XVI. *New Method of computing Logarithms. By* Thomas Manning, *Esq. Communicated by the Right Hon. Sir* Joseph Banks, *K. B. P. R. S.*

Read June 5, 1806.

If there already existed as full and extensive logarithmic tables as will ever be wanted, and of whose accuracy we were absolutely certain, and if the evidence for that accuracy could remain unimpaired throughout all ages, then any new method of computing logarithms would be totally superfluous so far as concerns the formation of tables, and could only be valuable indirectly, inasmuch as it might shew some curious and new views of mathematical truth. But this kind of evidence is not in the nature of human affairs. Whatever is recorded is no otherwise believed than on the evidence of testimony; and such evidence weakens by the lapse of time, even while the original record remains; and it weakens on a twofold account, if the record must from time to time be replaced by copies. Nor is this destruction of evidence arising from the uncertainty of the copy's being accurately taken, any where greater than in the case of copied numbers.

It is useful then to contrive new and easy methods for computing not only new tables, but even those we already have. It is useful to contrive methods by which any part of a table may be verified independently of the rest; for by

examining parts taken at random, we may in some cases satisfy ourselves of its accuracy, as well as by examining the whole.*

Among the various methods of computing logarithms, none, that I know of, possesses this advantage of forming them with tolerable ease independently of each other by means of a few easy bases. This desideratum, I trust, the following method will supply, while at the same time it is peculiarly easy of application, requiring no division, multiplication, or extraction of roots, and has its relative advantages highly increased by increasing the number of decimal places to which the computation is carried.

The chief part of the working consists in merely setting down a number under itself removed one or more places to the right, and subtracting, and repeating this operation; and consequently is very little liable to mistake. Moreover, from the commodious manner in which the work stands, it may be revised with extreme rapidity. It may be performed after a few minutes instruction by any one who is competent to subtract. It is as easy for large numbers as for small; and on an average about 27 subtractions will furnish a logarithm acccurately to 10 places of decimals. In general $9 \times \frac{n+1}{2}$ subtractions will be accurate to $2n$ places of decimals.

In computing hyperbolic logarithms by this method it is necessary to have previously establised the h. logs. of $\frac{10}{9}$, $\frac{100}{99}$, $\frac{1000}{999}$, &c. of 2 and of 10.

* For example, we may wish to know whether the editor of a table has been careless. We examine detached portions here and there to a certain extent; if we find no errors, we have a moral certainty that the editor was careful, and consequently a moral certainty that the edition is accurate.

With respect to the logs. of $\frac{10}{9}$, $\frac{100}{99}$, $\frac{1000}{999}$, &c. their computation is very easy,* they being the respective sums of the series

$$\frac{1}{10} + \frac{1}{2} \times \frac{1}{\overline{10}|^2} + \frac{1}{3} \times \frac{1}{\overline{10}|^3} + \frac{1}{4} \times \frac{1}{\overline{10}|^4} + \&c.$$

$$\frac{1}{100} + \frac{1}{2} \times \frac{1}{\overline{100}|^2} + \frac{1}{3} \times \frac{1}{\overline{100}|^3} + \frac{1}{4} \times \frac{1}{\overline{100}|^4} + \&c.$$

$$\frac{1}{1000} + \frac{1}{2} \times \frac{1}{\overline{1000}|^2} + \frac{1}{3} \times \frac{1}{\overline{1000}|^3} + \&c.$$

&c.

of which series each is more easily summed than the preceding.

With respect to the logarithms of 2 and 10 there are, it is well known, various ways of computing them, and the time requisite depends greatly on the practical habits of the calculator. Among other ways, they may be computed by the method given in this Paper, and with what degree of expedition, may be seen by the examples to the rules, where they are both of them worked.†

* Ex. H. l. $\frac{1000}{999} = \frac{1}{1000} + \frac{1}{2} \times \frac{1}{\overline{1000}|^2} + \frac{1}{3} \frac{1}{\overline{1000}|^3} + \&c.$

1st term = ,001

2d term = ,0000005

3d term = ,000000000333

Sum = ,001000500333, which is true to the last place of decimals.

† It may be seen there that the logarithm of 10 by this method requires very little work. The log. of 2 may also be computed from the log. of 10 as follows.

$2^{10} = 1024 = 1000 \times \left(1 + \frac{24}{1000}\right)$ therefore log. $2^{10} = 3 \times$ log. $10 +$ log. $\left(1 + \frac{24}{1000}\right) = 3 \times$ log. $10 + \left(\frac{24}{1000} - \frac{1}{2} \frac{\overline{24}|^2}{\overline{1000}|^2} + \frac{1}{3} \times \frac{\overline{24}|^3}{\overline{1000}|^3} - \&c.\right)$

TABLE, *containing the hyperbolic Logarithms of* $\frac{10}{9}$, $\frac{100}{99}$, $\frac{1000}{999}$, *&c. of* 2 *and of* 10; *together with the Reciprocal of the last or Modulus of the common Logarithms.*

H. l. of	$\frac{10}{9}$	$=$,105360515655
h. l.	$\frac{100}{99}$	$=$,010050335853
h. l.	$\frac{1000}{999}$	$=$,001000500333
h. l.	$\frac{10000}{9999}$	$=$,000100005 . . .
h. l.	$\frac{100000}{99999}$	$=$,00001000005
h. l.	$\frac{1000000}{999999}$	$=$,000001
h. l.	$\frac{10000000}{9999999}$	$=$,0000001
h. l.	$\frac{100000000}{99999999}$	$=$,00000001

1000)24(.024
2
48
12
$2 \times \overline{1000}|^2$)576(.000288
2
1152
12
$3 \times \overline{1000}|^3$)13824(.000004608
2
27648
12
$4 \times \overline{1000}|^4$)331776(.000000082944
2
663452
12
$5 \times \overline{1000}|^5$)7962624(.000000001592

Sum of odd terms	,024004609592
—— even ——	,000288082944
Difference	,023716526648
$3 \times$ log. 10	6,907755279648
Log. 2^{10} - -	6,931471806296. True to the 10th figure.

and so on; the unit receding regularly to the right.

h. l. 2 = ,693147180637

h. l. 10 = 2,302585093217

$\frac{1}{\text{h. l. } 10}$ = ,434294481861.

Certain Multiples of the above Numbers; viz. all those required in computing Logarithms by the subjoined Rules, and which are not evident upon Inspection.

Multiples of the h. logs. of $\frac{10}{9}$, $\frac{100}{99}$, &c.		Multiples of h. l. 2, h. l. 10, and $\frac{1}{\text{h. l. } 10}$. Of h. l. 2.	
Double h. l. of $\frac{10}{9}$	= ,210721031310	Double	=1,386294361274
triple	= ,316081546965	triple	=2,079441541911
quadruple	= ,421442062620	quadple.	=2,772588722548
5ple.	= ,526802578275	5ple.	=3,465735903185
6ple.	= ,632163093930	6ple.	=4,158883083822
7ple.	= ,737523609585	7ple.	4,852030264459
8ple.	= ,842884125240	8ple.	5,545177445096
9ple.	= ,948244640895	9ple.	6,238324625733

		Of h. l. 10.	
Double h. l. $\frac{100}{99}$	= ,020100671706	Double	=4,605170186434
triple	,030151007559	triple	=6,907755279651
quadruple	,040201343412	quadple.	9,210340372868
5ple.	,050251679265	5ple.	11,512925466085
6ple.	,060302015118	6ple.	13,815510559302
7ple.	,070352350971	7ple.	16,118095652519
8ple.	,080402686824	8ple.	18,420680745736
9ple.	,090453022677	9ple.	20,723265838953

		Of $\frac{1}{\text{h. l. } 10}$.	
Double h. l. $\frac{1000}{999}$	= ,002001000666	Double	= ,868588963722
triple	= ,003001500999	triple	=1,302883445583
quadruple	= ,004002001332	quadple.	=1,737177927444
5ple.	= ,005002501665	5ple.	=2,171472409305
6ple.	= ,006003001998	6ple.	=2,605766891166
7ple.	= ,007003502331	7ple.	=3,040061373027
8ple.	= ,008004002664	8ple.	=3,474355854888
9ple.	= ,009004502997	9ple.	=3,908650336749

I. *To find the hyperbolic Logarithm of any Number not exceeding* 2.

RULE. Set the number under itself, to be subtracted from itself, but removed so many places to the right as shall be necessary to make the remainder greater than 1; subtract. Proceed in the same manner with the remainder, and so on till the remainder becomes 1 followed by $\frac{1}{2}$ as many cyphers as the number of decimal places you work to; suppose at the end of the operation you find that you have removed *one* place to the right and subtracted b times; *two* places, c times; three places, d times, &c.; then $b \times \text{h. l.} \frac{10}{9} + c \times \text{h. l.} \frac{100}{99} + d \times \text{h. l.} \frac{1000}{999} +$ &c. $+$ decimal part of the last remainder $=$ h. l. sought.

And these numbers are collected together out of the Table; for b, c, d, &c. can none of them ever exceed 9.

Ex. 1. *To find the h. l. of 2 to 10 Places of Decimals.*

2.
.2
1.8
.18
1.62
162
1.458
.1458
1.3122
13122
1.18098
118098
1.062882 6
1062882
1.05225318
105225318
1.0417306482
10417306482
1.031313341718
103131334171
1.0210002083009
102100020830
1.0107902062179
101079020621
1.0006823042558 ...... 6.

$$6 \times \text{h. l.} \frac{10}{9} = .632163093930$$

$$6 \times \text{h. l.} \frac{100}{99} = .060302015118$$

$$6 \times \text{h. l.} \frac{10000}{9999} = .00060003 \ldots$$

$$8 \times \text{h. l.} \frac{\overline{10}|^5}{99999} = .0000800004 \ldots$$

$$2 \times \text{h. l.} \frac{\overline{10}|^6}{999999} = .000002 \ldots\ldots$$

decim. last rem. = .000000041189

h. l. 2 = 693147180637

Transferred to next page.

1.0006823042558
1000682304

1.0005822360254
1000582236

1.0004821778018
1000482177

1.0003821295841
1000382129

1.0002820913712
1000282091

1.0001820631621
1000182063

1.0000820449558 6
100008204

1.0000720441354
100007204

1.0000620434150
100006204

1.0000520427946
100005204

1.0000420422742
100004204

1.0000320418538
100003204

1.0000220415334
100002204

1.0000120413130
100001204

1.0000020411926 8
10000020

1.0000010411906
10000010

1.0000000411896 Last remainder.

Ex. II. *To find the h. l. of* 1.25.

1.25
125

1.125
.1125

1.0125 2
10125

1.002375 1
1002375

1.001372625
1001372625

1.000371252375 2
1000371252

1.0002712152498
1000271215

1.0001711881283
1000171188

1.0000711710095 3
100007117

1.0000611702978
100006117

1.0000511696861
100005116

1.0000411691745
100041116

1.0000311687629
100003116

1.0000211684513
100002116

1.0000111682397
100001116

1.0000011681281 7
10000011

1.0000001681270 last remainder.

$2 \times$ h. l. $\frac{10}{9}$	,210721031310
$1 \times$ h. l. $\frac{100}{99}$	,010050335853
$2 \times$ h. l. $\frac{1000}{999}$	,002001000666
$3 \times$ h. l. $\frac{\overline{10}\vert^{4}}{9999}$	,000300015000
$7 \times$ h. l. $\frac{\overline{10}\vert^{5}}{99999}$	,000070000350
$1 \times$ h. l. $\frac{\overline{10}\vert^{6}}{999999}$	,000001 ......
decimal of last remainder	,000000168127
log. required	,223143551306

II. *To find the h. l. of any Number, whole or mixt.*

RULE. Reduce the given number (if necessary) to a whole or mixt number less than 2, by setting the decimal point after the first significant figure, or if the given number be 10 or a power of 10 after the first 0; and then dividing by 2 (if necessary) till the integral part is 1.*

Find the h. l. of this reduced number by the last rule, and add to it or subtract from it as many times the h. l. of 10 as the decimal point was removed places to the left or right; also add to it as many times the h. l. of 2 as there were divisions by 2. The sum is the h. l. required.

EX. III. *To find the h. l. of* 10.

$\frac{10}{2^3} = 1.25$, whose h. log. is found in the last example to be

	,223143551306
h. l. 2^3 =	2,079441541911
h. l. 10	2,302585093217

EX. IV. *To find the h. l. of* 5548748 *to* 6 *Places of Decimals.*

4)5.548748	
1.387187	
1387187	
1.2484683	
12484683	
1.12362147	
112362147	
1.011259323 . . .	3
10112593	
1.001146730 . . .	1
1001146	
1.000145584	last remainder.

$3 \times$ h. l. $\frac{10}{9}$	=	,316081546
$1 \times$ h. l. $\frac{100}{99}$	=	,010050335
$1 \times$ h. l. $\frac{1000}{999}$	=	,001000500
decimal of last remainder	=	,000145584
h. log. 1.387187	=	,327277965
$6 \times$ h. l. 10	=	13,815510559
$2 \times$ h. l. 2	=	1,386294361
log. required		15,529082885

* Three divisions by 2 will always suffice.

Ex. v. *To find the h. l. of 7 to 5 Places of Decimals.*

	4)7.
	1.75
	175
	1.575
	1575
	1.4175
	14175
	1.27575
	127575
	1.148175
	1148175
5	1.0333575
	1033357
	1.02302393
	1023023
	1.01279370
	1012793
3	1.00266577
	100266
	1.00166311
	100166
last rem.	1.00066145

$5 \times$ h. l. $\frac{10}{9}$ ...	,5268025
$3 \times$ h. l. $\frac{100}{99}$ ...	,0301510
$2 \times$ h. l. $\frac{1000}{999}$ ...	,0020010
decim[l] of last rem[r] ...	,0006614
$2 \times$ h. l. 2 ...	1,3862943
log. 7 =	1,9459102

From the small number of subtractions that have been necessary, the log. of 7 must be correct to 6 places of decimals.

III. *To find the common Logarithm of any Number.*

Rule. Find the h. l. by the above rule, and multiply $\frac{1}{\text{h. l. } 10}$ into it.

Otherwise. Proceed by art. 2, omitting what concerns the

h. l. 10. Multiply into $\frac{1}{\text{h. l. } 10}$, and add or subtract as many units as the decimal point was removed to the left or right.

Note. The multiplication of a number by $\frac{1}{\text{h. l. } 10}$ is very expeditiously performed by means of the Table of multiples of $\frac{1}{\text{h. l. } 10}$.

The demonstration of the above rules is obvious. Setting the figures of a number one place to the right is dividing that number by 10; 2 places, by 100; 3 places, by 1000; and so on. And subtracting a number, so placed, from the number itself is subtracting a 10th, a 100th, a 1000th, &c. (in the respective cases) of the number from itself; and consequently the remainders are (respectively) $\frac{9}{10}$ths, $\frac{99}{100}$ths, $\frac{999}{1000}$ths, &c. of the numbers subtracted from. Let b, c, d, &c. denote as in the rule; then the original number $= \overline{\frac{10}{9}}\Big|^{b} \times \overline{\frac{100}{99}}\Big|^{c} \times \overline{\frac{1000}{999}}\Big|^{d} \times$ &c. $\times$ last remainder. Therefore the log. of the original number $= b \times \log. \frac{10}{9} + c \times \log. \frac{100}{99} + d \times \log. \frac{1000}{999} +$ &c. $+$ log. of last remainder. Now the last remainder being unity followed by a certain number of decimal cyphers, its correct hyp. log., as far as twice that number of places, is (as is well known) the decimal part itself of that remainder. Hence the rule is manifest.

A similar method, by addition only, by means of the ready computed logarithms of $\frac{11}{10}$, $\frac{101}{100}$, $\frac{1001}{1000}$, &c. might, in some cases, be used with advantage. Let N denote the given number, consisting of unity and a decimal whose h. l. is sought; and let P denote any number less than N, and whose h. l. is previously known. Set P under itself removed one

or more places to the right; add; and proceed with the sum in the same manner, till you have obtained a number, $N \pm a$, the difference between which and N shall be inconsiderable Let b, c, d, &c. denote as in the rule.

Then $P \times \overline{\tfrac{11}{10}}|^{b} \times \overline{\tfrac{101}{100}}|^{c} \times \overline{\tfrac{1001}{1000}}|^{d} \times$ &c. $= N \pm a$.

Therefore $\log. \overline{N \pm a} = \log. P + b \times \log. \tfrac{11}{10} + c \times \log. \tfrac{101}{100} + d \times \log. \tfrac{1001}{1000} +$ &c., which call B.

And $\log. N = \log. \frac{N}{N \pm a} + \log. \overline{N \pm a} = \log. \frac{N}{N \pm a} + B$.

Now we may either carry the operation so far that $\log. \frac{N}{N \pm a}$ may be neglected, or we may actually divide N by $N - a$, or $N + a$ by N (according as the sign is $-$ or $+$) and add or subtract the quotient from B.

Various artifices may be occasionally used to shorten the computation both in the method of subtraction, and in this of addition; and the two may sometimes be advantageously combined together.

It should be observed that, in setting down the numbers, the last figure set down ought to be increased by unity when the figure immediately following in the neglected part exceeds 4.

EXAMPLE *of the Method of Addition. To find the h. l. of* 2.

1.1	*Continued.*
11	1.991863
1.21	1992
121	1.993855
1.331	1994
1331	1.995849
1.4641	1996
14641	1.997845
1.61051	1998
161051	1.999843
1.771561	199
177156	2.000042
1.948717	
19487	
1.968204	
19682	
1.987886	
1988	
1.989874	
1989*	
1,991863	

From this operation it appears, that

$$\tfrac{11}{10} \times \overline{\tfrac{11}{10}}\big|^{6} \times \overline{\tfrac{101}{100}}\big|^{2} \times \overline{\tfrac{1001}{1000}}\big|^{6} \times \tfrac{10001}{10000} = 2.000042.$$

Consequently, $7 \times \text{h.l.}\ \frac{11}{10} + 2 \times \text{h.l.}\ \frac{101}{100} + 6 \times \text{h.l.}\ \frac{1001}{1000} +$ $\text{h. l.}\ \frac{10001}{10000} = \text{h. l. } 2{,}000042.$

$= \text{h. l. } 2 + \text{h. l.}\ \frac{2.000042}{2} = \text{h. l. } 2 + \text{h. l. } (1.000021) = \text{h. l. } 2 +$ $000021.$

The method by subtraction has many advantages over this

* Instead of this number 1989, it would be more correct to set down 1990, because the first figure of the neglected part, 874, exceeds 4.

by addition. It is more simple, and being more completely mechanical, may be confided to the most unskilful without danger of error. And though addition be an easier operation than subtraction, yet the greater facility arising from this circumstance will not be found sufficient to balance these and other advantages.