XVI. Newton's *Binomial Theorem legally demonstrated by Algebra.* *By the Rev.* William Sewell, *A. M.* *Communicated by Sir* Joseph Banks, *Bart. K. B. P. R. S.*

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Let m and n be any whole positive numbers; and $\overline{1+x}|^{\frac{m}{n}}$ a binomial to be expanded into a series, as $1 + Ax + Bx^2 + Cx^3 +$, &c. where A, B, C, D, &c. are the coefficients to be determined.

Assume $v^m = \overline{1+x}|^{\frac{m}{n}} = 1 + Ax + Bx^2 + Cx^3 + Dx^4 +$, &c.

And $z^m = \overline{1+y}|^{\frac{m}{n}} = 1 + Ay + By^2 + Cy^3 + Dy^4 +$, &c.

Then will $v^n = 1 + x$, and $z^n = 1 + y \therefore v^n - z^n = x - y$.

And $v^m - z^m = A \times \overline{x - y} + B \times \overline{x^2 - y^2} + C \times \overline{x^3 - y^3} + D \times \overline{x^4 - y^4} +$, &c.

Consequently $\frac{v^m - z^m}{v^n - z^n} = A + B \times \overline{x+y} + C \times \overline{x^2 + xy + y^2} + D \times \overline{x^3 + x^2 y + xy^2 + y^3} +$, &c. Now $v^m - z^m = \overline{v - z} \times : v^{m-1} + v^{m-2} z + v^{m-3} z^2 +$, &c.....$z^{m-1}$. Also $v^n - z^n = \overline{v - z} \times : v^{n-1} + v^{n-2} z + v^{n-3} z^2 +$, &c....$z^{n-1}$. Therefore $\frac{v^m - z^m}{v^n - z^n}$ reduces to, and becomes $= \frac{v^{m-1} + v^{m-2} z + v^{m-3} z^2 +, \&c.... z^{m-1}}{v^{n-1} + v^{n-2} z + v^{n-3} z^2 +, \&c.... z^{n-1}} = A + B \times \overline{x+y} + C \times \overline{x^2 + xy + y^2} + D \times \overline{x^3 + x^2 y + xy^2 + y^3} +$, &c.

The law is manifest; and it is likewise evident that the

numerator and denominator of the fraction, respectively terminate in m and n terms. Suppose then $x = y$; then will $v = z$; and our equation will become $\frac{mv^{m-1}}{nv^{n-1}}$, or $\frac{mv^{m-n}}{n} = A + 2Bx + 3Cx^2 + 4Dx^3 +$, &c.

But $v^n = 1 + x$, therefore by multiplying we have $\frac{mv^m}{n} = A + \overline{A + 2B}x + \overline{2B + 3C}x^2 + \overline{3C + 4D}x^3 +$, &c. Or $v^m = \overline{1 + x}|^{\frac{m}{n}} = \frac{nA}{m} + \frac{nA + 2nB}{m}x + \frac{2nB + 3nC}{m}x^2 + \frac{3nC + 4nD}{m}x^3 +$, &c. Compare this with the assumed series, to which it is similar and equal, and it will be

$$\frac{nA}{m} = 1$$

$$\frac{nA + 2nB}{m} = A$$

$$\frac{2nB + 3nC}{m} = B,$$

$$\text{\&c.} =, \text{\&c.}$$

$$\therefore A = \frac{m}{n};\ B = \frac{\overline{m - n}A}{1.2.n};\ C = \frac{\overline{m - 2n}B}{1.2.3.n};\ \text{\&c.}$$

Therefore $\overline{1 + x}|^{\frac{m}{n}} = 1 + \frac{m}{n}x + \frac{m \times \overline{m - n}}{1.2.n^2}x^2 + \frac{m \times \overline{m - n} \times \overline{m - 2n}}{1.2.3.n^3}x^3 +$, &c. the law is manifest, and agrees with the common form derived from other principles.

Sch. In the above investigation, it is obvious that unless m be a positive whole number, the numerator abovementioned does not terminate: it still remains, therefore, to shew how to derive the series when m is a negative whole number. In this case, the expression $(v^m - z^m)$ assumes this form, $\frac{1}{v^m} - \frac{1}{z^m}$, or its equal $\frac{z^m - v^m}{v^m z^m}$, which divided by $v^n - z^n$, as before, gives $\frac{1}{v^m z^m} \times \frac{z^m - v^m}{v^n - z^n} = \frac{1}{v^m z^m} \times \frac{-\overline{v - z} \times : v^{m-1} + v^{m-2}z + v^{m-3}z^2}{\overline{v -} \times : v^{n-1} + v^{n-2}z + v^{n-3}z^2 +}$, &c. $=$

$\frac{-1}{v^m z^m} \times \frac{v^{m-1}+v^{m-2}z+v^{m-3}z^2+,\ \&c.}{v^{n-1}+v^{n-2}z+v^{n-3}z^2+,\ \&c.} =$ (when $v = z$) $-\frac{mv^{m-1}}{v^{2m} \times nv^{n-1}}$ $= \frac{-mv^{-m-n}}{n}$, which is the same as the expression $\left(\frac{mv^{m-n}}{n}\right)$ before derived with only the sign of m changed. The remainder of the process being the same as before, shews that the series is general, or extends to all cases, regard being had to the signs. Q. E. D.