XXVIII. *On the Deflection of the Plumb-line at Arthur's Seat, and the Mean Specific Gravity of the Earth. Communicated by Lieutenant-Colonel* JAMES, *R.E., F.R.S., M.R.I.A. &c., Superintendent of the Ordnance Survey.*

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THE Royal Society has, from the very commencement of the Ordnance Survey of the United Kingdom, taken a deep interest in its progress. I have therefore great pleasure in announcing to the Society the completion of all the computations connected with the Primary Triangulation, the measurement of the Arcs of Meridians, and the figure and dimensions of the Earth.

The account of all the operations and calculations which have been undertaken and executed is now in the press, and will shortly be in the hands of the public.

After determining the most probable spheroid from all the astronomical and geodetic amplitudes in Great Britain, we find that the plumb-line is considerably deflected at several of our principal Trigonometrical Stations, and at almost every station the cause of the deflection is apparent in the configuration of the surrounding country.

The deflection of the plumb-line at Arthur's Seat is $5''\cdot25$, and at the Royal Observatory at Edinburgh it amounts to $5''\cdot63$ to the South. The unequal distribution of matter here, the great trough of the Firth of Forth being on the North, and the range of the Pentland on the South, presents a tangible cause for the deflection; but as the contoured plans of the county of Edinburgh are published, and we have the most perfect data that it is possible to obtain for estimating the amount of local attraction at Arthur's Seat and the Calton Hill, and as it appeared to me that an investigation of this matter was not only necessary to confirm and establish the results arrived at from the previous investigation of all the observed latitudes, but would also prove highly interesting to science, I decided on having observations taken with AIRY's Zenith Sector on the summit of Arthur's Seat, and at points near the meridian on the North and South of that mountain, at about one-third of its altitude above the surrounding country.

The observations were made by Serjeant-Major STEEL of the Royal Sappers and Miners, during the months of September and October last; 220 double observations of stars were taken at each Station, and the results have justified my confidence in him as an observer.

To Captain CLARKE, R.E., I entrusted all the reductions and computations connected with these observations, as well as the computations of the local attraction at

Calton Hill. The following communication has been drawn up by him, and I trust it will prove acceptable to the Royal Society, and do him credit as a mathematician.

I have myself examined the geological structure of Arthur's Seat and the whole of the county of Edinburgh, and have had the specific gravity of all the rocks ascertained, with the view of estimating the mean specific gravity of the whole mass; but although the geological structure of Arthur's Seat is well exposed, and we have deduced from its mean specific gravity (2·75) the mean specific gravity of the earth, viz. 5·316, it is not such a mountain as I should have selected for this special object.

Since these observations were made, on examining the correspondence connected with the Survey, with the view of drawing up an historical sketch of its progress for publication, I was agreeably surprised to find that the late Dr. MACCULLOCH had been employed for six years, from 1814 to 1819, in examining the whole of Scotland for the purpose of selecting a mountain which, from its homogeneous structure, size, and form, would be best suited for observations for the purpose of determining the mean specific gravity of the earth, and that he considered the Stack Mountain in Sutherlandshire admirably suited for the purpose. The transfer of the whole force of the Survey from the North of England and Scotland to Ireland, prevented the late General COLBY from undertaking this investigation; but as the Survey of Scotland is now in full progress, I purpose early in the spring to go down to the Stack Mountain, to have it and the surrounding country surveyed and contoured, and to have the observations taken for determining the attraction of its mass, and I trust at the close of the present year to lay the results before the Royal Society.

I forward herewith a model of Arthur's Seat, made from the contoured plan on the scale of 6 inches to a mile, and also an impression of the plan itself, with sections showing the geological structure of Arthur's Seat, and a table of the specific gravity of the rock of which it is composed.

HENRY JAMES,

Feb. 7, 1856. *Lieut.-Col. R.E.*

In deducing from the observations made at the three stations on Arthur's Seat, with the zenith sector, the latitudes of those stations, if we assign to the resulting latitude given by any one star a weight equal to the number of observations of that star, the final latitudes of the three stations will stand thus:—

Stations.	Designated.	Latitude.	Number of observations.
South Station	S	55° 56′ 26″·69	427
Arthur's Seat (summit*)	A	55 56 43·95	425
North Station	N	55 57 9·50	411

* The station on the summit of the hill was 14 feet from the Ordnance Trigonometrical Station, and bearing 18° North-west; the former is therefore 0″·13 North of the latter.

The latitudes thus obtained being affected by the errors of the assumed declinations of the stars, the amplitudes to be adopted as final are obtained in the following manner. Let $\phi_1\,\phi_2$ be the values of the amplitudes SA, AN to be determined, and let the stars observed at S and A only, give these values—

$$\phi_1=a,\quad \phi_1=a',\quad \phi_1=a''\ldots\ldots$$

Let stars observed at A and N only, give the values—

$$\phi_2=b,\quad \phi_2=b',\quad \phi_2=b''\ldots\ldots$$

Let stars observed at S and N only, give the values—

$$\phi_1+\phi_2=c,\quad \phi_1+\phi_2=c',\quad \phi_1+\phi_2=c''\ldots\ldots$$

And let stars observed at S, A, and N give the values—

$$\phi_1=a_1,\quad \phi_1=a_1',\quad \phi_1=a_1''\ldots\ldots$$

$$\phi_2=b_1,\quad \phi_2=b_1',\quad \phi_2=b_1''\ldots\ldots$$

Let d, e, and the same letters accented, be taken to denote the number of times the stars of the first set are observed at S and A respectively. Let f, g and h, k represent the same quantities for the stars of the second and third set; and let n, p, q, and the same letters accented, be taken to denote the numbers of times the stars of the fourth and last set are observed at S, A, N respectively.

The values of $\phi_1\,\phi_2$ adopted are those which render the quantity

$$\Sigma\left\{\frac{de}{d+e}(\phi_1-a)^2\right\}+\Sigma\left\{\frac{fg}{f+g}(\phi_2-b)^2\right\}+\Sigma\left\{\frac{hk}{h+k}(\phi_1+\phi_2-c)^2\right\}$$
$$+\Sigma\left\{\frac{np}{n+p}(\phi_1-a_1)^2\right\}+\Sigma\left\{\frac{pq}{p+q}(\phi_2-b_1)^2\right\}$$

a minimum. Making the differential coefficients of this quantity with respect to ϕ_1 and ϕ_2 respectively $=0$, we obtain

$$\mathrm{H}\phi_1+\mathrm{K}\phi_2-\mathrm{L}=0$$
$$\mathrm{K}\phi_1+\mathrm{M}\phi_2-\mathrm{N}=0,$$

in which equations

$$\mathrm{H}=\Sigma\left(\frac{de}{d+e}\right)+\Sigma\left(\frac{hk}{h+k}\right)+\Sigma\left(\frac{np}{n+p}\right)\qquad \mathrm{M}=\Sigma\left(\frac{fg}{f+g}\right)+\Sigma\left(\frac{hk}{h+k}\right)+\Sigma\left(\frac{pq}{p+q}\right)$$

$$\mathrm{L}=\Sigma\left(\frac{de}{d+e}\cdot a\right)+\Sigma\left(\frac{hk}{h+k}\cdot c\right)+\Sigma\left(\frac{np}{n+p}a_1\right)\quad \mathrm{N}=\Sigma\left(\frac{fg}{f+g}b\right)+\Sigma\left(\frac{hk}{k+k}c\right)+\Sigma\left(\frac{pq}{p+q}b_1\right)$$

$$\mathrm{K}=\Sigma\left(\frac{hk}{h+k}\right).$$

If μ be any number, the value of $\phi_1+\mu\phi_2$ is

$$\phi_1+\mu\phi_2=\frac{(-\mu\mathrm{K}+\mathrm{M})\mathrm{L}+(-\mathrm{K}+\mu\mathrm{H})\mathrm{N}}{\mathrm{HM}-\mathrm{K}^2},$$

hence the error of $\phi_1+\mu\phi_2$ depends upon the manner in which the errors of the quantities $a\,a_1\ldots b\,b_1\ldots c\ldots\ldots$ enter into this expression.

Let (γ_a) and (γ_b) be the sums of the errors of observation at S and A, of a star of the first set, the same quantities being accented for other stars. Let (α_b), (α_c) represent corresponding quantities for stars of the second set, (β_a), (β_c) and (ε_a), (ε_b), (ε_c) the same quantities for the third and fourth sets of stars.

Then L and N are affected with the errors

$$\mathrm{L} \quad \ldots \Sigma\frac{d(\gamma_b)-e(\gamma_a)}{d+e}+\Sigma\frac{k(\beta_c)-h(\beta_a)}{h+k}+\Sigma\frac{n(\varepsilon_b)-p(\varepsilon_a)}{n+p}$$

$$\mathrm{N} \quad \ldots \Sigma\frac{g(\alpha_c)-f(\alpha_b)}{f+g}+\Sigma\frac{k(\beta_c)-h(\beta_a)}{h+k}+\Sigma\frac{p(\varepsilon_c)-q(\varepsilon_b)}{p+q}.$$

From these expressions we may derive, finally, the following: if E be the probable error of an observation, the probable error of $\phi_1+\mu\phi_2$ is

$$\frac{\mathrm{E}}{\mathrm{MH}-\mathrm{K}^2}\Big\{\mathrm{M}(\mathrm{MH}-\mathrm{K}^2)+2\mathrm{PMK}-2\mu\big(\mathrm{K}(\mathrm{HM}-\mathrm{K}^2)+\mathrm{P}(\mathrm{HM}+\mathrm{K}^2)\big)+\mu^2(\mathrm{H}(\mathrm{HM}-\mathrm{K}^2)+2\mathrm{PKH})\Big\}^{\frac{1}{2}},$$

where

$$\mathrm{P}=\Sigma\frac{npq}{(n+p)(p+q)}.$$

The values of H, M, K, P, L and N are found to be

$$\mathrm{H}=168{\cdot}93 \quad \mathrm{M}=168{\cdot}52 \quad \mathrm{K}=46{\cdot}06$$

$$\mathrm{L}=362{\cdot}40 \quad \mathrm{N}=182{\cdot}20 \quad \mathrm{P}=49{\cdot}34,$$

whence we obtain

$$\phi_1=17''{\cdot}00 \quad \phi_2=25''{\cdot}53 \quad \phi_1+\phi_2=42''{\cdot}53.$$

Now the value of E is to be deduced from the differences between the individual and mean results given by the different stars. The sum of the squares of these errors is found from the whole of the observations to be 712·1, hence the mean square of an error of observation (1263 obs.) is 0·56, and the probable error of an observation consequently $=0''{\cdot}50$ $(={\cdot}67\sqrt{0{\cdot}56})$.

We have therefore the probable error of $\phi_1+\mu\phi_2$ equal to

$$\frac{0''{\cdot}50}{263{\cdot}4}\Big\{520{\cdot}48-544{\cdot}66\,\mu+522{\cdot}04\,\mu^2\Big\}^{\frac{1}{2}}=0''{\cdot}043\Big\{1-1{\cdot}046\,\mu+\mu^2\Big\}^{\frac{1}{2}},$$

so that the probable errors of ϕ_1 and ϕ_2 are each equal $0''{\cdot}043$.

As the differences of latitude are the quantities principally required, we may append these amplitudes to any one of the observed latitudes. Thus making use of the observed latitude of the South Station, namely 55° 56′ 26″·69, there will result by applying the above most probable amplitudes the following latitudes:—

Latitude of S = 55° 56′ 26″·69
" " A = 55° 56′ 43″·69
" " N = 55° 57′ 9″·22.

The last two latitudes differ from those in the first table by about a quarter of a second each.

The amplitudes derived from the latitudes in the first table, when compared with those we have considered as most probable, show the following differences:—

$$A-S \ldots\ldots +0''{\cdot}26$$
$$N-A \ldots\ldots +0''{\cdot}02$$
$$N-S \ldots\ldots +0''{\cdot}28.$$

Geodetical Amplitudes.

By means of a small network of triangulation connected with the secondary triangulation of the Ordnance Survey in the county of Edinburgh, the following results were obtained:—

From	To	Distance.	Bearing.
		ft.	° ′ ″
Arthur's Seat, Trigonometrical Station	S.	1426·7	179 42 7
	N.	2490·0	6 0 17

The bearings being reckoned from North round by East. The corresponding amplitudes are $14''{\cdot}06$ and $24''{\cdot}40$: in order, however, to the comparison of these with the amplitudes before considered, the quantity $0''{\cdot}13$ must be added to the first of the geodetical amplitudes and deducted from the second for the difference of the two stations on the summit of the hill. The geodetical amplitudes are therefore

$$A-S = 14''{\cdot}19$$
$$N-A = 24''{\cdot}27$$
$$N-S = 38''{\cdot}46.$$

By comparing these amplitudes with the actual astronomical amplitudes we find the following results:—

(1) Between the vertex of the hill and the South Station, the astronomical amplitude exceeds the geodetical by $2''{\cdot}81$.

(2) Between the vertex of the hill and the North Station, the astronomical amplitude exceeds the geodetical by $1''{\cdot}26$.

(3) Between the North Station and the South Station, the astronomical amplitude exceeds the geodetical by $4''{\cdot}07$.

Geodetical Latitudes.

The latitude of the Trigonometrical Station on the summit of Arthur's Seat, is, when referred to, or projected on that spheroidal surface which best represents all the astronomically determined points in Great Britain,

$$55° \; 56' \; 38''{\cdot}31,$$

from which, by the application of the geodetical amplitudes, we obtain the latitudes

of the other two points, shown in the following Table, in contrast with the observed latitudes:—

Station.	Astronomical Latitude.	Geodetical Latitude.	A–G.
S	55° 56′ 26″·69	55° 56′ 24″·25	2″·44
A	55 56 43·69	55 56 38·44	5·25
N	55 57 9·22	55 57 2·71	6·51

It might have been anticipated, that, on account of the attraction of the hill at the South Station, the deflection of the plumb-line would have been to the north, which by throwing the zenith to the south would have caused the observed latitude to be less than its true value. The contrary, however, takes place, for the observed latitude is greater than the geodetical. On proceeding next to the second station, namely, that on the summit of the hill, a similar anomaly is observed; there is an attraction or deflection to the south of more than five seconds, which can by no means be attributed to the hill, as its attraction upon any object at its vertex is very nearly equal north or south. A similar anomaly is visible at the North Station; there is a deflection to the south of 6″·5, which is considerably more than that due to the mass of the hill, as will appear hereafter.

It is clear, therefore, that there is some *other* disturbing force acting at each of these stations besides the attraction of Arthur's Seat, and which appears to produce a *general deflection* to the south of about five seconds.

The comparison of the observed and calculated latitudes of the observatory on the Calton Hill serves to corroborate this fact. The latitude of this observatory, as determined by observation, is

$$55° \ 57' \ 23''{\cdot}20.$$

The latitude of the Trigonometrical Station on this hill, when referred to the same spheroidal surface we have before mentioned, namely, that agreeing most nearly with all the astronomically determined points in Great Britain, is 55° 57′ 17″·51: the difference of latitude of these two points (taking the centre of the Altitude and Azimuth instrument of the observatory as the point whose latitude is above given) is 0″·06, so that the calculated latitude of the Calton Hill observatory is

$$55° \ 57' \ 17''{\cdot}57,$$

which is less than the astronomical by 5″·63; showing a deflection to the south of that amount in existence at the Calton Hill. Now the attraction of the mass of Arthur's Seat upon the Calton Hill is easily calculated to be between 0″·1 and 0″·2, consequently the deflection here visible is certainly not due to Arthur's Seat.

It seems therefore very probable that the general deflection of five seconds to the south, brought out at all these stations, is due to one and the same cause.

An explanation of this phenomenon immediately offers itself in the existence of the hollow of the Forth to the north, and the Pentland Hills and other high ground to

the south, but whether these may be sufficient to produce the effect observed will be considered hereafter.

Deflection caused by an Attracting Mass.

Let it be required to find the attraction exercised by a given mass placed on the surface of the earth upon a given point on the surface, the distance being supposed so small that the sphericity of the earth need not be considered. Let the position of any point of the attracting mass be determined by the coordinates r, θ, z; r and θ originating in the attracted point and being measured in the horizontal plane passing through that point, z being measured perpendicular to this plane. Let also the value of $\theta=0$ correspond to the meridian line, then the volume of an indefinitely small element of the attracting mass being $rd\theta.dr.dz$, if ϱ be its density, its attraction will be

$$\frac{r.\varrho dr.d\theta.dz}{r^2+z^2},$$

and therefore its attraction in the direction of the meridian is equal to this quantity multiplied by $r.(r^2+z^2)^{-\frac{1}{2}}.\cos\theta$; so that the attraction of the whole mass is equal to

$$\iiint \varrho \cos\theta d\theta \frac{r^2 dr.dz}{(r^2+z^2)^{\frac{3}{2}}}.$$

In order to perform the integrations here indicated, the equation of the surface of the attracting mass is required to determine the limits; this cannot be expressed, nor can ϱ, which is also a function of $r\theta z$. But it is easy to find the attraction of a mass of uniform density included within the following surfaces:—The horizontal planes $z=0$, $z=h$, the two cylindrical surfaces defined by the equations $r=r_1$, $r=r_2$, $r_1 r_2$ being constants, and two vertical planes determined by the equations $\theta=\theta_1$, $\theta=\theta_2$, $\theta_1\theta_2$ being constants; ϱ being supposed also constant. Integrating between these limits, the attraction of the mass under consideration is found to be

$$A=\varrho h(\sin\theta_2-\sin\theta_1)\log\frac{r_2+\sqrt{r_2^2+h^2}}{r_1+\sqrt{r_1^2+h^2}},$$

which being expanded is equal to (putting $r_1+r_2=2r$)

$$A=\varrho(r_2-r_1)(\sin\theta_2-\sin\theta_1)\frac{h}{\sqrt{r^2+h^2}}\left\{1+\frac{(r_2-r_1)^2}{24}\frac{2r^2-h^2}{(r^2+h^2)^2}+\dots\right\}.$$

Hence, by taking r_2-r_1 sufficiently small,

$$A=\varrho(r_2-r_1)(\sin\theta_2-\sin\theta_1)\frac{h}{(r^2+h^2)^{\frac{1}{2}}},$$

or if ε be the angle of elevation of the centre point of the upper horizontal surface of the mass in question, at the attracted point

$$A=\varrho(r_2-r_1)(\sin\theta_2-\sin\theta_1)\sin\varepsilon.$$

If h be small, so that its square may be neglected,

$$A=\varrho(\sin\theta_2-\sin\theta_1)\left(\log\frac{r_2}{r_1}\right)h.$$

The angle of deflection produced by any horizontal attracting force acting on the plumb-line is measured by the ratio of the attracting force to the force of gravity or the attraction of the earth.

The attraction of the earth upon any point on its surface in latitude λ is*

$$\frac{\mathrm{M}}{b^2}\left(1-e-\frac{3m}{2}+\left(\frac{5m}{2}-e\right)\sin^2\lambda\right),$$

where b is the polar semiaxis, e the ellipticity of the surface, and m the ratio of the centrifugal force at the equator to the equatorial gravity; if we put a for the radius of the equator, the attraction may also be expressed thus:

$$\frac{\mathrm{M}}{ab}\left(1-\frac{3m}{2}+\left(\frac{5m}{2}-e\right)\sin^2\lambda\right);$$

here $m=\frac{1}{289}$, $e=\frac{1}{300}$, $\sin^2\lambda=\frac{69}{100}$ nearly; whence it will follow that the term within the bracket will only influence the attraction by less than a six-hundredth part of its amount, and will therefore only affect the calculated deflection in that ratio. Therefore it is sufficiently exact to assume the attraction equal to that of a sphere whose radius is equal to the mean of the principal semidiameters of the earth, or 3956·1 miles: hence the attraction on any point on its surface $=\frac{4}{3}\pi.\delta(3956{\cdot}1)$, taking the mile as the unit of measure linear. The *deflection*, therefore (expressed in seconds), caused by any attracting force A on the surface of the earth may be taken as

$$\frac{\mathrm{A}}{\frac{4}{3}\pi.\delta.(3956{\cdot}1)\sin 1''}=\frac{\mathrm{A}}{\delta}\times 12''{\cdot}447,$$

δ being the mean density of the earth. Consequently the deflection caused by such a mass as we have been considering at the origin of coordinates or attracted point, is

$$\mathrm{D}=\frac{\varrho}{\delta}(r_2-r_1)s\sin\varepsilon\times 12''{\cdot}447,$$

or

$$\mathrm{D}=\frac{\varrho}{\delta}sh\log\frac{r_2}{r_1}\times 12''{\cdot}447,$$

where s is put for the difference of the sines.

The calculation of the attraction or deflection of the plumb-line at any point of the hill is easily effected by means of these formulæ. If through any one of the stations observed from, we draw on a contoured plan of the hill and surrounding country, a number of lines the sines of whose azimuth are successively $0, \frac{1}{12}, \frac{2}{12}, \ldots \frac{11}{12}, \frac{12}{12}$, counting from the south meridian in either direction, and from the north meridian in either direction; and draw also a number of circles whose radii are 500, 1000, 1500, 2000 .... feet, being in arithmetical progression with a common difference of 500 feet; the hill will be thus divided into a number of prisms, the deflection caused by any

* AIRY's Mathematical Tracts, pp. 167, 173.

one of which will be, putting x for the unknown ratio of the density of the hill to the density of the earth,

$$D = x\frac{500}{5280} \times 12''\cdot447 \times \frac{1}{12}\sin\varepsilon,$$

so that the total deflection is equal to

$$0''\cdot0982\ \Sigma(\sin\varepsilon)x.$$

At each of the three stations, the first ring of 500 feet was subdivided by rings at the distance of 100 feet; the result is shown in the following Table:—

South Station.

Sums of Sines.	1st Ring, 100 feet.	2nd Ring, 200 feet.	3rd Ring, 300 feet.	4th Ring, 400 feet.	5th Ring, 500 feet.
$\Sigma(\sin\varepsilon)$ for Prisms South of station	−1·796	−11·294	−11·808	−11·360	−10·001
$\Sigma(\sin\varepsilon)$ for Prisms North of station	+1·689	+ 5·448	+ 7·705	+ 9·038	+ 8·845
$\Sigma(\sin\varepsilon)=78\cdot984$ Deflect North.					
Arthur's Seat.					
$\Sigma(\sin\varepsilon)$ for Prisms North of station	−7·118	−11·281	−11·845	−12·014	−11·719
$\Sigma(\sin\varepsilon)$ for Prisms South of station	−3·524	− 9·164	− 8·603	− 7·436	− 6·421
$\Sigma(\sin\varepsilon)=18\cdot829$ Deflect South.					
North Station.					
$\Sigma(\sin\varepsilon)$ for Prisms South of station	+4·867	+4·347	+3·159	+2·179	+1·932
$\Sigma(\sin\varepsilon)$ for Prisms North of station	−2·002	−8·038	−8·856	−8·424	−6·973
$\Sigma(\sin\varepsilon)=50\cdot777$ Deflect South.					

By drawing twelve rings at 500 feet apart round the centre station, and sixteen rings round each of the other two stations, the results contained in the following Table are obtained:—

Stations.	Σ (sin ε).	2nd Ring.	3rd Ring.	4th Ring.	5th Ring.	6th Ring.	7th Ring.	8th Ring.	9th Ring.	10th Ring.	11th Ring.	12th Ring.	13th Ring.	14th Ring.	15th Ring.	16th Ring.
South Station.	Σ_n	+ 6·614	+3·187	+1·088	+0·025	−0·095	−0·088	−0·418	−0·926	−1·004	−1·037	−1·004	−0·908	−0·843	−0·828	−0·806
	Σ_s	− 6·299	−3·531	−2·536	−2·026	−1·713	−1·488	−1·309	−1·149	−0·985	−0·844	−0·729	−0·625	−0·539	−0·472	−0·434
Arthur's Seat.	Σ_n	−10·560	−7·996	−5·850	−4·656	−4·474	−4·570	−4·209	−3·839	−3·467	−3·119	−2·883				
	Σ_s	− 5·148	−6·634	−7·715	−6·425	−5·295	−4·549	−3·994	−3·587	−3·234	−2·931	−2·667				
North Station.	Σ_n	− 4·644	−3·094	−2·859	−2·361	−1·954	−1·623	−1·422	−1·258	−1·176	−1·117	−1·048	−0·996	−0·941	−0·935	−0·894
	Σ_s	+ 2·009	+1·695	+0·978	+1·028	+1·195	+0·745	−0·361	−0·747	−0·671	−0·604	−0·575	−0·558	−0·525	−0·488	−0·449

where Σ_n signifies $\Sigma(\sin\varepsilon)$ for the prisms north of the station,

Σ_s signifies $\Sigma(\sin\varepsilon)$ for the prisms south of the station.

Hence we obtain—

South Station.	Arthur's Seat.	North Station.
$\Sigma_n - \Sigma_s = +27{\cdot}636$	$\Sigma_s - \Sigma_n = +3{\cdot}441$	$\Sigma_s - \Sigma_n = +28{\cdot}994$.

In order to obtain the whole effect at each station, we must add to these the fifth part of the sum of the sines in the first ring of 500 feet at each of these stations: these are, respectively, 15·797, 3·766, 10·155; so that we have—

At South Station . . . $\Sigma(\sin \varepsilon) = 43{\cdot}433$

At Arthur's Seat . . . $\Sigma(\sin \varepsilon) = 7{\cdot}207$

At North Station . . . $\Sigma(\sin \varepsilon) = 39{\cdot}149$

Consequently,

Deflection at South Station $= 4''{\cdot}265\, x$ North

Deflection at Arthur's Seat $= 0''{\cdot}708\, x$ South

Deflection at North Station $= 3''{\cdot}845\, x$ South.

Comparison of observed and calculated Deflection.

We may now determine a value of x by the comparison of the observed effects of the action of the hill upon the amplitudes, with the calculated effects in terms of x. The equations thus obtained are

$$4{\cdot}973\, x = 2{\cdot}81$$
$$8{\cdot}110\, x = 4{\cdot}07,$$

whence

$$90{\cdot}503\, x = 46{\cdot}982$$
$$x = {\cdot}5191.$$

This solution contains tacitly the assumption that the effect of the *general* south deflection is equal at each of the three stations; if we put y for this quantity expressed in seconds, then the following equations will result from the comparison of the observed and geodetical latitudes, together with the calculated but unknown deflections in x,

$$y - 4{\cdot}265\, x - 2{\cdot}44 = 0$$
$$y + 0{\cdot}708\, x - 5{\cdot}25 = 0$$
$$y + 3{\cdot}845\, x - 6{\cdot}51 = 0,$$

which give $y = 4''{\cdot}68$ and $x = {\cdot}5076$.

These quantities give, when supplied in the equations, the following errors:

$$+0''{\cdot}08;\quad -0''{\cdot}21;\quad +0''{\cdot}13;$$

so nearly are the observations represented by these values of x and y.

Extension of the Calculation of Deflection.

The result just obtained, namely, that the ratio of the density of Arthur's Seat to

the mean density of the earth is equal to ·5076, is somewhat arbitrary, from this cause, that it is slightly dependent on the extent to which the calculation of the attraction is carried out. Had there not existed a marked difference in the mean height of the ground on the north side and on the south side of the hill, a smaller number of circles would have been sufficient. The existence of this attracting mass forbids our limiting the calculation to the visible extent of the hill; we must, therefore, in order to compare with what we have already obtained, extend the calculation to include a circle of about nine miles diameter round each station.

We shall now, instead of drawing the circles at 500 feet apart, make the radius of the $(n+1)$th circle equal to $\frac{7}{6}$ of the radius of the nth circle, so that they shall be in geometric progression. We have already drawn twelve circles round Arthur's Seat; the radius of the 13th circle will therefore be $\frac{7}{6}\cdot 500\cdot 12=(\frac{7}{6})\,6000$ feet, that of the 14th will be $(\frac{7}{6})^2\,6000$ feet, and so on. Around each of the other two stations sixteen circles have been drawn, the radii of the 17th and 18th will therefore be $(\frac{7}{6})\cdot 8000$ and $(\frac{7}{6})^2\cdot 8000$ feet, and so on.

Now if h be the height in feet of any one of the compartments thus formed, we have shown that the resulting deflection in seconds is

$$x\cdot\frac{h}{12}\log_e(\tfrac{7}{6})\,12\cdot 447\,\frac{1}{5280}$$
$$=0\cdot 00003027xh.$$

The following Table contains the sums of the heights of the surface for each of the additional rings:—

Station.	$\Sigma(h)$.	17th.	18th.	19th.	20th.	21st.	22nd.	23rd.			Total.
South Station.	$\Sigma(h)$ north	+2385	+1685	+ 935	+ 630	+ 465	+ 180	− 10	......	......	+ 6270
	$\Sigma(h)$ south	+6060	+6870	+8295	+9010	+8925	+8175	+10060	......	......	+57935
North Station.	$\Sigma(h)$ north	+ 520	+ 385	+ 277	− 15	− 200	− 320	− 375	......	......	+ 272
	$\Sigma(h)$ south	+4495	+5045	+5640	+6475	+8000	+8175	+ 8135	......	......	+45965
		13th.	14th.	15th.	16th.	17th.	18th.	19th.	20th.	21st.	
Arthur's Seat.	$\Sigma(h)$ north	+2643	+2245	+1624	+ 904	+ 548	+ 427	+ 148	− 92	− 132	+ 8315
	$\Sigma(h)$ south	+4645	+5025	+5460	+6080	+7460	+8485	+ 8845	+8550	+9595	+64125

Consequently the effective sums of the heights are,—

South Station . . . . . $\Sigma(h)=51665$
Arthur's Seat . . . . . $\Sigma(h)=55810$
North Station . . . . . $\Sigma(h)=45693$

And therefore, multiplying by $\cdot 00003027x$, the resulting deflections are,—

South Station . . . . . $1\cdot 565\,x$
Arthur's Seat . . . . . $1\cdot 691\,x$
North Station . . . . . $1\cdot 393\,x$

We see from this that the assumption of y being constant for the three stations was not very erroneous, though the difference is perceptible.

We shall now form the equations for x and y, remarking that y is not now the same quantity that was before represented by that symbol, and that the assumption of its being constant for the three stations is now almost unobjectionable. Taking into consideration the deflections before obtained, the total deflections *south* at each of the stations will be—

South Station	$(1{\cdot}565-4{\cdot}265)x=-2{\cdot}700x$
Arthur's Seat	$(0{\cdot}708+1{\cdot}691)x=\ \ 2{\cdot}399x$
North Station	$(3{\cdot}845+1{\cdot}393)x=\ \ 5{\cdot}238x$

Hence the equations are,—

$$y-2{\cdot}700x-2{\cdot}44=0$$
$$y+2{\cdot}399x-5{\cdot}25=0$$
$$y+5{\cdot}238x-6{\cdot}51=0,$$

which give for the most probable values of x and y,

$$x={\cdot}5173 \qquad y=3{\cdot}8820.$$

By substituting these values in the equations, they show the errors

$$+0''{\cdot}04; \quad -0''{\cdot}13; \quad +0''{\cdot}08,$$

showing that the above values agree very well with the observations. From a comparison of the errors of these equations with those previously solved, it would appear that the probable error of this value of x is considerably less than that of the value (·5076) then obtained. The two values, however, are as close as could be expected. We shall adopt, therefore, as most probable, so far as resulting from these observations,

$$x={\cdot}5173.$$

We may estimate the probable error of this quantity dependent upon the probable errors of the observed amplitudes thus; writing the three equations in the form

$$y+ax+a'=0$$
$$y+bx+b'=0$$
$$y+cx+c'=0,$$

we have

$$x=-\frac{(a-b)(a'-b')+(b-c)(b'-c')+(c-a)(c'-a')}{(a-b)^2+(b-c)^2+(c-a)^2}.$$

If now λ be the observed latitude of the South station, λ_1, λ_2, λ_3 the geodetic latitudes of the three stations,—

$$a'=\lambda-\lambda_1 \qquad b'=\lambda+\varphi_1-\lambda_2 \qquad c'=\lambda+\varphi_1+\varphi_2-\lambda_3$$
$$a'-b'=\lambda_2-\lambda_1-\varphi_1$$
$$b'-c'=\lambda_3-\lambda_2-\varphi_2$$
$$c'-a'=\lambda_1-\lambda_3+\varphi_1+\varphi_2.$$

The probable error of x depends on the probable errors of $a'-b'$, $b'-c'$, and $c'-a'$, that is supposing the geodetic amplitudes to be free of error, on the probable errors of ϕ_1 and ϕ_2. The part of x involving ϕ_1 and ϕ_2 is $\frac{1}{97{\cdot}07}\times(13{\cdot}037\phi_1+11{\cdot}070\phi_2)$: consequently the probable error of x is equal to the probable error of $\cdot1343(\phi_1+0{\cdot}85\phi_2)$, which, by means of the expression given for the probable error of $\phi_1+\mu\phi_2$, becomes (making $\mu=0{\cdot}85$)

$$\text{probable error of } x=\pm0{\cdot}0053.$$

Mean Density of the Earth.

Having now ascertained the *ratio* of the mean density of Arthur's Seat to the mean density of the earth, the knowledge of the latter results immediately from the knowledge of the former. Assuming as the result of observation $2{\cdot}75$ for the mean density of Arthur's Seat, it follows that

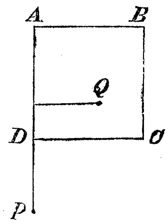
$$\text{Mean density of earth }=\frac{2{\cdot}75}{\cdot5173}=5{\cdot}316.$$

The probable error of the divisor $\cdot5173$ being $\cdot0053$, the probable error of the resulting mean density is $\pm\cdot054$, so that, considering no other cause of error than those of the zenith sector observations, we have

$$\text{Mean density of earth }=5{\cdot}316\pm\cdot054.$$

The General Deflection.

We proceed now to examine into the question of the sufficiency of the cause before mentioned, namely, the defect of matter to the north of Edinburgh and the accumulation of matter to the south, to produce the general deflection that is observed to the amount of $5''$, or rather more. In the first place, let it be required to find the attraction of a rectangular film ABCD, whose thickness is h and density ϱ, upon a point P in the production of one of its sides, AD. Measure x along PA and y perpendicular to it in the plane of the rectangle, then the mass of a small element is $\varrho h dx dy$, and therefore its attraction in the direction AP is



$$\frac{\varrho h x dx dy}{(x^2+y^2)^{\frac{3}{2}}};$$

the integral with respect to x between the limits aa' is

$$\varrho h\left\{\frac{dy}{(a^2+y^2)^{\frac{1}{2}}}-\frac{dy}{(a'^2+y')^{\frac{1}{2}}}\right\},$$

which being integrated from $y=0$ to $y=b$, is

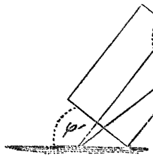
$$\varrho h \log_e \frac{\frac{y}{a}+\sqrt{1+\left(\frac{y}{a}\right)^2}}{\frac{y}{a'}+\sqrt{1+\left(\frac{y}{a}\right)}}.$$

If now we put the angles DCP$=\varphi$, ABP$=\varphi'$, we shall get, since

$$\frac{y}{a}=\cot\varphi \quad \frac{y}{a'}=\cot\varphi,$$

$$\text{Attraction } =\varrho h \log_e\left(\frac{\cot\frac{1}{2}\varphi}{\cot\frac{1}{2}\varphi'}\right).$$

We may thus determine a sufficiently close approximation to the effect of the hollow of the Forth. An examination of a map of Scotland, on a sufficiently large scale, will show that a rectangle of eighteen miles by twelve, having its longest side inclined 40° to the meridian, may be placed so as to cover the greater part of the Forth with some exactness, having also Edinburgh opposite to the middle point of the south side and Arthur's Seat nearly two miles from this side, as in the accompanying diagram. The angles φ and φ' will be found to be 73° and 18°, and therefore the attraction of a rectangular stratum of these dimensions with thickness h and density ϱ will be, (2·3025 being the reciprocal of the modulus of the common system of logarithms)



$$=2\varrho\frac{h}{5280}(2{\cdot}3025)\log\frac{\cot 9^\circ}{\cot(36^\circ\ 30')}$$

$$=2\varrho h\frac{2{\cdot}3025\times{\cdot}6695}{5280},$$

and therefore the corresponding or resulting deflection is

$$2\frac{2{\cdot}3025\times{\cdot}6695\times 12{\cdot}44{\cdot}7}{5280}\frac{\varrho}{\delta}h=0''{\cdot}00727\frac{\varrho}{\delta}h$$

in seconds, h to be expressed in feet, δ the mean density of the earth.

An inspection of a chart of the Forth will show that the depth may be taken at a very even average of 30 feet below mean water-level; so that the attraction of the water ($\varrho=1$) upon Arthur's Seat causes a deflection $=0''{\cdot}04$ to the north-east at mean water; the latitude of points in the neighbourhood is consequently variable to the amount of about $0''{\cdot}02$, depending on the tide.

We may now suppose the water to be removed and the hollow filled up with rock to a mean level of 70 feet. Then taking 2·5 for the mean density of the rock, the attraction of this stratum would be $0''{\cdot}36$, or resolved in the direction of the meridian, the deflection north would be $0''{\cdot}28$. If the hollow were filled up to a mean level of 150 feet, the deflection north would be $0''{\cdot}50$.

From this we conclude that the existence of the hollow of the Forth will account for but a small portion of the deflection of $5''$.

To the south of Edinburgh the country gradually rises, until at the southern boundary of the country the mean level is about 1000 feet with peaks rising to 1750 feet. The contours for the county of Peebles are not yet sufficiently advanced to permit the calculation of the attraction of the hills in the north of that county. We may however extend the calculation to the southern borders of Edinburghshire.

The number of circles already drawn round Arthur's Seat is 21, the last nine being drawn according to the law $r_{n+1}=\frac{7}{6}r_n$: if we draw seven more according to the same law, this will carry us slightly beyond the boundary of the county. The sums of the heights in the different rings will then be as follows:—

22nd.	23rd.	24th.	25th.	26th.	27th.	28th.
11850	12050	12850	15950	17100	18750	16800

the sum of which is 105350. The consequent deflection will be, using the same value of x, namely ·5173,

$$0''\cdot00003027\times105350\times\cdot5173=1''\cdot64.$$

To this we have to add the quantity obtained for the preceding nine rings, namely, $1''\cdot69x$ or $0''\cdot88$, making altogether $2''\cdot52$ due to the high ground to the south within the county of Edinburgh.

From the height of the country in the north of Peebleshire, it seems probable that when the calculation can be carried into that county, a sensible addition to the quantity above determined will be obtained, and that the whole of the $5''$ may *possibly* be accounted for.

In conclusion, the principal results arrived at are these:—

1st. The effect of the attraction of the Pentland Hills is observed in nearly equal amount at each of the three stations on Arthur's Seat.

2nd. The calculated attractions of the mass of Arthur's Seat at the three stations are,

South Station.	Arthur's Seat.	North Station.
$2''\cdot21$ North.	$0''\cdot37$ South.	$2''\cdot00$ South.

and, since the observed deflection at Arthur's Seat is $5''\cdot25$, the apparent effect of the Pentlands is $4''\cdot88$ at the summit of the hill.

3rd. Of this deflection of $4''\cdot88$, the computed attraction due to the configuration of the ground within a radius of fifteen miles accounts for about $2''\cdot5$; and, inasmuch as we know that the igneous rocks of Arthur's Seat and the Pentland Hills have an origin at a great depth below the surface of the earth, the difference between the observed and computed attraction is probably owing in part to the high specific gravity of the mass of rock beneath them.

4th. The deflection at the Royal Observatory, Calton Hill, being $5''\cdot63$ South, exceeds that at Arthur's Seat by $0''\cdot70$. Of this deflection, $0''\cdot60$ is due to the configuration of the ground comprised within a circle of a mile and a quarter round the Observatory.

5th. The latitude of Arthur's Seat or points in the neighbourhood varies to the amount of about $0''\cdot02$ between high and low water.

6th. The mean density of the earth determined from the observations at the three

stations on Arthur's Seat, is 5·316, with a probable error of $\pm$·054 (or one hundredth part) due to the probable errors of the astronomical amplitudes. If δ be the probable error of the assumed mean density of Arthur's Seat, the probable error of this determination of the mean density of the earth is

$$\pm\sqrt{3{\cdot}725\,\delta^2+{\cdot}003}.$$

Remarks.

In the original paper as read at the Meeting of the Royal Society on 21st February, the mean density was given as 5·14 with a probable error of $\pm$·07. In a subsequent revision of the calculations, the astronomical amplitudes and their probable errors were determined as herein explained. These amplitudes exceed those previously used by $0''{\cdot}02$, $0''{\cdot}01$, $0''{\cdot}03$, tending to increase the density. The attraction due to the ground within 100 feet round each of the stations, originally omitted, is now included, also tending to increase the density.

Explanation of the Plates.

PLATE XXXII.

Is the contoured plan of Arthur's Seat, on the scale of six inches to a mile: this is part of one of the sheets of the plan of Edinburghshire which has been published on that scale. The zenith sector stations and the lines of sections are marked on this plan. The contours furnish sufficient data to make a model.

PLATE XXXIII.

Contains geological sections taken on the three lines which are drawn on the plan, and also a table of the specific gravity of the rocks.

These plates have been engraved and electrotyped at the Ordnance Survey Office, and form part of the series of plates made to illustrate the account of the Trigonometrical Survey of Great Britain which is now in the press.

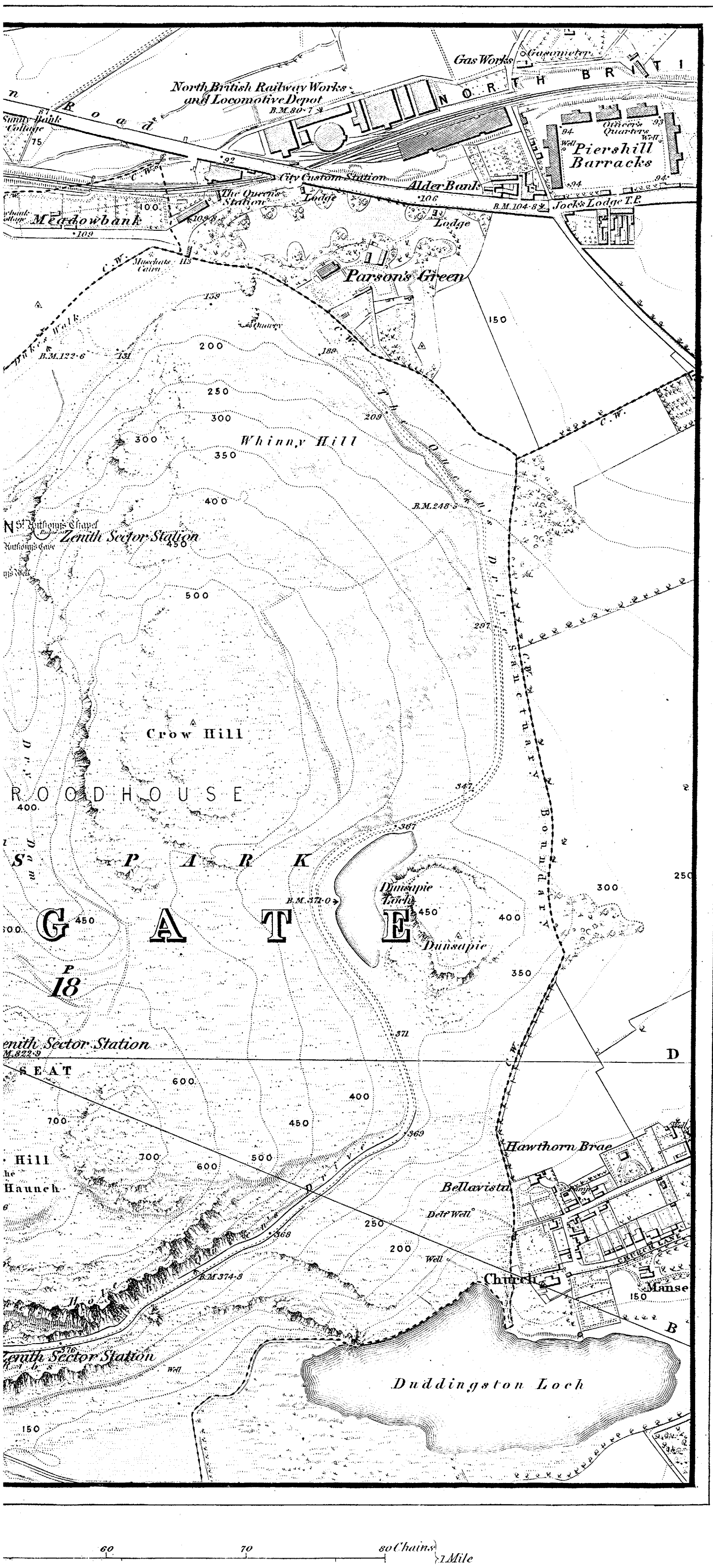
PLAN OF ARTHURS SEAT

Part of an Electrotype Copy of Sheet 2

EDINBURGHSHIRE

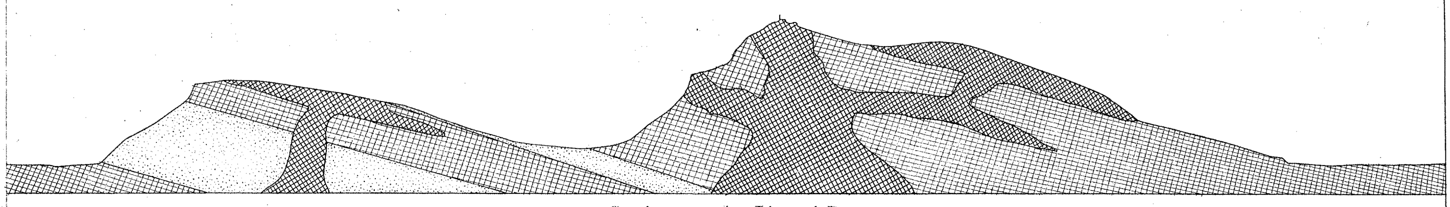
Scale _ Six Inches to One Mile

10 Chains 5 0 10 20 30 40 50

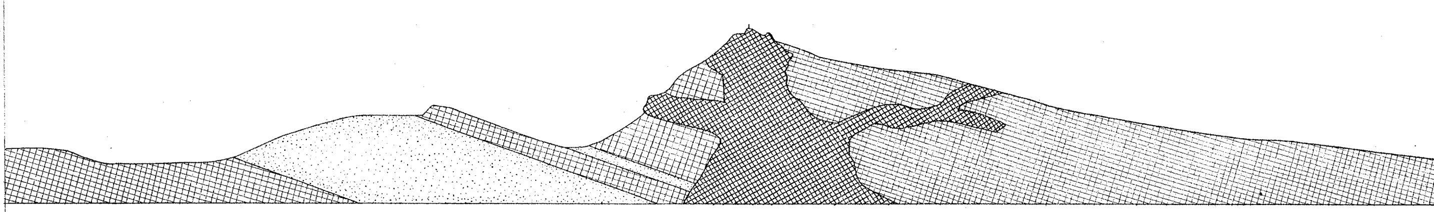

North British Railway Works and Locomotive Depot
Gas Works
Gasometer
NORTH BRITI
Piershill Barracks
Officers Quarters
City Custom Station
Alder Bank
Jock's Lodge T.P.
The Queen's Station
Lodge
Meadowbank
Parson's Green
Road
Quarry
Whinny Hill
St Anthony's Chapel
Zenith Sector Station
Crow Hill
ROODHOUSE
PARK
GATE
Dunsapie Loch
Dunsapie
Sanctuary Boundary
Drive
Zenith Sector Station
SEAT
Hill
Haunch
Hawthorn Brae
Bellavista
Doo Well
Well
Church
Manse
Zenith Sector Station
Duddingston Loch
D
B
P
18
60
70
80 Chains
1 Mile

SECTIONS THROUGH ARTHUR'S SEAT

 *Trap Rock* *Sandstone*



Section on the Line A.B.



Section on the Line C.D.

Section on the Line E.F.

Table of the Specific Gravities of the Rocks in Arthur's Seat

N.º 1 Trap tuff	2.577	N.º 6 Porphyry	2.679
2 Trap conglomerate	3.000	7 Greenstone (Salisbury Craigs)	2.689
3 Amygdaloidal trap	2.663	8 D.º very Compact	2.836
4 Basalt (S.t Anthony's)	2.830	9 Sandstone	2.676
5 Basalt under apex	2.698	10 D.º	2.454
		Mean	2.710

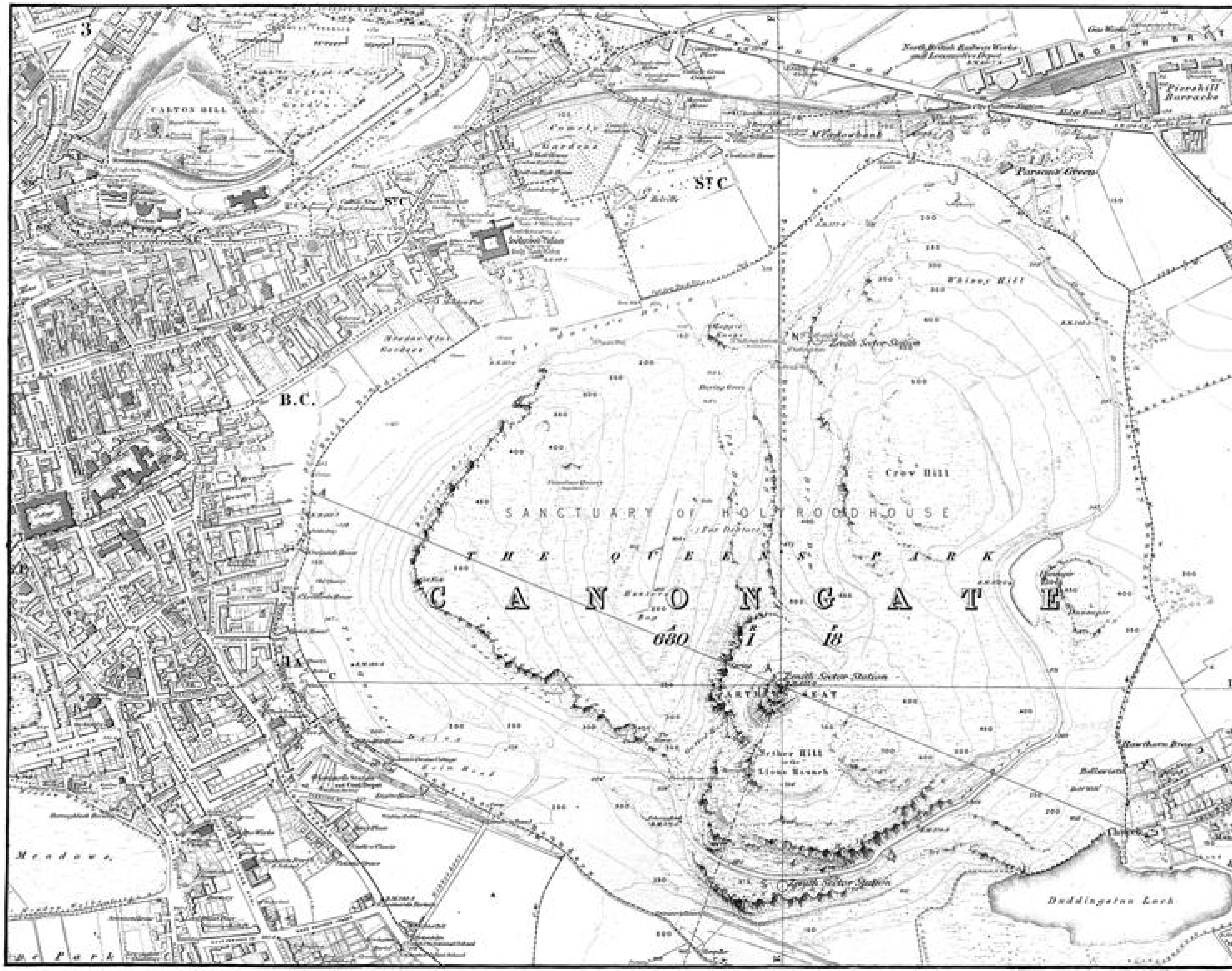
2.75 to be used as the Mean in the Computations

Engraved at the Ord. Map Office, Southampton.

PLAN OF ARTHUR'S SEAT

Part of an Electrotype Copy of Sheet 2

EDINBURGHSHIRE



Scale _ Six Inches to One Mile