XIII. *On the Calculus of Symbols.*—Second Memoir. *By* W. H. L. Russell, *Esq.*, *A.B.* *Communicated by* Arthur Cayley, *Esq.*, *F.R.S.*

Made up of two Memoirs: one received January 7,—Read January 30, 1862; the other received June 18,—Read June 19, 1862.

This Memoir is a continuation of one on the Calculus of Symbols which I had the honour to lay before the Society in December 1860, and which has since been published in the Philosophical Transactions. I commence this paper with some extensions of the method given in the former memoir for resolving functions of non-commutative symbols into binomial factors. I then explain a method, analogous to the process for extracting the square root in ordinary algebra, for resolving such functions into equal factors. I next investigate a process for finding the highest common internal divisor of two functions of non-commutative symbols, or, in other words, of finding if two linear differential equations admit of a common solution. After this, I give a rule for multiplying linear factors of non-commutative symbols, analogous to the ordinary algebraical rule for linear algebraical factors. I then resume the consideration of the binomial theorem explained in the former memoir. Two new forms of this binomial theorem are here given; and the method by which these forms are proved identical will, I hope, be considered an interesting portion of symbolical algebra, and as exhibiting in a remarkable manner its peculiar nature.

In the next place, I proceed to calculate the general values of the coefficients which occur in the form of the binomial theorem given in the first memoir; I then obtain an expression for the symbolical coefficient of the general term of the multinomial theorem as previously explained; and also a theorem for the multiplication of symbolical factors emanating from each other after a given law; lastly, I investigate a binomial theorem reciprocal to the binomial theorem already considered.

In the former memoir I explained a process by which the symbolical function

$$\varrho^n\varphi^n(\pi)+\varrho^{n-1}\varphi_{n-1}(\pi)+\varrho^{n-2}\varphi_{n-2}(\pi)+\ldots+\varphi_0(\pi)$$

could be resolved in all possible cases into factors of the form

$$(\varrho\psi_1{}^{(n)}\pi+\psi_0{}^{(n)}\pi)(\varrho\psi_1{}^{(n-1)}\pi+\psi_0{}^{(n-1)}\pi)\ldots(\varrho\psi_1\pi+\psi_0\pi).$$

I shall now give a method by which the same symbolical function may be resolved into factors of the form

$$(\varrho^\alpha\psi_\alpha^{(n)}\pi+\psi_0^{(n)}\pi)(\varrho^\beta\psi_\beta^{(n-1)}\pi+\psi_0^{(n-1)}\pi)\ldots(\varrho^\mu\psi_\mu(\pi)+\psi_0\pi).$$

By pursuing methods similar to those employed in the preceding paper, we find the following equations as the condition that $\varrho^2\psi_2(\pi)+\psi_0(\pi)$ may be an internal factor of

the given symbolical function,

$$\phi_0(\pi)-\frac{\psi_0(\pi)}{\psi_2(\pi-2)}\phi_2(\pi-2)+\frac{\psi_0\pi\psi_0(\pi-2)}{\psi_2(\pi-2)\psi_2(\pi-4)}\phi_4(\pi-4)-\ldots=0,$$

$$\phi_1\pi-\frac{\psi_0(\pi)}{\psi_1(\pi-2)}\phi_3(\pi+2)+\frac{\psi_0\pi\psi_0(\pi-2)}{\psi_2(\pi-2)\psi_2(\pi-4)}\phi_5(\pi-4)-\ldots=0.$$

Again, the conditions that $\varrho^2\psi_2(\pi)+\psi_0(\pi)$ may be an external factor of the same symbolical function will be given by the equations

$$\phi_0(\pi)-\frac{\psi_0\pi}{\psi_2\pi}\phi_2(\pi)+\frac{\psi_0\pi\psi_0(\pi+2)}{\psi_2\pi\psi_2(\pi+2)}\phi_4(\pi)-\ldots=0,$$

$$\phi_1(\pi)-\frac{\psi_0(\pi+1)}{\psi_2(\pi+2)}\phi_3(\pi)+\frac{\psi_0(\pi+1)\psi_0(\pi+3)}{\psi_2(\pi+1)\psi_2(\pi+3)}\phi_5(\pi)-\ldots=0.$$

We may also find in like manner the conditions that $\varrho^3\psi_3(\pi)+\psi_0(\pi)$, $\varrho^4\psi_4(\pi)+\psi_0(\pi)$ may be internal or external factors of the given symbolical function: in every case the number of equations of condition will be equal to the degree of (ϱ) in the given factor.

By applying the method of divisors, as explained in the former paper, we may ascertain the forms of $\psi_2(\pi)$, $\psi_0(\pi)$ in order that $\varrho^2\psi_2(\pi)+\psi_0(\pi)$ may be an internal factor of the given symbolical function. In the present case, however, $\psi_2(\pi)$ must be a divisor both of $\phi_n(\pi)$ and $\phi_{n-1}(\pi)$, $\psi_0(\pi)$ a divisor both of $\phi_1(\pi)$ and $\phi_0(\pi)$,—a consideration which will greatly simplify the process. We proceed, in like manner, should there be no internal factors of the form $\varrho^2\psi_2\pi+\psi_0\pi$, $\varrho^2\psi_2(\pi)+\psi_0(\pi)$, to ascertain if there are any of the form $\varrho^3\psi_2(\pi)+\psi_0(\pi)$, &c.; and continuing the investigation as before, we are able, in all possible cases, to resolve the given symbolical function

$$\varrho^n\phi_n(\pi)+\varrho^{n-1}\phi_{n-1}(\pi)+\ldots+\phi_0(\pi)$$

into binomial factors.

Hence, in any linear differential equation,

$$X_r\frac{d^ru}{dx^r}+X_{r-1}\frac{d^{r-1}u}{dx^{r-1}}+X_{r-2}\frac{d^{r-2}u}{dx^{r-2}}+\ldots=X,$$

if we put

$$\varrho=x,\ \pi=x\frac{d}{dx},$$

we shall be able in all possible cases to reduce it to a series of equations:—

$$\begin{aligned}\varrho^{\alpha}\psi_{\alpha}^{(n-1)}(\pi)u_{n-1}+\psi_0^{(n-1)}(\pi)u_{n-1}&=X,\\ \varrho^{\beta}\psi_{\beta}^{(n-2)}(\pi)u_{n-2}+\psi_0^{(n-2)}(\pi)u_{n-2}&=u_{n-1},\\ \&c.\qquad\qquad&=\&c.,\\ \varrho^{\mu}\psi_{\mu}\ \ (\pi)u\ \ +\psi_0\ \ (\pi)u\ \ &=u_1,\end{aligned}$$

a series of binomial equations, each of which may be treated by the methods due to Professor BOOLE. I shall now explain a process analogous to that denominated 'evolution' in ordinary algebra. To resolve the symbolical function

$$\varrho^{2n}\phi_{2n}(\pi)+\varrho^{2n-1}\phi_{2n-1}(\pi)+\ldots+\phi_0(\pi)$$

into two equal factors.

For this purpose, let us assume

$$\varrho^{2n}\phi_{2n}(\pi)+\varrho^{2n-1}\phi_{2n-1}(\pi)+\varrho^{2n-2}\phi_{2n-2}(\pi)+\ldots+\phi_0(\pi)$$
$$=(\varrho^n\theta_n(\pi)+\varrho^{n-1}\theta_{n-1}(\pi)+\varrho^{n-2}\theta_{n-2}(\pi)+\ldots+\theta_0(\pi))^2.$$

From this we find, equating the coefficients of ϱ^{2n},

$$\theta_n(\pi)\theta_n(\pi+n)=\phi_{2n}(\pi),$$

from whence

$$\theta_n(\pi+n)\theta_n(\pi+2n)=\phi_{2n}(\pi+n);$$

or

$$\theta_n(\pi+2n)=\frac{\phi_{2n}(\pi+n)}{\phi_{2n}(\pi)}\,\theta_n(\pi),$$

whence

$$\theta_n(\pi)=\frac{\phi_{2n}(\pi-n)\ \phi_{2n}(\pi-3n)\phi_{2n}(\pi-5n)\ldots}{\phi_{2n}(\pi-2n)\phi_{2n}(\pi-4n)\phi_{2n}(\pi-6n)\ldots}.$$

Again, equating the coefficients of ϱ^{2n-1}, we shall have

$$\theta_{n-1}(\pi+n)\theta_n\pi+\theta_n(\pi+n-1)\theta_{n-1}(\pi)=\phi_{2n-1}(\pi)\ldots; \quad \ldots\ldots \quad \text{(A.)}$$

$\theta_n(\pi)$, $\phi_{2n-1}(\pi)$ are known rational functions of (π), wherefore assume

$$\theta_n(\pi)=a+b\pi+c\pi^2+\ldots+k\pi^s,$$
$$\phi_{2n-1}(\pi)=\alpha+\beta\pi+\gamma\pi^2+\ldots+\varkappa\pi^r,$$

where a, b, c, ... α, β, γ, ... are known, and

$$\theta_{n-1}(\pi)=\mathrm{A}+\mathrm{B}\pi+\mathrm{C}\pi^2+\ldots+\mathrm{K}\pi^{r-s},$$

where A, B, C are known. They may be easily found by equating the coefficients of (π) in equation (A.) and $\theta_{n-1}(\pi)$ thus determined.

If we equate the coefficients of ϱ^{2n-2}, we shall have

$$\theta_{n-2}(\pi+n)\theta_n\pi+\theta_{n-2}(\pi)\theta_n(\pi+n-2)+\theta_{n-1}\pi\theta_{n-1}(\pi+n-1)=\phi_{2n-2}(\pi),$$

from which $\theta_{n-2}(\pi)$ may be determined in like manner. By this method we may in all possible cases reduce a proposed differential equation,

$$\mathrm{X}_{2r}\frac{d^{2r}u}{dx^{2r}}+\mathrm{X}_{2r-1}\frac{d^{2r-1}u}{dx^{2r-1}}+\ldots+\mathrm{X}_1\frac{du}{dx}+\mathrm{X}_0u=\mathrm{X},$$

to the form

$$\left\{\Xi_r\frac{d^r}{dx^r}+\Xi_{r-1}\frac{d^{r-1}}{dx^{r-1}}+\ldots+\Xi_0\right\}^2u=\mathrm{X},$$

when Ξ_r, Ξ_{r-1}, &c. are rational functions of (x).

The methods for finding the highest common divisor in ordinary algebra apply equally to the present Calculus, as will be seen by the following examples:—

To find the highest common internal divisor of the symbolical functions $\varrho^2+\varrho-\pi^2$ and $\varrho^2+\varrho(\pi^2+\pi+1)-\pi^3$.

Take $\varrho^2+\varrho-\varpi^2$ for a divisor, and proceed as follows:—

$$\begin{array}{l} \varrho^2+\varrho-\varpi^2)\varrho^2+\varrho(\varpi^2+\varpi+1)-\varpi^3 \\ \qquad\qquad\quad \underline{\varrho^2+\varrho \qquad\qquad -\varpi^2} \\ \qquad\qquad\qquad \varrho(\varpi^2+\varpi)-\varpi^3+\varpi^2 \end{array}$$

It is easily seen that the remainder may be written $\varpi(\varpi-1)(\varrho-\varpi)$.

Hence, taking $\varrho-\varpi$ as divisor,

$$\begin{array}{l} \varrho-\varpi)\varrho^2+\varrho-\varpi^2(\varrho+\varpi \\ \qquad\quad \underline{\varrho^2-\varrho\varpi} \\ \qquad\qquad \varrho(\varpi+1)-\varpi^2 \\ \qquad\qquad \underline{\varrho(\varpi+1)-\varpi^2} \end{array}$$

Hence $\varrho-\varpi$ is the highest common divisor of $\varrho^2+\varrho-\varpi^2$ and $\varrho^2+\varrho(\varpi^2+\varpi+1)-\varpi^3$.

Again, to find the highest common internal divisor of $\varrho^2\varpi(\varpi+1)+\varrho(\varpi^3+\varpi^2)+\varpi^2(\varpi-1)$ and $\varrho^3\varpi-2\varrho^2\varpi+\varrho(\varpi^2+\varpi)+\varpi^4$.

The first of these quantities is equivalent to $(\varpi-1)(\varrho^2\varpi+\varrho(\varpi^2+\varpi)+\varpi^2)$.

We take $\varrho^2\varpi+\varrho(\varpi^2+\varpi)+\varpi^2$ for divisor, and proceed as follows:—

$$\begin{array}{l} \varrho^2\varpi+\varrho(\varpi^2+\varpi)+\varpi^2)\varrho^3\varpi-2\varrho^2\varpi+\varrho(\varpi^2+\varpi)+\varpi^4(\varrho-(\varpi+1) \\ \qquad\qquad\qquad\qquad \underline{\varrho^3\varpi+\varrho^2(\varpi^2+\varpi)+\varrho\varpi^2} \\ \qquad\qquad\qquad\qquad -\varrho^2(\varpi^2+3\varpi)+\varrho\varpi+\varpi^4 \\ \qquad\qquad\qquad\qquad \underline{-\varrho^2(\varpi^2+3\varpi)-\varrho(\varpi^3+3\varpi^2+2\varpi)-\varpi^2(\varpi+1)} \\ \qquad\qquad\qquad\qquad\quad \varrho(\varpi^3+3\varpi^2+3\varpi)+\varpi^4+\varpi^3+\varpi^2 \end{array}$$

The remainder is equivalent to $(\varpi^2+\varpi+1)(\varrho\varpi+\varpi^2)$.

Hence, taking $\varrho\varpi+\varpi^2$ for a new divisor,

$$\begin{array}{l} \varrho\varpi+\varpi^2)\varrho^2\varpi+\varrho(\varpi^2+\varpi)+\varpi^2(\varrho+1 \\ \qquad\qquad \underline{\varrho^2\varpi+\varrho\varpi^2} \\ \qquad\qquad\qquad \varrho\varpi+\varpi^2 \\ \qquad\qquad\qquad \underline{\varrho\varpi+\varpi^2} \end{array}$$

Hence $\varrho\varpi+\varpi^2$ is the highest common internal divisor of

$$\varrho^2\varpi(\varpi+1)+\varrho(\varpi^3+\varpi^2)+\varpi^2(\varpi-1)$$

and

$$\varrho^3\varpi-2\varrho^2\varpi+\varrho(\varpi^2+\varpi)+\varpi^4.$$

It is evident, as was mentioned in the introduction to this memoir, that this process is equivalent to finding the conditions that two linear differential equations may have a common solution.

I shall next proceed to find the general term of the continued product

$$(\varrho+\theta_1\varpi)(\varrho+\theta_2\varpi)(\varrho+\theta_3\varpi)\ldots(\varrho+\theta_n\varpi).$$

This product, when developed, will be of the form

$$\varrho^n\varphi_n(\pi)+\varrho^{n-1}\varphi_{n-1}(\pi)+\ldots+\varrho^m\varphi_m(\pi)+\ldots+\varrho\varphi_1\pi+\varphi_0(\pi).$$

Then $\varphi_m(\pi)$ is given by the following rule:—

Write down the following symbolical product:

$$\theta_1(\pi+m)\theta_2(\pi+m)\theta_3(\pi+m)\ldots\ldots\theta_{n-m}(\pi+m);$$

take every possible combination of the quantities $1, 2, 3, \ldots n$ taken $(n-m)$ at a time, and substitute them as the weight* of θ in this continued product, diminishing (m) in each factor by the increment of the weight of the factor; add all the results together, and we obtain the value of $\varphi_m(\pi)$. The truth of this rule will be manifest to every one who will consider the following result obtained by actual multiplication:—

$$\begin{aligned}
&(\varrho+\theta_1\pi)(\varrho+\theta_2\pi)(\varrho+\theta_3\pi)(\varrho+\theta_4\pi)(\varrho+\theta_5\pi)=\varrho^5\\
&+\varrho^4\{\theta_1(\pi+4)+\theta_2(\pi+3)+\theta_3(\pi+2)+\theta_4(\pi+1)+\theta_5(\pi)\}\\
&+\varrho^3\{\theta_1(\pi+3)\theta_2(\pi+3)+\theta_1(\pi+3)\theta_3(\pi+2)+\theta_2(\pi+2)\theta_3(\pi+2)+\theta_1(\pi+3)\theta_4(\pi+1)\\
&+\theta_2(\pi+2)\theta_4(\pi+1)+\theta_3(\pi+1)\theta_4(\pi+1)+\theta_1(\pi+3)\theta_5\pi+\theta_2(\pi+2)\theta_5(\pi)+\theta_3(\pi+1)\theta_5\pi+\theta_4\pi\theta_5\pi\}\\
&+\varrho^2\{\theta_1(\pi+2)\theta_2(\pi+2)\theta_3(\pi+2)+\theta_1(\pi+2)\theta_2(\pi+2)\theta_4(\pi+1)+\theta_1(\pi+2)\theta_3(\pi+1)\theta_4(\pi+1)\\
&+\theta_2(\pi+1)\theta_3(\pi+1)\theta_4(\pi+1)+\theta_1(\pi+2)\theta_2(\pi+2)\theta_5\pi+\theta_1(\pi+2)\theta_3(\pi+1)\theta_5\pi\\
&+\theta_2(\pi+1)\theta_3(\pi+1)\theta_5\pi+\theta_1(\pi+2)\theta_4\pi\theta_5\pi+\theta_2(\pi+1)\theta_4\pi\theta_5\pi+\theta_3\pi\theta_4\pi\theta_5\pi\}\\
&+\varrho\{\theta_1(\pi+1)\theta_2(\pi+1)\theta_3(\pi+1)\theta_4(\pi+1)+\theta_1(\pi+1)\theta_2(\pi+1)\theta_3(\pi+1)\theta_5\pi\\
&+\theta_1(\pi+1)\theta_2(\pi+1)\theta_4\pi\theta_5\pi+\theta_1(\pi+1)\theta_3\pi\theta_4\pi\theta_5\pi+\theta_2\pi\theta_3\pi\theta_4\pi\theta_5\pi\}\\
&+\theta_1\pi\theta_2\pi\theta_3\pi\theta_4\pi\theta_5\pi.
\end{aligned}$$

I now come to the investigation of the two new forms of the binomial theorem as explained in the former memoir.

It is evident, in the first place, that in multiplying any binomial $(\varrho^2+\varrho\theta(\pi))^n$ by $\varrho^2+\varrho\theta(\pi)$, the result in this case will be the same whether we employ internal or external multiplication.

Let

$$(\varrho^2+\varrho\theta(\pi))^n=\varrho^{2n}+\varrho^{2n-1}\varphi_{2n-1}(\pi)+\varrho^{2n-2}\varphi_{2n-2}\pi+\ldots,$$

where $\varphi_{2n-1}(\pi)$, $\varphi_{2n-2}(\pi)$, $\varphi_{2n-3}(\pi)$ are unknown functions of (π) which we seek to determine.

Then multiplying externally and internally by $(\varrho^2+\varrho\theta(\pi))$

$$\begin{aligned}
(\varrho^2+\varrho\theta(\pi))^{n+1}&=\varrho^{2n+2}+\varrho^{2n+1}\varphi_{2n-1}(\pi)\quad+\varrho^{2n}\varphi_{2n-2}(\pi)\\
&\qquad+\varrho^{2n+1}\theta(\pi+2n)\quad+\varrho^{2n}\theta(\pi+2n-1)\varphi_{2n-1}(\pi)+\ldots\\
&=\varrho^{2n+2}+\varrho^{2n+1}\varphi_{2n-1}(\pi+2)+\varrho^{2n}\varphi_{2n-2}(\pi+2)+\ldots\\
&\qquad+\varrho^{2n+1}\theta(\pi)\qquad+\varrho^{2n}\theta\pi\varphi_{2n-1}(\pi+1)+\ldots
\end{aligned}$$

* The use I have here made of the term 'weight' will be familiar to every one who is conversant with the modern Higher Algebra.

From whence, by equating the coefficients of (ϱ),

$$\phi_{2n-1}(\pi)+\theta(\pi+2n)=\phi_{2n-1}(\pi+2)+\theta(\pi),$$

$$\therefore \quad \left(\varepsilon^{2\frac{d}{d\pi}}-1\right)\phi_{2n-1}(\pi)=\left(\varepsilon^{2n\frac{d}{d\pi}}-1\right)\theta(\pi),$$

whence

$$\phi_{2n-1}(\pi)=\frac{\varepsilon^{2n\frac{d}{d\pi}}-1}{\varepsilon^{2\frac{d}{d\pi}}-1}\theta(\pi).$$

Again,

$$\phi_{2n-2}(\pi)+\theta(\pi+2n-1)\phi_{2n-1}(\pi)=\phi_{2n-2}(\pi+2)+\phi_{2n-1}(\pi+1)\theta\pi,$$

$$\therefore \quad \phi_{2n-2}(\pi)=\frac{1}{\varepsilon^{2\frac{d}{d\pi}}-1}\left\{\frac{\varepsilon^{2n\frac{d}{d\pi}}-1}{\varepsilon^{2\frac{d}{d\pi}}-1}\theta(\pi)\right\}\left\{\varepsilon^{(2n-1)\frac{d}{d\pi}}\theta\pi\right\}-\frac{1}{\varepsilon^{2\frac{d}{d\pi}}-1}\left\{\frac{\varepsilon^{(2n+1)\frac{d}{d\pi}}-\varepsilon^{\frac{d}{d\pi}}}{\varepsilon^{2\frac{d}{d\pi}}-1}\theta\pi\right\}\theta\pi.$$

We shall now investigate another form of this expansion, by which we shall be able to obtain a remarkable expression for the general term. We shall express the unknown functions by a notation slightly differing from that which we have just employed. The reason for so doing will be easily seen by the reader.

Let

$$(\varrho^2+\varrho\theta\pi)^n=\varrho^{2n}+\varrho^{2n-1}\phi_1^{(n)}\pi+\varrho^{2n-2}\phi_2^{(n)}(\pi)+\ldots$$

Then

$$(\varrho^2+\varrho\theta(\pi))^{n+1}=\varrho^{2n+2}+\varrho^{2n+1}\phi_1^{(n)}(\pi+2)+\varrho^{2n}\phi_2^{(n)}(\pi+2)+\ldots$$
$$+\varrho^{2n+1}\theta(\pi)+\varrho^{2n}\theta(\pi)\phi_1^{(n)}(\pi+1)+\ldots$$
$$=\varrho^{2n+2}+\varrho^{2n+1}\phi_1^{(n+1)}(\pi)+\varrho^{2n}\phi_2^{(n+1)}(\pi)+\ldots;$$

$$\therefore \quad \phi_1^{(n+1)}(\pi)=\phi_1^{(n)}(\pi+2)+\theta(\pi),$$

or

$$\phi_1^{(n+1)}(\pi)-\varepsilon^{2\frac{d}{d\pi}}\phi_1^{(n)}\pi=\theta(\pi).$$

Wherefore, solving this equation in finite differences, we have

$$\phi_1^{(n)}(\pi)=\varepsilon^{2(n-1)\frac{d}{d\pi}}\Sigma\varepsilon^{-2n\frac{d}{d\pi}}\theta(\pi).$$

Again,

$$\phi_2^{(n+1)}\pi=\phi_2^{(n)}(\pi+2)+\phi_1^{(n)}(\pi+1)\theta\pi,$$

whence

$$\phi_2^{(n+1)}\pi-\varepsilon^{2\frac{d}{d\pi}}\phi_2^{(n)}\pi=\theta\pi\varepsilon^{\frac{d}{d\pi}}\phi_1^{(n)}\pi.$$

Hence

$$\phi_2^{(n)}(\pi)=\left\{\varepsilon^{2(n-1)\frac{d}{d\pi}}\Sigma\varepsilon^{-2n\frac{d}{d\pi}}\right\}\left(\theta(\pi)\varepsilon^{\frac{d}{d\pi}}\right)\left\{\varepsilon^{2(n-1)\frac{d}{d\pi}}\Sigma\varepsilon^{-2n\frac{d}{d\pi}}\right\}\theta(\pi)$$
$$=\varepsilon^{-\frac{d}{d\pi}}\left\{\varepsilon^{(2n-1)\frac{d}{d\pi}}\Sigma\varepsilon^{-2n\frac{d}{d\pi}}\right\}\theta(\pi)\left\{\varepsilon^{(2n-1)\frac{d}{d\pi}}\Sigma\varepsilon^{-2n\frac{d}{d\pi}}\right\}\theta(\pi)$$
$$=\varepsilon^{-\frac{d}{d\pi}}\left\{\varepsilon^{(2n-1)\frac{d}{d\pi}}\Sigma\varepsilon^{-2n\frac{d}{d\pi}}\theta(\pi)\right\}^2;$$

and similarly,

$$\phi_r^{(n)}\pi=\varepsilon^{-\frac{d}{d\pi}}\left\{\varepsilon^{(2n-1)\frac{d}{d\pi}}\Sigma\varepsilon^{-2n\frac{d}{d\pi}}\theta(\pi)\right\}^r,$$

where, however, a proper correction must be added after each performance of the symbol Σ.

As there are some peculiarities connected with this form, I shall calculate its value at length for the values $r=1$ and $r=2$.

First, let $r=1$. Then

$$\phi_1^{(n)}(\pi)=\varepsilon^{2(n-1)\frac{d}{d\pi}}\Sigma\varepsilon^{-2n\frac{d}{d\pi}}\theta(\pi)$$

$$=\varepsilon^{2(n-1)\frac{d}{d\pi}}\Pi_1-\frac{\theta(\pi)}{\varepsilon^{2\frac{d}{d\pi}}-1}.$$

Let $n=1$. Then

$$\phi_1^{(1)}\pi=\theta\pi,\quad\text{and}\quad \Pi_1=\frac{\varepsilon^{2\frac{d}{d\pi}}\theta(\pi)}{\varepsilon^{2\frac{d}{d\pi}}-1}.$$

Consequently

$$\phi_1^{(n)}(\pi)=\frac{\varepsilon^{2n\frac{d}{d\pi}}-1}{\varepsilon^{2\frac{d}{d\pi}}-1}\theta(\pi),$$

which coincides with the value of the coefficient of ϱ^{2n-1} obtained by the former process.

Again,

$$\phi_2^{(n)}(\pi)=\varepsilon^{2(n-1)\frac{d}{d\pi}}\Sigma\varepsilon^{-2n\frac{d}{d\pi}}\theta(\pi)\varepsilon^{(2n-1)\frac{d}{d\pi}}\Sigma\varepsilon^{-2n\frac{d}{d\pi}}\theta(\pi)$$

$$=\varepsilon^{2(n-1)\frac{d}{d\pi}}\Sigma\varepsilon^{-2n\frac{d}{d\pi}}\theta(\pi)\varepsilon^{\frac{d}{d\pi}}\frac{\varepsilon^{2n\frac{d}{d\pi}}-1}{\varepsilon^{2\frac{d}{d\pi}}-1}\theta(\pi)$$

$$=\varepsilon^{2(n-1)\frac{d}{d\pi}}\Sigma\theta(\pi-2n)\varepsilon^{\frac{d}{d\pi}}.\frac{\varepsilon^{2n\frac{d}{d\pi}}-1}{\varepsilon^{2\frac{d}{d\pi}}-1}\theta(\pi-2n)$$

$$=\varepsilon^{2(n-1)\frac{d}{d\pi}}\Sigma\left(\varepsilon^{-2n\frac{d}{d\pi}}\theta(\pi)\right)\varepsilon^{\frac{d}{d\pi}}\frac{\varepsilon^{2n\frac{d}{d\pi}}-1}{\varepsilon^{2\frac{d}{d\pi}}-1}\left(\varepsilon^{-2n\frac{d}{d\pi}}\theta(\pi)\right)$$

$$=\varepsilon^{2(n-1)\frac{d}{d\pi}}\Sigma\left\{\left(\varepsilon^{-2n\frac{d}{d\pi}}\theta(\pi)\right)\left(\frac{\varepsilon^{\frac{d}{d\pi}}}{\varepsilon^{2\frac{d}{d\pi}}-1}\theta(\pi)\right)\right\}$$

$$-\varepsilon^{2(n-1)\frac{d}{d\pi}}\Sigma\varepsilon^{-2n\frac{d}{d\pi}}\left\{\theta(\pi).\frac{\varepsilon^{\frac{d}{d\pi}}}{\varepsilon^{2\frac{d}{d\pi}}-1}\theta(\pi)\right\}$$

$$=\varepsilon^{2(n-1)\frac{d}{d\pi}}\Pi_2+\varepsilon^{2(n-1)\frac{d}{d\pi}}\left(\frac{\varepsilon^{-2n\frac{d}{d\pi}}}{\varepsilon^{-2\frac{d}{d\pi}}-1}\theta(\pi)\right)\left(\frac{\varepsilon^{\frac{d}{d\pi}}}{\varepsilon^{2\frac{d}{d\pi}}-1}\theta(\pi)\right)$$

$$-\varepsilon^{2(n-1)\frac{d}{d\pi}}\frac{\varepsilon^{-2n\frac{d}{d\pi}}}{\varepsilon^{-2\frac{d}{d\pi}}-1}.\left(\theta(\pi)\frac{\varepsilon^{\frac{d}{d\pi}}}{\varepsilon^{2\frac{d}{d\pi}}-1}\theta(\pi)\right)$$

$$=\varepsilon^{2(n-1)\frac{d}{d\pi}}\Pi_2-\left(\frac{\theta(\pi)}{\varepsilon^{2\frac{d}{d\pi}}-1}\right)\left(\frac{\varepsilon^{(2n-1)\frac{d}{d\pi}}}{\varepsilon^{2\frac{d}{d\pi}}-1}\theta(\pi)\right)$$

$$+\frac{1}{\varepsilon^{2\frac{d}{d\pi}}-1}\left(\theta(\pi)\frac{\varepsilon^{\frac{d}{d\pi}}}{\varepsilon^{2\frac{d}{d\pi}}-1}\theta(\pi)\right),$$

where Π_2 is a function of (π) to be determined.

For this purpose put $n=1$, then $\varphi_2^{(n)}(\pi)=0$, and

$$\Pi_2=\left(\frac{\theta(\pi)}{\varepsilon^{2\frac{d}{d\pi}}-1}\right)\left(\frac{\varepsilon^{\frac{d}{d\pi}}\theta(\pi)}{\varepsilon^{2\frac{d}{d\pi}}-1}\right)-\frac{1}{\varepsilon^{2\frac{d}{d\pi}}-1}\left(\theta(\pi)\frac{\varepsilon^{\frac{d}{d\pi}}}{\varepsilon^{2\frac{d}{d\pi}}-1}\theta(\pi)\right);$$

$$\therefore\ \varphi_1^{(n)}(\pi)=\left(\frac{\varepsilon^{2(n-1)\frac{d}{d\pi}}-1}{\varepsilon^{2\frac{d}{d\pi}}-1}\theta(\pi)\right)\left(\frac{\varepsilon^{(2n-1)\frac{d}{d\pi}}}{\varepsilon^{2\frac{d}{d\pi}}-1}\theta(\pi)\right)-\frac{\varepsilon^{2(n-1)\frac{d}{d\pi}}-1}{\varepsilon^{2\frac{d}{d\pi}}-1}\left(\theta(\pi)\frac{\varepsilon^{\frac{d}{d\pi}}}{\varepsilon^{2\frac{d}{d\pi}}-1}\theta(\pi)\right).$$

We next proceed to show the identity of this value of the coefficient of ϱ^{2n-2} with that formerly obtained.

The truth of the following theorem is easily seen:—

$$\left(\varepsilon^{2\frac{d}{d\pi}}-1\right)f_1(\pi)f_2(\pi)=\left(\varepsilon^{2\frac{d}{d\pi}}-1\right)f_1(\pi)\left(\varepsilon^{2\frac{d}{d\pi}}-1\right)f_2(\pi)+f_2(\pi)\left(\varepsilon^{2\frac{d}{d\pi}}-1\right)f_1(\pi)+f_1(\pi)\left(\varepsilon^{2\frac{d}{d\pi}}-1\right)f_2(\pi).$$

Hence

$$\left(\varepsilon^{2\frac{d}{d\pi}}-1\right)\varphi_1^n\pi=\left\{\left(\varepsilon^{2(n-1)\frac{d}{d\pi}}-1\right)\theta(\pi)\varepsilon^{(2n-1)\frac{d}{d\pi}}\theta(\pi)\right\}+\left\{\left(\varepsilon^{2(n-1)\frac{d}{d\pi}}-1\right)\theta(\pi)\right\}\left\{\frac{\varepsilon^{(2n-1)\frac{d}{d\pi}}}{\varepsilon^{2\frac{d}{d\pi}}-1}\theta(\pi)\right\}$$

$$+\left\{\left(\varepsilon^{(2n-1)\frac{d}{d\pi}}\right)\theta\pi\right\}\left\{\left(\frac{\varepsilon^{2(n-1)\frac{d}{d\pi}}-1}{\varepsilon^{2\frac{d}{d\pi}}-1}\right)\theta(\pi)\right\}-\left(\varepsilon^{2(n-1)\frac{d}{d\pi}}-1\right)\left(\theta(\pi)\frac{\varepsilon^{\frac{d}{d\pi}}}{\varepsilon^{2\frac{d}{d\pi}}-1}\theta(\pi)\right).$$

But

$$\left(\varepsilon^{2(n-1)\frac{d}{d\pi}}-1\right)\left(\theta\pi.\frac{\varepsilon^{\frac{d}{d\pi}}}{\varepsilon^{2\frac{d}{d\pi}}-1}\theta(\pi)\right)=\left(\varepsilon^{2(n-1)\frac{d}{d\pi}}\theta(\pi)\right)\left(\frac{\varepsilon^{(2n-1)\frac{d}{d\pi}}}{\varepsilon^{2\frac{d}{d\pi}}-1}\theta(\pi)\right)-\theta\pi\frac{\varepsilon^{\frac{d}{d\pi}}}{\varepsilon^{2\frac{d}{d\pi}}-1}\theta(\pi);$$

$$\therefore\quad \left(\varepsilon^{2\frac{d}{d\pi}}-1\right)\varphi_1^{(n)}\pi=\left(\varepsilon^{2(n-1)\frac{d}{d\pi}}\theta(\pi)\right)\left(\varepsilon^{(2n-1)\frac{d}{d\pi}}\theta(\pi)\right)+\left(\frac{\varepsilon^{2(n-1)\frac{d}{d\pi}}-1}{\varepsilon^{2\frac{d}{d\pi}}-1}\theta(\pi)\right)\left(\varepsilon^{(2n-1)\frac{d}{d\pi}}\theta(\pi)\right)$$

$$-\theta(\pi)\varepsilon^{(2n-1)\frac{d}{d\pi}}\theta\pi-\theta(\pi).\frac{\varepsilon^{(2n-1)\frac{d}{d\pi}}}{\varepsilon^{2\frac{d}{d\pi}}-1}\theta\pi+\theta(\pi).\frac{\varepsilon^{\frac{d}{d\pi}}}{\varepsilon^{2\frac{d}{d\pi}}-1}\theta\pi$$

$$=\left\{\frac{\varepsilon^{2n\frac{d}{d\pi}}-1}{\varepsilon^{2\frac{d}{d\pi}}-1}\theta\pi\right\}\left\{\varepsilon^{(2n-1)\frac{d}{d\pi}}\theta(\pi)\right\}-\left\{\frac{\varepsilon^{(2n+1)\frac{d}{d\pi}}-\varepsilon^{\frac{d}{d\pi}}}{\varepsilon^{2\frac{d}{d\pi}}-1}\theta(\pi)\right\}\theta(\pi),$$

or

$$\varphi_1^{(n)}\pi=\frac{1}{\varepsilon^{2\frac{d}{d\pi}}-1}\left\{\frac{\varepsilon^{2n\frac{d}{d\pi}}-1}{\varepsilon^{2\frac{d}{d\pi}}-1}\theta\pi\right\}\left\{\varepsilon^{(2n-1)\frac{d}{d\pi}}\theta\pi\right\}-\frac{1}{\varepsilon^{2\frac{d}{d\pi}}-1}\left\{\frac{\varepsilon^{(2n+1)\frac{d}{d\pi}}-\varepsilon^{\frac{d}{d\pi}}}{\varepsilon^{2\frac{d}{d\pi}}-1}\theta(\pi)\right\}\theta\pi,$$

which agrees with the value of the symbolical coefficient of ϱ^{2n-2} as before obtained.

It is proper to add that the same method of investigation applies to all binomials of the form $(\varrho^{\alpha}+\varrho^{\beta}\theta(\pi))^{n}$, of which I have, for the sake of simplicity, selected the case $(\varrho^{2}+\varrho\theta\pi)^{n}$.

I now come to the calculation of the coefficients of the general term of the form of the binomial theorem as given in the first memoir.

Let us assume

$$\begin{aligned}(\varrho^2+\varrho\theta(\pi))^n= & \quad \varrho^{2n} + A_1^{(0)}\varrho^{2n-1} + A_2^{(0)}\varrho^{2n-2} + \&c. \\ & + (A_1^{(1)}\varrho^{2n-1} + A_2^{(1)}\varrho^{2n-2} + A_3^{(1)}\varrho^{2n-3} + \&c.)\pi \\ & + (A_1^{(2)}\varrho^{2n-1} + A_2^{(2)}\varrho^{2n-2} + A_3^{(2)}\varrho^{2n-3} + \&c.)\pi^2 + \&c. \\ & + (A_1^{(r)}\varrho^{2n-1} + A_2^{(r)}\varrho^{2n-2} + A_3^{(r)}\varrho^{2n-3} + \&c.)\pi^r + \&c.\end{aligned}$$

Then we have, writing $\theta_m(\pi)$ for $\frac{\theta^{(m)}(\pi)}{1.2.3\ldots.m}$,

$$A_1^{(0)}=\Sigma\theta(2n)$$

$$A_2^{(0)}=\Sigma\theta(2n-1)\Sigma\theta(2n)$$

$$A_3^{(0)}=\Sigma\theta(2n-2)\Sigma\theta(2n-1)\Sigma\theta 2n$$

$$\&c.=\&c.$$

$$A_s^{(0)}=\Sigma\theta(2n-s+1)\Sigma\theta(2n-s+2)\ldots.\Sigma\theta 2n$$

$$A_{(1)}^{(1)}=\Sigma\theta_1(2n)$$

$$A_2^{(1)}=\Sigma\theta(2n-1)\Sigma\theta_1(2n)+\Sigma\theta_1(2n-1)\Sigma\theta(2n)$$

$$\begin{aligned}A_3^{(1)}= & \Sigma\theta_1(2n-2)\Sigma\theta(2n-1)\Sigma\theta(2n) \\ & +\Sigma\theta(2n-2)\Sigma\theta_1(2n-1)\Sigma\theta(2n) \\ & +\Sigma\theta(2n-2)\Sigma\theta(2n-1)\Sigma\theta_1(2n)\end{aligned}$$

$$\&c.=\&c.$$

$$\begin{aligned}A_s^{(1)}= & \Sigma\theta_1(2n-s+1)\Sigma\theta(2n-s+2)\ldots\Sigma\theta(2n) \\ & +\Sigma\theta(2n-s+1)\Sigma\theta_1(2n-s+2)\ldots\Sigma\theta(2n) \\ & +\ldots \\ & +\Sigma\theta(2n-s+1)\Sigma\theta(2n-s+2)\ldots\Sigma\theta_1(2n)\end{aligned}$$

$$A_1^{(2)}=\Sigma\theta_2(2n)$$

$$\begin{aligned}A_2^{(2)}= & \Sigma\theta_2(2n-1)\Sigma\theta 2n+\Sigma\theta_1(2n-1)\Sigma\theta_1(2n) \\ & +\Sigma\theta(2n-1)\Sigma\theta_2(2n)\end{aligned}$$

$$\begin{aligned}A_s^{(2)}= & \Sigma\theta_2(2n-s+1)\Sigma\theta(2n-s+2)\ldots\Sigma\theta(2n) \\ & +\Sigma\theta(2n-s+1)\Sigma\theta_2(2n-s+2)\ldots\Sigma\theta(2n) \\ & +\ldots \\ & +\Sigma\theta(2n-s+1)\Sigma\theta(2n-s+2)\ldots\Sigma\theta_2(2n) \\ & +\Sigma\theta_1(2n-s+1)\Sigma\theta_1(2n-s+2)\ldots\Sigma\theta(2n) \\ & +\Sigma\theta_1(2n-s+1)\Sigma\theta(2n-s+2)\ldots\Sigma\theta_1(2n) \\ & +\ldots\end{aligned}$$

Where there are (s) terms in the first part of this expression, and $s\cdot\frac{s-1}{2}$ in the second:

$$\begin{aligned}A_s^{(r)}= & \Sigma\theta_{\alpha_1}(2n-s+1)\Sigma\theta_{\beta_1}(2n-s+2)\Sigma\theta_{\gamma_1}(2n-s+3)\ldots\Sigma\theta_{\nu_1}(2n) \\ & +\Sigma\theta_{\alpha_2}(2n-s+1)\Sigma\theta_{\beta_2}(2n-s+2)\Sigma\theta_{\gamma_2}(2n-s+3)\ldots\Sigma\theta_{\nu_2}(2n) \\ & +\Sigma\theta_{\alpha_3}(2n-s+1)\Sigma\theta_{\beta_3}(2n-s+2)\Sigma\theta_{\gamma_2}(2n-s+3)\ldots\Sigma\theta_{\nu_2}(2n) \\ & +\ldots\end{aligned}$$

where $\alpha_1, \beta_1, \gamma_1, \ldots \nu, \alpha_2, \beta_2, \gamma_2, \ldots \nu_2$, &c. are all the whole numbers which satisfy the equation

$$\alpha+\beta+\gamma+\ldots+\nu=r.$$

In the preceding investigation we have used $\theta_m(\pi)$ as an abbreviation for $\frac{\theta^{(m)}\pi}{1.2.3\ldots m}$, where $\theta^{(m)}\pi$ is the mth function derived from $\theta(\pi)$. In the following investigation, it is proper to remark that $\theta_1(\pi)$, $\theta_2(\pi)$, $\theta_3(\pi)\ldots$ are any rational and entire functions of (π) whatever.

To find an expression for the general term of the multinomial theorem, by which

$$(\varrho^\alpha+\varrho^{\alpha-1}\theta_1(\pi)+\varrho^{\alpha-2}\theta_2(\pi)+\varrho^{\alpha-3}\theta_3(\pi)+\ldots)^n$$

is expanded in powers of (ϱ).

Let us assume

$$(\varrho^\alpha+\varrho^{\alpha-1}\theta_1(\pi)+\varrho^{\alpha-2}\theta_2(\pi)+\ldots)^n=\varrho^{\alpha n}+\varrho^{\alpha n-1}\varphi_1^{(n)}(\pi)+\varrho^{\alpha n-2}\varphi_2^{(n)}(\pi)+\varrho^{\alpha n-3}\varphi_3^{(n)}(\pi)+\ldots$$

Then multiplying internally by the factor

$$\varrho^\alpha+\varrho^{\alpha-1}\theta_1(\pi)+\varrho^{\alpha-2}\theta_2(\pi)+\varrho^{\alpha-3}\theta_3\pi+\ldots,$$

and equating coefficients of like powers of (ϱ), we have the following series of equations:—

$$\varphi_1^{(n+1)}(\pi)-\varphi_1^{(n)}(\pi+\alpha)=\theta_1(\pi),$$

$$\varphi_2^{(n+1)}(\pi)-\varphi_2^{(n)}(\pi+\alpha)=\theta_1(\pi)\varphi_1^{(n)}(\pi+\alpha-1)+\theta_2(\pi),$$

$$\varphi_3^{(n+1)}(\pi)-\varphi_3^{(n)}(\pi+\alpha)=\theta_1(\pi)\varphi_2^{(n)}(\pi+\alpha-1)+\theta_2(\pi)\varphi_1^{(n)}(\pi+\alpha-2)+\theta_3(\pi),$$

and thus we proceed: hence we have, putting the symbol $\varepsilon^{(n+1)\alpha\frac{d}{d\pi}}\Sigma\varepsilon^{-(n+1)\alpha\frac{d}{d\pi}}=\Pi$,

$$\varphi_1^{(n)}(\pi)=\varepsilon^{-\alpha\frac{d}{d\pi}}\Pi\theta_1\pi,$$

$$\varphi_2^{(n)}\pi=\varepsilon^{-\alpha\frac{d}{d\pi}}\Pi\theta_1(\pi)\varepsilon^{-\frac{d}{d\pi}}\Pi\theta_1(\pi)+\varepsilon^{-\alpha\frac{d}{d\pi}}\Pi\theta_2(\pi)$$

$$\varphi_3^{(n)}\pi=\varepsilon^{-\alpha\frac{d}{d\pi}}\Pi\theta_1(\pi)\varepsilon^{-\frac{d}{d\pi}}\Pi\theta_1(\pi)\varepsilon^{-\frac{d}{d\pi}}\Pi\theta_1\pi+\varepsilon^{-\alpha\frac{d}{d\pi}}\Pi\theta_1(\pi)\varepsilon^{-\frac{d}{d\pi}}\Pi\theta_2(\pi)+\varepsilon^{-\alpha\frac{d}{d\pi}}\Pi\theta_2(\pi)\varepsilon^{-2\frac{d}{d\pi}}\Pi\theta_1\pi+\varepsilon^{-\alpha\frac{d}{d\pi}}\Pi\theta_3\pi$$

$$\varphi_4^{(n)}(\pi)=\varepsilon^{-\alpha\frac{d}{d\pi}}\Pi\theta_1(\pi)\varepsilon^{-\frac{d}{d\pi}}\Pi\theta_1(\pi)\varepsilon^{-\frac{d}{d\pi}}\Pi\theta_1(\pi)\varepsilon^{-\frac{d}{d\pi}}\Pi\theta_1(\pi)$$

$$+\varepsilon^{-\alpha\frac{d}{d\pi}}\Pi\theta_1(\pi)\varepsilon^{-\frac{d}{d\pi}}\Pi\theta_1(\pi)\varepsilon^{-\frac{d}{d\pi}}\Pi\theta_2\pi+\varepsilon^{-\alpha\frac{d}{d\pi}}\Pi\theta_1\pi\varepsilon^{-\frac{d}{d\pi}}\Pi\theta_2(\pi)\varepsilon^{-2\frac{d}{d\pi}}\Pi\theta_1\pi+\varepsilon^{-\alpha\frac{d}{d\pi}}\Pi\theta_2(\pi)\varepsilon^{-\frac{d}{d\pi}}\Pi\theta_1\pi\varepsilon^{-\frac{d}{d\pi}}\Pi\theta_1\pi$$

$$+\varepsilon^{-\alpha\frac{d}{d\pi}}\Pi\theta_1(\pi)\varepsilon^{-\frac{d}{d\pi}}\Pi\theta_3(\pi)+\varepsilon^{-\alpha\frac{d}{d\pi}}\Pi\theta_3(\pi)\varepsilon^{-3\frac{d}{d\pi}}\Pi\theta_1(\pi)+\varepsilon^{-\alpha\frac{d}{d\pi}}\Pi\theta_2\pi\varepsilon^{-2\frac{d}{d\pi}}\Pi\theta_2\pi+\varepsilon^{-\alpha\frac{d}{d\pi}}\Pi\theta_4(\pi).$$

We easily see that the general term may be expressed thus: construct the formula

$$\varepsilon^{-\alpha\frac{d}{d\pi}}\Pi\theta_a(\pi)\varepsilon^{-a\frac{d}{d\pi}}\Pi\theta_b(\pi)\varepsilon^{-b\frac{d}{d\pi}}\Pi\theta_c(\pi)\varepsilon^{-c\frac{d}{d\pi}}\ldots\Pi\theta_e(\pi),$$

and give to $\alpha, \beta, \gamma, \ldots \nu$ all the values which satisfy the equation

$$a+b+c+\ldots+e=r,$$

where $\alpha n-r$ is the index of (ϱ). Then the sum of all the terms so formed will be the required result.

To determine the product of the factors

$$(\varrho+\chi_n(\pi))(\varrho+\chi_{n-1}(\pi))\ldots(\varrho+\chi(\pi))\ldots(\varrho+\chi_{n-1}(\pi))(\varrho+\chi_n\pi),$$

whence $\chi_1(\pi)$, $\chi_2(\pi)$, $\chi_3(\pi)$ emanate from each other after a given law.

Let
$$(\varrho+\chi_n(\pi))\dots(\varrho+\chi\pi)\dots\varrho+\chi_n(\pi)=\varrho^{2n+1}+\varrho^{2n}\phi_1^{(n)}\pi+\varrho^{2n-1}\phi_2^{(n)}\pi+\dots$$
Then multiplying internally and externally by $\varrho+\chi_{n+1}(\pi)$,
$$\begin{aligned}\varrho^{2n+3}&+\varrho^{2n+2}\phi_1^{(n+1)}\pi+\varrho^{2n+1}\phi_2^{(n+1)}\pi+\&c.\\ &=\varrho^{2n+3}+\varrho^{2n+2}\{\phi_1^{(n)}(\pi+1)+\chi_{n+1}(\pi+2n+2)+\chi_{n+1}(\pi)\}\\ &\quad+\varrho^{2n+1}\{\phi_2^{(n)}(\pi+1)+\phi_1^{(n)}(\pi+1)\chi_{n+1}(\pi+2n+1)\\ &\quad+\phi_1^{(n)}(\pi)\chi_{n+1}(\pi)+\chi_{n+1}(\pi+2n+1)\chi_{n+1}(\pi)\}.\end{aligned}$$
Hence
$$\phi_1^{(n+1)}\pi-\phi_1^{(n)}(\pi+1)=\chi_{n+1}(\pi+2n+2)+\chi_{n+1}\pi,$$
whence
$$\phi_1^{(n)}(\pi)=\varepsilon^{n\frac{d}{d\pi}}\Sigma\left(\varepsilon^{(n+1)\frac{d}{d\pi}}+\varepsilon^{-(n+1)\frac{d}{d\pi}}\right)\chi_{n+1}(\pi),$$
and also
$$\begin{aligned}\phi_2^{(n+1)}(\pi)-\phi_2^{(n)}(\pi+1)&=\phi_1^{(n)}(\pi+1)\chi_{n+1}(\pi+2n+1)\\ &\quad+\phi_1^{(n)}(\pi)\chi_{n+1}(\pi)+\chi_{n+1}(\pi+2n+1)\chi_{n+1}(\pi);\end{aligned}$$
$$\begin{aligned}\therefore\quad \phi_2^{(n)}(\pi)=\varepsilon^{n\frac{d}{d\pi}}\Sigma\varepsilon^{-(n+1)\frac{d}{d\pi}}&\left\{\varepsilon^{(2n+1)\frac{d}{d\pi}}\chi_{n+1}(\pi)\right\}\left\{\varepsilon^{n\frac{d}{d\pi}}\Sigma\left(\varepsilon^{(n+1)\frac{d}{d\pi}}-\varepsilon^{-(n+1)\frac{d}{d\pi}}\right)\varepsilon^{\frac{d}{d\pi}}\chi_{n+1}(\pi)\right\}\\ &+\varepsilon^{n\frac{d}{d\pi}}\Sigma\varepsilon^{-(n+1)\frac{d}{d\pi}}\chi_{n+1}(\pi)\varepsilon^{n\frac{d}{d\pi}}\Sigma\left(\varepsilon^{(n+1)\frac{d}{d\pi}}-\varepsilon^{-(n+1)\frac{d}{d\pi}}\right)\chi_{n+1}(\pi)\\ &+\varepsilon^{n\frac{d}{d\pi}}\Sigma\varepsilon^{-(n+1)\frac{d}{d\pi}}\chi_{n+1}(\pi)\left\{\varepsilon^{(2n+1)\frac{d}{d\pi}}\chi_{n+1}(\pi)\right\};\end{aligned}$$
and in like manner we find the values of the succeeding symbolical coefficients.

I now come to the form of the binomial theorem which is reciprocal to that previously investigated.

To expand $(\pi^2+\theta(\varrho).\pi)^n$ in powers of (π).

Let us assume
$$(\pi^2+\theta(\varrho).\pi)^n=\pi^{2n}+\phi_{2n-1}(\varrho).\pi^{2n-1}+\phi_{2n-2}(\varrho).\pi^{2n-2}+\dots,$$
we know that
$$\pi^r\theta(\varrho)=\theta(\varrho).\pi^r+r\left(\varrho\frac{d}{d\varrho}\right)\theta(\varrho).\pi^{r-1}+r.\frac{r-1}{2}\left(\varrho\frac{d}{d\varrho}\right)^2\theta\varrho\pi^{r-2}+\dots$$

Hence, multiplying internally and externally by $\pi^2+\theta(\varrho)\pi$, we shall have
$$\begin{aligned}&\pi^{2n+2}+\phi_{2n-1}(\varrho)\pi^{2n+1}+\phi_{2n-2}(\varrho).\pi^{2n}+\phi_{2n-3}(\varrho).\pi^{2n-1}+\dots\\ &\quad+\left\{\theta(\varrho)\pi^{2n}+2n\left(\varrho\frac{d}{d\varrho}\right)\theta(\varrho)\pi^{2n-1}+2n.\frac{2n-1}{2}\left(\varrho\frac{d}{d\varrho}\right)^2\theta(\varrho)\pi^{2n-2}+\dots\right\}\pi\\ &\quad+\phi_{2n-1}(\varrho)\left\{\theta(\varrho).\pi^{2n-1}+(2n-1)\left(\varrho\frac{d}{d\varrho}\right)\theta(\varrho)\pi^{2n-2}+\dots\right\}\pi\\ &\quad+\phi_{2n-2}(\varrho)\left\{\theta(\varrho).\pi^{2n-2}+(2n-2)\left(\varrho\frac{d}{d\varrho}\right)\theta(\varrho).\pi^{2n-3}+\dots\right\}\pi\\ &\quad+\&c.\\ &=\pi^{2n+2}+\left\{\phi_{2n-1}(\varrho).\pi^2+2\left(\varrho\frac{d}{d\varrho}\right)\phi_{2n-1}(\varrho).\pi+\left(\varrho\frac{d}{d\varrho}\right)^2\phi_{2n-1}(\varrho)\right\}\pi^{2n-1}\\ &\quad+\left\{\phi_{2n-2}(\varrho).\pi^2+2\left(\varrho\frac{d}{d\varrho}\right)\phi_{2n-2}(\varrho).\pi+\left(\varrho\frac{d}{d\varrho}\right)^2\phi_{2n-2}(\varrho)\right\}\pi^{2n-2}\end{aligned}$$

$$+\ \&c.\ +\theta\varrho\pi^{2n+1}+\theta(\varrho).\left(\phi_{2n-1}(\varrho).\pi+\left(\varrho\frac{d}{d\varrho}\right)\phi_{2n-1}(\varrho)\right)\pi^{2n-1}$$

$$+\theta(\varrho)\left(\phi_{2n-2}(\varrho).\pi+\left(\varrho\frac{d}{d\varrho}\right)\phi_{2n-2}(\varrho)\right)\pi^{2n-2}+\ \&c.$$

Hence equating coefficients of the powers of (ϱ),

$$\phi_{2n-2}(\varrho)+2n\left(\varrho\frac{d}{d\varrho}\right)\theta(\varrho).+\phi_{2n-1}(\varrho).\theta(\varrho)=2\left(\varrho\frac{d}{d\varrho}\right)\phi_{2n-1}(\varrho)+\phi_{2n-2}(\varrho)+\theta(\varrho)\phi_{2n-1}(\varrho),$$

whence

$$\phi_{2n-1}(\varrho)=n\theta(\varrho),$$

$$\phi_{2n-3}(\varrho)+2n.\frac{2n-1}{2}\left(\varrho\frac{d}{d\varrho}\right)^2\theta(\varrho)+(2n-1)\phi_{2n-1}(\varrho)\left(\varrho\frac{d}{d\varrho}\right)\theta(\varrho)+\phi_{2n-2}(\varrho)\theta\varrho$$

$$=\left(\varrho\frac{d}{d\varrho}\right)^2\phi_{2n-1}(\varrho)+2\left(\varrho\frac{d}{d\varrho}\right)\phi_{2n-2}(\varrho)+\theta(\varrho)\left(\varrho\frac{d}{d\varrho}\right)\phi_{2n-1}(\varrho)+\theta(\varrho)\phi_{2n-2}(\varrho)+\phi_{2n-3}(\varrho),$$

whence

$$\phi_{2n-2}(\varrho)=n(n-1)\left(\varrho\frac{d}{d\varrho}\right)\theta(\varrho)+\frac{n(n-1)}{2}(\theta(\varrho))^2;$$

$$\therefore\quad(\pi^2+\theta(\varrho).\pi)^n=\pi^{2n}+n\theta\varrho.\pi^{2n-1}+\left\{n(n-1)\left(\varrho\frac{d}{d\varrho}\right)\theta(\varrho)+\frac{n(n-1)}{2}(\theta(\varrho))^2\right\}\pi^{2n-2}+\ldots,$$

the required expansion.