

XXX. *On Simultaneous Differential Equations of the First Order in which the Number of the Variables exceeds by more than one the Number of the Equations.* By GEORGE BOOLE, F.R.S., Professor of Mathematics in Queen's College, Cork.

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It is a fundamental proposition of analysis that a system of n differential equations of the first order containing $n+1$ variables admits of n integrals, each of which is expressed by a function of the variables equated to an arbitrary constant.

But when a system of n differential equations of the first order connects $n+r$ variables, r being greater than unity, no existing theory assigns in a general manner the number of theoretically possible integrals of the above species, or shows us how to discover them. Yet such cases are of great importance.

I wish to develop here the theory of a method for the solution of the above classes of equations, which was published by me in the 'Proceedings of the Royal Society' for March 6th of the present year, and which enables us to assign the number of theoretically possible integrals, and to reduce their discovery to the solution of a system of simultaneous differential equations equal in number to the number of integrals, and expressible as exact differential equations.

The solution of the problem as thus reduced may be effected by known methods, but I have thought it desirable to discuss this part of the subject also in direct sequence to the other, and in conformity with its method.

Of the Connexion between ordinary and partial Differential Equations.

It has been found convenient, in researches bearing upon the general theory of differential equations, to use the term 'integral' in two distinct senses, viz. to denote, as above, a relation satisfying the differential equation or system of equations, and expressed by the equating of a function of the variables to a constant, and to denote the function itself. The particular sense intended will always be shown by the connexion.

With this convention two systems of differential equations will be said to be equivalent when they have in either of the above senses (and the one implies the other) the same system of integrals. This will explain the meaning of the following proposition.

PROPOSITION I.—*A system of n ordinary differential equations of the first order connecting $n+r$ variables may be converted into an equivalent system of r linear partial differential equations of the first order.*

Let $x_1, x_2, \dots x_{n+r}$ be the variables, then the supposed given system of differential equa-

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shows that by an exactly similar rule any system of n partial differential equations of the first order, the terms of which consist of the differential coefficients of P multiplied by functions of the independent variables x_1, x_2, \dots, x_{n+r} , may be converted into an equivalent system of r common differential equations of the first order.

For the proposed system of partial differential equations is by algebraic reduction expressible in the form

$$\left. \begin{aligned} \frac{dP}{dx_1} &= A_{11} \frac{dP}{dx_{n+1}} + A_{12} \frac{dP}{dx_{n+2}} \dots + A_{1r} \frac{dP}{dx_{n+r}}, \\ \frac{dP}{dx_2} &= A_{21} \frac{dP}{dx_{n+1}} + A_{22} \frac{dP}{dx_{n+2}} \dots + A_{2r} \frac{dP}{dx_{n+r}}, \\ \frac{dP}{dx_n} &= A_{n1} \frac{dP}{dx_{n+1}} + A_{n2} \frac{dP}{dx_{n+2}} \dots + A_{nr} \frac{dP}{dx_{n+r}}. \end{aligned} \right\} \dots \dots \dots (III.)$$

If the values of $\frac{dP}{dx_1}, \frac{dP}{dx_2}, \dots, \frac{dP}{dx_n}$ in this system be substituted in the previous general equation, and the coefficients of the differential coefficients $\frac{dP}{dx_{n+1}}, \frac{dP}{dx_{n+2}}, \dots, \frac{dP}{dx_{n+r}}$ in the result be separately equated to 0, we shall have

$$\left. \begin{aligned} dx_{n+1} + A_{11} dx_1 + A_{21} dx_2 \dots + A_{n1} dx_n &= 0, \\ dx_{n+2} + A_{12} dx_1 + A_{22} dx_2 \dots + A_{n2} dx_n &= 0, \\ dx_{n+r} + A_{1r} dx_1 + A_{2r} dx_2 \dots + A_{nr} dx_n &= 0. \end{aligned} \right\} \dots \dots \dots (IV.)$$

These equations may in like manner be formed by inspection from the columns of the second member of (III.), by writing for $\frac{dP}{dx_{n+i}}$ in any column dx_{n+i} , adding to this dx_1, dx_2, \dots, dx_n multiplied by the descending coefficients of the column, and equating the final sum to 0. The rule for the one case differs from that for the other only in that differentials take the place of differential coefficients.

As an objection may be felt to the legitimacy of that step of the above process in which, the differential coefficients $\frac{dP}{dx_1}, \frac{dP}{dx_2}, \dots, \frac{dP}{dx_n}$ being eliminated, the coefficients of the remaining ones are separately equated to 0, I will point out another mode of procedure which leads to the same result, and which is founded upon LAGRANGE'S method of solution. Let the equations of the system (III.) be added together after having been multiplied respectively by $\lambda_1, \lambda_2, \dots, \lambda_n$, which are to be regarded as indeterminate functions of the variables x_1, x_2, \dots, x_{n+r} . The result will be a linear partial differential equation of the first order, of which the Lagrangian auxiliary system of ordinary differential equations will be

$$\frac{dx_1}{\lambda_1} = \frac{dx_2}{\lambda_2} \dots = \frac{dx_n}{\lambda_n} = \frac{-dx_{n+1}}{A_{11}\lambda_1 \dots + A_{n1}\lambda_n} \dots = \frac{-dx_{n+r}}{A_{1r}\lambda_1 \dots + A_{nr}\lambda_n}.$$

Hence, eliminating $\lambda_1, \lambda_2, \dots, \lambda_n$, or more strictly speaking the ratios

$$\frac{\lambda_1}{\lambda_n}, \frac{\lambda_2}{\lambda_n}, \dots, \frac{\lambda_{n-1}}{\lambda_n},$$

The proof of this will consist of two parts:—1st. It will be shown that, if a system of p integrals exists, the conditions represented by (8.) will be identically satisfied.

2ndly. It will be shown that, when the conditions represented by (5.) are identically satisfied, the solution either of the final system of partial differential equations (4.), or of the corresponding system of ordinary differential equations, by a system of p integrals is theoretically possible.

It will follow from these conjoined, that the number of actually existing integrals is exactly p .

1st. The system of ordinary differential equations corresponding to (4.) may be expressed in the form

[illegible]

Now suppose this system to have p integrals. Then, by means of these, $x_1, x_2, \dots x_p$ can be eliminated from the coefficients $H_{11}, \&c.$ in the second members, which will thus become exact differentials of those functions of the variables $x_{p+1} \dots x_{p+m}$ which express the values of $x_1, \dots x_p$. Hence we shall have the system of conditions

[illegible]

k representing any integer from 1 to p , and i, j any integers from 1 to m , and the bracketed symbols of differentiation referring to H_{kj} , H_{ki} as transformed. Hence, the unbracketed symbols referring to the prior state of the functions, we have

$$\begin{aligned} \left(\frac{d}{dx_{p+i}} \right) &= \frac{d}{dx_{p+i}} + \frac{dx_1}{dx_{p+i}} \frac{d}{dx_1} \cdots + \frac{dx_p}{dx_{p+i}} \frac{d}{dx_p} \\ &= \frac{d}{dx_{p+i}} + H_{i1} \frac{d}{dx_1} \cdots + H_{ip} \frac{d}{dx_p} \\ &= \Delta_i. \end{aligned}$$

In the same way

$$\left(\frac{d}{dx_{p+j}}\right) = \Delta_j;$$

so that (7.) becomes

$$\Delta_i H_{ki} - \Delta_j H_{ji} = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8.)$$

Now if we construct, in analogy with (4.), Prop. II., the developed form of the conditional equation (5.) of the present section, we shall have

$$(\Delta_i H_{ij} - \Delta_j H_{ji}) \frac{dP}{dx_1} \dots + (\Delta_i H_{pj} - \Delta_g H_{pi}) \frac{dP}{dx_n} = 0,$$

or, Σ denoting summation from $k=1$ to $k=p$,

$$\sum (\Delta_i H_{kj} - \Delta_j H_{ki}) \frac{dp}{dx_k} = 0.$$

Now as by (8.) the coefficients vanish identically, the equation, and generally the system of conditional equations of which it is the type, will be identically satisfied.

2ndly. We proceed to show that the system of m linear partial differential equations (7.), represented under the form

$$\Delta_1 P = 0, \Delta_2 P = 0, \dots \Delta_m P = 0,$$

and satisfying identically the system of conditions represented by

$$(\Delta_i \Delta_j - \Delta_j \Delta_i) P = 0,$$

will admit of p integrals expressing distinct values of P ; and the system of ordinary differential equations (6.) corresponding to the above system of partial differential equations will be expressible as a system of exact differential equations, and will by integration give the above systems of integrals.

Beginning with the first partial differential equation of the system (4.), and forming the corresponding Lagrangian system of ordinary differential equations

$$\begin{aligned} dx_{p+1} &= \frac{dx_1}{H_{11}} \dots = \frac{dx_p}{H_{p1}}, \\ dx_{p+2} &= 0, \quad dx_{p+m} = 0, \end{aligned}$$

we see that the integrals of this system will be of the form

$$\begin{aligned} u_1 &= c_1 \quad \dots \quad u_p = c_p, \\ x_{p+2} &= c_{p+2} \dots x_{p+m} = c_{p+m}, \end{aligned}$$

$u_1 \dots u_p$ being functions of all the variables x_1, \dots, x_{p+m} , among which, by virtue of the integrals of the second line, x_{p+2}, \dots, x_{p+m} may be regarded as constant. The general integral will be

$$F(u_1, \dots, u_p, x_{p+2}, \dots, x_{p+m}) = 0,$$

the form of F being perfectly arbitrary.

Now the general form of *any* equation of the system (4.) is

$$\frac{dP}{dx_{p+i}} + H_{1i} \frac{dP}{dx_1} \dots + H_{pi} \frac{dP}{dx_p} = 0 \dots \dots \dots (9.)$$

Let us transform this by assuming

$$u_1 \dots u_p, \quad x_{p+1} \dots x_{p+m}$$

as independent variables. Then referring the right-hand members of the following equations to the new, the left-hand to the old system of variables, we have

$$\begin{aligned} \frac{dP}{dx_{p+i}} &= \frac{dP}{dx_{p+i}} + \frac{dP}{du_1} \frac{du_1}{dx_{p+i}} \dots + \frac{dP}{du_p} \frac{du_p}{dx_{p+i}}, \\ \frac{dP}{dx_1} &= \frac{dP}{du_1} \frac{du_1}{dx_1} \dots + \frac{dP}{du_p} \frac{du_p}{dx_1}, \\ &\dots \dots \dots \\ \frac{dP}{dx_p} &= \frac{dP}{du_1} \frac{du_1}{dx_p} \dots + \frac{dP}{du_p} \frac{du_p}{dx_p}; \end{aligned}$$

$j=1$ and $P=u_s$, we have

$$(\Delta_i \Delta_1 - \Delta_1 \Delta_i) u_s = 0.$$

But $\Delta_1 u_s = 0$; therefore, by the above,

$$\Delta_1 \Delta_i u_s = 0.$$

Now $\Delta_i u_s$ is expressible at most as a function of $u_1, \dots, u_p, x_{p+1}, \dots, x_{p+m}$. But this transformation converts, as has been seen, Δ_1 into $\frac{d}{dx_{p+1}}$. Thus we have

$$\frac{d}{dx_{p+1}} \Delta_i u_s = 0,$$

so that $\Delta_i u_s$ is free from x_{p+1} . Thus the system (11.) is free from x_{p+1} .

Lastly, since by the above transformation $\Delta_2 \dots \Delta_p$ are converted into $\Delta'_2 \dots \Delta'_p$, the system of conditions $(\Delta_i \Delta_j - \Delta_j \Delta_i) P = 0$ is converted into $(\Delta'_i \Delta'_j - \Delta'_j \Delta'_i) P = 0$.

It is thus seen that the system of p partial differential equations

$$\Delta_1 P = 0, \Delta_2 P = 0, \dots, \Delta_p P = 0,$$

containing $p+m$ independent variables x_1, x_2, \dots, x_{p+m} , and satisfying the conditions

$$(\Delta_i \Delta_j - \Delta_j \Delta_i) P = 0,$$

is convertible into a system of $p-1$ partial differential equations,

$$\Delta'_2 P = 0, \Delta'_3 P = 0, \dots, \Delta'_p P = 0,$$

containing $p+m-1$ independent variables $u_1, \dots, u_p, x_{p+2}, \dots, x_{p+m}$, and satisfying the condition

$$(\Delta_i \Delta_j - \Delta_j \Delta_i) P = 0.$$

And as this system possesses the same character as that upon which the previous transformation depended, it will admit of transformation into a system of $p-2$ partial differential equations containing $p+m-2$ independent variables; and so on until we arrive at a single final partial differential equation containing $p+1$ independent variables, and having therefore p distinct integrals, which will be the common integrals of the primary system of partial differential equations as well as of the system of ordinary differential equations to which they correspond.

Cor. The property of the coefficients $\Delta_i u_1$, &c. of the system (11.), of being free from the variable x_{p+1} , enables us, by properly determining the integrals of the partial differential equation $\Delta_1 P = 0$, to reduce the system to a form of great simplicity.

Let $\Delta_i u_j$ be any one of those coefficients. Its developed form is

$$\left(\frac{d}{dx_{p+i}} + H_{1i} \frac{d}{dx_1} + \dots + H_{pi} \frac{d}{dx_p} \right) u_j \dots \dots \dots (12.)$$

Now as this expression will, after the performance of the differentiations, be free from x_{p+1} , and as the differentiations are none of them with respect to x_{p+1} , we can give to x_{p+1} in it any particular value before differentiation without affecting the final result. Let us then suppose that in H_{1i}, \dots, H_{pi} , and in u_j , x_{p+1} is made equal to 0. Now it is possible so to determine the integrals u_j as functions of the variables x_1, \dots, x_{p+r} , that

If *all* such prove to be identities, the given system of differential equations admits of n integrals, and is reducible to a system of exact differential equations. But if *any* such equation is not an identity, it will constitute a new partial differential equation of the form

$$B_1 \frac{dP}{dx_1} + B_2 \frac{dP}{dx_2} \dots + B_n \frac{dP}{dx_n} = 0.$$

And this, combined with the previous ones, will enable us to form a system of $r+1$ partial differential equations, in which $\frac{dP}{dx_n}, \frac{dP}{dx_{n+1}}, \dots, \frac{dP}{dx_{n+r}}$ appear each in only one equation and with coefficient unity. Upon this system let the same process be repeated as upon the previous system of r partial differential equations, and so continually repeated until we arrive at a final system of partial differential equations such that, if that system be represented in the form

$$\Delta_1 P = 0 \dots \Delta_m P = 0,$$

the condition

$$(\Delta_i \Delta_j - \Delta_j \Delta_i) P = 0$$

shall be identically satisfied for every pair.

Then, the number of such partial differential equations being m , the number of integrals of the original system of partial differential equations will be $n+r-m$, i. e. it will be equal to the number of the original variables diminished by the number of final partial differential equations.

And if by that final system we eliminate m of the differential coefficients from

$$\frac{dP}{dx_1} dx_1 + \frac{dP}{dx_2} dx_2 \dots + \frac{dP}{dx_{n+r}} dx_{n+r} = 0,$$

and equate to 0 the coefficients of the remaining differential coefficients, we shall have a system of $n+r-m$ differential equations expressible as exact differential equations for the determination of the integrals.

Actually to determine these, we should endeavour in the first instance to reduce the final system of differential equations, as such reduction is theoretically possible, to a system of exact differential equations. If the means of doing this are not obvious, the method of the variation of parameters or the equivalent methods of Prop. III. must be applied.

Lastly, if the process which consists in the application of the theorem $(\Delta_i \Delta_j - \Delta_j \Delta_i) P = 0$ do not stop with the formation of the final system of partial differential equations, but lead to algebraic relations among the variables, the given system of differential equations will have no integrals properly so called, but it may admit of *solutions* analogous to those the theory of which has been developed by PFAFF, JACOBI, and others for the differential equation

$$X_1 dx_1 + X_2 dx_2 \dots + X_n dx_n = 0.$$

Applications.

1st. Suppose it required to find the number of integrals of the form $P=c$, which the system of differential equations

$$dz=(t+xy+xz)dx+(xzt+y-xy)dy,$$

$$dt=(y+z-3x)dx+(zt-y)dy$$

admits, and to determine such integrals.

Eliminating dz and dt from the equation

$$\frac{dP}{dx}dx+\frac{dP}{dy}dy+\frac{dP}{dz}dz+\frac{dP}{dt}dt=0,$$

and equating to 0 the coefficients of dz and dt in the result, we have

$$\frac{dP}{dx}+(t+xy+xz)\frac{dP}{dz}+(y+z-3x)\frac{dP}{dt}=0, \quad \dots \dots \dots (1.)$$

$$\frac{dP}{dy}+(xzt+y-xy)\frac{dP}{dz}+(zt-y)\frac{dP}{dt}=0. \quad \dots \dots \dots (2.)$$

Hence writing

$$\Delta=\frac{d}{dx}+(t+xy+xz)\frac{d}{dz}+(y+z-3x)\frac{d}{dt},$$

$$\Delta'+\frac{d}{dy}+(xzt+y-xy)\frac{d}{dz}+(zt-y)\frac{d}{dt},$$

and forming the equation

$$(\Delta\Delta'-\Delta'\Delta)P=0,$$

we have, on rejecting a common algebraic factor,

$$x\frac{dP}{dz}+\frac{dP}{dt}=0. \quad \dots \dots \dots (3.)$$

By substituting in (1.) and (2.) the value of $\frac{dP}{dt}$ hence obtained, we have the system of three equations,

$$\frac{dP}{dx}+(3x^2+t)\frac{dP}{dz}=0,$$

$$\frac{dP}{dy}+y\frac{dP}{dz}=0,$$

$$\frac{dP}{dt}+x\frac{dP}{dz}=0.$$

Now if upon any two of these we repeat the same process as upon (1.) and (2.), we obtain as the result $0=0$. Thus the system of partial differential equations is complete.

As then there are three equations in this final system, while the number of original variables was four, the primary system will admit of one integral of the form $P=c$.

To obtain this integral, eliminate $\frac{dP}{dx}$, $\frac{dP}{dy}$, $\frac{dP}{dt}$ from the above equations and

$$\frac{dP}{dx}dx+\frac{dP}{dy}dy+\frac{dP}{dz}dz+\frac{dP}{dt}dt=0,$$

and equate to 0 the coefficient of $\frac{dP}{dz}$ in the result. We find

$$dz - (t + 3x^2)dx - ydy - xdt = 0,$$

the integral of which is

$$z - xt - x^3 - y^2 = c.$$

2nd. The solution of the partial differential equation

$$Rr + Ss + Tt + U(s^2 - rt) = V,$$

as well as of the special equations

$$Rr + Ss + Tt = V,$$

$$Rr + Ss + Tt + U(s^2 - rt) = 0,$$

the theory of which constitutes an exception to that of the more general form, depends in general upon the integration of three simultaneous differential equations between five variables. To this integration the method of the foregoing sections is applicable.

The only cases for which the theory of the ultimate solution can be said to be complete, are those in which the auxiliary system of common differential equations admits either three integrals of the form $P = c$, or two integrals of that form.

We may apply the method of the foregoing sections, not only to the determination of the integrals, but also to the discovery of the *a priori* conditions connecting the coefficients R, S, T, U, V in order that each of these species of integration may be possible.

For example, the solution of the equation

$$Rr + Ss + Tt + (s^2 - rt) = V$$

depending upon the integration of the system

$$dq = -m_1 dx + R dy,$$

$$dp = -m_2 dy + T dx,$$

$$dz = p dz + q dy,$$

in which m_1 and m_2 are roots of the equation

$$m^2 - Sm + RT - V = 0,$$

let it be required to determine the conditions under which the system admits three integrals.

Eliminating dq, dp, dz between the above equations and

$$\frac{dP}{dx} dx + \frac{dP}{dy} dy + \frac{dP}{dz} dz + \frac{dP}{dp} dp + \frac{dP}{dq} dq = 0,$$

and equating to 0 the coefficients of dx and dy in the result, we obtain two partial differential equations which may be thus represented, viz.

$$\Delta P = 0, \quad \Delta' P = 0,$$

in which

$$\Delta = \frac{d}{dx} - m_1 \frac{d}{dq} + T \frac{d}{dp} + p \frac{d}{dz},$$

$$\Delta' = \frac{d}{dy} + R \frac{d}{dq} - m_2 \frac{d}{dp} + q \frac{d}{dz}.$$

(including the original equations) will be complete when no new equation arises from the combination of any one of the equations with any one of the equations of the original system.

I will illustrate this by investigating the conditions of integrability of the expression

$$\phi\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \dots \frac{d^ny}{dx^n}\right)dx.$$

If this expression admit of an integral V , it is easy to see that V will satisfy the two partial differential equations

$$\frac{dV}{dx} + y_1 \frac{dV}{dy} + y_2 \frac{dV}{dy_1} \dots + y_n \frac{dV}{dy_{n-1}} = \phi, \quad . \quad . \quad . \quad . \quad . \quad . \quad (2.)$$

$$\frac{dV}{dy_n} = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad (3.)$$

in which

$$y_1 = \frac{dy}{dx}, \quad y_2 = \frac{d^2y}{dx^2} \dots\dots,$$

and ϕ stands for

$$\phi(x, y, y_1, \dots y_n).$$

The above are, in fact, the partial differential equations which we should obtain by Prop. I. as the equivalents of the system of ordinary differential equations,

$$dV = \phi dx,$$

$$dy = y_1 dx, \quad dy_1 = y_2 dx, \quad \dots \quad dy_{n-1} = y_n dx.$$

If we write

$$\left(\frac{d}{dx}\right) = \frac{d}{dx} + y_1 \frac{d}{dy} + y_2 \frac{d}{dy_1} \dots + y_n \frac{d}{dy_{n-1}},$$

the above partial differential equations become

$$\left(\frac{d}{dx}\right)V = \phi \dots (I.), \quad \frac{dV}{dy_n} = 0 \dots (II.)$$

The combination of (I.) with (II.) (by the theorem (1.)), then of (I.) with the result, and so on, gives a series of equations which may be thus expressed:—

$$\frac{dV}{dy_{n-1}} = \Delta_n \phi, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (III.)$$

$$\frac{dV}{dy_{n-2}} = \Delta_{n-1} \phi, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (IV.)$$

$$. \quad . \quad . \quad . \quad .$$

$$\frac{dV}{dy_0} = \Delta_1 \phi, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (V.)$$

$$0 = \Delta_0 \phi, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (VI.)$$

in which

$$\Delta_r = \frac{d}{dy_r} - \left(\frac{d}{dx}\right) \frac{d}{dy_{r+1}} + \left(\frac{d}{dx}\right)^2 \frac{d}{dy_{r+2}} - \&c.,$$

$$\Delta_0 = \frac{d}{dy} - \left(\frac{d}{dx}\right) \frac{d}{dy_1} + \left(\frac{d}{dx}\right)^2 \frac{d}{dy_2} - \&c.$$

The combination of (II.) with (III.) ... (V.) gives the series of conditions

$$\frac{d}{dy_n} \Delta_n \phi = 0, \frac{d}{dy_n} \Delta_{n-1} \phi = 0, \dots \frac{d}{dy_n} \Delta_1 \phi = 0. \quad \dots \quad \text{(VII.)}$$

The conditions of integrability are expressed by (VI.) and (VII.). These satisfied, the equations (III.), (IV.), ... (V.) show that ϕdx can be expressed as an exact differential with respect to x, y, y_1, y_{n-1} , in the form

$$\begin{aligned} \phi dx = & (\phi - y_1 \Delta_1 \phi - y_2 \Delta_2 \phi \dots - y_n \Delta_n \phi) dx \\ & + \Delta_1 \phi dy + \Delta_2 \phi dy_1 \dots + \Delta_n \phi dy_{n-1}, \end{aligned}$$

a result first established by M. SARRUS.