,.\,a'}{r} + \frac{nf\,.\,\overline{a'+a''}}{r^2} + \&c.} + \frac{S}{3abc} \times \overline{\frac{a'md}{r} + \frac{a''ne}{r^2} + \frac{a'''of}{r^3} + \&c.}$$

$$+ \frac{S}{abcr} \times \overline{\frac{a'ne}{r} + \frac{\overline{a'+a''}\,.\,of}{r^2} + \&c.} + \frac{S}{2acr} \times \overline{\frac{\overline{d-e}\,.\,a'}{r} + \frac{\overline{e-f}\,.\,\overline{a'+a''}}{r^2} + \&c.}$$

$$+ \frac{S}{2abr} \times \overline{\frac{\overline{m-n}\,.\,a'}{r} + \frac{\overline{n-o}\,.\,\overline{a'+a''}}{r^2} + \&c.} = S \text{ into } \frac{\overline{r-1}\,.}{r} \times$$

$$\overline{\frac{V - ABC}{3} - \frac{B + C}{2} + BC - \frac{AB + AC}{2r}} + \frac{m}{2br} \times \overline{\overline{1 + AP} - \frac{PC - APC}{3}}$$

$$+ \frac{d}{2cr} \times \overline{\overline{1 + AT} - \frac{BT - ABT}{3}} - \frac{4m\,.\,\overline{1+APT}}{3b} - \frac{\beta\,.\,\overline{FC - AFC}}{3b} - \frac{x}{3c} \times$$

$$\overline{\overline{BK - ABK} + \frac{\beta\,.\,AFK}{b}}.$$

If B and C are both of them older than A, and also are nearly of the same age, this general rule will be sufficiently correct. But if the ages of B and C differ much from each other, it is evident that the annuity on the single life of the younger of them (suppose C), and on the joint lives of AC and AT, ought to be continued only for as many years as are equal to the difference between the age of B and of the oldest life in the table of observations. In this case also there is a further value of S, after the necessary extinction of the life of B, arising from the contingency of that life's having failed after the life of A, and of C's having failed after both of them. Let $x$, $\pi$, C′, and A′C′ respectively denote the same

values as in the third part of the first problem, and let $A'T'$ denote the value of an annuity on the joint lives of A and T for $x$ years, $P'$ the value of an annuity on the life of P for the same term, $C^x$ the value of an annuity on a life $x$ years older than C, and $k$ the number of persons living in the table at that age, then will the value of the given sum be in this case

$$= \text{S into } \frac{\overline{r-1}\,.\,\overline{BC-ABC}}{3r} + \frac{x}{3c} \times \overline{\frac{\beta\,.\,\overline{FK-AFK}}{b} - \overline{BK-ABK} - \frac{\beta\,.\,\overline{FC-AFC}}{3b}} + \frac{m}{2br} \times \overline{\frac{4d\,.\,\overline{PT-APT}}{3c} - \frac{PC-APC}{3} - \overline{P'-A'P'} - \frac{d}{2cr} \times \frac{\overline{BT-ABT}}{3} + \overline{T'-A'T'}} + \frac{B+C'-\overline{AB+A'C'}}{2r} + \frac{\pi\,.\,k\,.\,\overline{r-1}}{er^{x+1}} \times \overline{V-C^x}.$$

*If A be the oldest of the three lives*, it will be necessary to substitute $\overline{a-s}$, $\overline{s-t}$, $\overline{t-u}$, &c. for their equals $a'$, $a''$, $a'''$, &c. and $b'$, $b''$, $b'''$, &c. for their equals $\overline{b-m}$, $\overline{m-n}$, $\overline{n-o}$, &c. In this case let C be supposed the oldest of B and C, and the series expressing the value of the reversion during the life of A will become

$$\frac{2\,.\,S}{3abc} \times \overline{\frac{adb'}{r} + \frac{esb''}{r^2} + \frac{ftb'''}{r^3} + \&c.} + \frac{S}{2abcr} \times \overline{\frac{esb'}{r} + \frac{ft\,.\,\overline{b'+b''}}{r^2} + \&c.} + \frac{S}{3abc} \times \overline{\frac{dsb'}{r} + \frac{etb''}{r^2} + \frac{fub'''}{r^3} + \&c.} - \frac{S}{6abc} \times \overline{\frac{acb'}{r} + \frac{dsb''}{r^2} + \frac{etb'''}{r^3} + \&c.} - \frac{S}{2abcr} \times \overline{\frac{dsb'}{r} + \frac{et\,.\,\overline{b'+b''}}{r^2} + \&c.} - \frac{S}{3abc} \times \overline{\frac{csb'}{r} + \frac{dtb''}{r^2} + \frac{eub'''}{r^3} + \&c.} + \frac{S}{2b} \times \overline{\frac{b'}{r} + \frac{b''}{r^2} + \frac{b'''}{r^3} + \&c.} - \frac{S}{2ab} \times \overline{\frac{ab'}{r} + \frac{sb''}{r^2} + \frac{tb'''}{r^3} + \&c.} + \frac{S}{2bc} \times \overline{\frac{cb'}{r} + \frac{db''}{r^2} + \frac{eb'''}{r^3} + \&c.} - \frac{S}{bc} \times \overline{\frac{db'}{r} + \frac{eb''}{r^2} + \frac{fb'''}{r^3} + \&c.} + \frac{S}{2bcr} \times \overline{\frac{\overline{d-e}\,.\,b'}{r} + \frac{\overline{e-f}\,.\,\overline{b'+b''}}{r^2} + \&c.}$$

Let $y$ represent the difference between the ages of A and of the oldest person in the table, let $K'$, $C'$, $B'$, $T'$, $B'C'$, $B'K'$,

and $B'T'$, respectively denote the values of annuities on those single and joint lives for $y$ years, then will the first and second series be $= \frac{2\alpha . \overline{HC - HBC}}{3a} - \frac{d}{6cr} \times \overline{AT - ATB}$, the third series $= \frac{AC - ABC}{3} - \frac{ds}{3acr} \times \overline{NT - NBT} = \frac{ds}{3acr} \times \overline{1 + NBT} - \frac{ABC}{3}$, the fourth and fifth series $= \frac{-\alpha\varkappa . \overline{HK - HBK}}{6ac} - \frac{AC - ABC}{3r} = \frac{\alpha\varkappa . HBK}{6ac} - \frac{1}{6r} + \frac{ABC}{3r} - \frac{AC}{2r}$, the sixth series $= \frac{-\varkappa . \overline{AK - ABK}}{3c} + \frac{s . \overline{NC - NBC}}{3ar}$, the seventh series $= \frac{\overline{r - 1} . \overline{V - B'}}{2r}$, the eighth series $= \frac{-\alpha . \overline{H - HB}}{2a} + \frac{A - AB}{2r} = \frac{\alpha . HB}{2a} - \frac{1 + AB}{2r}$, the ninth series $= \frac{\varkappa . \overline{K' - B'K'}}{2c} - \frac{C' - B'C'}{2r}$, the tenth series $= -\overline{C' - B'C'} + \frac{d}{cr} \times \overline{T' - B'T'}$, and the eleventh series $= \frac{C' - B'C'}{2r} - \frac{d}{2cr} \times \overline{T' - B'T'}$. In order to obtain the value of S after the necessary extinction of the life of A, let $\pi$ and $\mu$ denote the probability that B and C respectively die after A; $k$, $\delta$, $\epsilon$, $\zeta$, &c. the number of persons living in the table opposite the age of C at the end of $y$, $\overline{y + 1}$, $\overline{y + 2}$, &c. years; $p$ the number of persons living opposite the age of B at the end of $y$ years; and $\beta'$, $\beta''$, $\beta'''$, &c. the decrements of life at the age of B after $\overline{y + 1}$, $\overline{y + 2}$, $\overline{y + 3}$, &c. years. In the $\overline{y + 1}$st year the given sum may be received, provided either of three events shall have happened. 1st, If B and C shall have both died in that year. 2dly, If C only shall have died, B having died after A in the first $y$ years. 3dly, If B only shall have died, C having died after A in the first $y$ years. The value of S depending on these contingencies will be $= \frac{S}{r^y} \times \frac{\beta . \overline{k - \delta}}{bcr} + \frac{S . \pi . \overline{k - \delta}}{cr^{y+1}} + \frac{S . \mu . \beta'}{br^{y+1}}$. In the $\overline{y + 2d}$,

$\overline{y+3}$d, &c. years, the given sum may be received, provided either of five events shall have happened. 1st, If both the lives of B and C shall have become extinct in the year. 2dly, If C only shall have failed, B having died after A in the first $y$ years. 3dly, If B only shall have failed, C having died after A in the first $y$ years. 4thly, If B shall have failed, C having died in any of the preceding years after the first $y$ years: and 5thly, If C shall have failed, B having died in any of the preceding years after the first $y$ years. The series therefore expressing the value of S after the necessary extinction of A's life will be $\frac{S}{bcr^y} \times \overline{\frac{\overline{k-\delta}\,.\,\beta'}{r} + \frac{\overline{\delta-\varepsilon}\,.\,\beta''}{r^2} + \frac{\overline{\varepsilon-\zeta}\,.\,\beta'''}{r^3} + \&c.} + \frac{S\,.\,\pi}{cr^y} \times \overline{\frac{\overline{k-\delta}}{r} + \frac{\overline{\delta-\varepsilon}}{r^2} + \frac{\overline{\varepsilon-\zeta}}{r^3} + \&c.} + \frac{S\,.\,\mu}{br^y} \times \overline{\frac{\beta'}{r} + \frac{\beta''}{r^2} + \frac{\beta'''}{r^3} + \&c.} + \frac{S}{bcr^y} \times \overline{\frac{\overline{\delta-\varepsilon}\,.\,\beta'}{r^2} + \frac{\overline{\varepsilon-\zeta}\,.\,\overline{\beta'+\beta''}}{r^3} + \&c.} + \frac{S}{bcr^y} \times \overline{\frac{\overline{k-\delta}\,.\,\beta''}{r^2} + \frac{\overline{k-\varepsilon}\,.\,\beta'''}{r^3} + \&c.}$ $= \frac{\overline{S\,.\,\pi}}{cr^y} \times \overline{\frac{\overline{k-\delta}}{r} + \frac{\overline{\delta-\varepsilon}}{r^2} + \&c.} + \frac{S\mu}{br^y} \times \overline{\frac{\beta'}{r} + \frac{\beta''}{r^2} + \frac{\beta'''}{r^3} + \&c.} + \frac{Sk}{cr^y} \times \overline{\frac{\beta'}{r} + \frac{\beta''}{r^2} + \frac{\beta'''}{r^3} + \&c.} - \frac{S}{bcr^y} \times \overline{\frac{\delta\beta'}{r} + \frac{\varepsilon\beta''}{r^2} + \frac{\zeta\beta'''}{r^3} + \&c.} + \frac{S}{bcr^y} \times \overline{\frac{\overline{\delta-\varepsilon}\,.\,\beta'}{r^2} + \frac{\overline{\varepsilon-\zeta}\,.\,\overline{\beta'+\beta''}}{r^3} + \&c.}$ The three first of these series are $= \frac{S\,.\,\pi k\;\overline{r-1}\,.\,\overline{V-C^y}}{cr^{y+1}} + \frac{Sp\,.\,\overline{r-1}}{br^{y+1}} \times \overline{\mu + \frac{k}{c}} \times \overline{V-B^y}$, supposing $C^y$ and $B^y$ respectively to be the values of annuities on the single lives of persons $y$ years older than C and B. The other two series are a continuation of the tenth and eleventh series in the former part of this solution, so that the sum of those four series will be $= \frac{d}{cr} \times \overline{T-BT} - \overline{C-BC} + \frac{C-BC}{r} - \frac{d}{cr} \times \overline{T-BT} - \frac{C'-B'C'}{2r} + \frac{d\,.\,\overline{T'-B'T'}}{2cr} = \frac{d}{2cr} \times \overline{T'-B'T'} - \frac{C'-B'C'}{2r} -$

$\frac{\overline{r-1}\,.\,\overline{C-BC}}{r}$, and the whole value of the given sum will be =

S into $\frac{\overline{r-1}\,.\,\overline{V-L}}{3r} + \frac{B'C'-C'}{r} - \frac{\overline{r-1}\,.\,\overline{V+B'+AB+AC}}{2r} + \frac{\varkappa}{2c} \times \frac{\overline{\alpha\,.\,HBK}}{3a}$

$\overline{+ K' - B'K' - \frac{2\,.\,\overline{AK-ABK}}{3}} + \frac{\alpha}{2a} \times \overline{HB + \frac{4\,.\,\overline{HC-HBC}}{3}} + \frac{s}{3ar}$

$\times \overline{NC - NBC} + \frac{d}{2cr} \times \overline{T' - B'T' + \frac{2s\,.\,\overline{1+NBT}}{3c} - \frac{AT-ABT}{3}} +$

$\frac{\pi\,.\,k\,.\,\overline{r-1}\,.}{cr^{y+1}} \times \overline{V-C^y} + \mu + \frac{k}{c} \times \frac{p\,.\,\overline{r-1}}{br^{y+1}} \times \overline{V-B^y}.$

If *the three lives be equal,* the two first rules become S into $\frac{\overline{r-1}\,.\,\overline{V-L}}{3r} + \frac{d}{cr} \times \overline{1 + \frac{2CT+CCT}{3}} - \frac{2dd}{3ccr} \times \overline{1 + CTT} - \frac{\varkappa\varkappa}{3cc} \times$ CKK $- \frac{2\varkappa}{3c} \times \overline{CK - CCK}$, and the last rule becomes S into $\frac{\overline{r-1}\,.\,\overline{V-L}}{3r} + \frac{\varkappa\varkappa\,.\,CKK}{6cc} + \frac{\varkappa}{3c} \times \overline{CK - CCK} - \frac{d}{3cr} \times CT - \frac{d}{6cr}$ $\times CCT - \frac{d}{2cr} + \frac{dd}{3ccr} \times \overline{1 + CTT}$. If all the expressions, except the first, in these rules be resolved into their respective series, they will be found to destroy each other, and the general rule in both cases will become simply $= \frac{S\,.\,\overline{r-1}\,.\,\overline{V-L}}{3r}$, which is known from self-evident principles to express the true value in this particular case. The same general rule may also be obtained immediately from the series which denote the value of S in each year, for in the first year its value will in this case be $= \frac{S\,.\,\overline{c-d}|^3}{3c^3r}$, in the second year $= \frac{S\,.\,\overline{d-e}|^3}{3c^3r^2} + \frac{S\,.\,\overline{c-d}\,.\,\overline{d-e}|^2}{c^3r^2}$ $+ \frac{S\,.\,\overline{c-d}|^2\,.\,\overline{d-e}}{c^3r^2}$, in the third year $= \frac{S\,.\,\overline{e-f}|^3}{3c^3r^3} + \frac{S\,.\,\overline{c-e}\,.\,\overline{e-f}|^2}{c^3r^3} +$ $\frac{S\,.\,\overline{c-e}|^2\,.\,\overline{e-f}}{c^3r^3}$, and so on; hence the whole value will be =

$\frac{S}{3r} - \frac{S}{c} \times \overline{\frac{d}{r} + \frac{e}{r^2} + \frac{f}{r^3} + \&c.} + \frac{S}{cc} \times \overline{\frac{dd}{r} + \frac{ee}{r^2} + \frac{ff}{r^3} + \&c.} - \frac{S}{c^3}$

$$\times\ \overline{\frac{d^3}{r} + \frac{e^3}{r^2} + \frac{f^3}{r^3} + \&c.} + \frac{S}{cr} \times \overline{\frac{d}{r} + \frac{e}{r^2} + \frac{f}{r^3} + \&c.} - \frac{S}{ccr} \times \overline{\frac{dd}{r} + \frac{ee}{r^2} + \frac{ff}{r^3} + \&c.} + \frac{S}{c^3 r} \times \overline{\frac{d^3}{r} + \frac{e^3}{r^2} + \frac{f^3}{r^3} + \&c.} = \frac{S . \overline{r-1}}{3r} \times \overline{V - L}.$$ Q. E. D.

### PROBLEM VI.

To find the value of a given sum payable on the death of C, provided A should be the first, B the second, and C the third that shall fail of the three lives A, B, and C.

### SOLUTION.

*When C is the oldest of the three lives.* To receive the given sum in the first year, it is only necessary that the three lives should become extinct in the order specified in this problem, and therefore the value of S for this year will be $= \frac{S . \overline{b-m} . \overline{c-d} . a'}{6abcr}$. In the second year the given sum may be received, provided either of three events shall take place; 1st, That all the lives fail in the order required by the problem. 2dly, That B dies after A in the first year, and C dies in the second year. 3dly, That A only dies in the first year, and C dies after B in the second year. Hence the value of S for this year will be $= \frac{S . \overline{m-n} . \overline{d-e} . a''}{6abcr^2} + \frac{S . \overline{b-m} . \overline{d-e} . a'}{2abcr^2} + \frac{S . \overline{m-n} . \overline{d-e} . a'}{2abcr^2}$. To receive the given sum in the third year either of the same events must take place. 1st, The three lives must drop in the order stated above; or 2dly, B must die after A in the first or second year, and C die in the third year; or

3dly, A must die in the first or second year, and C die after B in the third year. The value therefore of S for this year will be $= \frac{S.\overline{n-o}.\overline{e-f}.a'''}{6abcr^3} + \frac{S.\overline{b-n}.\overline{e-f}.\overline{a'+a''}}{2abcr^3} + \frac{S.\overline{n-o}.\overline{e-f}.\overline{a'+a''}}{2abcr^3}$. By pursuing the same steps during C's life, the whole value may be found

$$= \frac{S}{6abc} \times \overline{\frac{a'bc}{r} + \frac{a''md}{r^2} + \frac{a'''ne}{r^3} + \&c.} - \frac{S}{6abc} \times \overline{\frac{a'me}{r} + \frac{a''nd}{r^2} + \frac{a'''oe}{a^3} + \&c.} - \frac{S}{2abcr} \times \overline{\frac{nd.a'}{r} + \frac{oe.\overline{a'+a''}}{r^2} + \&c.} - \frac{S}{6abc} \times \overline{\frac{a'bd}{r} + \frac{a''me}{r^2} + \frac{a'''nf}{r^3} + \&c.} + \frac{S}{6abc} \times \overline{\frac{a'md}{r} + \frac{a''ne}{r^2} + \frac{a'''of}{r^3} + \&c.} + \frac{S}{2abcr} \times \overline{\frac{ne.a'}{r} + \frac{of.\overline{a'+a''}}{r^2} + \&c.} + \frac{S}{2acr} \times \overline{\frac{\overline{d-e}.a'}{r} + \frac{\overline{e-f}.\overline{a'+a''}}{r^2} + \&c.} = S \text{ into } \frac{\kappa}{6c} \times \overline{\frac{\beta.\overline{FK-AFK}}{b} - \overline{BK-ABK}} + \frac{\overline{r-1}.\overline{BC-ABC}}{6r} + \frac{m}{3br} \times \overline{\frac{d.\overline{PT-APT}}{c} - \overline{PC-APC}} - \frac{\beta.\overline{FC-AFC}}{6b} + \frac{C-AC}{2r} + \frac{d}{2cr} \times \overline{\frac{BT-ABT}{3} - \overline{T-AT}}.$$

*When B is the oldest of the three lives,* it is evident that none of the foregoing series ought to be continued beyond the extinction of B's life, and that after this period the payment of the given sum will depend simply upon the failure of C's life in each of the remaining years, A having previously been survived by B. Let the difference between the age of B and of the oldest person in the table of observations be denoted by $x$, the probability that B dies after A by $\pi$, the value of an annuity on the life of a person $x$ years older than C by $C^x$, the number of living at this age by $k$, and the values of annuities on the single and joint lives of A, C, and T for $x$ years by C′, T′, A′C′ and A′T′, then will the required value in this case be $=$

$$S \text{ into } \frac{\kappa}{6c} \times \overline{\frac{\beta.\overline{FK-AFK}}{b} - \overline{BK-ABK}} + \frac{\overline{r-1}.\overline{BC-ABC}}{6r} + \frac{m}{3br}$$

$\times \overline{\dfrac{d\ \overline{\mathrm{PT}-\mathrm{APT}}}{c} - \overline{\mathrm{PC}-\mathrm{APC}}} - \dfrac{\beta\ .\ \overline{\mathrm{FC}-\mathrm{AFC}}}{6b} + \dfrac{\mathrm{C}'-\mathrm{A}'\mathrm{C}'}{2r} + \dfrac{d}{2rc} \times$
$\overline{\dfrac{\overline{\mathrm{BT}-\mathrm{ABT}}}{3} - \overline{\mathrm{T}'-\mathrm{A}'\mathrm{T}'}} + \dfrac{\pi k\ .\ \overline{r-1}}{cr^{x+1}} \times \overline{\mathrm{V}-\mathrm{C}^x}.$

*When A is the oldest and B the youngest of the three lives*, let the symbols be changed which denote the decrements and probabilities of life of A and B; let $z$ be the difference between the age of A, and the oldest person in the table, and the whole value of the given sum during the life of A will be $= \dfrac{\mathrm{S}}{2bc} \times$
$\overline{\dfrac{cb'}{r} + \dfrac{db''}{r^2} + \dfrac{eb'''}{r^3} + \ldots .(z)} + \dfrac{\mathrm{S}}{2bcr} \times \overline{\dfrac{db'}{r} + \dfrac{e\ .\ \overline{b'+b''}}{r^2} + \ldots .(z)}$
$- \dfrac{\mathrm{S}}{2bc} \times \overline{\dfrac{db'}{r} + \dfrac{eb''}{r^2} + \dfrac{fb'''}{r^3} + \ldots .(z)} - \dfrac{\mathrm{S}}{2bcr} \times \overline{\dfrac{eb'}{r} + \dfrac{f\ .\ \overline{b'+b''}}{r^2} +}$
$\overline{\ldots .(z)} - \dfrac{\mathrm{S}}{3abc} \times \overline{\dfrac{acb'}{r} + \dfrac{sdb''}{r^2} + \dfrac{te.b'''}{r^3} + \&c.} - \dfrac{\mathrm{S}}{2abcr} \times \overline{\dfrac{sd.b'}{r} +}$
$\overline{\dfrac{te\ .\ \overline{b'+b''}}{r^2} + \&c.} + \dfrac{\mathrm{S}}{3abc} \times \overline{\dfrac{adb'}{r} + \dfrac{se\ .\ b''}{r^2} + \dfrac{tf.b'''}{r^3} + \&c.} + \dfrac{\mathrm{S}}{2abcr} \times \overline{\dfrac{seb'}{r}}$
$\overline{+ \dfrac{tf\ .\ \overline{b'+b''}}{r^2} + \&c.} - \dfrac{\mathrm{S}}{6abc} \times \overline{\dfrac{scb'}{r} + \dfrac{td\ .\ b''}{r^2} + \dfrac{ue\ .\ b'''}{r^3} + \&c.} + \dfrac{\mathrm{S}}{6abc}$
$\times \overline{\dfrac{dsb'}{r} + \dfrac{et\ .\ b''}{r^2} + \dfrac{fu\ .\ b'''}{r^3} + \&c.}$ Let $k$, $\delta$, $\varepsilon$, $\zeta$, &c. denote the number of persons living opposite the age of C in the table at the end of $z$, $\overline{z+1}$, $\overline{z+2}$, &c. years, $\beta'$, $\beta''$, $\beta'''$, &c. the decrements of life opposite the age of B at the end of those years respectively, $\pi$ the probability that B dies after A, and $\mathrm{C}^z$ the value of an annuity on a life $z$ years older than C; then will the value of S in the $\overline{z+1}$st year (depending on the contingency of C's dying after B in that year, or of C's dying in that year, B having died after A in either of the foregoing $z$ years) be expressed by $\dfrac{\mathrm{S}\ .\ \overline{k-\delta}\ .\ \beta}{2bcr^{z+1}} + \dfrac{\pi\ .\ \mathrm{S}\ .\ \overline{k-\delta}}{cr^{z+1}}$. In the $\overline{z+2}$d, $\overline{z+3}$d, &c. years, the payment of S will depend upon either

of three events: 1st, Of C's dying after B in that particular year. 2dly, Of C only dying in that year, B having died in either of the preceding $\overline{z+1}$, $\overline{z+2}$, &c. years: or, 3dly, Of C only dying in the year, B having died after A in the first $z$ years. Hence the whole value of S, after the necessary extinction of the life of A by the table, will be $= \frac{S}{2bcr^{z}} \times \overline{\frac{\overline{k-\delta}.\beta'}{r} + \frac{\overline{\delta-\varepsilon}.\beta''}{r^2} + \frac{\overline{\varepsilon-\zeta}.\beta'''}{r^3} + \&c.} + \frac{S}{bcr^{z+1}} \times \overline{\frac{\overline{\delta-\varepsilon}.\beta'}{r} + \frac{\overline{\varepsilon-\zeta}.\overline{\beta'+\beta''}}{r^2} + \&c.} + \frac{S.\pi}{cr^{z}} \times \overline{\frac{\overline{k-\delta}.}{r} + \frac{\overline{\delta-\varepsilon}}{r^2} + \frac{\overline{\varepsilon-\zeta}}{r^3} + \&c.}$ The last of these series is $= \frac{S.\pi.k.\overline{r-1}}{cr^{z+1}} \times \overline{V-C^{z}}$; the other two series being added to the four first series in this solution, their sum will be found $= \frac{\overline{r-1}.\overline{V-C+BC}}{2r} + \frac{B'C'}{2r} - \frac{\varkappa.BK}{2c} + \frac{d}{2cr} \times \overline{T'-B'T'+\overline{1-BT}}$ (T′, B′T′, and B′C′ denoting the values of annuities on those single and joint lives respectively for $z$ years.) The fifth and sixth series in the solution are $= \frac{\alpha\varkappa.HBK}{3ac} - \frac{AC}{2r} - \frac{\overline{r-1}.\overline{2V-ABC}}{6r}$, the seventh and eighth series are $= \frac{\alpha.\overline{HC-HBC}}{3a} + \frac{d}{6cr} \times \overline{AT-ABT}$, the ninth is $= \frac{s}{6ar} \times \overline{NC-NBC} - \frac{\varkappa}{6c} \times \overline{AK-ABK}$, and the tenth is $= \frac{ds}{6acr} \times \overline{1+NTB} - \frac{ABC}{6}$.

Hence the whole value of the given sum in this case is $=$ S into $\frac{\overline{r-1}.\overline{V-L}}{6r} + \frac{B'C' - \overline{AC+C'}}{2r} - \frac{\varkappa}{2c} \times \overline{BK + \frac{AK-ABK}{3}} + \frac{d}{2cr} \times \overline{\frac{AT-ABT}{3} + 1 + T' - BT + B'T'} + \frac{s}{6ar} \times \overline{NC-NBC} + \frac{d.\overline{1+NBT}}{c} + \frac{\alpha}{3a} \times \overline{HC-HBC} + \frac{\varkappa.HBK}{c} + \frac{\pi k.\overline{r-1}}{cr^{z+1}} \times \overline{V-C^{z}}$.

*When A is the oldest and C the youngest of the three lives,* the

symbols $c'$, $c''$, $c'''$, &c. must be substituted for $\overline{c-d}$, $\overline{d-e}$, $\overline{e-f}$, &c. and the symbols $\overline{b-m}$, $\overline{m-n}$, $\overline{n-o}$, &c. for $b'$, $b''$, $b'''$, &c. and the value of the given sum for the first $z$ years, or during A's life, will be $= \frac{S}{6abc} \times \overline{\frac{abc'}{r} + \frac{smc''}{r^2} + \frac{tnc'''}{r^3} + \&c.} - \frac{S}{6abc} \times \overline{\frac{bsc'}{r} + \frac{mtc''}{r^2} + \frac{nu \cdot c'''}{r^3} + \&c.} + \frac{S}{3abc} \times \overline{\frac{amc'}{r} + \frac{snc''}{r^2} + \frac{toc'''}{r^3} + \&c.} + \frac{S}{6abc} \times \overline{\frac{msc'}{r} + \frac{ntc''}{r^2} + \frac{ouc'''}{r^3} + \&c.} + \frac{S}{2c} \times \overline{\frac{c'}{r} + \frac{c''}{r^2} + \frac{c'''}{r^3} + \ldots (z)} - \frac{S}{2ac} \times \overline{\frac{ac'}{r} + \frac{sc''}{r^2} + \frac{tc'''}{r^3} + \&c.} - \frac{S}{2bc} \times \overline{\frac{mc'}{r} + \frac{nc''}{r^2} + \frac{oc'''}{r^3} + \&c. \ldots (z)}$. Let $p$, $\mu$, $\nu$, $\xi$, &c. represent the number of persons living in the table opposite the age of B at the end of $z$, $\overline{z+1}$, $\overline{z+2}$, &c. years, and $\varkappa'$, $\varkappa''$, $\varkappa'''$, &c. the decrements of life opposite the age of C at the end of those years respectively; then, by reasoning as in the foregoing case, the value of S after the necessary extinction of the life of A will be $= \frac{S}{2bcr^z} \times \overline{\frac{\overline{p-\mu} \cdot \varkappa'}{r} + \frac{\overline{\mu-\nu} \cdot \varkappa''}{r^2} + \frac{\overline{\nu-\xi} \cdot \varkappa'''}{r^3} + \&c.} + \frac{S}{bcr^{z+1}} \overline{\frac{\overline{p-\mu} \cdot \varkappa''}{r} + \frac{\overline{\mu-\nu} \cdot \varkappa'''}{r^2} + \frac{\overline{\nu-\xi} \cdot \varkappa''''}{r^3} + \&c.} + \frac{S \cdot \pi \cdot}{cr^z} \times \overline{\frac{\varkappa'}{r} + \frac{\varkappa''}{r^2} + \frac{\varkappa'''}{r^3} + \&c.}$ This last series is $= \frac{S \cdot \pi \cdot k \cdot \overline{r-1}}{cr^{z+1}} \times \overline{V - C^z}$. The other two series may be resolved into $\frac{S \cdot p}{cbr^z} \times \overline{\frac{\varkappa'}{r} + \frac{\varkappa''}{r^2} + \frac{\varkappa'''}{r^3} + \&c.} - \frac{S}{2bcr^z} \times \overline{\frac{p\varkappa'}{r} + \frac{\mu\varkappa''}{r^2} + \frac{\nu\varkappa'''}{r^3} + \&c.} - \frac{S}{2bcr^z} \times \overline{\frac{\mu\varkappa'}{r} + \frac{\nu\varkappa''}{r^2} + \frac{\xi\varkappa'''}{r^3} + \&c.}$ The first of these is $= \frac{S \cdot p \cdot \overline{r-1}}{br^{z+1}} \times \overline{V - C^z}$; the second (supposing F′C′ to denote the value of the joint lives of F and C for $z$ years) is $= \frac{\beta \cdot \overline{FC - F'C'}}{2b} - \frac{\overline{BC - B'C'}}{2r}$; the third is a continuation of the seventh series in this solution, and therefore the whole

of that series is $= \frac{BC}{2} - \frac{m\cdot\overline{1+PC}}{2br}$ . . . The first series in this solution is $= \frac{\overline{1+ABC}}{6r} - \frac{\alpha\beta\cdot HFC}{6ab}$, the second is $= \frac{s}{6ar} \times \overline{NB - NBC} - \frac{\beta\cdot\overline{AF - AFC}}{6b}$, the third is $= \frac{\alpha}{3a} \times \overline{HB - HBC} - \frac{m\cdot\overline{AP - APC}}{3br}$, the fourth is $= \frac{ms}{6abr} \times \overline{1 + NPC} - \frac{ABC}{6}$, the fifth is $= \frac{\overline{r-1}\cdot\overline{V - C'}}{2r}$, and the sixth is $= \frac{\alpha\cdot HC}{2a} - \frac{\overline{1+AC}}{2r}$. The whole value therefore of the given sum may be found $=$ S . into $\frac{\overline{r-1}}{6r} \times \overline{V - 3C' + 3BC - ABC} + \frac{B'C' - AC}{2r} + \frac{\beta}{2b} \times \overline{FC - F'C' - \frac{AF - AFC}{3} - \frac{\alpha\cdot HFC}{3a}} + \frac{\alpha}{2a} \times \overline{\frac{2\cdot\overline{HB - HBC}}{3} + HC} + \frac{s}{6ar} \times \overline{NB - NBC + \frac{m\cdot\overline{1+NPC}}{b}} - \frac{m}{2br} \times \overline{1 + PC + \frac{2\cdot\overline{AP - APC}}{3}} + \overline{\pi} + \frac{p}{b} \times \frac{k\cdot\overline{r-1}}{cr^{\varkappa+1}} \times \overline{V - C^{\varkappa}}$.

*When the lives are all equal,* the expression $\frac{\beta\varkappa\cdot FK}{6bc}$ in the first and second rules becomes $= \frac{\overline{1+CC}}{6r}$, the expression $\frac{md\cdot PT}{3ccr}$ becomes $= \frac{CC}{3} - \frac{dd}{3ccr}$, and the expression $\frac{C}{2r} - \frac{d\cdot T}{2cr}$, or $\frac{C'}{2r} - \frac{dT'}{2cr}$ becomes $= \frac{d}{2cr} - \frac{\overline{r-1}\cdot C}{2r}$, so that those rules become $=$ S into $\frac{\overline{r-1}\cdot\overline{V-L}}{6r} - \frac{\varkappa}{3c} \times \overline{CK - CCK} + \frac{d}{2cr} \times \overline{1 + \frac{CCT}{3}} - \frac{\varkappa\varkappa\cdot CKK}{6cc} - \frac{dd}{3ccr} \times \overline{1 + CTT}$. In the third rule the expressions $\frac{d}{2cr} \times \overline{T' - B'T'} - \frac{C' - B'C'}{2r}$ become $= \frac{CC}{2r} - \frac{d}{2cr} \times \overline{1 - CT}$, so that in this case the value is $=$ S into $\frac{\overline{r-1}\cdot\overline{V-L}}{6r} + \frac{d}{3cr} \times \overline{CT - CCT} - \frac{\varkappa}{3c} \times \overline{CK + \frac{CCK}{2}} + \frac{\varkappa\varkappa\cdot CKK}{3cc} + \frac{dd}{6ccr} \times \overline{1 + CTT}$,

and by the fourth rule it becomes $=$ S into $\frac{\overline{r-1}\,.\,\overline{V-L}}{6r}+\frac{\varkappa}{3c}\times\overline{2CK-\frac{CCK}{2}-\frac{\varkappa\varkappa\,.\,CKK}{6cc}-\frac{d}{2cr}}-\frac{d}{3cr}\times\overline{2CT-\frac{CCT}{2}}+\frac{dd}{6ccr}\times\overline{1+CTT}$. If the values of the joint lives in each of those rules be resolved into their respective series, all the expressions, except the first, will be found to destroy each other, and the general rule in all of them will become simply $=\frac{S\,.\,\overline{r-1}\,.\,\overline{V-L}}{6r}$, which from self-evident principles in this particular case, is known to be the true value. A similar result may likewise be immediately obtained from the series themselves; for the value of S for the first year is easily found in this case to be $=\frac{S}{r}\times\overline{\frac{1}{6}-\frac{d}{2c}+\frac{dd}{2cc}-\frac{d^3}{6c^3}}$, for the second year $=\frac{S}{r^2}\times\overline{\frac{ee}{2cc}-\frac{dd}{2cc}+\frac{d}{2c}-\frac{e}{2c}+\frac{d^3}{2c^3}-\frac{e^3}{2c^3}}$, for the third year $=\frac{S}{r^3}\times\overline{\frac{ff}{2cc}-\frac{ee}{2cc}+\frac{e}{2c}-\frac{f}{2c}+\frac{e^3}{2c^3}-\frac{f^3}{2c^3}}$, and so on for the other years. Hence the whole value is $=\frac{S\,.\,\overline{r-1}\,.\,\overline{V-L}}{6r}$. . . . . Q. E. D.

It is to be observed, that the fractions $\frac{\overline{b-m}\,.\,\overline{d-e}\,.\,a'}{2abcr^2}$, $\frac{\overline{b-n}\,.\,\overline{e-f}\,.\,\overline{a'+a''}}{2abcr^3}$, &c. do not accurately express the value of S on the second contingency in this problem; but that according to the lemma they should have been $\frac{\overline{b-n}\,.\,\overline{d-e}\,.\,a'}{2abcr^2}$, $\frac{\overline{\overline{b-n}\,.\,a'+\overline{m-n}\,.\,\overline{a'+a''}}\times\overline{e-f}}{2abcr^3}$, &c. In order to determine how near the former* approach to the

* When B is the oldest these fractions are $=\frac{C'-A'C'-BC+ABC}{2r}+\frac{d}{2cr}\times\overline{BT-ABT-T'-A'T'}$. When A is the oldest they are $=\frac{C'-B'C'-AC+ABC}{2r}+\frac{d}{2cr}\times\overline{AT-ABT-T-B'T'}$.

true values, I have in the following examples undergone the labour of separately computing each of those latter fractions, and the results appear to differ so little from the approximated values, that I think a greater degree of accuracy need not be required.

*Value of £ 100, payable on the contingency in this problem, computed from the Northampton table, at* 4 per cent.

| Ages of A. | Ages of B. | Ages of C. | Value by the rule. | Correct value. | Difference. |
|---|---|---|---|---|---|
| 10 | 85 | 80 | 1.467 | 1.438 | 0.029 |
| 15 | 75 | 73 | 2.233 | 2.150 | 0.083 |
| 15 | 75 | 35 | 2.761 | 2.589 | 0.172 |
| 15 | 75 | 78 | 1.698 | 1.513 | 0.185 |
| 20 | 65 | 64 | 3.031 | 2.912 | 0.119 |
| 20 | 65 | 70 | 2.588 | 2.580 | 0.008 |
| 70 | 80 | 78 | 9.457 | 9.068 | 0.389 |
| 70 | 80 | 35 | 10.109 | 9.618 | 0.491 |

I have chosen those cases in which the approximation was likely to have been most inaccurate; for if the ages of A and B are either both younger, or differ less from each other than they do in these examples, it is obvious that the foregoing rules must be still nearer the truth. I have also uniformly supposed the life of B to be older than that of A, and of consequence the approximated value always errs in excess; if the life of A had been the older of the two, it would have been found to have erred in defect, and nearly to the same amount. But as, in this latter case, the value of the reversion is greater

than when B is the older life, the error must necessarily bear a less proportion to the whole value than it does in the preceding examples.

With regard to the fifth problem, the error in some cases is greater, in others less than in the present problem. If B and C are both older than A it will be nearly twice as great. If one is older and the other younger, it will be altogether inconsiderable; for the fractions which express the probability of the older of B and C dying after A will be as much above the truth, as the other fractions expressing the probability that the younger of these two lives die after A will be below it, and thus the errors of one correct those of the other, and render the computation almost perfectly accurate. I have not given any examples to that problem, not only as the correctness of its rules may be inferred from the examples which have been given to those of the present problem, but as I wished to make as few additions as possible to a paper, which having engaged a large portion of my time and attention for the last three years, has already become too long, and for which my only apology is the attempt to give correct, and not very laborious, solutions to some of the most difficult and complicated cases in the doctrine of survivorships.