XXIV. *On the Tangential of a Cubic.* By Arthur Cayley, *Esq., F.R.S.*

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In my "Memoir on Curves of the Third Order*," I had occasion to consider a derivative which may be termed the "tangential" of a cubic, viz. the tangent at the point (x, y, z) of the cubic curve $(*\between x, y, z)^3=0$ meets the curve in a point (ξ, η, ζ), which is the tangential of the first-mentioned point; and I showed that when the cubic is represented in the canonical form $x^3+y^3+z^3+6lxyz=0$, the coordinates of the tangential may be taken to be $x(y^3-z^3):y(z^3-x^3):z(x^3-y^3)$. The method given for obtaining the tangential may be applied to the general form $(a, b, c, f, g, h, i, j, k, l\between x, y, z)^3$: it seems desirable, in reference to the theory of cubic forms, to give the expression of the tangential for the general form†; and this is what I propose to do, merely indicating the steps of the calculation, which was performed for me by Mr. Creedy.

The cubic form is

$$(a, b, c, f, g, h, i, j, k, l\between x, y, z)^3,$$

which means

$$ax^3+by^3+cz^3+3fy^2z+3gz^2x+3hx^2y+3iyz^2+3jzx^2+3kxy^2+6lxyz;$$

and the expression for ξ is obtained from the equation

$$\begin{aligned} x^2\xi=&(b, f, i, c\between(j, f, c, i, g, l\between x, y, z)^2, -(h, b, i, f, l, k\between x, y, z)^2)^3\\ &-(a, b, c, f, g, h, i, j, k, l\between x, y, z)^3(\mathfrak{C}x+\mathfrak{D}),\end{aligned}$$

where the second line is in fact equal to zero, on account of the first factor, which vanishes. And $\mathfrak{C}$, $\mathfrak{D}$ denote respectively quadric and cubic functions of (y, z), which are to be determined so as to make the right-hand side divisible by x^2; the resulting value of ξ may be modified by the adjunction of the evanescent term

$$(2x+hy+gz)(a, b, c, f, g, h, i, j, k, l\between x, y, z)^3,$$

where a, g, h are arbitrary coefficients; but as it is not obvious how these coefficients should be determined in order to present the result in the most simple form, I have given the result in the form in which it was obtained without the adjunction of any such term.

Write for shortness

$$\begin{aligned} \mathrm{P}&=(k,\ l\quad \between y, z),\\ \mathrm{Q}&=(b,\ f,\ i\quad \between y, z)^2,\end{aligned}$$

* Philosophical Transactions, vol. cxlvii. 1857.

† At the time when the present paper was written, I was not aware of Mr. Salmon's theorem (Higher Plane Curves, p. 156), that the tangential of a point of the cubic is the intersection of the tangent of the cubic with the first or line polar of the point with respect to the Hessian; a theorem, which at the same time that it affords the easiest mode of calculation, renders the actual calculation of the coordinates of the tangential less important. Added 7th October, 1858.—A. C.

$$
\begin{aligned}
\mathrm{R} &= (l,\ g\ \between y,\ z)\,,\\
\mathrm{S} &= (f,\ i,\ c\ \between y,\ z)^2,\\
\mathrm{B} &= (h,\ j\ \between y,\ z)\,,\\
\mathrm{C} &= (k,\ l,\ g\ \between y,\ z)^2,\\
\mathrm{D} &= (b,\ f,\ i,\ c\between y,\ z)^3,
\end{aligned}
$$

so that

$$
\begin{aligned}
(h,\ b,\ i,\ f,\ l,\ k\ &\between x,\ y,\ z)^2 = (h\ ,\ \mathrm{P},\ \mathrm{Q}\ \between x,\ 1)^2,\\
(j,\ f,\ c,\ i,\ g,\ l\ &\between x,\ y,\ z)^2 = (j\ ,\ \mathrm{R},\ \mathrm{S}\ \between x,\ 1)^2,\\
(a,\ b,\ c,\ f,\ g,\ h,\ i,\ j,\ k,\ l\ &\between x,\ y,\ z)^3 = (a\ ,\ \mathrm{B},\ \mathrm{C},\ \mathrm{D}\between x,\ 1)^3.\\
\mathfrak{C}x+\mathfrak{D}\ & = (\mathfrak{C},\ \mathfrak{D}\ \between x,\ 1),
\end{aligned}
$$

and then for greater convenience writing $(h,\ 2\mathrm{P},\ \mathrm{Q}\between x,\ 1)^2$, &c. for $(h,\ \mathrm{P},\ \mathrm{Q}\between x,\ 1)^2$, &c., and omitting the $(x,\ 1)^2$, &c. and the arrow-heads, or representing the functions simply by $(h,\ 2\mathrm{P},\ \mathrm{Q})$, &c., we have

$$
\begin{aligned}
x^2\xi = &\ \ b(j,\ 2\mathrm{R},\ \mathrm{S})^3\\
&-3f(j,\ 2\mathrm{R},\ \mathrm{S})^2.(h\ ,\ 2\mathrm{P},\ \mathrm{Q})\\
&+3i(j,\ 2\mathrm{R},\ \mathrm{S})\,.(h\ ,\ 2\mathrm{P},\ \mathrm{Q})^2\\
&-\ c\ .(h\ ,\ 2\mathrm{P},\ \mathrm{Q})^3\\
&-\ (a,\ 3\mathrm{B},\ 3\mathrm{C},\ \mathrm{D})\,.(\mathfrak{C},\ \mathfrak{D}\ ),
\end{aligned}
$$

which can be developed in terms of the quantities which enter into it. The conditions, in order that the coefficients of x, x^0 may vanish, are thus seen to be

$$
\begin{aligned}
\mathrm{D}\mathfrak{D} &= b\mathrm{S}^3-3f\mathrm{S}^2\mathrm{Q}+3i\mathrm{S}\mathrm{Q}^2-c\mathrm{Q}^3,\\
\mathrm{D}\mathfrak{C}-3\mathrm{C}\mathfrak{D} &= b(6\mathrm{R}\mathrm{S}^2)-3f(2\mathrm{S}^2\mathrm{P}+4\mathrm{R}\mathrm{S}\mathrm{Q})+3i(2\mathrm{R}\mathrm{Q}^2+4\mathrm{S}\mathrm{P}\mathrm{Q})-c6\mathrm{P}\mathrm{Q}^2,
\end{aligned}
$$

and from these we obtain

$$
\mathfrak{C}=\left(\begin{array}{l|l|l}
-3\ bck & +6\ big & +3\ bcg\\
+6\ bil & -6\ cfk & -6\ cfl\\
+3\ fik & -6\ f^2g & -3\ fgi\\
-6\ f^2l & +6\ i^2k & +6\ i^2l
\end{array}\between y,\ z\right)^2
$$

$$
\mathfrak{D}=\left(\begin{array}{l|l|l|l}
-1\ b^2c & -3\ bcf & +3\ bci & +1\ bc^2\\
+3\ bfi & +6\ bi^2 & -6\ cf^2 & -3\ cfi\\
-2\ f^3 & -3\ f^2i & +3\ fi^2 & +2\ i^3
\end{array}\between y,\ z\right)^3
$$

and substituting these values, the right-hand side of the equation divides by x^2, and throwing out this factor we have the value of ξ; and the values of η, ζ may be thence deduced by a mere interchange of letters. The value for ξ is

x^4	x^3y	x^3z	x^2y^2	x^2yz	x^2z^2	xy^3	xy^2z	xyz^2	xz^3	y^4	y^3z	y^2z^2	yz^3	z^4
$+1\ bj^3$	$+6\ bj^2l$	$+6\ bgj^2$	$+3\ abck$	$-6\ abgi$	$-3\ abcg$	$+1\ ab^2c$	$+3\ abcf$	$-3\ abci$	$-1\ abc^2$	$+3\ b^2ij$	$+3\ b^2cj$	$+9\ bcfj$	$-3\ bc^2h$	$+3\ bcg^2$
$-1\ ch^3$	$-6\ ch^2k$	$-6\ ch^2l$	$-6\ abil$	$+6\ acfk$	$+6\ acfl$	$-3\ abfi$	$-6\ abi^2$	$+6\ acf^2$	$+3\ acfi$	$-3\ bck^2$	$-3\ bcfh$	$-9\ bchi$	$+6\ bcgl$	$-3\ c^2fh$
$-3\ fhj^2$	$-12\ fhjl$	$-12\ fghj$	$+6\ af^2l$	$+6\ af^2g$	$+3\ afgi$	$+2\ af^3$	$+3\ af^2i$	$-3\ afi^2$	$-2\ ai^3$	$-3\ bf^2j$	$-3\ bckl$	$+18\ bgil$	$+3\ bcij$	$-6\ cfgl$
$+3\ h^2ij$	$-6\ fj^2k$	$-6\ fj^2l$	$-3\ afik$	$-6\ ai^2k$	$-6\ ai^2l$	$-3\ bchk$	$-12\ bchl$	$-9\ bcgh$	$+3\ bcgj$	$-3\ bfhi$	$+3\ bfij$	$-18\ cfkl$	$+6\ bg^2i$	$+3\ cfij$
	$+6\ h^2il$	$+6\ gh^2i$	$-3\ bch^2$	$+24\ bgjl$	$+3\ bcj^2$	$-6\ bhil$	$+9\ bcjk$	$+12\ bcjl$	$+8\ bg^3$	$+6\ bikl$	$+3\ bgik$	$-18\ f^2gl$	$+6\ cf^2j$	$+3\ chi^2$
	$+12\ hijk$	$+12\ hijl$	$+6\ bhij$	$+6\ bij^2$	$+12\ bg^2j$	$+12\ bijk$	$-6\ bghi$	$+24\ bg^2l$	$-12\ cfgh$	$+3\ f^3h$	$-6\ bhi^2$	$-9\ f^2ij$	$-6\ cfgk$	$-3\ fg^2i$
			$+12\ bjl^2$	$-6\ cfh^2$	$-6\ cfhj$	$+8\ bl^3$	$+24\ bgl^2$	$+6\ bgij$	$+3\ cfjl$	$-6\ f^2kl$	$+12\ bil^2$	$+9\ fhi^2$	$-3\ cfhi$	$+6\ gi^2l$
			$-12\ chk^2$	$-24\ chkl$	$-24\ fjl^2$	$-8\ ck^3$	$+18\ bijl$	$-18\ cfhl$	$-8\ cl^3$	$+3\ fik^2$	$-6\ cfk^2$	$+18\ i^2kl$	$-12\ cfl^2$	$-3\ i^3j$
			$-6\ f^2hj$	$-6\ f^2j^2$	$-12\ fg^2h$	$+6\ f^2hl$	$-6\ cfhk$	$+6\ cfjk$	$-24\ fg^2l$		$-6\ f^3j$		$-6\ f^2g^2$	
			$-3\ fh^2i$	$-24\ fghl$	$-24\ fgjl$	$-12\ f^2jk$	$-24\ ck^2l$	$-24\ ckl^2$	$-3\ fgij$		$-6\ f^2gk$		$-6\ fgil$	
			$-12\ fhl^2$	$-24\ fgjk$	$-3\ fij^2$	$+3\ fhik$	$+6\ f^2gh$	$-6\ f^2gj$	$+12\ ghi^2$		$+9\ f^2hi$		$-9\ fi^2j$	
			$-24\ fjkl$	$-24\ fjl^2$	$+24\ ghil$	$-24\ fkl^2$	$-18\ f^2jl$	$-24\ fg^2k$	$+24\ gil^2$		$-12\ f^2l^2$		$+6\ gi^2k$	
			$+24\ hikl$	$+24\ ghik$	$+6\ hi^2j$	$+24\ ik^2l$	$-48\ fghl$	$+9\ fghi$	$-6\ i^2jl$		$+6\ fikl$		$+6\ hi^3$	
			$+12\ ijk^2$	$+6\ h^2i^2$	$+12\ ijl^2$		$+12\ fhil$	$-48\ fgl^2$			$+6\ i^2k^2$		$+12\ i^2l^2$	
				$+24\ hil^2$			$-9\ fijk$	$-12\ fijl$						
				$+24\ ijkl$			$-24\ fl^3$	$+48\ gikl$						
							$+24\ gik^2$	$+18\ hi^2l$						
							$+6\ h^2k$	$-6\ i^2jk$						
							$+48\ ikl^2$	$+24\ il^3$						

And it is not necessary to write down the corresponding values for η, ζ.