XV. *On the Conditions of Equilibrium of an Incompressible Fluid, the Particles of which are acted upon by Accelerating Forces.* By James Ivory, *K.H. M.A. F.R.S. L. & E., Instit. Reg. Sc. Paris, Corresp. et Reg. Sc. Gottin. Corresp.*

Received May 21,—Read June 20, 1839.

Experience shows that physical problems of difficulty are never solved in a satisfactory manner but after reiterated attempts. The examples that might be adduced in support of this remark, are too obvious and numerous to need particular mention. A remarkable instance is the problem of which it is proposed to treat in this paper, namely, that relating to the figure of equilibrium of a mass of fluid, the particles of which are subjected to the action of accelerating forces. This problem, suggested by the inquiry into the figure of the planets, was first treated of by Newton and Huyghens; it then passed into the hands of Maclaurin, Clairaut, and D'Alembert; and it finally occupied the attention of Euler, Lagrange, and Laplace, by whose researches it is declared on high authority* that the solution is completed, leaving no difficulties, except of a mathematical kind, in applying it to any case that may be proposed. The theory thus finally settled is imposing by its great generality and apparent simplicity; it succeeds in solving a certain class of problems, although not on sound principles; but in other instances no degree of mathematical skill has been able to obtain satisfactory results. A candid inquirer who will endeavour to form just notions of the conditions required for the equilibrium of a fluid, will not fail to have his attention arrested by much that is inconsistent and obscure in the usual manner in which this subject is treated. This seems to imply some imperfection in the grounds of the theory; and the best way of removing all difficulties is to mount up to the origin of the inquiry, and to trace it with careful examination through all its successive steps. In this manner we may detect what is defective or erroneous; and having arrived at physical conditions not liable to objection or uncertainty, the theory may be placed on a firm foundation.

It will not be necessary to say a word on the importance of a theory which has occupied the attention of so many eminent geometers, and which is the subject of no small part of what has been written on the system of the universe. As it treats of the figure of a fluid, it seems to suppose that the earth and planets were originally in a state of fluidity, either by the solution of their solid parts in a liquid, or by the effect of heat. Now as we have no knowledge of the primitive condition of the bodies of our system, it may be objected that the problem, whatever ingenuity may be re-

* M. Poisson.

quired to overcome its difficulties, is merely speculative and hypothetical. But the matter may be viewed in a different light. No small progress has already been made in the investigation of the figure of the earth; and our knowledge in this respect may be made more perfect by assiduous observation and discussion: we are also acquainted with all the forces, whether attractive or centrifugal, that urge every particle of the matter of which our globe is composed; and hence, reversing the usual question, the inquiry may be, whether a change in the actual figure of the earth would necessarily take place if the bonds that hold together its solid parts were loosened, and a state of fluidity induced upon the whole or any portion. A speculation of this kind at any time, and in every revolution imposed by fashion on scientific research, may be deemed not altogether uninteresting, and may be useful in studying the changes that take place on the surface of our globe.

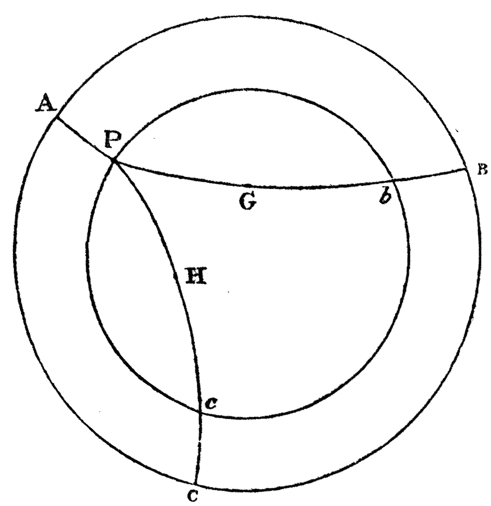
1. It is obvious that a homogeneous body of fluid, the particles of which are not agitated by extraneous forces, but are left freely to their mutual action on one another, will ultimately assume the figure of a perfect sphere. In treating of the figure of the earth, NEWTON supposes that this sphere is made to revolve about one of its diameters; in consequence of which the centrifugal force will cause the fluid to recede from the axis of rotation, and to subside in the direction parallel to that axis. He makes no inquiry into the nature of this new figure, but immediately concludes, without assigning a reason, that it is an oblate elliptical spheroid turning about the less axis.

It would be in vain to inquire on what grounds NEWTON inferred that the revolving sphere is changed into an exact oblate spheroid. The flattening at the poles suggests a resemblance of the two figures; and we may add that, by making the two axes of the spheroid more and more unequal, it will pass through all degrees of oblateness, and may be supposed, in some one of its forms, to coincide with the flattened sphere, if not exactly, at least with a sufficient approximation.

Having guessed the form of equilibrium, the main difficulties of the research were overcome; for it is much easier to investigate the properties of a known figure, than to determine the form itself which is required for an equilibrium. He begins with laying down this principle, that in a fluid spheroid in equilibrium, the weights, or efforts, of all the small rectilineal columns extending from the centre to the surface, must balance one another round the centre. Assuming a spheroid, of which the axes are very nearly equal, namely, 101 and 100, he computes, by means of his assumed principle, and some propositions in his immortal work, the weights at the centre, caused by the attraction of the matter of the spheroid, of two columns, one drawn in the plane of the equator, and the other to the pole. The spheroid must next be supposed to revolve about the polar axis, so that the centrifugal force, by diminishing the weight of the equatorial column, may equalise the weights of both columns at the centre, as an equilibrium requires; and when this condition is fulfilled, it is found that the oblateness of the spheroid, or the difference of the two semi-axes in parts of

the equatorial radius, is to the centrifugal force at the equator in parts of the gravity, as 5 to 4. Now this proportion of 5 to 4 is common to all spheroids of which the axes are nearly equal; and as the centrifugal force at the earth's equator is $\frac{1}{289}$ of gravity, the oblateness of the terrestrial spheroid will be $\frac{5}{4} \times \frac{1}{289} = \frac{1}{230}$, making the proportion of the polar axis to the diameter of the equator as 229 to 230. Such would be the true figure of the earth if its matter were homogeneous, and if NEWTON'S reasoning were liable to no objections. All that the present purpose requires to be noticed in his very able investigation, is the principle of the equiponderance of the central columns which he introduced.

NEWTON had occasion to consider no figure but the elliptical spheroid, in which, from its symmetry, the equiponderance of the central columns is self-evident, every column being counteracted by an equal and similar column diametrically opposite. In a body of fluid at liberty and in equilibrium by the action of accelerating forces, is there always a central point round which the efforts of the whole mass balance one another? It is obvious that all canals extending from a particle to the surface of the fluid, will impel the particle with equal intensity; for otherwise the particle would not be at rest. What is it, then, that distinguishes the equal pressures upon a particle in any situation, from the like pressures on the central point, if there be such a point? In a research in which there have occurred so many inadvertencies arising from hypothetical admissions, it is necessary to inquire in what manner the pressures are distributed among the particles.



Let A B C represent a mass of fluid in equilibrium; P any particle; P A, P B, P C, small canals diverging to the surface of the fluid: as P is at rest, the efforts of all the canals will balance one another. In passing along any canal A P B ending both ways in the surface, the pressure, which is zero at A, will first increase to a certain point G, after which it will decrease to zero at B. Because the pressure from A to G is equal and contrary to the pressure from B to G, there is always a part B *b* at one end, which presses inward with an intensity equal to the like effort of the part A P at the other end. Thus, of the three parts of the canal, the forces which urge the fluid in the two extreme parts transmit equal and contrary pressures to P; but the forces acting on the fluid in the middle part P *b*, destroy one another's effects, and cause no pressure on P. The same thing is true of any other canal A P C; the effort of the part A P on one side of the point of maximum-pressure H being balanced by the

contrary effort of the part C *c* on the other side of that point. Now if we suppose a curve surface to be drawn through P, and through all the points *b*, *c*, &c., it follows from what has been proved, that the pressures which impel P with equal intensity in all directions, are caused solely by the forces which urge the portion of fluid on the outside of that surface; for any canal being drawn between P and the upper surface, it is only the part of such canal between the two surfaces that transmits an effective pressure to P.

Such interior surfaces as P *b c* are called level surfaces. Every level surface is pressed at all its points with the same intensity by the exterior fluid; for the forces acting on the particles contained in any canal between a level surface and the upper surface, produce the same effort directed inward.

If we conceive that the interior surface P *b c* gradually lessens in its dimensions by the increasing depth, it will finally contract into a point; which point, or centre, is distinguished from every other point P, by this, that the pressures which impel it equally in all directions are produced by the forces which urge all the particles of the body of fluid.

The central point is distinguished by another property peculiar to it: for the pressure being a maximum, the partial differentials of the pressure, or the forces, will be zero: so that if a particle in the centre be removed a little in any direction, there will be no sensible change of the pressure upon it. Thus the centre is in stable equilibrium with respect to the action of the whole mass of fluid. Although any other particle, as P, is pressed equally on all sides, yet, as the forces in action are not zero, the change of pressure which it undergoes when moved a little from its place, will be different according to the direction of its motion; unless it be moved so as to continue upon the level surface P *b c*, in which case the pressure, being still produced by the action of the same portion of the whole mass, will not vary.

What has been said establishes NEWTON's equiponderance of the central columns as a general principle of equilibrium that holds in every case of a mass of fluid at liberty, the particles of which are urged by accelerating forces.

Enough has also been said to demonstrate the insufficiency of the principle of equality of pressure for determining the figure of equilibrium of a fluid. For it has been shown that the equal pressures which a particle sustains have no other effect than to make it immoveable by the action of the portion of fluid on the outside of the level surface that passes through it: but from this it does not follow in all cases that the particle is reduced to a state of rest relatively to the whole mass. The same theory tacitly assumes that every body of fluid contained within a level surface will maintain its form and position, merely by the equal pressures of the exterior fluid; not adverting to the necessity of taking into account all the forces of whatever description that act on the particles. In the foregoing investigation it is clearly proved on the supposition of an equilibrium, that the forces in action must be without effect to cause pressure either way in any canal, as P *b* or P *c*, within a level surface.

2. Some years before the publication of the Principia, it had been ascertained by observation that the same mass of matter has not the same weight at all the points of the earth's surface. A pendulum clock regulated by mean time at Paris was found by M. RICHER to lose two minutes a day at Cayenne, within 5° of the equator. Now the length of a pendulum that oscillates in a given time is an exact measure of gravity; and the fact observed by M. RICHER proved that, if a heavy body were carried from Paris to Cayenne, it would lose some part of its weight. On learning this fact, HUYGHENS conjectured that it was caused by the centrifugal force arising from the daily revolution of the earth; the intensity of this force varying in different latitudes at different distances from the axis of rotation. Combining this observed variation of gravity with a principle, which is indisputably true, namely, that a plumb-line freely suspended is perpendicular to the surface of standing water, or to the surface of the earth supposed entirely fluid, he drew an argument, that the earth is not exactly spherical. Were the earth a perfect sphere, the attraction of its mass would be perpendicular to the surface at every point: the centrifugal force, directed at right angles from the axis of rotation, is oblique to the surface: wherefore gravity, being the resultant of both forces, and consequently not coinciding in its direction with either, would not be perpendicular to the surface, which is contrary to the admitted principle. It will readily appear that the resultant of the two forces, or the true direction of the plumb-line, would always make a small angle with the radius of the sphere on the side towards the equator: so that a horizontal plane perpendicular to this direction, would necessarily fall within the sphere towards the pole: which is a direct proof that the surface of the earth, formed by all such horizontal planes, is depressed at the poles. These speculations of HUYGHENS are contained in his dissertation, *De causa gravitatis*: in an addition to that dissertation which was published after the author's death, he proceeds to investigate the oblateness of the earth, or the difference between the equatorial radius and the polar semiaxes, caused by the centrifugal force. By this time the Principia was published: but the doctrine of the mutual gravitation of all matter, was at first generally objected to, and forced its way very slowly to universal approbation. HUYGHENS, rejecting the principle that every particle of matter gravitates to every other, substituted in its place a tendency of all the parts of the earth supposed in a fluid state, to the common centre of the mass, the central force acting with the same intensity at all distances. Following the method devised by NEWTON for the solution of the same problem, the simple law of gravity adopted by HUYGHENS leads to an easy solution: because a narrow rectilineal canal of any length drawn from the centre, will have a weight proportional to the quantity of fluid it contains. If a fluid spheroid revolve about the polar axis with a circular velocity capable of impressing on the particles a centrifugal force the intensity of which is to that of gravity as n to 1, it is easy to prove that the centrifugal force will diminish the weight of a canal in the plane of the equator, and equal to the radius of that circle, or to 1, by the quantity $\frac{n}{2} \times 1$: so that the whole weight

of the canal from being equal to 1 will be reduced to $1 - \frac{n}{2}$; which will therefore be the length of a canal reaching from the centre to the pole equiponderant to the equatorial canal. Applying this result to the earth, we have $n = \frac{1}{289}$, and the proportion of the radius of the equator to the polar semi-axis equal to 578 to 577, the oblateness being much less than in the Newtonian Theory.

HUYGHENS next attempts to investigate, what form the perpendicularity of gravity to the earth's surface requires the terrestrial meridians to have. But in this part of his researches no result is obtained which it would be useful to notice. He finds indeed a curve which, in his law of gravity, answers the mathematical conditions: but this curve is a paraboloid consisting of two infinite branches that diverge continually from one another; a form irreconcilable with the continuous surface of the earth, every meridian of which is an oval curve returning into itself. It appears from what has been said that the contribution of HUYGHENS to the theory of the equilibrium of fluids, must be limited to the perpendicularity of the resultant of the forces to the surface, which principle he was the first to suggest.

3. We have now two properties that must be verified in every mass of fluid at liberty and in equilibrium by forces acting on its particles, namely, the equiponderance of the central columns of which NEWTON is the author, and the perpendicularity to the surface of the resultant of all the forces urging a particle, which was proposed by HUYGHENS. But it is one thing to detect particular properties of an equilibrium, however general in their application, and another thing to fix with precision the conditions necessary for inducing that state on a mass of fluid acted upon by given forces. In solving problems, geometers sometimes made use of one principle, and sometimes of the other. It was soon found that a figure obtained by means of one principle, did not in all cases verify the other; and even that the concurrence of both in the same mass of fluid was not sufficient in some instances to ensure an equilibrium. From all this it could only be inferred that the problem was still involved in obscurity, and required to be further discussed.

4. In the Principia NEWTON has completely determined the attraction of spheres. He has also given methods for determining the attraction of other bodies; which methods, although sufficient for obtaining numerical results, fail for the most part in ascertaining the law according to which the attractive force of the mass will vary when the attracted point changes its place. MACLAURIN, by a happy application of the ancient geometry, determined this law in elliptical spheroids of revolution, for all particles within the solid or in its surface. He found that the mass of the spheroid attracted a particle so situated, in directions perpendicular to the plane of the equator and to the axis of rotation, with forces respectively proportional to the distances from the plane and from the axis*. Now the centrifugal force is directly proportional to

* Although MACLAURIN's demonstration rests on this property, yet this property itself is essentially dependent on another property, which the author has demonstrated in his Fluxions, § 630.

the distance from the axis of rotation: and thus was known every force tending to move any particle of an elliptical spheroid revolving about its axis in a given time. The difficulty of estimating the forces and pressures in different parts of the spheroid, obliged NEWTON to confine his attention to the central columns. The discovery of MACLAURIN removed this difficulty, and enabled him to ascertain whether the spheroid fulfilled any proposed property of equilibrium, or not. He first determines the relative dimensions of the spheroid, which are necessary for making the resultant of the attractive and centrifugal forces perpendicular to the surface at every point, as is required by the principle of HUYGHENS; and he demonstrates that the same figure verifies NEWTON's principle of the equiponderance of the central columns. Taking now any particle, or small portion of the fluid, and conceiving that it is pressed on every side by rectilineal canals standing upon it and terminating in the surface, he showed that the pressures of these canals impel the particle equally in all directions. These several points are demonstrated with the utmost elegance, and with all the rigour of EUCLID or ARCHIMEDES.

Such is the celebrated demonstration of MACLAURIN, which adds the property of every particle being pressed equally in all directions, to what NEWTON and HUYGHENS had before shown to be necessary for an equilibrium. But on reflection it is not quite clear how this new property is to be understood or what use is to be made of it. There is no doubt that most authors infer, from the equal pressures which a particle sustains on every side, that it it is necessarily brought to a state of rest within the spheroid; and hence the equality of pressure has been erected into a general principle, on which is founded the usual theory of equilibrium. But what MACLAURIN does really demonstrate amounts to this*, that a particle placed on an elliptical surface similar and concentric to the surface of the spheroid, is impelled by any rectilineal canal standing upon it and terminating in the surface of the spheroid, with a pressure equal to the effort of a given canal having for its length the difference of the polar semi-axes of the two similar surfaces. Now the proper inference certainly is, that the particles at every point of the interior surface press upon one another, and upon the surface in which they are placed, with the same intensity. To say that a particle is pressed equally in all directions, is tantamount to saying that it is placed on a level surface; every particle on such a surface being urged equally on all sides by the exterior fluid, either by direct action, or by the efforts transmitted through the fluid contained within the surface. But it cannot be reasonably inferred, as is done in the theory of equilibrium founded on equality of pressure, that a particle is reduced to a state of rest relatively to the whole of a body of fluid, merely because it is pressed equally in all directions by a portion only of the mass.

In the treatise on Fluxions published in 1742 MACLAURIN does not affirm explicitly that a particle is at rest within the spheroid, because it is pressed equally on all sides; although it is undoubtedly implied that this is true: and he concludes his investiga-

* Fluxions, § 639.

tion* with saying, that the surfaces similar and concentric to the surface of the spheroid, are the true level surfaces at all depths; which is alone sufficient for the equilibrium, and is indeed the simple and direct and the only exact ground of the demonstration, agreeing perfectly with what has been advanced. In the Dissertation presented by the author to the Academy of Sciences in 1740, the matter is differently stated. Having proved, in his first proposition or *Theorema Fundamentale*, that any particle is impelled equally in all directions by a certain force depending only on the position of the particle, or rather on the surface passing through the particle similar and concentric to the surface of the spheroid, he adds, "*quæ* (*particula*) *cum æqualiter urgeatur, fluidum est ubique in equilibrio.*" Here it is unequivocally asserted that the equal pressures which a particle sustains, reduce it to a state of rest within the spheroid. This would be correct if it were proved that the equal pressures are produced by the action of the whole mass of the spheroid. That a particle is pressed equally by the surrounding fluid, and that it is at rest within the spheroid, are two distinct propositions, of which the second is not necessarily a consequence of the first: for the equal pressures may be caused by the action of only a part of the fluid; whereas the effect of the forces that act upon all the particles must be taken into account, in order to prove that a particle is at rest relatively to the whole mass. What MACLAURIN has accurately proved of one particle, holds equally of all the particles situated in any surface similar and concentric to the surface of the spheroid, the pressure on all such particles being the same; and the proper inference to be drawn is what the author has stated in his Fluxions, namely, that all such interior surfaces are the true level surfaces at all depths.

5. In proving that the pressure upon any interior particle of the spheroid is equal in all directions, MACLAURIN used rectilineal canals; but it is evident that the effect must be the same, whether the canals be rectilineal, or have curvilineal figures varied in any manner. From observing that, in a fluid at rest, the pressure of a canal will be the same when its extreme points are the same, however its form may be varied, CLAIRAUT deduced a relation between the figure of a fluid in equilibrium and the mathematical expression of the pressure, or of the forces which produce the pressure. This consideration greatly simplified and improved the theory of equilibrium, as it made it unnecessary to seek after such artifices of investigation as MACLAURIN was obliged to have recourse to. This property is enunciated in the following theorem.

Theorem.—In a fluid at rest by the action of accelerating forces on its particles, the mathematical expression of the pressure at any point of the mass can be no other than a function of the three co-ordinates of the point, these co-ordinates being considered as independent and unrelated quantities.

Demonstration.—Let a communication be opened between any two points of the mass of fluid by means of a canal of any figure; because the fluid is supposed to be

* Fluxions, § 640.

at rest, the pressure of the fluid in the whole length of the canal will be equal to the difference of the pressures in the body of fluid at the two orifices, which pressure will therefore remain the same, however the figure of the canal be varied. It is easy to ascertain that this property will be verified when the pressure at any point of the fluid is represented by a function of the co-ordinates, such as is described in the theorem. Thus the co-ordinates of the two orifices of the canal being represented by a, b, c and a', b', c'; and the pressures at the same points by $\varphi\,(a, b, c)$ and $\varphi\,(a', b', c')$; through whatever gradations the figure of a canal requires that the independent variables a, b, c be made to pass so as finally to become equal to a', b', c', the function $\varphi\,(a, b, c)$ will always be changed into $\varphi\,(a', b', c')$.

But the theorem may be regularly demonstrated in the manner following. Let the variables x, y, z represent the co-ordinates, and $\varphi\,(x, y, z)$ or φ the pressure at a point of the fluid: if the co-ordinates vary in the curve of the canal, the sum of the differentials in the whole length of the canal will be

$$\int\left(\frac{d\,\phi}{d\,x}\,d\,x+\frac{d\,\phi}{d\,y}\,d\,y+\frac{d\,\phi}{d\,z}\,d\,z\right);$$

and the sum of the variations of all these differentials in another canal very near the first and between the same extreme points, will be

$$\int\delta\,.\left(\frac{d\,\phi}{d\,x}\,d\,x+\frac{d\,\phi}{d\,y}\,d\,y+\frac{d\,\phi}{d\,z}\,d\,z\right);$$

and the condition that the pressure caused by the efforts of the fluid in one canal is not different from the like pressure in the other canal, is thus expressed:

$$0=\int\delta\,.\left(\frac{d\,\phi}{d\,x}\,d\,x+\frac{d\,\phi}{d\,y}\,d\,y+\frac{d\,\phi}{d\,z}\,d\,z\right).$$

When this last expression is integrated by parts, we obtain

$$\begin{aligned}0=\frac{d\,\phi}{d\,x}\,\delta\,x+\int d\,x\,\delta\,.\frac{d\,\phi}{d\,x}-\int\delta\,x\,d\,.\frac{d\,\phi}{d\,x}\\+\frac{d\,\phi}{d\,y}\,\delta\,y+\int d\,y\,\delta\,.\frac{d\,\phi}{d\,y}-\int\delta\,y\,d\,.\frac{d\,\phi}{d\,y}\\+\frac{d\,\phi}{d\,z}\,\delta\,z+\int d\,z\,\delta\,.\frac{d\,\phi}{d\,z}-\int\delta\,z\,d\,.\frac{d\,\phi}{d\,z}.\end{aligned}$$

Now the terms without the sign of integration in this expression are zero when the integrals are extended to the whole lengths of the canals; for the extreme points being fixed, the variations of the co-ordinates at these points are zero. We thus obtain by an easy reduction,

$$\begin{aligned}0=\int(d\,x\,\delta\,y-\delta\,x\,d\,y)\cdot\frac{d\,d\,\phi}{d\,x\,d\,y}+\int(d\,x\,\delta\,z-\delta\,x\,d\,z)\cdot\frac{d\,d\,\phi}{d\,x\,d\,z}\\+\int(d\,y\,\delta\,x-\delta\,y\,d\,x)\cdot\frac{d\,d\,\phi}{d\,y\,d\,x}+\int(d\,y\,\delta\,z-\delta\,y\,d\,z)\cdot\frac{d\,d\,\phi}{d\,y\,d\,z}\end{aligned}$$

$$+\int (dz\,\delta x - \delta z\,dx)\cdot\frac{dd\phi}{dz\,dx} + \int (dz\,\delta y - \delta z\,dy)\cdot\frac{dd\phi}{dz\,dy}:$$

which equation will be verified only by means of these formulas, viz.

$$\frac{dd\phi}{dx\,dy} = \frac{dd\phi}{dy\,dx},\quad \frac{dd\phi}{dx\,dz} = \frac{dd\phi}{dz\,dx},\quad \frac{dd\phi}{dy\,dz} = \frac{dd\phi}{dz\,dy}.$$

Now these equations express the well-known conditions that the variables of the function $\varphi\,(x, y, z)$ are independent quantities; which sort of function is therefore alone fit to represent with consistency the pressure of a fluid at rest.

The nature of the pressure in a fluid at rest being ascertained, the expressions of the forces which produce the pressure follow as a corollary; for these expressions are necessarily no other than the partial differentials of the pressure. Supposing always, for the sake of simplicity, that the density is constant and equal to unit, let X, Y, Z represent the forces which, acting in the respective directions of the co-ordinates, produce the pressure $\varphi\,(x, y, z)$ or φ; we shall have

$$X = \frac{d\phi}{dx},\quad Y = \frac{d\phi}{dy},\quad Z = \frac{d\phi}{dz}.$$

From the foregoing equations between the differentials of φ, the following formulas are easily deduced, viz.

$$\frac{dX}{dy} = \frac{dY}{dx},\quad \frac{dX}{dz} = \frac{dZ}{dx},\quad \frac{dY}{dz} = \frac{dZ}{dy}.$$

These equations are the known conditions for the integrability of the differential

$$X\,dx + Y\,dy + Z\,dz:$$

and they must be verified in every problem by the proposed forces, otherwise the equilibrium of the fluid will be impossible.

If p represent the pressure estimated on a given surface at the point $(x\,y\,z)$ of the fluid, we shall have,

$$p = \varphi\,(x, y, z), \quad \ldots\ldots\ldots \quad \text{(A.)}$$

$$p = \int (X\,dx + Y\,dy + Z\,dz),$$

the integral extending from the point to the surface of the mass. We next obtain these differential equations,

$$dp = \frac{d\phi}{dx}\,dx + \frac{d\phi}{dy}\,dy + \frac{d\phi}{dz}\,dz,$$

$$dp = X\,dx + Y\,dy + Z\,dz:$$

and if we suppose $d\,p = 0$, or that the co-ordinates vary in a surface at every point of which the pressure is the same, that is, in a level surface, the surface of the mass being included, there will result

$$0 = \frac{d\phi}{dx}\,dx + \frac{d\phi}{dy}\,dy + \frac{d\phi}{dz}\,dz,$$

$$0 = X\,dx + Y\,dy + Z\,dz.$$

Now these last equations prove that the forces urging a particle in any level surface have no effect to move the particle in any direction upon a plane touching the surface; from which it follows that the resultant of such forces is perpendicular to the surface in which they act. It is also obvious that the resultant is always directed towards a level surface and towards the surface of the mass.

The equations that have been investigated are common to every mass of fluid at liberty and in equilibrium by the action of accelerating forces on its particles. The same equations contain all that is taught in the usual manner of treating this subject. If we suppose that the pressure p remains constant, while the co-ordinates vary, the equation (A.) will determine a level surface. If the point at which the pressure is p remains fixed, while the co-ordinates vary in any canal terminating in the surface of the fluid, the same equation (A.) proves that all the canals will exert the same pressure p upon the point. Every point in a level surface is impelled in all directions with the same intensity of pressure; and nothing is gained, but the hazard of misconception is incurred, by applying equality of pressure to isolated points.

Maclaurin's demonstration will always be admired; but that geometer was unfortunate in considering the pressure upon an isolated point. If he had observed, what follows from his reasoning, that every particle in a surface similar and concentric to the surface of the spheroid is pressed equally on all sides, he would have been led to the property of the level surfaces of which he has ultimately made use in his Fluxions, and which is the true principle of the equilibrium of a fluid, namely, that the level surfaces at all depths must have determinate figures.

Although the equation (A.) is common to every fluid in equilibrium, it does not follow that every problem can be solved by one equation. The level surfaces depend upon the proposed forces; and they require for the determination of their figure as many independent equations as these given forces that derive their origin from independent sources.

The not observing that in every canal terminating both ways in the surface of the fluid there are always two points pressed inwards in contrary directions with the same intensity, and consequently an intermediate part which presses neither way, has occasioned the misconception, from which much confusion has arisen, that the equal pressures of the surrounding fluid upon a particle necessarily reduce it to a state of rest within a body of fluid in equilibrium.

6. The following Problems are added for elucidating the principles that have been investigated.

Problem I.

To determine the figure of equilibrium of an incompressible fluid when the forces are such functions of the co-ordinates as are susceptible of only one value for any proposed values of the co-ordinates.

Solution.—Let x, y, z denote the co-ordinates of a particle of the fluid, p the

pressure, and X, Y, Z the forces acting in the respective directions of x, y, z; from the equation (A.) we obtain

$$p = \int (\mathrm{X}\, d\, x + \mathrm{Y}\, d\, y + \mathrm{Z}\, d\, z);$$

or, as the integration may always be effected, if $\varphi\,(x, y, z)$ represent the integral

$$p = \varphi\,(x, y, z).$$

When p is made constant, this equation will determine a level surface, which, in the simple hypothesis of this problem, is a curve susceptible of only one form for every value assigned to p. Now if p passes through all gradations from its maximum value at the centre of the mass to zero at the upper surface, all the level surfaces will be ascertained; this determines the form of the mass; and the equilibrium follows from the consideration that the level surfaces at all depths are determinate curves.

Of this problem an example is added, which is not undeserving of notice on its own account.

Example.—To determine the figure of equilibrium of an incompressible fluid at liberty, the particles being supposed to attract one another with a force proportional to the distance, at the same time that they are urged by a centrifugal force caused by revolving about an axis.

Solution.—On account of the mutual attraction of the particles it may at first be surmised, that the problem here proposed does not come under the head of which it is given as an example. It does not immediately appear that the forces urging a particle depend entirely on the place of the particle, and are explicit functions of its co-ordinates. Such, however, is the case, owing to a property peculiar to the supposed law of attraction, which Newton has demonstrated in the 88th proposition of the first book of his Principia. The property alluded to consists in this, that the resultant of the accumulated attractions of the mass upon a particle is directed to the centre of gravity of the mass, and is the same as it would be if the whole attracting matter were collected in that point. Of this a succinct investigation is as follows.

The origin of the co-ordinates being at the centre of gravity of the whole body of fluid, let x, y, z denote the co-ordinates of an attracted particle, and x', y', z' those of $d\,m$, an element of the mass; the attraction of $d\,m$ upon the particle at the distance f will be $f\,d\,m$; and as the cosines of the angles which f makes with x, y, z are

$$\frac{x - x'}{f},\quad \frac{y - y'}{f},\quad \frac{z - z'}{f},$$

the partial attractions of $d\,m$ in the directions in which x, y, z decrease, will be

$$d\,m\,(x - x'),\quad d\,m\,(y - y'),\quad d\,m\,(z - z');$$

and by summing the attractions of all the elements, the partial attractions of the whole mass in the same directions will be

$$x \int d\,m - \int x'\, d\,m,\quad y \int d\,m - \int y'\, d\,m,\quad z \int d\,m - \int z'\, d\,m.$$

But by the property of the centre of gravity,

$$\int x' \, dm = 0, \int y' \, dm = 0, \int z' \, dm = 0;$$

wherefore the attractions of the whole mass upon the particle in the same directions as before, will be

$$x \times m, \quad y \times m, \quad z \times m.$$

Now the resultant of these forces is directed to the origin of the co-ordinates, and is equal to $\sqrt{x^2 + y^2 + z^2} \times m$, which is NEWTON's proposition.

Having proved that the example comes under the foregoing problem, it is next to be observed, that the axis of rotation, supposed parallel to x, will pass through the centre of gravity of the mass: for that point must be at rest by the action of all the forces. Assuming $m \times 1$ to represent the central force, and $\varepsilon \times m \times 1$ to denote the centrifugal force at the distance 1 from the axis of rotation; according to what has been shown, the central forces on a particle will be

$$m \times x, \quad m \times y, \quad m \times z;$$

and the centrifugal forces

$$\varepsilon \times m \times y, \quad \varepsilon \times m \times z;$$

wherefore, by equation (A.),

$$X = m \times x, \quad Y = m \times (1 - \varepsilon) y, \quad Z = m \times (1 - \varepsilon) z,$$

$$\text{Const.} = x^2 + (1 - \varepsilon) . (y^2 + z^2);$$

so that all the level surfaces are elliptical curves; and the figure of equilibrium of the fluid is an oblate spheroid.

The radius of the equator is to the polar semi-axis as 1 to $\sqrt{1 - \varepsilon}$; and if $\varepsilon = \frac{1}{289}$, as in the case of the earth, the proportion is 578 to 577, agreeing with what HUYGHENS found.

PROBLEM II.

To determine the figure of equilibrium of a homogeneous fluid at liberty, the particles attracting one another in the inverse proportion of the square of the distance, at the same time that they are urged by a centrifugal force caused by revolving about an axis.

Solution.—If we adopt for the unit of mass a sphere of the given fluid having its radius equal to a, the attractive force at the surface of the sphere will be $g \times a$, the value of g being the same for all spheres of the same matter; and if the time of one entire revolution about the axis of rotation be denoted by T, the centrifugal force at the surface of the sphere will be $\frac{4\pi^2}{T^2} \times a$; so that the known quantity

$$\varepsilon = \frac{4\pi^2}{T^2} \cdot \frac{1}{g}$$

will denote the centrifugal force estimated in parts of the attractive force.

The origin of the co-ordinates being at the central point, or point of maximum-pressure, let x, y, z represent the co-ordinates of a particle of the fluid, x being parallel to the axis of rotation; and put P, Q, R for the attractions of the whole mass upon the particle in the respective directions of x, y, z: the centrifugal force is equal to $\varepsilon \times \sqrt{y^2 + z^2}$ at the distance $\sqrt{y^2 + z^2}$ from the axis of rotation; and the partial forces parallel to y and z, are therefore $\varepsilon \times y$ and $\varepsilon \times z$: wherefore, taking the total forces acting in the respective directions of x, y, z, we shall have, according to the equation (A.),

$$X = P, \quad Y = Q + \varepsilon y, \quad Z = R + \varepsilon z,$$

$$p = \int (P\,dx + Q\,dy + R\,dz) + \tfrac{\varepsilon}{2}(y^2 + z^2), \quad . \quad . \quad . \quad . \quad (M.)$$

which equation must be verified by every level surface the upper surface included; p being constant in every level surface, and equal to zero in the upper surface.

Now the equation just found is not sufficient to solve the problem: first, because all the forces that act upon the particles in a level surface, and tend to change its figure, are not taken into account; secondly, because the equation is indeterminate, and incapable of ascertaining a level surface which admits of only one form.

In regard to the first point it is to be observed, that the pressure upon a level surface is the effect of all the forces that urge the particles of the exterior fluid; and in the present instance a part of these forces is the attraction of the fluid within the level surface. But, as action is always attended with reaction, if the fluid within the level surface attract the fluid on the outside and cause it to press, the fluid on the outside will react, and, by its attraction, urge the particles within the level surface to move from their places. It is not necessary to investigate in what manner all the particles within a level surface are acted upon by the exterior fluid: it is sufficient to consider the forces acting upon the particles in the surface itself; because the form of equilibrium of the mass will be ascertained, when the figure of every level surface is determined. Now the nature of a level surface consists in this, that the resultant of all independent forces urging a particle in it, must be perpendicular to it. Wherefore if, as before, x, y, z represent the co-ordinates of a particle in a level surface; and if P′, Q′, R′ denote the partial attractions parallel to x, y, z, of the stratum of fluid exterior to the level surface, we must have this equation,

$$P'\,dx + Q'\,dy + R'\,dz = 0,$$

which expresses the condition that the resultant of the attractions upon the particle is perpendicular to the surface; and in order that the same thing may be true of every point in the surface, we must have the equation

$$\text{const.} = \int (P'\,dx + Q'\,dy + R'\,dz) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (N.)$$

The problem is completely solved by the two equations (M.) and (N.). These equations together take into account all the forces tending to move a particle in a level

surface; and what is wanting in the equation (M.) for giving a determinate figure to any such surface, is supplied by the equation (N.).

If we apply the foregoing solution to the oblate elliptical spheroid, it will immediately appear, according to what is proved in the Principia, Lib. 1. Prop. 91. Cor. 3., that the equation (N.) is verified by any surface similar and concentric to the surface of the spheroid. A further simplification arises from the same property; for we may substitute the attraction of the matter within the interior surface, for the attraction of all the matter of the spheroid: so that the solution of the problem is reduced to the single condition of finding an elliptical surface to which the resultant of the attractive and centrifugal forces shall be perpendicular.

It may not be here improper to draw attention to the difference between the analysis of a problem, and its synthetic demonstration. In an analysis it is necessary to mount up to the essential principles of a problem, which always occupy a prominent place in the investigation; whereas a synthetic demonstration may proceed on properties previously investigated, and may be read and understood, although the essential grounds of the problem may never be brought into view. There is no doubt that it is the property cited above from the Principia, which makes the elliptical spheroid, exclusively of all other figures, the form of equilibrium of a homogeneous mass of fluid revolving about an axis; yet of this property no mention is made in MACLAURIN's demonstration. Nay, it has been contended on high authority that the property in question is merely accidental, and not essential to the equilibrium*. A little patience to have traced the property on which MACLAURIN's reasoning rests to its ultimate foundation, would have shown that, however the processes of investigation may be varied, they all originate from one source†.

Having now found the equations for determining *à priori* the figure of equilibrium of an incompressible fluid revolving about an axis, it remains to solve these equations.

PROBLEM III.

To solve the equations of the last problem.

Solution.—These equations are not easily solved without complicated calculations, at least if we proceed by direct methods.

The co-ordinates of a particle in a level surface being x, y, z the partial attractions parallel to x, y, z of the whole mass upon the particle, are represented in the equation (M.) by P, Q, R: and in the equation (N.), P', Q', R' are the like partial attractions of the exterior stratum upon the particle: wherefore if P'', Q'', R'' denote the like partial attractions of the body of fluid within the level surface upon the particle, we shall have to solve these two equations,

$$\text{const.} = \int (P'' dx + Q'' dy + R'' dz) + \frac{\varepsilon}{2}(y^2 + z^2) \quad . \quad . \quad . \quad . \quad (M'.)$$

$$\text{const.} = \int (P' dx + Q' dy + R' dz) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (N.)$$

the sum of which is the equation (M.).

* POISSON, Traité de Mécanique, No. 593.

† Vide Note, p. 248.

Let $d\mu$ be a molecule of the body of fluid within the level surface; x', y', z' the co-ordinates of $d\mu$; f the distance of $d\mu$ from the attracted particles in the level surface: then

$$f = \sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}.$$

The direct attraction of $d\mu$ upon the attracted particle is

$$\frac{d\mu}{f^2}.$$

the cosines of the angles which f makes with x, y, z, are respectively

$$\frac{x-x'}{f}, \frac{y-y'}{f}, \frac{z-z'}{f}:$$

wherefore the partial attractive forces parallel to x, y, z, acting upon the particle in the level surface in the directions in which the co-ordinates decrease, will be

$$d\mu \cdot \frac{x-x'}{f^3}, \quad d\mu \cdot \frac{y-y'}{f^3}, \quad d\mu \cdot \frac{z-z'}{f^3};$$

or, which are equivalent,

$$-d\mu \cdot \frac{d \cdot \frac{1}{f}}{dx}, \quad -d\mu \cdot \frac{d \cdot \frac{1}{f}}{dy}, \quad -d\mu \cdot \frac{d \cdot \frac{1}{f}}{dz}:$$

and, observing that x, y, z, are independent of $d\mu$, the like partial forces of all the molecules of the mass of fluid within the level surface, may be thus expressed:

$$-\frac{d \cdot \int \frac{d\mu}{f}}{dx}, \quad -\frac{d \cdot \int \frac{d\mu}{f}}{dy}, \quad -\frac{d \cdot \int \frac{d\mu}{f}}{dz},$$

the integral extending to all the molecules within the level surface. Now these forces are the same in quantity, but have contrary directions to the forces represented by P'', Q'', R'' in the equation (M'.); so that by substituting and then integrating, that equation will be thus transformed:

$$\text{const.} = \int \frac{d\mu}{f} + \frac{\varepsilon}{2}(y^2 + z^2).$$

Now the inspection of this equation is alone sufficient to show that, if it be verified in one curve surface, it will be verified in every curve surface similar and similarly posited about the central point. For assume two attracted points similarly situated in two such similar surfaces; divide the matter within the surfaces into the same number of infinitesimal molecules proportional to the masses within the surfaces; then, taking any molecules similarly situated with respect to the attracted points, the differential

$$\frac{d\mu}{f}$$

will be proportional to the square of the linear dimensions of the surfaces. This is evident; for the numerators are as the masses, or in the triplicate ratio of the linear

dimensions; and f representing similar lines of the surfaces, the denominators are simply as the linear dimensions. Wherefore the integral

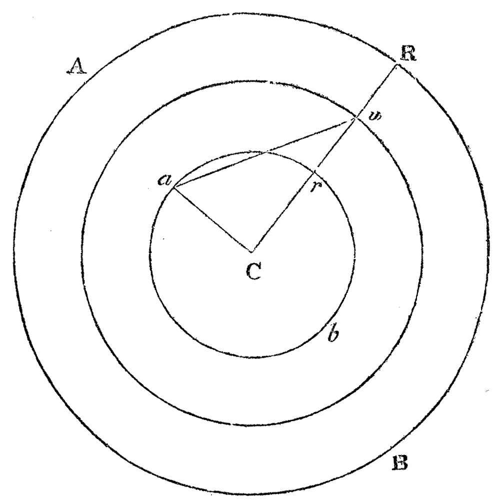
$$\int \frac{d\mu}{f},$$

extended to all the molecules within the surfaces, will be proportional to the square of the linear dimensions. Further, because the attracted points are similarly placed in the two surfaces, their co-ordinates will be similar lines; consequently

$$y^2 + z^2$$

will be as the squares of the linear dimensions of the surfaces. From what has been proved we learn that, for attracted points similarly situated in the surfaces, C will vary from one surface to another proportionally to the square of the linear dimensions of the surfaces; wherefore if C be constant at all the points of any one surface, it will be constant at all the points of every surface similar and similarly posited about the central point.

The solution of the equation (M′.) with respect to all the level surfaces, is now reduced to the verification of that equation by the upper surface of the fluid. When this condition is fulfilled, all the interior level surfaces, it has been demonstrated, are similar to the upper surface, and similarly posited about the central point.



We come next to turn our attention to the equation (N). Let A R B represent the surface of the fluid in equilibrium; C the central point; $a\,r\,b$ a level surface similar to A R B, and similarly situated about C; further, a being an attracted point in the level surface, and u a point of the stratum between the two surfaces, put x, y, z for the co-ordinates of a, and x', y', z' for those of u; then f being the distance from a to u, we shall have

$$f = \sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}.$$

Let $d\mu'$ represent a molecule of the stratum at the point u; the direct attraction of $d\mu'$ upon a particle at a, will be

$$\frac{d\mu'}{f^2}:$$

and by proceeding as before it will be found that the partial forces parallel to x, y, z, caused by the attractions of all the molecules of the stratum upon the particle, and estimated in the directions in which the co-ordinates increase, are as follows:

$$\frac{d\,.\int\frac{d\mu'}{f}}{dx}, \quad \frac{d\,.\int\frac{d\mu'}{f}}{dy}, \quad \frac{d\,.\int\frac{d\mu'}{f}}{dz}.$$

Now these are the forces represented by P′, Q′, R′ in the equation (N.): so that, by substituting and integrating, that equation will be changed into this which follows:

$$\text{const.} = \int \frac{d\mu'}{f},$$

the integral extending to all the molecules of the stratum.

The co-ordinates at the points a and u, are thus expressed:

$$\begin{array}{ll} \mathrm{C}\,a = r = \sqrt{x^2 + y^2 + z^2} & \mathrm{C}\,u = s = \sqrt{x'^2 + y'^2 + z'^2} \\ x = r\cos\theta & x' = s\cos\theta' \\ y = r\sin\theta\cos q & y' = s\sin\theta'\cos q' \\ z = r\sin\theta\sin q & z' = s\sin\theta'\sin q', \end{array}$$

in which formulas, θ, θ' are the angles that r and s make with the axis of rotation; aud q, q' the angles that determine the position of the projections of r and s upon the plane of $y\,z$. By substituting the values of the co-ordinates we obtain

$$f = \sqrt{s^2 - 2\gamma r.s + r^2},$$
$$\gamma = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(q - q').$$

If the variables s, θ', q' change to $s + d\,s$, $\theta' + d\,\theta'$, $q' + d\,q'$, the three small lines $d\,s$, $s\,d\,\theta'$, $s\sin\theta'\,d\,q'$, will be perpendicular to one another; and, as the density is unit, we shall have

$$d\mu' = s^2\,d\,s\,.\,d\,\theta'\,d\,q'\sin.\theta'.$$

The foregoing values being substituted, this result will be obtained,

$$\int\frac{d\mu'}{f} = \int\int d\,\theta'\,d\,q'\sin\theta' \int \frac{s^2\,d\,s}{\sqrt{s^2 - 2\gamma r.s + r^2}},$$

the integrations extending from $\gamma = 1$ to $\gamma = -1$; and from $\theta' = 0$, $q' = 0$, to $\theta' = \pi$, $q' = 2\pi$; and from $s = r'$, to $s = \mathrm{R}$, r' and R being the values of s at the interior and upper surfaces.

Since r' and R have the same ratio in every position of the line C r' R, we may put $s = (1 + \alpha) \times r'$, $d\,s = d\,\alpha \times r'$; by which substitutions the last equation will become,

$$\int\frac{d\mu}{f} = \int\int d\,\theta'\,d\,q'\sin\theta'.\int(1+\alpha)\,d\,\alpha\,.\,\frac{r'^2}{\sqrt{1 - 2\gamma\frac{r}{r'(1+\alpha)} + \frac{r^2}{r'^2(1+\alpha)^2}}},$$

the quantity α being constant at all the points of an exterior surface similar to the level surface, but varying from one exterior surface to another. The radical quantity must next be expanded in a series of the powers of $\frac{r}{r'(1+\alpha)}$, viz.

$$1 + \mathrm{C}^{(1)}\cdot\frac{r}{r'(1+\alpha)} + \mathrm{C}^{(2)}\cdot\frac{r^2}{r'^2(1+\alpha)^2} + \mathrm{C}^{(3)}\cdot\frac{r^3}{r'^3(1+\alpha)^3} + \text{\&c.},$$

the coefficients being determined by the well-known formula,

$$C^{(i)} = \frac{1}{2^i} \cdot \frac{d^i (\gamma^2 - 1)}{1 . 2 . 3 \ldots i \times d \gamma^i}.$$

This expansion is admissible; for $(1 + \alpha)\, r'$ being the radius of a surface exterior to the level surface, $\frac{r}{r'(1 + \alpha)}$ will be less than unit, and the series will always converge. By substituting the series, we get,

$$\begin{aligned} \int \frac{d\mu'}{f} = & \int_0^\alpha (1 + \alpha)\, d\alpha \,. \iint d\theta'\, dq' \sin\theta' \,.\, r'^2 \\ & + r \,.\, \alpha \iint d\theta'\, dq' \sin\theta' \,.\, r'\, C^{(1)} \\ & + r^2 \int_0^\alpha \frac{d\alpha}{1 + \alpha} \cdot \iint d\theta'\, dq' \sin\theta' \,.\, C^{(2)} \\ & + r^3 \int_0^\alpha \frac{d\alpha}{(1 + \alpha)^2} \cdot \iint d\theta'\, dq' \sin\theta' \,.\, \frac{C^{(3)}}{r'} \\ & \quad \vdots \\ & r^i \int_0^\alpha \frac{d\alpha}{(1 + \alpha)^{i-1}} \cdot \iint d\theta'\, dq' \sin\theta' \,.\, \frac{C^{(i)}}{r'^{(i-2)}}. \end{aligned}$$

As $\int \frac{d\mu'}{f}$ must have the same value at all the points of the level surface, it must be independent of r, which varies with the position of the attracted point; and hence we learn that all the terms of the foregoing series containing r, or any power of r, must be separately equal to zero. This brings the question to the two following equations: first,

$$\int \frac{d\mu'}{f} = \int_0^\alpha (1 + \alpha)\, d\alpha \,. \iint d\theta'\, dq' \sin\theta' \,.\, r'^2,$$

which ascertains the quantity of $\int \frac{d\mu'}{f}$, resulting from the attraction of a stratum contained between the similar surfaces, of which the radii are r' and $r' . (1 + \alpha)$, and such that $\int \frac{d\mu'}{f}$ has the same value for all positions of the attracted point in the level surface; secondly, the equation

$$0 = \iint d\theta'\, dq' \sin\theta' \cdot \frac{C^{(i)}}{r'\,(i - 2)}, \quad \ldots\ldots\ldots \quad (O.)$$

for all values of i from 1 to ∞, which determines the expression of r', and the nature of the spheroid. It now only remains to solve this equation.

Whatever figure the spheroid sought may be supposed to have, the equation of its

surface will be a function of the three co-ordinates of a point of that surface: and the expression of a radius r' deduced from that equation, can only be a function of the three quantities

$$\cos\theta', \quad \sin\theta' \cos q', \quad \sin\theta' \sin q', \quad \text{or } a', b', c',$$

which determine the position of r'. We may suppose that r', or any power of r' positive or negative, is either exactly or approximately a rational function of a', b', c'; and the terms of every such function may be so arranged as to come under this form of expression:

$$U^{(0)} + U^{(1)} + U^{(2)} \ldots . + U^{(n)},$$

$U^{(0)}$ being a constant, and $U^{(n)}$ representing generally a homogeneous function of a', b', c' of n dimensions. By means of this notation we obtain the following theorems:

$$\text{I.} \quad 0 = \int_0^{\pi} \int_0^{2\pi} d\theta' \, dq' \sin\theta' . U^{(n)} C^{(i)}$$

in all cases when n is less than i:

$$\text{II.} \quad 0 = \int_0^{\pi} \int_0^{2\pi} d\theta' \, dq' \sin\theta' \, U^{(n)} C^{(i)}$$

when n is an even number, and i an odd number.

These theorems relate to a branch of analysis that has been much cultivated; and as they are easily deduced from well-known properties, the demonstrations are omitted for the sake of abridging.

According to what has been said,

$$r' = U^{(0)} + U^{(1)} + U^{(2)} \ldots . U^{(n)};$$

and if we make $i = 1$, $C^{(1)} = \gamma$, the equation (O.) will become

$$0 = \int\int d\theta' \, dq' \sin\theta' . \{\gamma U^0 + \gamma U^{(1)} + \gamma U^{(2)} \ldots . + \gamma U^{(n)}\}.$$

Now, by the second theorem, all the terms in which n is an even number will be zero; and hence, in order to make the whole expression zero, all the other terms in which n is an odd number must be exterminated. Thus the value of r' that will verify the equation (O.) is as follows:

$$r' = U^0 + U^{(2)} + U^{(4)} \ldots . + U^{(2n)}.$$

Every power of r' positive or negative will be of the like form, so that

$$\frac{1}{r'^{i-2}} = U^0 + U^{(2)} + U^{(4)} \ldots . + U^{(2n)};$$

and, by the second theorem, this expression will verify the equation (O.) in all cases when i is an odd number.

As the equation (O.) does not contain r' when $i = 2$, it is obviously verified in that case. When $i = 4$ we have

$$\frac{1}{r'^2} = U^0 + U^{(2)} + U^{(4)} +, \&c.;$$

and the equation (O.) will take this form,

$$0 = \int\int d\,\theta'\,d\,q' \sin\theta' . \{U^0\,C^{(4)} + U^{(2)}\,C^{(4)} + U^{(4)}\,C^4 +, \&c.\};$$

of which expression the two first terms are zero by the first theorem; but as the succeeding terms have all determinate values, they must be cancelled, which limits the expression of $\frac{1}{r'^2}$ that will verify the equation (O.) to

$$\frac{1}{r'^2} = U^{(0)} + U^{(2)}.$$

We have now only to inquire whether the expression of $\frac{1}{r'^2}$ thus found will verify the equation (O.) in all cases when i is an even number greater than 4: now

$$\int\int d\,\theta'\,d\,q' \sin\theta' \frac{C^{(i)}}{r'^{(i-2)}} = \int\int d\,\theta'\,d\,q' \sin\theta' . C^i \;(U^0 + U^{(2)})^{\frac{i}{2}-1}:$$

and i being any even number, if the binomial quantity be expanded and arranged in homogeneous functions, it will be of this form,

$$U^{(0)} + U^{(2)} + U^{(4)} \ldots + U^{i-2},$$

the substitution of which will produce a series of terms, every one of which will vanish by the first theorem.

The foregoing investigation proves that every figure capable of fulfilling the conditions required for the equilibrium of an incompressible fluid subject to the law of attraction that prevails in nature, and revolving about an axis, is comprehended in the formula

$$\frac{1}{r'^2} = U^{(0)} + U^{(2)}.$$

Taking the most general expression of $U^{(2)}$, which stands for a homogeneous expression of two dimensions, of $\cos\theta'$, $\sin\theta'\cos q'$, $\sin\theta'\sin q'$, or a', b', c'; and observing that the constant

$$U^0 = U^0 . a'^2 + U^0 . b'^2 + U^0 . c'^2,$$

may be blended with the expression of $U^{(2)}$, we shall have

$$\frac{1}{r'^2} = A\,a'^2 + B\,b'^2 + C\,c'^2 + D\,a'\,b' + E\,a'\,c' + F\,b'\,c':$$

and as $r'\,a'$, $r'\,b'$, $r'\,c'$ are the co-ordinates of a point in the surface of the fluid, we obtain this general equation:

$$1 = A\,x'^2 + B\,y'^2 + C\,z'^2 + D\,x'\,y' + E\,x'\,z + F\,y'\,z',$$

which is that of an ellipsoid, the co-ordinates being parallel to any three diameters intersecting at right angles. This general equation is modified by the rotatory mo-

tion; for it is easy to prove that the axis about which the fluid revolves, or the diameter parallel to the co-ordinate x', must be perpendicular to the surface of the fluid, and consequently it must be one of the axes of the ellipsoid: and as nothing hinders from assuming the other two axes for the diameters to which the co-ordinates y and z are parallel, the foregoing equation will take this more simple form:

$$1 = \mathrm{A}\,x'^2 + \mathrm{B}\,y'^2 + \mathrm{C}\,z'^2 = \frac{x'^2}{k^2} + \frac{y'^2}{k'^2} + \frac{z'^2}{k''^2},$$

the three semi-axes of the ellipsoid being k, k', k'', of which k is the axis of rotation. Thus the ellipsoid comprehends every possible figure of equilibrium, the rotatory motion being performed about one of the axes. When the centrifugal force is given, the particular figure of equilibrium is found by making the resultant of the attractive and centrifugal forces perpendicular to the surface of the ellipsoid.