X. *On the Distribution of Surfaces of the Third Order into Species, in reference to the absence or presence of Singular Points, and the reality of their Lines. By Dr.* SCHLÄFLI, *Professor of Mathematics in the University of Berne. Communicated by* ARTHUR CAYLEY, *F.R.S.**

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THE theory of the 27 lines on a surface of the third order is due to Mr. CAYLEY and Dr. SALMON; and the effect, as regards the 27 lines, of a singular point or points on the surface was first considered by Dr. SALMON in the paper "On the triple tangent planes of a surface of the third order," Camb. and Dub. Math. Journ. vol. iv. pp. 252–260 (1849). The theory as regards the reality or non-reality of the lines on a general surface of the third order, is discussed in Dr. SCHLÄFLI'S paper, "An attempt to determine the 27 lines &c.," Quart. Math. Journ. vol. ii. pp. 56–65, and 110–120. This theory is reproduced and developed in the present memoir under the heading, I. General cubic surface of the third order and twelfth class; but the greater part of the memoir relates to the singular forms which are here first completely enumerated, and are considered under the headings II., III. &c. to XXII., viz. II. Cubic surface with a proper node, and therefore of the tenth class, &c., down to XXII. Ruled surface of the third order. Each of these families is discussed generally (that is, without regard to reality or non-reality), by means of a properly selected canonical form of equation; and for the most part, or in many instances, the reciprocal equation (or equation of the surface in plane-coordinates) is given, as also the equation of the Hessian surface and those of the Spinode curve; and it is further discussed and divided into species according to the reality or non-reality of its lines and planes. The following synopsis may be convenient:—

I. General cubic surface, or surface of the third order and twelfth class. Species I. 1, 2, 3, 4, 5.

II. Cubic surface with a proper node, and therefore of the tenth class. Species II. 1, 2, 3, 4, 5.

III. Cubic surface of the ninth class with a biplanar node. Species III. 1, 2, 3, 4.

IV. Cubic surface of the eighth class with two proper nodes. Species IV. 1, 2, 3, 4, 5, 6.

V. Cubic surface of the eighth class with a biplanar node. Species V. 1, 2, 3, 4.

* Dr. SCHLÄFLI authorized me to make any alterations in the phraseology of his memoir, and to add remarks which might appear to me desirable. Passages in [ ], or distinguished by my initials, are by me, but I have not thought it necessary to distinguish alterations which are merely verbal or of trifling importance.—A. C.

I. *General cubic surface, or surface of the third order and twelfth class.*

Art. 1. As the system of coordinates undergoes various transformations (sometimes imaginary ones), it becomes necessary to adhere to an invariable system of a real meaning, for instance the usual one of three rectangular coordinates. We shall call this the system of *fundamental coordinates*, and define it by the condition that the coordinates of every real point (or the ratios of them, if they be four in number) shall be real. Consequently any system of rational and integral equations, expressed in variables of a real meaning, and where all the coefficients are real, will be termed a *real system* (of equations), whether there be real solutions or none, provided that the number of equations do not exceed that of the variables, or of the quantities to be determined. The degree of the system will be the number of solutions of it when augmented by a sufficient number of arbitrary linear equations; and such degree will generally be the product of the degrees of the single equations. It is obvious that the system, whenever its degree is *odd*, represents a *real* continuum of as many dimensions as there are

independent variables; for instance, every real quaternary cubic represents a real surface.

It is known* that on the surface of the third order there are 27 lines which form 45 triangles in such manner that through each line there pass five planes meeting the surface in this line and two other lines, or say five triangle-planes. Lines not intersecting each other may be termed *independent lines*, as far as a surface of the third order is capable of containing all of them; the greatest number of such lines is four; that is to say, in whatever manner we may choose two, three, or four not intersecting lines on the surface, the system has always the same properties. Let two independent lines I. and II. on the surface be given, and imagine any one of the five triangle-planes passing through I.; then II. must intersect one of the two other sides of this triangle; in other words, this triangle affords a line cutting both I. and II., and a line cutting I. alone. Hence it appears that there are five lines cutting both I. and II., five lines cutting I. only, five lines cutting II. only, and ten lines cutting neither I. nor II.

[The theory of the 27 lines depends on the expression of the equation of the surface in the form $P-Q=0$, where P and Q are real or imaginary cubics breaking up into linear factors; in fact, if the equation be so expressed, it is at once seen that each of the planes $P=0$ meets each of the planes $Q=0$ in a line on the surface, so that the form gives at once 9 out of the 27 lines. The three planes represented by the equation $P=0$ (or $Q=0$) are termed a *Trihedral* of the surface.]

Art. 2. PROP. *It is always possible to find a trihedral represented by a real quaternary cubic.*

The truth of this proposition is evident when all the 27 lines are real. But when some of them are imaginary, these are conjugate by pairs. As the case when two conjugate lines intersect one another is fitter for our purpose, we begin with the other case when two conjugate lines do not intersect each other.

The problem, then, of finding the five lines intersecting such pair of conjugate lines depends on a real system. Hence among the five lines there will be an odd number of real ones; and imaginary ones, when existing, will be conjugate by pairs. Call the given two independent and conjugate lines I. and II., and the five lines intersecting each of them a, b, c, d, e. If d and e be imaginary and conjugate, the plane containing I. and d will be conjugate to that containing II. and e, and these two planes will not intersect in a line of the surface (for if they did, a line of the surface would unite the intersection of II. and d with that of I. and e; and it is obviously a great loss of generality if *three* lines of the surface meet in a point). But if all the five lines a, b, c, d, e be real, then—because they can be intersected simultaneously only by the lines I. and II., and because through each of the five lines there passes at least one real triangle-plane—*it must be possible to choose* among all the real planes each passing through any one of the real lines a, b, c, d, e, *two real triangle-planes not intersecting in a line of the surface.*

* See Cambridge and Dublin Math. Journ. vol. iv. p. 118, the original memoirs of Messrs. CAYLEY and SALMON on the triple tangent-planes of the cubic surface.

As to the easier case first mentioned, when there are on the surface two conjugate lines intersecting each other, it is plain at first sight that they afford us four pairs of conjugate triangle-planes not intersecting in a line of the surface.

Now whether we have two conjugate planes, or two real planes not intersecting in a line of the surface, the third plane completing them to a trihedral is singly determined by a real system and is therefore real; and hence the trihedral is represented by a real cubic.

Art. 3. PROP. *A real cubic surface of the twelfth class* (or, what is the same thing, without nodes) *can always be represented by* $\mathrm{uvw}+\mathrm{xyz}=0$, *where both* uvw *and* xyz *are real cubics breaking up into linear factors.*

Let $\lambda\mathrm{A}+\mathrm{B}=0$ be a cubic equation expressed in the fundamental coordinates with real coefficients, λ a numerical factor imaginary if possible, A, B cubics each decomposible into linear factors, but A real and B imaginary if possible, and let λ', B' be respectively conjugate to λ, B. Then $(\lambda-\lambda')\mathrm{A}+\mathrm{B}-\mathrm{B}'=0$ must be an identical equation, and each solution satisfying the system $\mathrm{A}=0$, $\mathrm{B}=0$ will therefore also satisfy $\mathrm{B}'=0$. But it would be a loss of generality if, through the nine lines in which the two trihedrals A and B intersect each other, there should pass a third trihedral B'. Therefore we must have $\lambda=\lambda'$, $\mathrm{B}=\mathrm{B}'$; in other words, if one trihedral of a pair is represented by a real cubic, its associate is also so, and the trihedral-pair equation does not imply any imaginary numerical factor. We are therefore justified in asserting that a real cubic surface (without nodes) can always be exhibited in one of these three trihedral-pair forms $uvw+xyz=0$; 1. u, v, w, x, y, z are all real; 2. u, v, w, x are real, y is conjugate to z; 3. u, x are real, v is conjugate to w, and y to z.

Art. 4. To save the reader the trouble of consulting my paper in vol. ii. of the Quarterly Mathematical Journal, I will give here a scheme which serves to determine and denote the twenty-seven lines. In space, only four linear functions can be independent; any fifth one will be a linear and homogeneous function of these four linear functions. Hence it is plain that in the identical equation

$$\mathrm{A}u+\mathrm{B}v+\mathrm{C}w+\mathrm{D}x+\mathrm{E}y+\mathrm{F}z=0$$

the coefficients are linear and homogeneous functions of two arbitrary constants; and of course only their ratio is here of importance. The identical equation

$$\mathrm{A}u(\mathrm{B}v+\mathrm{D}x)(\mathrm{C}w+\mathrm{D}x)+\mathrm{D}x(\mathrm{A}u+\mathrm{E}y)(\mathrm{A}u+\mathrm{F}z)=\mathrm{ABC}uvw+\mathrm{DEF}xyz$$

then suggests the propriety of making the six coefficients subject to the condition $\mathrm{ABC}=\mathrm{DEF}$, because we have then a transformation of the original trihedral-pair form into another like form. But the condition (being a cubic equation) has three roots, according to which we may put

$$\Sigma au=au+bv+cw+dx+ey+fz=0,\quad abc=def;$$
$$\Sigma a'u=0,\quad a'b'c'=d'e'f';\quad \Sigma a''u=0,\quad a''b''c''=d''e''f''.$$

We denote the line $(u=0,\ x=0)$ by $\overline{ux}$, and so on for all the nine lines arising from

the intersection of the two trihedrals uvw, xyz. Again, since there are twenty-seven forms of the equation of the surface such as

$$au(bv+dx)(cw+dx)+dx(au+ey)(au+fz)=0,$$

the equations $au+dx=0$, $bv+ey=0$, $cw+fz=0$ belong to a line of the surface which we denote by l, while (ux), $(ux)'$, $(ux)''$ respectively represent the triangle-planes

$$au+dx=0,\ a'u+d'x=0,\ a''u+d''x=0,$$

and so on. Now in the scheme of the nine initial lines

	x	y	z
u	$\overline{ux}$	$\overline{uy}$	$\overline{uz}$
v	$\overline{vx}$	$\overline{vy}$	$\overline{vz}$
w	$\overline{wx}$	$\overline{wy}$	$\overline{wz}$

we may first perform all the positive permutations of the columns, and then deduce from these the negative ones by permuting only y and z. In each permutation we keep in view only the lines placed in the principal diagonal. We thus obtain the following easily intelligible scheme

through	$\overline{ux}$,	$\overline{vy}$,	$\overline{wz}$	pass	l,	l',	l'',	through	$\overline{ux}$,	$\overline{vz}$,	$\overline{wy}$	pass	p,	p',	p'',
„	$\overline{uy}$,	$\overline{vz}$,	$\overline{wx}$	„	m,	m',	m'',	„	$\overline{uz}$,	$\overline{vy}$,	$\overline{wx}$	„	q,	q',	q'',
„	$\overline{uz}$,	$\overline{vx}$,	$\overline{wy}$	„	n,	n',	n'',	„	$\overline{uy}$,	$\overline{vx}$,	$\overline{wz}$	„	r,	r',	r''.

The plane (ux) contains the lines $\overline{ux}$, l, p, and so on; and the plane containing l, m', n'' may be represented by (lmn), and so on.

I do not think it worth while to show that the equation ABC=DEF, when explicitly written, always has real coefficients, and that each of the cases hereafter coming into consideration can be *constructed*, and that it therefore *exists*.

Art. 5. *First case.*—u, v, w, x, y, z are all of them real.

A. The cubic condition (ABC=DEF) has three real roots. It is then at once plain that all the twenty-seven lines and all the forty-five triangle-planes are real. *First species*, I., 1.

B. The cubic condition has but one real root, to which let belong the coefficients a, b, ... Each geometrical form then changes into its conjugate by merely permuting the two accents $'$ and $''$. So the nine initial lines and the six lines l, m, n, p, q, r (together fifteen) are real, and the remaining lines are imaginary and form a double-six

$$\begin{pmatrix} l', & m', & n', & p'', & q'', & r'' \\ l'', & m'', & n'', & p', & q', & r' \end{pmatrix}$$

where any two corresponding lines are also conjugate. Fifteen lines and fifteen planes are real. *Second species*, I., 2.

Art. 6. *Second case.*—y and z only are imaginary, and therefore conjugate.

A. The cubic condition has three real roots. Each form changes into its conjugate by merely permuting y and z. Therefore, in the trihedral-pair scheme, only the first column contains real lines, the two other columns are conjugate; and as to the eighteen remaining lines, their two schemes are conjugate in the above-mentioned order. Three lines and thirteen planes are real; for there is one real triangle through each side of which there pass, besides the plane of the triangle, four other real planes. *Fourth species*, I., 4.

B. The cubic condition has but one real root to which let belong the coefficients a, b, ... Each form changes into its conjugate one by permuting at once y, z and the two accents $'$ and $''$. Three lines and seven planes are real. The real lines form a triangle, through each side of which there pass, besides the plane of the triangle, two other real planes. *Fifth species*, same as third case B, *infra.*

Third case.—v is conjugate to w; y to z; and u, x are real.

A. The cubic condition has three real roots. Each form changes into its conjugate by permuting at once v, w, and y, z. The three above-mentioned schemes (each of nine lines) change hereby respectively into

$$\begin{array}{ccc|ccc|ccc} \overline{ux} & \overline{uz} & \overline{uy} & l & l' & l'' & p & p' & p'' \\ \overline{wx} & \overline{wz} & \overline{wy} & n & n' & n'' & r & r' & r'' \\ \overline{vx} & \overline{vz} & \overline{vy} & m & m' & m'' & q & q' & q''. \end{array}$$

The comparison shows that only $\overline{ux}$, l, l', l'', p, p', p'' keep their places, and are therefore real. Of the planes, only u, x, (ux), $(ux)'$, $(ux)''$ are real. Seven lines and five planes are real; namely, through a real line there pass five real planes, three of which, (ux), $(ux)'$, $(ux)''$, contain real triangles. *Third species*, I., 3.

B. The cubic condition has but one real root. To find the form conjugate to a given one, we must at once permute v, w, also y, z, and lastly the two accents $'$ and $''$. The three schemes of lines by this process become

$$\begin{array}{ccc|ccc|ccc} \overline{ux} & \overline{uz} & \overline{uy} & l & l'' & l' & p & p'' & p' \\ \overline{wx} & \overline{wz} & \overline{wy} & n & n'' & n' & r & r'' & r' \\ \overline{vx} & \overline{vz} & \overline{vy} & m & m'' & m' & q & q'' & q' \end{array}$$

Only $\overline{ux}$, l, p keep their places, and therefore are real. Besides the planes u, x (ux), also the planes (lmn), (lnm), (pqr), (prq) are real. The three real lines form a triangle, through each side of which there pass two more real planes. *Fifth species*, I., 5.

Art. 7. *How many kinds of nodes can exist on a cubic surface?*

Considering in the first instance the theory of an ordinary node or conical point, let us imagine a surface of the nth order with a node, at which we are allowed to place the point of reference $\frac{\partial}{\partial w}$*. Let then an arbitrary line be given, through which tangent

* As to this mode of expression, see foot-note to art. 8.—A. C.

planes to the surface are to pass, and through this line draw the planes of reference $z=0$ (through the node) and $w=0$ (not passing through the node). The equation of the surface will then take the form

$$F=Pw^{n-2}+Qw^{n-3}+Rw^{n-4}+\&c.=0,$$

where

$$P=(x, y, z)^2,\ Q=(x, y, z)^3,\ R=(x, y, z)^4,\ \&c.,$$

and the points of contact of tangent planes passing through the given line ($z=0$, $w=0$) must satisfy the conditions $\frac{\partial F}{\partial x}=0$, $\frac{\partial F}{\partial y}=0$. In the proximity of the node the system of the three equations reduces itself to $P=0$, $\frac{\partial P}{\partial x}=0$, $\frac{\partial P}{\partial y}=0$ $\left(\text{or, what is the same thing, } \frac{\partial P}{\partial x}=0,\ \frac{\partial P}{\partial y}=0,\ z\frac{\partial P}{\partial z}=0\right)$, if none of these equations be a necessary consequence of the other two. The node $\frac{\partial}{\partial w}$ then represents two solutions, because the equations are respectively of the degrees 2, 1, 1 [or, what is the same thing, among the tangent planes through the line the plane passing through the node counts for two tangent planes; that is, the class of the surface is diminished by 2]. The exception $\left(\frac{\partial P}{\partial x}=0,\ \frac{\partial P}{\partial y}=0,\ z=0\right)$ is inadmissible; for should the plane $z=0$ touch the cone $P=0$, the line ($z=0$, $w=0$) would not be arbitrarily chosen. The only possible exception is when the three equations

$$\frac{\partial P}{\partial x}=0,\ \frac{\partial P}{\partial y}=0,\ \frac{\partial P}{\partial z}=0$$

can be simultaneously satisfied. Consequently so long as the nodal cone $P=0$ does not break up into a pair of planes, there are two solutions, or the class is diminished by 2.

In the excepted case, where the nodal cone $P=0$ breaks up into a pair of planes, we may assume $P=xy$ (or $P=x^2$, to be discussed in the sequel); and since now the equations $xy=0$, $x=0$, $y=0$, are no longer independent, we must go on to consider also

$$Q=az^3+Lz^2+Mz+N,$$

where

$$L=(x, y),\quad M=(x, y)^2,\quad N=(x, y)^3.$$

For the sake of shortness, let $w=1$. We then have

$$xy+az^3+Lz^2+Mz+N+\&c.=0,$$

$$y\ +\frac{\partial L}{\partial x}z^2+\frac{\partial M}{\partial x}z+\frac{\partial N}{\partial x}+\&c.=0,$$

$$x\ +\frac{\partial L}{\partial y}z^2+\frac{\partial M}{\partial y}z+\frac{\partial N}{\partial y}+\&c.=0,$$

and unless the constant a vanish, the system (in the proximity of the node) reduces itself to $x=0$, $y=0$, $z^3=0$; that is to say, a biplanar node, in general, counts for three solutions, or diminishes the class by 3.

Next it remains to put $a=0$, $L=ax+by$, when the system becomes

$$x+bz^2+\ldots=0,\quad y+az^2+\ldots=0,\quad xy+(ax+by)z^2+\ldots+Kz^4+\ldots=0,$$

where Kz^4 is borrowed from R; and the last equation of the system reduces itself by means of the others to $(K-ab)z^4+\ldots.=0$. The node here unites four solutions, unless $K-ab$ should vanish; that is to say, if the nodal edge $(x=0, y=0)$ lie on the cone $Q=0$, the biplanar node lowers the class of the surface by 4, unless the portion of the surface surrounding the node be, in the first approximation, represented by the form $(x+bz^2)(y+az^2)+$ terms of the fifth order in regard to z, $=0$.

The further supposition would be $K-ab=0$; but let us now assume a cubic surface, that we may have $K=0$, and therefore $ab=0$. Selecting the case $b=0$, we put

$$Q=axz^2+(bx^2+cxy+dy^2)z+N,$$

whence

$$x+(cx+2dy)z+\ldots=0, \quad y+az^2+\ldots.=0,$$

or neglecting higher orders than here come into consideration, $y=-az^2$, $x=2adz^3$, whereby $F=0$ becomes $a^2dz^5+\ldots=0$, so that the system is reduced to $x=0$, $y=0$, $a^2dz^5=0$. That is to say, if one of the nodal planes touch the surface along the nodal edge, the biplanar node lowers the class of the surface by 5, unless the cone $Q=0$ have that line of contact either for a double line (if $a=0$), or for a line of inflexion (if $d=0$).

The exceptional supposition then to be made separates itself into $a=0$ and $d=0$. But $a=0$ would cause all the terms of F to be of the second degree, at least in respect to x, y, so that the surface would have $(x=0, y=0)$ for a double line. Assuming then $d=0$, we may put

$$Q=xz^2+(ax^2+bxy)z+cx^3+dx^2y+exy^2+fy^3,$$

when the system reduces itself to $x=0$, $y=0$, $-fz^6=0$. That is to say, if one of the nodal planes osculate the surface along the nodal edge, the biplanar node lowers the class by 6. Here we must stop; for if we suppose $f=0$, the cubic F becomes divisible by x.

We go on to the case where the nodal cone becomes a pair of coincident planes, or say where we have a uniplanar node. The equation of the surface is

$$F=x^2w+ay^3+3by^2z+3cyz^2+dz^3+x(ey^2+fyz+gz^2)+x^2(hy+jz)+Kx^3=0.$$

For indefinitely small values of x, y, z, the equation $\frac{\partial F}{\partial x}=0$ causes x to be of the second order in respect to y, z. The system of conditions for the point of contact (in the proximity of the node) of a tangent-plane passing through the line $(z=0, w=0)$ reduces itself therefore to

$$x=0, \quad ay^2+2byz+cz^2=0, \quad ay^3+3by^2z+3cyz^2+dz^3=0,$$

unless the discriminant of the last-mentioned cubic should vanish. Except in this case, the system shows that the node counts for six solutions of

$$\left(F=0, \quad \frac{\partial F}{\partial x}=0, \quad \frac{\partial F}{\partial y}=0\right),$$

or, what is the same thing, that a uniplanar node lowers in general the class by 6.

But if the binary cubic $ay^3+3by^2z+3cyz^2+dy^3$ contain a squared factor, we may denote

this by y^2, and then write

$$F=x^2w+ay^3+by^2z+(cy^2+dyz+ez^2)x=0$$

for the equation of the surface; for it is plain that we are allowed to disregard the subsequent terms divisible by x^2. On forming the equation in plane-coordinates, it is immediately seen that this surface is of the fifth class, unless $b=0$; that is, in the general case, the class is diminished by 7.

Lastly, if $b=0$, then we have

$$F=x^2w+ay^3+(cy^2+dyz+ez^2)x=0;$$

and by forming the equation in plane-coordinates, the surface would be found to be of the fourth class, that is, the class of the surface is diminished by 8.

A closer discussion of the last two cases is reserved for a fit occasion.

In the whole we are to distinguish eight kinds of nodes on the cubic surface: 1, the proper node, which lowers the class by *two*; 2, the biplanar node, where the nodal edge does not belong to the surface and which lowers the class by *three*; 3, the biplanar node, where a plane different from both nodal planes touches the surface along the nodal edge and which lowers the class by *four*; 4, the biplanar node, where one of the two nodal planes touches the surface along the nodal edge and which lowers the class by *five*; 5, the biplanar node, where one nodal plane osculates the surface along the nodal edge and which lowers the class by *six*; 6, the uniplanar node, where the nodal plane intersects the surface in three distinct lines and which lowers the class by six; 7, the uniplanar node, where the nodal plane touches the surface along a line and which lowers the class by *seven*; 8, the uniplanar node, where the nodal plane osculates the surface along a line and which lowers the class by *eight*.

Art. 8. *On the case of two nodes on the cubic surface.*

Let f be the quaternary cubic of the surface, P, Q the symbols* of two different nodes on it; then P^2f, Q^2f will identically vanish. If now R be the symbol of any third point, the symbol $\alpha P+\beta Q+\gamma R$, where α, β, γ denote arbitrary multipliers, will belong to a point in the same plane with the points P, Q, R, and the equation

$$(\alpha P+\beta Q+\gamma R)^3f=6\alpha\beta\gamma PQRf+3\gamma^2(\alpha P+\beta Q)R^2f+\gamma^3R^3f=0$$

will represent the section of the surface made by the plane. Then if the point R satisfy the condition $PQRf=0$, the equation will become divisible by γ^2, that is to say, the equation $PQRf=0$, in respect to the elements of R, represents a plane touching the surface along the line joining the nodes P and Q, and besides intersecting it in a line represented by

$$3\alpha PR^2f+3\beta QR^2f+\gamma R^3f=0,$$

if here α, β, γ are regarded as planimetrical coordinates, and the point R as fixed. In the sequel I shall sometimes term the former line *axis* and the latter *transversal*.

* If x', y', z', w' are the coordinates of a node, x', y', z', w' current coordinates, then the symbol P of the node is $=x'\partial_x+y'\partial_y+z'\partial_z+w'\partial_w$ and P^2f is $=P'f'=(x\partial_{x'}+y\partial_{y'}+z\partial_{z'}+w\partial_{w'})f'$, which vanishes identically, that is independently of x, y, z, w, in virtue of the equations $\partial_{x'}f'=0$, &c. satisfied at the node.—A. C.

If $Pw+Q=0$, where $P=(x, y, z)^2$, $Q=(x, y, z)^3$, be the equation of a cubic surface with a node, I shall call the six lines represented by the system $P=0$, $Q=0$ *nodal rays.* They belong to the surface, and it is plain that two of them at least must coincide in order that the surface may have another node, and this will lie on the line uniting two or more rays of the first node.

II. *Cubic surface with a proper node, and therefore of the tenth class.*

Art. 9. The equation of this surface can always be thrown into the form $Pw+Q=0$, where $P=(x, y, z)^2$, $Q=(x, y, z)^3$.

Let l be a linear and homogeneous function of x, y, z, then

$$P(w-l)+Q+lP=0$$

is the same equation. But we may in fifteen different ways dispose of the three coefficients in l, so that $Q+lP$ breaks up into three linear factors, and are therefore allowed to write

$$\varphi=(ax^2+by^2+cz^2+2dyz+2ezx+2fxy)w+2xyz=0$$

as an equation of the surface, where, for the sake of shortness, the ternary quadric ax^2+&c. of the nodal cone may be denoted by χ, and the derivatives of $\frac{1}{2}\chi$ by X, Y, Z. Again, let

$$\Delta=abc-ad^2-be^2-cf^2+2def,\quad A=bc-d^2,\quad B=ca-e^2,\quad C=ab-f^2,$$
$$D=ef-ad,\quad E=fd-be,\quad F=de-cf,$$

and determine the constant λ by the quadratic equation $(a\lambda-D)^2-BC=0$, then $\chi+2\lambda yz$ will break up into two linear factors, and $\varphi=(\chi+2\lambda yz)w+2(x-\lambda w)yz$ will be a trihedral-pair form of the surface. Its particularity is sufficiently determined by the condition that an edge of one trihedral intersects an edge of the other trihedral, the point of intersection being the node. I wished only to notice the connexion of such form with the presence of a proper node, yet will no longer dwell upon it, because I prefer to select hereafter one of those ten trihedral-pairs in which no plane passes through the node, for investigating by its help the position of the 27 lines.

Let p, q, r, s denote plane-coordinates such that $px'+qy'+rz'+sw'=0$ shall be the equation in point-coordinates x', y', z', w' of a tangent plane to the surface $\varphi=0$. To find then the *reciprocal equation* of the surface, we are concerned with the system

$$\varphi=0,\quad \frac{\partial\varphi}{\partial x}:\frac{\partial\varphi}{\partial y}:\frac{\partial\varphi}{\partial z}:\frac{\partial\varphi}{\partial w}=p:q:r:s,$$

where the first equation may also be replaced by $l+sw=px+qy+rz+sw=0$. The equations

$$\tfrac{1}{2}\frac{\partial\varphi}{\partial x}=Xw+yz,\quad \tfrac{1}{2}\frac{\partial\varphi}{\partial y}=Yw+xz,\quad \tfrac{1}{2}\frac{\partial\varphi}{\partial z}=Zw+xy,\quad \frac{\partial\varphi}{\partial w}=\chi$$

lead to the system

$$p\chi+2lX-2syz=0,\quad q\chi+2lY-2szx=0,\quad r\chi+2lZ-2sxy=0,$$

the equations whereof are the derivatives of

$$l\chi-2sxyz\left(=(px+qy+rz)\chi-2sxyz\right)=0$$

with respect to x, y, z. The reciprocal equation of the surface therefore is of the form $\Omega=0$, where Ω is a decimic function of (p, q, r, s), which multiplied by s^2 is the discriminant of the ternary cubic

$$\begin{aligned}&(3ap)x^3+3.(aq+2fp)x^2y+3.(ar+2ep)x^2z+3.(bp+2fq)xy^2\\&\quad+6.(dp+eq+fr-s)xyz+3.(cp+2er)xz^2+(3bq)y^3+3.(br+2dq)y^2z\\&\quad+3.(cq+2dr)yz^2+(3cr)z^3.\end{aligned}$$

Hence to work out the decimic in question we may use the invariants of the fourth and sixth order which Dr. SALMON* denotes by S and T, only that we replace the latter notation by $-8\mathrm{T}$. Putting, then,

$$\Phi=\mathrm{A}p^2+\mathrm{B}q^2+\mathrm{C}r^2+2\mathrm{D}qr+2\mathrm{E}rp+2\mathrm{F}pq,\quad \tfrac{1}{2}d\Phi=\mathrm{P}dp+\mathrm{Q}dq+\mathrm{R}dr,$$

$$t=dp+eq+fr,\quad \mathrm{U}=adqr+berp+cfpq,\quad \mathrm{V}=2\Delta pqr-aqr\mathrm{P}-brp\mathrm{Q}-cpq\mathrm{R},$$

$$\mathrm{W}=a^2\mathrm{A}q^2r^2+b^2\mathrm{B}r^2p^2+c^2\mathrm{C}p^2q^2+2pqr(bc\mathrm{D}p+ca\mathrm{E}q+ab\mathrm{F}r),$$

$$\mathrm{L}=s^2-2ts-\Phi,\quad \mathrm{M}=\mathrm{U}s+\mathrm{V},\quad \mathrm{N}=2abcpqrs+\mathrm{W},\quad \mathrm{S}^3-\mathrm{T}^2=108s^2\Omega,$$

we find

$$\mathrm{S}=\mathrm{L}^2-12s\mathrm{M},\quad \mathrm{T}=\mathrm{L}^3-18s\mathrm{LM}-54s^2\mathrm{N},$$

$$\begin{aligned}\Omega&=\mathrm{L}^3\mathrm{N}+\mathrm{L}^2\mathrm{M}^2-18s\mathrm{LMN}-16s\mathrm{M}^3-27s^2\mathrm{N}^2\\&=2abcpqrs^7+\{abc\Sigma aq^2r^2+2pqr\Sigma bc(2ef-7ad)p\}s^6\\&\quad+2\{\Sigma bc^2(ef-3ad)p^3q^2+pqr\Sigma bc(-3abc+21ad^2+be^2+cf^2-12def)p^2\\&\quad+2pqr\Sigma a(-8abcd+16bcef-6d(be^2+cf^2)+2d^2ef)qr\}s^5\\&\quad+(\ \ )s^4+(\ \ )s^3+(\ \ )s^2+(\ \ )s\\&\quad-\Delta\Phi^2(cq^2-2dqr+br^2)(ar^2-2erp+cp^2)(bp^2-2fpq+aq^2),\end{aligned}$$

and $\Omega=0$ is the reciprocal equation of the surface.

If $-16\mathrm{H}$ denote the Hessian of the cubic φ, then

$$\mathrm{H}=\Delta\chi w^2+2(\chi\Sigma\mathrm{D}x-3\Delta xyz)w-\Sigma a^2x^4+2\Sigma bcy^2z^2+4xyz\Sigma(ad+ef)x;$$

the spinode curve therefore is represented by the system

$$\varphi=0,\quad 8\Delta xyzw-8xyz\Sigma adx+\Sigma a^2x^4-2\Sigma bcy^2z^2=0;$$

hence it is a complete curve of the twelfth degree, and has the node of the cubic surface for a sixfold point, where the six nodal rays are tangents to the curve.

Art. 10. Starting from a trihedral-pair form $uvw+xyz=0$, where no four of the six

* Higher Plane Curves, pp. 184 and 186.

planes have a point in common, and letting $\mathfrak{D}$ be a linear differentiation symbol signifying that the differentials of the four fundamental coordinates may be replaced by arbitrary quantities ($\mathfrak{D}=\alpha\partial_x+\beta\partial_y+\gamma\partial_z+\delta\partial_w$, if for the moment (x, y, z, w) are the fundamental coordinates), we see that at the node the differential equation $\mathfrak{D}(uvw+xyz)=0$ ought not to be different from the general identical equation

$$A\mathfrak{D}u+B\mathfrak{D}v+C\mathfrak{D}w+D\mathfrak{D}x+E\mathfrak{D}y+F\mathfrak{D}z=0;$$

hence the coefficients of the differentials in both equations must be proportional. But since in the former the coefficients vw, uw, uv, yz, xz, xy satisfy the equation

$$vw.uw.uv=yz.xz.xy,$$

or, which is the same thing,

$$(uvw+xyz)(uvw-xyz)=0,$$

the coefficients in the latter differential equation belong to one of the roots of the well-known cubic condition. Let them, for instance, be a', b', c', d', e', f'; then in consequence of the equation of the surface the proportions in question become

$$a'u=b'v=c'w=-d'x=-e'y=-f'z;$$

or, because without any loss of generality (since the linear functions u, v, ... imply arbitrary numerical factors) we may replace a', b', c', d', e', f' by 1, 1, 1, 1, 1, 1, more simply

$$u=v=w=-x=-y=-z$$

at the node. Hence, and from the first and third identical relations, we get

$$a+b+c=d+e+f,\quad abc=def,\quad a''+b''+c''=d''+e''+f'',\quad a''b''c''=d''e''f''.$$

But we may put

$$a''=\lambda a+\mu,\quad b''=\lambda b+\mu, \text{ \&c.}$$

and we then obtain

$$(bc+ca+ab-ef-fd-de)\lambda^2\mu=0.$$

The factor within the brackets, if vanishing, would require one of the six cases such as $a=d$, $b=e$, $c=f$, and leave λ, μ indeterminate. Avoiding so great a restriction, and keeping to the proper meaning of the auxiliary cubic condition, we find that it has two equal roots $\lambda=0$, and a single root $\mu=0$. Consequently the constants corresponding to the single root are a, b, c, d, e, f, and satisfy the equations

$$a+b+c=d+e+f,\quad abc=def;$$

the constants in the accented sets are all of them equal to unity. Hence the line l' coincides with l'', m' with m'', and so on, and all these six pairs of coincident lines pass through the node. It may also be observed that they formed in the general case a double-six, and that now the corresponding lines (in both sixes) also coincide. Moreover, since the three independent lines l', m', n' (in the general case) are intersected by each of the three independent lines p'', q'', r'', all these six lines lie (in the general case) upon a quadratic surface; and now that all the six lines meet in a point, the quadratic surface must degenerate into a cone. Let

$$P=(v+x)(w+x)-(u+y)(u+z),\quad Q=(u+x)(u+y)(u+z),$$

then

$$u\mathrm{P}+\mathrm{Q}=ux(u+v+w+x+y+z)+(uvw+xyz);$$

and because $u+v+w+x+y+z=0$ is the second (or third) identical relation, and $uvw+xyz=0$ the equation of the surface, the latter is changed into $u\mathrm{P}+\mathrm{Q}=0$, which form shows the nodal cone $\mathrm{P}=0$, the equation of which may also be exhibited under the symmetrical form

$$vw+wu+uv-yz-zx-xy=0.$$

Art. 11. *Distribution into species.*—It is plain that a single node of a real cubic surface cannot but be a real point. We may therefore draw through it three (real) fundamental planes (which call x, y, z) and take the fourth plane at pleasure (call it w); the equation of the surface then is $w\mathrm{P}+\mathrm{Q}=0$, where the functions P, Q contain only x, y, z, and therefore represent cones respectively of the second and third orders; and it is obvious that as well in P as in Q all the coefficients will be real. Hence as to the six nodal rays ($\mathrm{P}=0$, $\mathrm{Q}=0$), all of them may be real, or four, or two, or none. So we might distinguish four species of the cubic surface with a single proper node; but in the last of the mentioned cases (when the node is an isolated point of the surface) the cone $\mathrm{P}=0$ may be real or imaginary. Let us therefore distinguish five species.

First species, II. 1. *All six nodal rays are real.*—The surface is constructed, when we assume six constants and six linear functions of the fundamental coordinates, all of them real, and satisfy the equations

$$a+b+c=d+e+f,\quad abc=def,\quad u+v+w+x+y+z=0,\quad au+bv+cw+dx+ey+fz=0,$$

where $bc+ca+ab-ef-fd-de$ must not vanish. Then $uvw+xyz=0$ is the equation of the surface. Not passing through the node, there are fifteen simple real lines, which form fifteen triangles, each line being common to three simple triangle-planes. Of the fifteen planes to be twice counted, each contains one of the simple lines and two nodal rays. This species constitutes the transition from the first to the second species of the general surface*.

Second species, II. 2. *Only four nodal rays are real.*—While we keep to the same system of equations as before, it is possible to dispose of the constants and linear functions in such manner that a, b, c are respectively conjugate to d, e, f, and u, v, w to x, y, z. Then by permuting i and $-i$, the three schemes

$$\begin{array}{ccc|ccc|ccc}
\overline{ux} & \overline{uy} & \overline{uz} & l & (l' & l'') & p & (p' & p'') \\
\overline{vx} & \overline{vy} & \overline{vz} & m & (m' & m'') & q & (q' & q'') \\
\overline{wx} & \overline{wy} & \overline{wz} & n & (n' & n'') & r & (r' & r'')
\end{array}$$

change into

$$\begin{array}{ccc|ccc|ccc}
\overline{ux} & \overline{vx} & \overline{wx} & l & (l' & l'') & p & (p' & p'') \\
\overline{uy} & \overline{vy} & \overline{wy} & n & (n' & n'') & q & (q' & q'') \\
\overline{uz} & \overline{vz} & \overline{wz} & m & (m' & m'') & r & (r' & r'')
\end{array}$$

* Viz. from I. 1 to I. 2, and so in other cases where the species of the general surface are referred to.—A. C.

Hence the four nodal rays (l', l''), (p', p''), (q', q''), (r', r'') and the remaining ones (m', m''), (n', n'') are conjugate. Of the simple lines seven only, viz. $\overline{ux}$, $\overline{vy}$, $\overline{wz}$, l, p, q, r are real and form three real triangles which have the line l in common. Besides these three simple planes there are seven real planes to be twice counted, each of which passes through the node and one of the seven real simple lines. When the two equal roots of the cubic condition separate themselves into real roots, the four real nodal rays become eight real lines, and the surface changes into the general one of the second species. In the other case, only the plane passing through the two conjugate nodal rays resolves itself into two real planes (in the former case into two conjugate planes), so that there arises a general surface of the third species.

Third species, II. 3. *Only two nodal rays are real.*—It is possible to satisfy the above system in such manner that the constants a, d are real, b conjugate to c, and e to f; again, that the planes u, x are real, v conjugate to w, and y to z. By the change of i into $-i$ the three original schemes then change into

$$\begin{array}{ccc|ccc|ccc} \overline{ux} & \overline{uz} & \overline{uy} & l & (l' & l'') & p & (p' & p'') \\ \overline{wx} & \overline{wz} & \overline{wy} & n & (n' & n'') & r & (r' & r'') \\ \overline{vx} & \overline{vz} & \overline{vy} & m & (m' & m'') & q & (q' & q'') \end{array}$$

The two nodal rays (l', l''), (p', p'') alone are real; and (not passing through the node) only the lines $\overline{ux}$, l, p, forming a triangle, are real. Besides the three simple planes u, x, (ux) the only real planes are the three planes (to be twice counted), which pass through the node and through one of the real simple lines. This case forms the transition from the third to the fifth species of the general surface.

Fourth and fifth species, II. 4, and II. 5. *Three pairs of conjugate nodal rays.*—The above system is compatible with the condition that e shall be conjugate to f, and the plane y to z, while all the other constants and planes are real. Then in the first of the three original schemes of lines the second and third columns interchange, and the second and third schemes interchange. Hence the nodal rays (l', l''), (m', m''), (n', n'') are respectively conjugate to (p', p''), (q', q''), (r', r''), and of the simple lines only $\overline{ux}$, $\overline{vx}$, $\overline{wx}$, forming a triangle, are real. Of simple planes only the seven x; u, (ux); v, (vx); w, (wx) are real, and of planes to be twice counted only those joining two conjugate nodal rays, therefore three in number. The case is intermediate between the fourth and fifth species of the general surface.

To decide the question, when is the nodal cone real or not? We throw its quadric $P=(v+x)(w+x)-(u+y)(u+z)$ into the form

$$-(d-b)(d-c)P=\{(d-c)(v+x)+(a-f)(u+y)\}\{(d-c)(v+x)+(a-e)(u+z)\}$$
$$+[(d-b)(d-c)-(a-e)(a-f)](u+y)(u+z).$$

On the right-hand side the first term is positive as a product of two conjugate factors, and in the second term $(u+y)(u+z)$ is positive for the same reason. Hence the cone is real when $(d-b)(d-c)-(a-e)(a-f)$ is negative; in the opposite case it is imaginary.

But if we eliminate a and d by the help of the equations

$$a+b+c=d+e+f,\ abc=def,$$

the expression becomes

$$(b-e)(b-f)(c-e)(c-f):(bc-ef),$$

where the numerator is positive, since its factors are conjugate by pairs. The nodal cone is therefore real when the denominator $bc-ef$ is *negative* (*fourth species*, II. 4), but imaginary when $bc-ef$ is *positive* (*fifth species*, II. 5).

III. *Cubic surface of the ninth class with a biplanar node.*

Art. 12. The equation $xyw+z^3=0$, where, in the proximity of the node, only w remains finite, when discussed under both suppositions of x, y being real or conjugate, gives a preliminary view of the biplanar node at the point $\frac{\partial}{\partial w}$. A plane turning about its edge ($x=0$, $y=0$) cuts the surface in a curve with a cusp, which changes its direction into the opposite one whenever the turning plane has passed one of the two real nodal planes; or always keeps its direction if the nodal planes be conjugate, so that in the latter case the surface here terminates in the form of a thorn [viz. in such a form as is generated by the revolution of a semicubical parabola about the cuspidal tangent].

The equation of the surface is $uvw+Q=0$, where u, v are linear functions and Q a cubic one of x, y, z. Denote the three nodal rays ($u=0$, $Q=0$ by 1, 2, 3, and the three ($v=0$, $Q=0$) by 4, 5, 6. Then each combination such as (14, 25, 36) gives a determinate position of the plane $w=0$, in virtue of which the cone Q breaks up into three planes. Keeping to the order of 1, 2, 3 and permuting only 4, 5, 6, we see there are six such transformations. But whenever $Q=xyz$, the surface contains a simple triangle ($w=0$, $xyz=0$); and it is also easy to see that the three positive permutations give one trihedral, and that three negative ones give the other trihedral of a trihedral-pair where no four of the six planes meet in a point, the only possible trihedral-pair of such kind.

If in art. 9 we put $\chi=2(lx+my+nz)(l'x+m'y+n'z)$,

$$\begin{vmatrix} l, & m, & n \\ l', & m', & n' \\ p, & q, & r \end{vmatrix} =\lambda p+\mu q+\nu r=\sigma,\ \Sigma ll'(mn'+m'n)qr=v,\ \Sigma ll'\lambda qr=\psi,$$

then we have

$$A=-\lambda^2,\ B=-\mu^2,\ C=-\nu^2,\ D=-\mu\nu,\ E=-\nu\lambda,\quad F=-\lambda\mu,$$

$$\Delta=0,\quad t=\Sigma(mn'+m'n)p,\ U=2v,\quad V=2\sigma\psi,\ W=-4\psi^2,\ L=s^2-2ts+\sigma^2,$$

$$M=2(vs+\sigma\psi),\quad N=4(4lmnl'm'n'pqrs-\psi^2),$$

$$\frac{\Omega}{4s}=4lmnl'm'n'pqr\{L^3-36sL(vs+\sigma\psi)+216s^2\psi^2-432lmnl'm'n'pqrs^3\}$$

$$+L^2\{s(v^2-\psi^2)+2\sigma v\psi+2t\psi^2\}+36L\psi^2(vs+\sigma\psi)-32(vs+\sigma\psi)^3-108s\psi^4.$$

The first term of the expression according to the descending powers of s is

$$4lmnl'm'n'pqrs^6,$$

and the last is

$$-4\lambda\mu\nu\sigma^3(nq-mr)(lr-np)(mp-lq)(n'q-m'r)(l'r-n'p)(m'p-l'q).$$

The system

$$(lx+my+nz)(l'x+m'y+n'z)w+xyz=0,$$
$$\Sigma l^2l'^2x^4-2\Sigma mm'nn'y^2z^2-4xyz\Sigma ll'(mn'+m'n)x=0$$

represents the spinode curve, which is therefore a complete curve of the twelfth degree and has the node for an eightfold point, where the tangents are determined by the system

$$(\Sigma lx)(\Sigma l'x)=0,\ \Sigma\lambda^2y^2z^2-2xyz\Sigma\mu\nu x=0,$$

since the cone drawn from the node through the spinode curve may also be thrown into the form

$$\Sigma lx\,.\,\Sigma l'x\,.\,\{\Sigma ll'x^2-\Sigma(mn'+m'n)yz\}+\Sigma\lambda^2y^2z^2-2xyz\Sigma\mu\nu x=0.$$

Art. 13. Let us represent by $uvw+xyz=0$ the only possible trihedral-pair no plane of which passes through the node, and considering this as a particular case of art. 10, let

$$u+v+w+x+y+z=0$$

be that identical relation which answers to the two equal roots which we know must exist of the cubic condition, and

$$\mathrm{A}u+\mathrm{B}v+\mathrm{C}w+\mathrm{D}x+\mathrm{E}y+\mathrm{F}z=0$$

any other identical relation. Then the coefficients in the relation corresponding to the single root of the cubic condition will be

$$a=\lambda\mathrm{A}+\mu,\ b=\lambda\mathrm{B}+\mu,\ \&c.;$$

and since this condition

$$(\lambda\mathrm{A}+\mu)(\lambda\mathrm{B}+\mu)(\lambda\mathrm{C}+\mu)-(\lambda\mathrm{D}+\mu)(\lambda\mathrm{E}+\mu)(\lambda\mathrm{F}+\mu)=0$$

must be divisible by λ^2, it follows

$$\mathrm{A}+\mathrm{B}+\mathrm{C}=\mathrm{D}+\mathrm{E}+\mathrm{F},\ a=(\mathrm{A}-\mathrm{D})(\mathrm{A}-\mathrm{E})(\mathrm{A}-\mathrm{F}),\ \&c.,\ d=(\mathrm{A}-\mathrm{D})(\mathrm{B}-\mathrm{D})(\mathrm{C}-\mathrm{D}),\ \&c.$$

Again, at the end of art. 11 we had a form of the nodal cone P containing only the three variables $v+x$, $u+y$, $u+z$, in respect to which the discriminant of P is

$$\frac{(b-d)(c-d)-(a-e)(a-f)}{(b-d)^2}=\frac{\Sigma\mathrm{BC}-\Sigma\mathrm{EF}}{(\mathrm{B}-\mathrm{D})^2}.$$

Now in order that the nodal cone may break up into two planes, we must have

$$\mathrm{BC}+\mathrm{CA}+\mathrm{AB}=\mathrm{EF}+\mathrm{FD}+\mathrm{DE},$$

which reduces the cubic condition to

$$(\mathrm{ABC}-\mathrm{DEF})\lambda^3=0.$$

Rejecting the solution

$$\mathrm{ABC}=\mathrm{DEF}$$

as giving rise to

$$\mathrm{A}=\mathrm{D}.\ \mathrm{B}=\mathrm{E}.\ \mathrm{C}=\mathrm{F}$$

for instance, and thus bringing

$$u+x=0,\ v+y=0,\ w+z=0$$

into one and the same plane, we infer that if a trihedral-pair form, explicitly not singular, belong to a cubic surface of the ninth class, the cubic condition inherent to such a trihedral-pair must have three equal roots.

Reciprocally, let $uvw+xyz=0$ be the equation of the surface, and

$$u+v+w+x+y+z=0,\ \mathrm{A}u+\mathrm{B}v+\mathrm{C}w+\mathrm{D}x+\mathrm{E}y+\mathrm{F}z=0$$

identical relations, where

$$\mathrm{A+B+C=D+E+F,\ BC+CA+AB=EF+FD+DE},$$

but where ABC$-$DEF is different from zero, then we have a set of proportions such as

$$\frac{\mathrm{A-E}}{\mathrm{B-D}}=\frac{\mathrm{C-D}}{\mathrm{A-F}};$$

and since at the node $u=v=w=-x=-y=-z$, the nodal cone is represented by

$$\left(\frac{\partial}{\partial u}+\frac{\partial}{\partial v}+\frac{\partial}{\partial w}-\frac{\partial}{\partial x}-\frac{\partial}{\partial y}-\frac{\partial}{\partial z}\right)(uvw-xyz)=vw+uw+uv-yz-zx-xy=0.$$

But because the equation

$$\begin{aligned}&\{(\mathrm{B-E})(u+x)-(\mathrm{A-D})(v+y)\}\{(\mathrm{B-F})(u+x)-(\mathrm{A-D})(v+z)\}\\ &=[(\mathrm{B-E})(\mathrm{B-F})-(\mathrm{A-D})(\mathrm{C-D})](u+x)^2\\ &-(\mathrm{A+B+C-D-E-F})(\mathrm{A-D})(u+x)(v+y+z)\\ &+(\mathrm{A-D})[(\mathrm{A-D})(u+v)+\mathrm{C}(u+x)](u+v+w+x+y+z)\\ &-(\mathrm{A-D})(u+x)(\mathrm{A}u+\mathrm{B}v+\mathrm{C}w+\mathrm{D}x+\mathrm{E}y+\mathrm{F}z)\\ &-(\mathrm{A-D})^2(vw+uw+uv-yz-xz-xy)\end{aligned}$$

is *explicitly* identical, therefore the equation

$$\begin{aligned}&-(\mathrm{A-D})^2(vw+wu+uv-yz-zx-xy)\\ &=\{(\mathrm{B-E})(u+x)-(\mathrm{A-D})(v+y)\}\{(\mathrm{B-F})(u+x)-(\mathrm{A-D})(v+z)\}\end{aligned}$$

is identical in respect to the fundamental coordinates; in other words, the nodal cone breaks up into a pair of planes. The nodal edge may be represented by

$$u=s+\mathrm{A}t,\ v=s+\mathrm{B}t,\ w=s+\mathrm{C}t,\ x=-s-\mathrm{D}t,\ y=-s-\mathrm{E}t,\ z=-s-\mathrm{F}t,$$

where s, t denote independent variables.

Now it is plain that the equation $u+x=0$, for instance, represents at once the three planes previously denoted by (ux), $(ux)'$, $(ux)''$, wherefore now the three lines l, l', l'' coincide, and so on. Each of the six nodal rays thus unites three (independent) lines of the surface; only the lines $uvw=0$, $xyz=0$ are simple lines. We have in all $6\cdot3+9\cdot1=27$ lines. One nodal plane unites all the six planes such as (lmn), and the other all the six planes such as (pqr). Of the nine planes joining any ray of the one nodal plane with any ray of the other, each unites three planes such as (ux), $(ux)'$, $(ux)''$;

only the six planes of the trihedral-pair here chosen are simple triangle-planes. There are in all $2 \cdot 6 + 9 \cdot 3 + 6 \cdot 1 = 45$ triangle-planes.

Art. 14. There are four species.

First species, III. 1.—u, v, w, x, y, z are real. Everything is then real.

Second species, III. 2.—u is conjugate to x, v to y, and w to z. Both nodal planes are real; one of them contains the real ray l and the two conjugate rays m, n; the other nodal plane contains the three real rays p, q, r. Of the nine simple lines three only, $\overline{ux}$, $\overline{vy}$, $\overline{wz}$, are real.

Third species, III. 3.—u, x are real, v is conjugate to w, y to z. Both nodal planes are real, and each of them contains a real and two conjugate rays; for l and p are real, and m is conjugate to n, q to r. Of the nine simple lines one only, $\overline{ux}$, is real.

Fourth species, III. 4.—u, v, w, x are real, y is conjugate to z. The two nodal planes are conjugate; for l, m, n are respectively conjugate to p, q, r. Of the nine simple lines three only, forming the triangle ($x=0$, $uvw=0$), are real.

The enumeration is complete, because all cases that can happen in respect to the nodal rays are exhausted.

IV. *Cubic surface of the eighth class with two proper nodes.*

Art. 15. From art. 8 we already know that the line joining the two nodes, or *axis*, unites two and the same rays of each node, and that there is a singular tangent plane which touches the surface, and therefore also each nodal cone along the *axis*, and besides intersects the surface in a single line which we have termed the *transversal*. Since then, besides the axis, each nodal cone has four rays not passing through the other node, there are in all ten nodal rays which represent twenty lines of the surface (considered as though it were general), so that there remain only seven simple lines, one of which is the transversal above mentioned. Because this transversal is not intersected by the eight disengaged nodal rays, but only by the axis, that is by four lines, it must meet all the six other lines, and will therefore form with them three triangles. Besides such triangle, there pass through each of the six lines four other planes, which are of course those passing through one or the other node, each of them counting for two triangle-planes. Again, a plane through the axis and a disengaged ray of one node must intersect the surface in a third line, which cannot but be a disengaged ray of the other node. Such plane counts for four triangle-planes; for any one of the four disengaged rays of one node, since it determines with each of the three remaining rays three triangle-planes, must determine with the axis two such planes; and because it is made up of two independent lines of the surface, the two planes must be twice counted. As to the singular tangent plane, it counts twice, because through the transversal there already pass three simple triangle-planes. The surface thus has a line representing four lines, viz. the axis; eight other nodal rays, each representing two lines; and seven simple lines, viz. the transversal and the remaining sides of the three simple triangles standing upon it; in all $1 \cdot 4 + 8 \cdot 2 + 7 \cdot 1 = 27$ lines. Again, the surface has four planes each representing

four triangle-planes of the surface, viz. those passing through the axis and one ray of either node; thirteen planes each representing two triangle-planes, viz. the singular tangent plane and the twice six other planes each of them through two disengaged rays of the same node; lastly, the three simple triangle-planes passing through the transversal; in all $4\cdot4+13\cdot2+3\cdot1=45$ triangle-planes.

We proceed now to reduce the equation of the surface in question to its simplest form. Let $x=0$ be the equation of the singular tangent-plane, and let the plane $y=0$ pass through the axis, while the planes $z=0$ and $w=0$ touch respectively the nodal cones in lines belonging to the plane $y=0$, then the term yzw and those divisible by z^2, w^2, xz, xw will disappear, and we may therefore write

$$xzw+y^2(z+w)+ax^3+4bx^2y+6cxy^2+4dy^3=0.$$

But this cubic if multiplied by x becomes

$$(xz+y^2)(xw+y^2)-(y-dx)^4+6(c+d^2)x^2y^2+4(b-d^3)x^3y+(a+d^4)x^4,$$

while

$$xz+y^2=x(z+2dy-d^2x)+(y-dx)^2,\quad xw+y^2=x(w+2dy-d^2x)+(y-dx)^2.$$

Now it will be readily seen that the equation of the surface can in but one way be reduced to the form

$$xzw+y^2(z+w)+ax^3+bx^2y+cxy^2=0,$$

where we might also put unity instead of one of the three constants a, b, c. In respect to the fundamental coordinates, the equation implies seventeen constant elements, as it should do, since the two nodes take away two disposable constants from the full number 19.

Let us attempt to form the equation reciprocal to this. We have

$$\theta p=zw+3ax^2+2bxy+cy^2,\quad \theta q=2y(z+w)+bx^2+2cxy,\quad \theta r=xw+y^2,\quad \theta s=xz+y^2.$$

Putting then

$$\phi=px^2+qxy-(r+s)y^2,\quad \chi=ax^4+bx^3y+cx^2y^2-y^4,$$

regarding p, q, r, s in respect to $\frac{\partial}{\partial x}$, $\frac{\partial}{\partial y}$ as constants, and eliminating z, w by the help of the original equation of the surface, we find

$$\theta^2rs=-\chi,\quad \theta\frac{\partial\phi}{\partial x}=\frac{\partial\chi}{\partial x},\quad \theta\frac{\partial\phi}{\partial y}=\frac{\partial\chi}{\partial y},$$

whence $\theta\phi=2\chi$, $2\theta rs=-\phi$; and lastly, on eliminating θ,

$$\frac{\partial}{\partial x}(\phi^2+4rs\chi)=0,\quad \frac{\partial}{\partial y}(\phi^2+4rs\chi)=0,$$

that is to say, the discriminant of the binary quartic

$$\left(px^2+qxy-(r+s)y^2\right)^2+4rs(ax^4+bx^3y+cx^2y^2-y^4)$$

must vanish, and divided by r^2s^2 it will give the reciprocal equation required.

Denoting the Hessian of the primitive cubic by 4H, we have

$$\begin{aligned}\mathrm{H}=&xzw(z+w)+y^2(z-w)^2+cx^2zw-(z+w)(3ax^3+3bx^2y+2cxy^2)\\&+(b^2-3ac)x^4+bcx^3y+(c^2+12a)x^2y^2+4bxy^3.\end{aligned}$$

Hence arises for the spinode curve the system

$$\left.\begin{aligned}xzw+y^2(z+w)+ax^3+bx^2y+cxy^2=0,\\-4y^2zw-4(z+w)(ax^3+bx^2y+cxy^2)+(b^2-4ac)x^4+12ax^2y^2+4bxy^3=0,\end{aligned}\right\}$$

where the axis ($x=0$, $y=0$) counts for two solutions; therefore the spinode curve is a partial curve of the tenth degree, and each node of the original surface is a quadruple point of the curve, the nodal rays at such point being tangents to the curve.

Art. 16. We proceed to determine the lines and triangle-planes of the surface. The transversal is ($x=0$, $z+w=0$). The nodal cones are $xz+y^2=0$, $xw+y^2=0$; besides touching one another along the axis, they intersect in a conic the plane of which is $z-w=0$. This plane and the transversal therefore cut the axis harmonically in regard to the two nodes.

Cutting the surface by the plane $y-\lambda x=0$, and omitting the solution $x=0$, we obtain the equation

$$(z+\lambda^2x)(w+\lambda^2x)+(a+b\lambda+c\lambda^2-\lambda^4)x^2=0;$$

and in order that this break up into factors, the condition $\lambda^4-c\lambda^2-b\lambda-a=0$ must be fulfilled, and the equation of the section then becomes $(z+\lambda^2x)(w+\lambda^2x)=0$. Now, as is well known, the solution of the quartic condition depends upon that of the cubic equation

$$\mathrm{X}^3-2c\mathrm{X}^2+(c^2+4a)\mathrm{X}-b^2=0.$$

Accordingly, in order to avoid irrationalities, we put

$$2c=\alpha^2+\beta^2+\gamma^2,\quad c^2+4a=\beta^2\gamma^2+\gamma^2\alpha^2+\alpha^2\beta^2,\quad b=\alpha\beta\gamma,$$

and, for the sake of shortness, $\sigma=\frac{1}{2}(\alpha+\beta+\gamma)$, whence

$$a=\sigma(\sigma-\alpha)(\sigma-\beta)(\sigma-\gamma),\quad c=\tfrac{1}{2}(\alpha^2+\beta^2+\gamma^2),$$

and λ has the four values σ, $\alpha-\sigma$, $\beta-\sigma$, $\gamma-\sigma$. Hence the four triangle-planes passing through the axis are

$$\sigma x-y=0,\quad(\sigma-\alpha)x+y=0,\quad(\sigma-\beta)x+y=0,\quad(\sigma-\gamma)x+y=0.$$

The plane

$$\tfrac{1}{2}(-\alpha^2+\beta^2+\gamma^2)x+z+w=0$$

passing through the transversal, cuts the surface in the trilateral

$$x\{(\sigma-\alpha)\,x+\alpha y\;+z\qquad\}\{(\sigma-\beta)(\sigma-\gamma)x+\alpha y-z\}=0,$$

or

$$x\{(\sigma\;-\beta)(\sigma-\gamma)x+\alpha y-w\}\{\qquad\sigma(\sigma-\alpha)x+\alpha y+w\}=0,$$

in each of which representations the consecution of the three sides is the same, while in the first all the planes (or factors) pass through the node W, and in the second through the node Z. Denoting the sides of the triangle corresponding to the constant α in the

same order by *axis*, a, a, and the planes passing through them and the nodes by (Wa), (Wa), (Za), (Za), we see that through each nodal ray there pass three planes, as follows:—

(Wa), (Wb), (Wc) through the ray $\sigma x-y=0,\quad \sigma^2x+z=0,$

(Wa), (Wb), (Wc) through the ray $(\sigma-\alpha)x+y=0,\quad (\sigma-\alpha)^2x+z=0,$

(Wb), (Wc), (Wa) through the ray $(\sigma-\beta)x+y=0,\quad (\sigma-\beta)^2x+z=0,$

(Wc), (Wa), (Wb) through the ray $(\sigma-\gamma)x+y=0,\quad (\sigma-\gamma)^2x+z=0.$

If we permute the nodes W and Z, we must in this scheme also permute a with a, b with b, c with c, and z with w.

Art. 17. In order to get a trihedral-pair form, let

$$P=(\sigma-\beta)(\sigma-\gamma)x+\alpha y-z,\quad Q=(\sigma-\gamma)(\sigma-\alpha)x+\beta y-z,$$
$$R=(\sigma-\alpha)(\sigma-\beta)x+\gamma y-z,\quad S=w-z,$$

and

$$(\beta-\gamma)(\gamma-\alpha)(\alpha-\beta)=\theta;$$

then it will be found that

$$\Sigma\alpha(Q-R)P(P-S)+2\sigma(Q-R)(R-P)(P-Q)=\theta\{xzw+y^2(z+w)+ax^3+bx^2y+cxy^2\};$$

but the left-hand side of this identical equation is equal to

$$-\Sigma QR\{(\beta-\gamma)S+(\gamma+\alpha)Q-(\alpha+\beta)R\},$$

where

$$(\beta-\gamma)S+(\gamma+\alpha)Q-(\alpha+\beta)R=(\beta-\gamma)\{\sigma(\sigma-\alpha)x+\alpha y+w\}.$$

Put therefore

$$p=(\beta-\gamma)\{\sigma(\sigma-\alpha)x+\alpha y+w\},\quad q=(\gamma-\alpha)\{\sigma(\sigma-\beta)x+\beta y+w\},$$
$$r=(\alpha-\beta)\{\sigma(\sigma-\gamma)x+\gamma y+w\},$$

and then

$$pQR+qRP+rPQ=0$$

will be the equation of the surface, where the six linear functions fulfil the identical relations

$$p+q+r=0,\quad \alpha p+\beta q+\gamma r+(\beta^2-\gamma^2)P+(\gamma^2-\alpha^2)Q+(\alpha^2-\beta^2)R=0.$$

If h denote a number which is ultimately made to vanish, this equation may be exhibited under the form

$$(P+hp)(Q+hq)(R+hr)-(P-hp)(Q-hq)(R-hr)=0.$$

Let $p=\lambda P$, $q=\mu Q$, $r=\nu R$; then $\lambda+\mu+\nu=0$ in virtue of the equation of the surface. Again, if for shortness we put

$$f=\alpha\lambda+\beta\mu+\gamma\nu,\quad g=\alpha^2\lambda+\beta^2\mu+\gamma^2\nu,\quad h=\sigma f-g,$$

from the foregoing relations it will next be found

$$S=g+\lambda\mu\nu,\quad P=g+(\beta-\gamma)\mu\nu,\ \&c.,\quad \theta x=-\Sigma(\beta-\gamma)P,$$
$$\theta y=\Sigma(\beta-\gamma)(\sigma-\alpha)P,\quad \theta z=\Sigma(\beta-\gamma)(\sigma-\alpha)^2P,$$

and then

$$\theta x=f^2,\quad \theta y=fh,\quad \theta z=-\theta g-h^2,\quad \theta w=\theta\lambda\mu\nu-h^2;$$

the coordinates of a point of the surface are thus expressed in terms of two independent variables; only the values $\lambda=\beta-\gamma$, $\mu=\gamma-\alpha$, $\nu=\alpha-\beta$ are inadmissible. To verify the equation of the surface we have

$$\theta(\sigma x-y)=fg,\quad (\beta-\gamma)\{(\sigma-\alpha)x+y\}=\lambda f,\ \&c.,$$

whence

$$\theta^2(ax^4+bx^3y+cx^2y^2-y^4)=\lambda\mu\nu f^4g;$$

and on the other side

$$\theta(xz+y^2)=-gf^2,\quad \theta(xw+y^2)=\lambda\mu\nu f^2.$$

This gives indeed

$$(xz+y^2)(xw+y^2)+ax^4+bx^3y+cx^2y^2-y^4=0;$$

but the values of the nodal cone quadratics show that three rays of the node W and one ray of the node Z cannot be expressed.

We have still to divide this sort of surface into species. Whether both nodes be real (when z, w are real) or conjugate (z, w conjugate), there are but three cases to be distinguished.

1. α, β, γ are real. Then the four triangle-planes passing through the axis and the three passing through the transversal are real. IV. 1, and IV. 4.

2. α is real, β, γ are complex and conjugate. Then of the planes passing through the axis only two are real, the two others are conjugate; and of those passing through the transversal but one is real and two are conjugate. IV. 2, and IV. 5.

3. α is real, β, γ are lateral (according to the denomination proposed by GAUSS, that is to say, β^2 and γ^2 are negative). Then the planes passing through the axis are conjugate by pairs; and those passing through the transversal are all of them real. IV. 3, and IV. 6.

Hence arise six species.

V. *Cubic surface of the eighth class with a biplanar node.*

Art. 18. From art. 7 it appears that the equation of this surface can be written

$$xyw+(x+y)z^2+2(ax^2+by^2)z+cx^3+dy^3=0,$$

since all the terms divisible by xy may be joined to the first term. But giving this equation the form

$$\begin{aligned}&xy[w-2(a+b)z-(2ab+c)x-(2ab+d)y]\\&\quad+(x+y)[(z+ax+by)^2+(c-a^2)x^2+(d-b^2)y^2]=0,\end{aligned}$$

we see that more briefly it may also be thus written,

$$2xyw+(x+y)(z^2-ax^2-by^2)=0.$$

The equation reciprocal to this is contained in the discriminant of the binary quartic

$$r^2x^2y^2+2s(px+qy)xy(x+y)+s^2(ax^2+by^2)(x+y)^2.$$

If we put

$$\mathrm{L}=(a+b)s^2+2(p+q)s+r^2,\quad \mathrm{M}=(bp+aq)s+pq,\quad \mathrm{N}=abr^2-bp^2-aq^2,$$

and denote by S, T the same invariants as are found in Dr. SALMON'S 'Higher Algebra,' p. 100, then we have

$$12S=L^2-12s^2M,\quad 216T=-L^3+18s^2LM+54s^4N,\quad 16(S^3-27T^2)=s^4\Theta,$$

and ultimately

$$\begin{aligned}\Theta=&ab(a+b)^2\{(a+b)r^2-(p-q)^2\}s^6\\ &+2ab\{3(a+b)[(a-2b)p+(-2a+b)q]r^2+(p-q)^2[(-3a+5b)p+(5a-3b)q]\}s^5\\ &+\{3ab(a^2-7ab+b^2)r^4+[b(9a^2+26ab-b^2)p^2-26ab(a+b)pq+a(-a^2+26ab\\ &\quad+9b^2)q^2]r^2+(p-q)^2[b(-12a+b)p^2+22abpq+a(a-12b)q^2]\}s^4\\ &+2\{3ab[(2a-b)p+(-a+2b)q]r^4+[b(-2a+5b)p^3+b(3a-2b)p^2q\\ &\quad+a(-2a+3b)pq^2+a(5a-2b)q^3]r^2+2(p-q)^2[-2bp^3+bp^2q+apq^2-2aq^3]\}s^3\\ &+\{3ab(a+b)r^6+[b(9a-2b)p^2+8abpq+a(-2a+9b)q^2]r^4\\ &\quad+2[-6bp^4+bp^3q-(a+b)p^2q^2+apq^3-6aq^4]r^2+4p^2q^2(p-q)^2\}s^2\\ &+2\{3ab(p+q)r^6-(3bp^3+2bp^2q+2apq^2+3aq^3)r^4+4p^2q^2(p+q)r^2\}s\\ &+r^4(ar^2-p^2)(br^2-q^2)=0\end{aligned}$$

is the reciprocal equation required.

Let 16H be the Hessian of the primitive function, then

$$H=2(x+y)xyw+(x-y)^2z^2+(x+y)(3ax^3-ax^2y-bxy^2+3by^3),$$

whence the system

$$\left\|\begin{matrix} xy, & ax^2+by^2-z^2, & ax^3+by^3\\ x+y, & 2w, & z^2\end{matrix}\right\|=0$$

will represent the spinode curve, which is therefore a partial curve of the tenth degree, and in which there pass through the node six branches, in lowest approximation represented by the systems $(2xw+z^2=0,\ 2by^3w+z^4=0)$, $(2yw+z^2=0,\ 2ax^3w+z^4=0)$, and having the axis for a common tangent with a singular kind of contact. Any plane passing through the node intersects here the curve in six coincident points, any plane passing through the axis in eight, and each nodal plane in ten coincident points.

Art. 19. Let $a=\alpha^2$, $b=\beta^2$, $U=2\alpha\beta(x+y)-W$, $V=z+\alpha x+\beta y$, $W=z-\alpha x-\beta y$, $X=2\alpha\beta(x+y)+W$, $Y=-z-\alpha x+\beta y$, $Z=-z-\alpha x+\beta y$; then the original equation takes the form $UVW+XYZ$, and the six new functions fulfil the two identical relations $V+W+Y+Z=0$, $U-(\alpha+\beta)V+(\alpha+\beta)W+X-(\alpha-\beta)Y+(\alpha-\beta)Z=0$. Imagine instead of these the relations

$$hU+V+W+hX+Y+Z=0,\quad AhU+BV+CW+DhX+EY+FZ=0,$$

where

$$A=\frac{1}{h}-(\alpha+\beta),\quad B=-(\alpha+\beta)+h(\alpha+\beta)^2,\quad C=\alpha+\beta-h(\alpha-\beta)^2,$$

$$D=\frac{1}{h}-(\alpha+\beta)+4\alpha\beta h,\quad E=-(\alpha-\beta)\{1-h(\alpha-\beta)\},\quad F=-E.$$

Because the six constants fulfil the equations

$$A+B+C=D+E+F,\quad BC+CA+AB=EF+FD+DE,$$

the cubic condition inherent to the trihedral pair $h\mathrm{UVW}+h\mathrm{XYZ}=0$ has three equal roots. Let then h vanish, and the former system will be reproduced. At the same time such equations of triangle-planes as in art. 13 were $u+x=0$, $u+y=0$, $v+x=0$ will now become respectively $\mathrm{U}+\mathrm{X}=0$, $\mathrm{Y}=0$, $\mathrm{V}=0$, and so on; but we shall continue to denote them by (ux), (uy), (vx) as before, yet omit accents, since all three accents coincide. So we get the following survey of the twenty-seven lines on the surface, showing in what manner they coincide:—

The nodal edge (or here *axis*, since the surface is along it touched by a plane) ($x=0$, $y=0$) unites the six lines l, p. The four other nodal rays unite each of them four lines such as follow, $(\overline{vy}, r)$, $(\overline{wz}, q)$; $(\overline{vz}, n)$, $(\overline{wy}, m)$. The transversal $\overline{ux}$, and the other sides of the two simple triangles standing upon it, $\overline{uy}$, $\overline{uz}$, $\overline{vx}$, $\overline{wx}$, are the only five simple lines. In all $1\cdot 6+4\cdot 4+5\cdot 1=27$ lines.

Each nodal plane unites twelve triangle-planes, viz. $x=0$ unites (vz), (wy), (lmn); and $y=0$ unites (vy), (wz), (pqr). The four planes, joining a ray of one nodal plane with a ray of the other nodal plane, unite each of them four triangle-planes, viz.

$$\mathrm{V}=0\ \{v, (vx)\},\quad \mathrm{W}=0\ \{w, (wx)\},\quad \mathrm{Y}=0\ \{y, (uy)\},\quad \mathrm{Z}=0\ \{z, (uz)\}.$$

The singular tangent plane $x+y=0$ unites the three planes (ux). Lastly, there are but two simple triangle planes, those passing through the transversal $\mathrm{U}=0$, and $\mathrm{X}=0$. In the whole $2\cdot 12+4\cdot 4+1\cdot 3+2\cdot 1=45$ triangle-planes.

Since the functions z, w, $x+y$, xy must always be real, there are four species.

1. All is real, and a, c are positive. V. 1.

2. x, y are real, a is positive, and b negative. The only real lines are the axis, two rays in only one nodal plane, and the transversal. V. 2.

3. x, y are real, a and b are negative. The axis and transversal are the only real lines. Each nodal plane contains two conjugate rays. V. 3.

4. x, y are conjugate, and so also are a, b. The axis and transversal are real; of the two real planes passing through the transversal, one only contains a real triangle; these four lines only are real. V 4.

VI. *Cubic surface of the seventh class with a biplanar and a proper node.*

Art. 20. If a cubic surface have two nodes, chosen for points of reference $\frac{\partial}{\partial z}$, $\frac{\partial}{\partial w}$, its equation necessarily takes the form $lzw+mz+nw+p=0$, where $l=(x, y)$; $m, n=(x, y)^2$, $p=(x, y)^3$; and if $\frac{\partial}{\partial w}$ be a biplanar node, $lz+n$ must break up into factors, whence l must divide n, so that $lz+n$ may then be replaced by xz. And joining the terms in mz, which are divisible by xz, to the term xzw, we may write

$$xzw+xy^2z+ax^3+3bx^2y+3cxy^2+dy^3=0$$

as the equation of the surface.

The equation reciprocal to this is found by dividing the discriminant of the binary quartic $(px^2+qxy-sy^2)^2+4rsx(ax^3+3bx^2y+3cxy^2+dy^3)$ by r^2s^3 and equating the quotient to zero. Let

$$\begin{aligned}&\mathrm{L}=q^2+4(p+3cr)s,\ \mathrm{M}=-dpq+3(-2cp+bq-2bdr)s+2as^2,\\&\mathrm{N}=d^2p^2+2d(3bp-2aq+2adr)s+3(3b^2-4ac)s^2,\\&12\mathrm{S}=\mathrm{L}^2+24rs\mathrm{M},\ -216\mathrm{T}=\mathrm{L}^3+36rs\mathrm{LM}+216r^2s^2\mathrm{N},\\&\mathrm{M}^2-\mathrm{LN}=4s\mathrm{P},\ \mathrm{S}^3-27\mathrm{T}^2=r^2s^3\Theta,\end{aligned}$$

then S, T are the two invariants of the quartic in question, and

$$\Theta=\mathrm{L}^2\mathrm{P}+8r\mathrm{M}^3-9r\mathrm{LMN}-27r^2s\mathrm{N}^2=0$$

is the equation of the surface in plane-coordinates. If

$$\delta=3b\frac{\partial}{\partial a}+2c\frac{\partial}{\partial b}+d\frac{\partial}{\partial c}+q\frac{\partial}{\partial p}-2s\frac{\partial}{\partial q},$$

then $\delta\mathrm{N}=-2d\mathrm{M}$, $\delta\mathrm{M}=-d\mathrm{L}$, $\delta\mathrm{L}=12drs$, whence $\delta\mathrm{S}=0$, $\delta\mathrm{T}=0$, $\delta\Theta=0$.

The quartic function, the Hessian of the original cubic, is

$$\begin{aligned}&\{z+3(cx+dy)\}(xzw+y^2z+ax^3+3bx^2y+3cxy^2+dy^3)-4z(ax^3+3bx^2y+3cxy^2+dy^3)\\&\quad-3\{(4ac-3b^2)x^4+4adx^3y+6bdx^2y^2+4cdxy^3+d^2y^4\}.\end{aligned}$$

The spinode curve is therefore a partial curve of the ninth degree, which has the biplanar node for a quintuple and the proper node for a triple point. The tangents at the latter node are the three disengaged nodal rays; but of those at the former node one tangent is $(x=0,\ 3dy+4z=0)$, and the four remaining tangents are

$$z=0,\ (4ac-3b^2)x^4+4adx^3y+6bdx^2y^2+4cdxy^3+d^2y^4=0.$$

Art. 21. If d vanish, the edge of the biplanar node would belong to the surface, and its class would therefore sink to six, contrary to the supposition. We are therefore allowed to change z into dz and write

$$xzw+y^2z+(y+\alpha x)(y+\beta x)(y+\gamma x)=0$$

as the equation of the surface. Again, let

$$\begin{aligned}\mathrm{P}&=w-\beta\gamma x-(\beta+\gamma)y,\\\mathrm{Q}&=w-\gamma\alpha x-(\gamma+\alpha)y,\\\mathrm{R}&=w-\alpha\beta x-(\alpha+\beta)y,\\p&=(\beta-\gamma)(\alpha x+y+z),\\q&=(\gamma-\alpha)(\beta x+y+z),\\r&=(\alpha-\beta)(\gamma x+y+z);\end{aligned}$$

the six new linear functions will satisfy the identical relations

$$\begin{aligned}&p+q+r=0,\quad(\beta-\gamma)\mathrm{P}+(\gamma-\alpha)\mathrm{Q}+(\alpha-\beta)\mathrm{R}+\alpha p+\beta q+\gamma r=0,\\&\quad p\mathrm{QR}+q\mathrm{RP}+r\mathrm{PQ}=-(\beta-\gamma)(\gamma-\alpha)(\alpha-\beta)\{xzw+y^2z+(y+\alpha x)(y+\beta x)(y+\gamma x)\};\end{aligned}$$

and the equation of the surface is now changed into

$$pQR+qRP+rPQ=0.$$

Introducing then a number h which is ultimately made to vanish, we put

$$U=P+hp,\ V=Q+hq,\ W=R+hr,\ X=-P+hp,\ Y=-Q+hq,\ Z=-R+hr,$$

whereby the equation of the surface becomes $UVW+XYZ$; and if, for the sake of shortness, we put

$$A=\alpha+(\beta-\gamma)h,\quad B=\beta+(\gamma-\alpha)h,\quad C=\gamma+(\alpha-\beta)h,$$
$$D=\alpha-(\beta-\gamma)h,\quad E=\beta-(\gamma-\alpha)h,\quad F=\gamma-(\alpha-\beta)h,$$

the above-mentioned two identical relations become

$$U+V+W+X+Y+Z=0,\quad AU+BV+CW+DX+EY+FZ=0,$$

where the six constants satisfy the relations

$$A+B+C=D+E+F,\quad BC+CA+AB=EF+FD+DE$$

strictly, while $ABC-DEF$ is different from zero. All three roots, therefore, of the cubic condition inherent to this trihedral-pair form coincide, and the corresponding relation $U+V+W+X+Y+Z=0$ counts for three such intersections.

Hence the axis $(x=0,\ y=0)$ unites six lines, viz. the lines m, n. Each of the remaining four rays of the biplanar node unites three lines; viz. $(x=0,\ y+z=0)$ unites the three lines l, $(z=0,\ \alpha x+y=0)$ unites the three lines p, $(z=0, \beta x+y=0)$ unites the three lines q, $(z=0,\ \gamma x+y=0)$ unites the three lines r. Each of the remaining three rays of the proper node unites two lines, viz. $(\alpha x+y=0,\ w-\alpha y=0)$ unites $\overline{vz}$, $\overline{wy}$; $(\beta x+y=0,\ w-\beta y=0)$ unites $\overline{wx}$, $\overline{uz}$; $(\gamma x+y=0,\ w-\gamma y=0)$ unites $\overline{uy}$, $\overline{vx}$. Three lines are simple viz. $(P=0,\ p=0)$ or $\overline{ux}$, $(Q=0,\ q=0)$ or $\overline{vy}$, $(R=0,\ r=0)$ or $\overline{wz}$. In the whole $1\cdot6+4\cdot3+3\cdot2+3\cdot1=27$ lines.

Of the following five triangle-planes each counts for six. The singular tangent plane $x=0$ unites all the six planes (lmn); the other plane $z=0$ of the biplanar node unites the six planes (pqr); of the three further planes which (besides $x=0$) pass through the axis, the plane $\alpha x+y=0$ unites the two triads (vz), (wy), the plane $\beta x+y=0$ unites the two triads (wx), (uz), and the plane $\gamma x+y=0$ unites the two triads (uy), (vx). The three planes which combine the single ray l of the biplanar node with any one of the rays p, q, r in the opposite nodal plane count each of them for three triangle-planes, viz. $p=0$ unites the three planes (ux), $q=0$ unites the three planes (vy), $r=0$ unites the three planes (wz). Lastly, the three planes which combine any two of the three disengaged rays of the proper node count each of them for two triangle-planes, viz. $P=0$ unites the planes u, x; $Q=0$ unites v, y; $R=0$ unites w, z; they are the planes of the two coinciding trihedrals. In all $5\cdot6+3\cdot3+3\cdot2=45$ triangle-planes.

Art. 22. As to the reality of the linear functions in the original equation, it appears that both x and z must be real, since the two planes of the biplanar node play different

parts, and that it is always allowed to assume y as real, since the corresponding plane may be turned about the real axis; but w will then also be real, and of the three constants α, β, γ one at least must be real. There are therefore but two species.

1. All is real. VI. 1.

2. α is real, β and γ are conjugate. Only the axis ($x=0$, $y=0$), two rays of the biplanar node ($x=0$, $y+z=0$) and ($z=0$, $\alpha x+y=0$), one ray of the proper node ($\alpha x+y=0$, $w-\alpha y=0$), and the simple line ($P=0, p=0$) are real. VI. 2.

VII. *Cubic surface of the seventh class with a biplanar node.*

Art. 23. According to art. 7 we put

$$xyw+xz^2+(2ax^2+by^2)z+cx^3+dy^3=0$$

as an equation of the surface in question, since all the terms divisible by xy can be carried into the single term xyw. The mark of this sort of biplanar node is that one of its planes (here $x=0$) *touches* the surface along the nodal edge; if b were to vanish it would osculate the surface, and then the class would sink to six. Since therefore b is not allowed to vanish, we may put the above equation under the form

$$xy\left(w-2\frac{d}{b}(z+ax)-\left(ab+\frac{d^2}{b^2}\right)y\right)+x\left(z+ax+\frac{d}{b}y\right)^2+by^2\left(z+ax+\frac{d}{b}y\right)+(c-a^2)x^3=0,$$

or more simply

$$xyw+xz^2+y^2z-ax^3=0.$$

We shall in the sequel retain the constant a, because its being positive or negative decides as to reality or non-reality. But now that we are concerned with the reciprocal equation of the surface, we may, on putting $a=\lambda^4$, change y, z, w respectively into λy, $\lambda^2 z$, $\lambda^3 w$, and we get

$$xyw+xz^2+y^2z-x^3=0.$$

The reciprocal septic Θ, when multiplied by s^5, is the discriminant S^3-27T^2 of the binary quartic

$$y^2(rx-sy)^2+4sx^2(sx^2+pxy+qy^2);$$

hence

$$12S=L^2+24s^2M, \quad -216T=L^3+36s^2LM+216s^4N,$$

where

$$L=r^2+4qs, \quad M=pr+2s^2, \quad N=p^2-4qs;$$

and

$$\begin{aligned}\Theta&=L^2.\frac{M^2-LN}{4s}+8sM^3-9sLMN-27s^3N\\ &=64s^7+32(3pr-4q^2)s^5+16q(5r^2+9p^2)s^4\\ &\quad+(r^4+30p^2r^2+160pq^2r-27p^4+64q^4)s^3\\ &\quad+4q(11pr^3+12q^2r^2-9p^3r-4p^2q^2)s^2\\ &\quad+r^2(pr^3+12q^2r^2-p^3r-8p^2q^2)s+qr^4(r^2-p^2)=0\end{aligned}$$

is the equation of the surface in plane-coordinates.

The quartic function, the Hessian of the cubic $f=xyw+xz^2+y^2z-x^3$, is

$$xf-4xy^2z+4x^4+y^4;$$

and since the system

$$(f=0, \quad 4x^4+y^4-4xy^2z=0)$$

contains the axis $(x=0,\ y=0)$ four times, the spinode curve of the original surface is a partial curve of the *eighth* degree. An arbitrary plane passing through the node intersects the quartic cone in four lines, each of which also cuts the cubic surface in a point distinct from the node. This arbitrary plane thus intersects the spinode curve in four points distinct from the node, so that this must be a quadruple point of the curve, since it unites the remaining points of intersection. One of the four branches passing through the node is (at the lowest approximation) represented by

$$yw=-5z^2, \quad xw^2=\frac{25}{4}z^3,$$

and therefore osculates the nodal plane which is a singular tangent plane to the surface, and merely touches the other nodal plane. If t denote a very small variable number, the three other branches may be represented by

$$z=t^3w, \quad x=t^5w, \quad y=-t^6w.$$

Art. 24. The nodal plane $x=0$, which touches the surface along the nodal edge or *axis*, contains only a single disengaged ray (call it f), the other nodal plane $y=0$ contains two rays (call them g, h); and the planes combining the former ray with any one of the two latter rays are (fg) or $z+x=0$, and (fh) or $z-x=0$. It is manifest that, besides the nodal planes and these two planes, there pass no other triangle-planes through the node. The planes (fg) and (fh) intersect the surface respectively in the simple lines $(z+x=0,\ w-y=0)$ or j and $(z-x=0,\ w+y=0)$ or k. Now as a plane containing the node and any distinct and therefore simple line of the surface must be a triangle-plane and therefore combine two nodal rays, there cannot, on all such planes being exhausted, be found any other simple line of the surface. Hence these distinct lines j and k are the only simple lines of the surface, and it is obvious that they do not intersect each other. Again, since two independent lines are cut by five lines, and these lines (j and k) are simultaneously cut only by the ray f, this ray f unites five lines of the general surface; and then, because each of the simple lines j and k must besides be cut by five more lines, each of the nodal rays g and h also unites five lines. But all the lines thus far mentioned count as $3\cdot5+2\cdot1=17$ lines. Therefore the nodal edge (or *axis*) unites ten lines of the general surface, precisely those *ten* (as we know from art. 1) lines disengaged from the two independent lines j and k.

It is already proved that the two planes passing through the simple lines and the node count each of them as five. No one of the five lines united in the ray f intersects any other of them; wherefore no two of them can lie in the same triangle-plane. But two triangle-planes passing through any one of them have been already spoken of, viz. (fg) and (fh); the three remaining ones must therefore coincide with the nodal plane $x=0$;

hence this plane counts as fifteen. The ray g unites the five lines intersecting j but not k, the ray h unites the five lines intersecting k but not j, and each of the former five lines is (as may be inferred from the consideration of a simple triangle) cut by four of the latter five, which determine with it four different triangles. Therefore twenty triangle-planes coincide in the nodal plane $y=0$. In all $1\cdot20+1\cdot15+2\cdot5=45$ triangle-planes.

The same consequences may be derived from a trihedral-pair form. Let

$$\mathrm{U}=x+z-h(w-y),\quad \mathrm{V}=-y-h(x+z),\quad \mathrm{W}=x-z,$$
$$\mathrm{X}=x-z+h(w+y),\quad \mathrm{Y}=\ \ y-h(x-z),\quad \mathrm{Z}\ =x+z,$$

then

$$\mathrm{UVW}+\mathrm{XYZ}=2h\{xyw+xz^2+y^2z-x^3+hy(z^2-x^2)\},\quad \mathrm{V}+h\mathrm{W}+\mathrm{Y}+h\mathrm{Z}=0,$$
$$\mathrm{U}+h\mathrm{V}-(1+h^2)\mathrm{W}+\mathrm{X}-h\mathrm{Y}-(1-h^2)\mathrm{Z}=0.$$

If the number h vanish, the equation $\mathrm{UVW}+\mathrm{XYZ}=0$ will, at the limit, exhibit the present surface, and the former of the linear relations, by reason of the latter, counts for three relations answering to the cubic condition. Omitting the three accents as in (ux) and the permutations as in (lmn), we then get the following survey of the manner of coincidence of the 27 lines and 45 triangle-planes of the general surface.

The axis $(x=0,\ y=0)$ unites the ten lines $(\overline{vy},\ l,\ p,\ r)$. The nodal rays unite each of them five lines; viz. the ray $(x=0,\ z=0)$ unites $(\overline{ux},\ \overline{wz},\ q)$, the ray $(y=0,\ x+z=0)$ unites $(\overline{uy},\ \overline{vz},\ n)$, the ray $(y=0,\ x-z=0)$ unites $(\overline{vx},\ \overline{wy},\ m)$; and there remain but two simple lines $(x+z=0,\ w-y=0)$ or $\overline{uz}$, and $(x-z=0,\ w+y=0)$ or $\overline{wx}$.

The nodal plane $y=0$ unites the twenty triangle-planes v, y, (uy), (vx), (vz), (wy), (lmn); the nodal plane $x=0$ unites the fifteen planes (ux), (vy), (wz), (pqr); the two remaining planes unite each of them five triangle-planes, viz. $x+z=0$ unites u, z, (uz); and $x-z=0$ unites w, x, (wx).

Art. 25. In the equation $xyw+xz^2+y^2z-ax^3=0$ the vanishing of the constant a would give rise to a second node $\frac{\partial}{\partial x}$. Therefore we have here only the two cases when a is positive and when it is negative. Since no two of the linear functions x, y, z, w play a like part in the equation, we are obliged to suppose them all real. So there are only two species.

1. a is positive; all is real. VII. 1.

2. a is negative; the two simple lines are conjugate, and so also the two rays in the nodal plane $y=0$. VII. 2.

VIII. *Cubic surface of the sixth class with three proper nodes.*

Art. 26. If we place the points of reference $\frac{\partial}{\partial y}$, $\frac{\partial}{\partial z}$, $\frac{\partial}{\partial w}$ at the three nodes, the equation of the surface will contain the terms x^3, x^2y, x^2z, x^2w, xzw, xwy, xyz, yzw; but the last term is capable of taking up the three next preceding terms; in other words, the three singular tangent planes which touch the surface along the *axes* (or lines joining two nodes) may be chosen for the planes $y=0$, $z=0$, $w=0$; then we are at liberty to write

$$x^3+(y+z+w)x^2+ayzw=0$$

as an equation of the surface. The term x^3, if disappearing, would not alter the class, but would merely form a particular case of the sort of surface here to be considered, which case might readily be restored from the more general form by changing a, x respectively into h^2, hx, dividing by h^2 and letting h vanish. But the term x^2y cannot disappear without bringing the class down to *five*; for the point $\frac{\partial}{\partial y}$ would then become a biplanar node. Nor is the constant a allowed to be zero or -4; for in the former case the cubic would be divisible by x^2, in the latter it would be half the expression

$$x\Sigma(x+2z)(x+2w)-(x+2y)(x+2z)(x+2w),$$

which shows a fourth proper node at the point

$$-2\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}+\frac{\partial}{\partial w}.$$

If we denote the surface by $f=0$, and let

$$\frac{\partial f}{\partial x}=pu,\ \frac{\partial f}{\partial y}=qu,\ \frac{\partial f}{\partial z}=ru,\ \frac{\partial f}{\partial w}=su,\ x^2=t,\ \chi=\tfrac{1}{4}at(t-pu)^2+(t-qu)(t-ru)(t-su),$$

where t, u are to be regarded as the independent variables, then we have not only

$$\chi=af\{x^3+x^2(y+z+w)-ayzw\},$$

but also

$$\frac{\partial\chi}{\partial t}=a(y+z+w)f;$$

whence

$$\frac{\partial\chi}{\partial t}=0,\quad \frac{\partial\chi}{\partial u}=0,$$

whenever $f=0$. That is to say, the equation reciprocal to $f=0$ is the discriminant Θ of the binary cubic χ, when equated to zero. Putting, for shortness,

$$\alpha=q+r+s,\quad \beta=rs+sq+qr,\quad \gamma=qrs,$$

we find

$$\begin{aligned}27\Theta=&\tfrac{1}{16}a^3p^3(p-q)(p-r)(p-s)\\ &+\tfrac{1}{16}a^2\{(12\beta-\alpha^2)p^4-4(2\alpha\beta+9\gamma)p^3+2(15\alpha\gamma+4\beta^2)p^2-36\beta\gamma p+27\gamma^2\}\\ &+\ \tfrac{1}{2}a\{(6\beta^2-\alpha^2\beta-9\alpha\gamma)p^2+2(6\alpha^2\gamma-\alpha\beta^2-9\beta\gamma)p+2\beta^3+27\gamma^2-9\alpha\beta\gamma\}\\ &-\ (r-s)^2(s-q)^2(q-r)^2=0,\end{aligned}$$

as the equation of the surface in plane-coordinates.

The Hessian of the primitive cubic is

$$4a^2\{ayzw(3x+y+z+w)+x^2(y^2+z^2+w^2-2zw-2wy-2yz)\}.$$

Hence the spinode curve is a complete curve of the sixth degree represented by the system

$$\left.\begin{aligned}&x^3+x^2(y+z+w)+ayzw=0,\\ &\tfrac{3}{4}x^2+x\,(y+z+w)+zw+wy+yz=0,\end{aligned}\right\}$$

which shows that the nodes are double points of the curve, and that at these points the (disengaged) nodal rays of the surface are tangents to the curve.

Art. 27. By what has been said in art. 15 we can at once judge of the disposition of

the lines and triangle-planes. The three transversals are the only simple lines, and form a triangle ($x+y+z+w=0$, $yzw=0$), the plane whereof is the only simple triangle-plane. The planes determined by a transversal and the opposite node intersect the surface in thrice two (disengaged) nodal rays, each of which unites two lines. Each of the three axes unites four lines. Together $3\cdot1+6\cdot2+3\cdot4=27$ lines. The singular tangent planes $y=0$, $z=0$, $w=0$ count each of them twice, and so also does each of the three planes passing through a transversal and the opposite node. Through each axis and two nodal rays there pass two planes, together six planes, each of which counts four times. Lastly, the plane $x=0$, containing the three nodes, counts eight times. Together $1\cdot1+6\cdot2+6\cdot4+1\cdot8=45$ triangle-planes.

If we assume the trihedral-pair form $\mathrm{UVW}+\mathrm{XYZ}=0$, where on putting $a=\frac{(\alpha-1)^2}{\alpha}$ we have

$$\begin{array}{lll} \mathrm{U}=-(\alpha-1)(x+y+z+w), & \mathrm{V}=-\alpha x-(\alpha-1)y, & \mathrm{W}=x-(\alpha-1)y, \\ \mathrm{X}=\ \ (\alpha-1)y, & \mathrm{Y}=(\alpha-1)(x+y+w), & \mathrm{Z}=(\alpha-1)(x+y+z), \end{array}$$

then the constants in the auxiliary relations

$$a\mathrm{U}+\ldots=0,\ a'\mathrm{U}+\ldots=0,\ a''\mathrm{U}+\ldots=0 \text{ are } a=0,\ b=1,\ c=\alpha,\ d=\alpha+1,\ e=0,\ f=0$$

(therefore when e, f are imagined to be indefinitely small, a is of the order ef, whence, for instance, $a\mathrm{U}+f\mathrm{Y}=0$ reduces itself to $\mathrm{Y}=0$),

$$a'=b'=c'=d'=e'=f',\quad a''=b''=c''=d''=e''=f'',$$

and at length we get the following survey: viz., the lines are

$$\begin{array}{ll} (x=0,\ y=0) & [\overline{vx},\ \overline{wx},\ l,\ p], \\ (x=0,\ z=0) & [m',\ m'',\ r',\ r''], \\ (x=0,\ w=0) & [n',\ n'',\ q',\ q''], \\ (x:\ z:w=\alpha-1:-\alpha:1) & [l',\ l''], \\ (x:\ z:w=\alpha-1:1:-\alpha) & [p',\ p'']; \\ (x:w:\ y=\alpha-1:-\alpha:1) & [\overline{wy},\ r], \\ (x:w:\ y=\alpha-1:1:-\alpha) & [\overline{vy},\ m]; \\ (x:\ y:\ z=\alpha-1:-\alpha:1) & [\overline{vz},\ q], \\ (x:\ y:\ z=\alpha-1:1:-\alpha) & [\overline{wz},\ n]; \\ (x+z+w=0,\ y=0) & [\overline{ux}], \\ (x+y+w=0,\ z=0) & [\overline{uy}], \\ (x+y+z=0,\ w=0) & [\overline{uz}]; \end{array}$$

and the planes are

$$\begin{array}{ll} (x=0) & [(vx)',\ (vx)'',\ (wx)',\ (wx)'',\ (lmn),\ (lnm),\ (pqr),\ (prq)], \\ (\ \alpha x+(\alpha-1)y=0) & [v,\ (vy),\ (vz),\ (wx)], \\ (-\ x+(\alpha-1)y=0) & [w,\ (vx),\ (wy),\ (wz)], \\ (\ \alpha x+(\alpha-1)z=0) & [(wz)',\ (wz)'',\ (nlm),\ (nml)], \end{array}$$

$(-\ x+(\alpha-1)z=0)$ $[(vz)', (vz)'', (qpr), (qrp)]$,
$(\ \ \alpha x+(\alpha-1)w=0)$ $[(wy)', (wy)'', (rpq), (rqp)]$,
$(-\ x+(\alpha-1)w=0)$ $[(vy)', (vy)'', (mln), (mnl)]$,
$(y=0)$ $[x, (ux)]$, $(z=0)$ $[(uy)', (uy)'']$, $(w=0)$ $[uz)', (uz)'']$,
$(x+z+w=0)$ $[(ux)', (ux)'']$,
$(x+y+w=0)$ $[y, (uy)]$, $(x+y+z=0)$ $[z, (uz)]$,
$(x+y+z+w=0)$ $[u]$.

Art. 28. One node at least must be real, for instance $\frac{\partial}{\partial y}$, and then the two others may be real or conjugate. Accordingly x is always real, and while we keep y real, z and w may be either real or conjugate. On the other hand the constant a may be between -4 and 0, or beyond these limits. From these two reasons of partition there arise four species of the surface with three proper nodes. But we prefer to distinguish five species. For if z, w be conjugate, the nodal cone $x^2+azw=0$ becomes imaginary or real, according as $a>0$ or $a<-4$.

1. z, w are real; $a(a+4)>0$, and therefore α real. All is real. VIII. 1.

2. z,w real; $-4<a<0$. Let $a=-4\sin^2\frac{\theta}{2}$, then $\alpha=\varepsilon^{i\theta}$. The real lines are the three *axes* and the three transversals. The real planes are the plane of the three nodes, the three singular tangent planes, the plane passing through a transversal and the opposite node, and the transversal plane. VIII. 2.

3. z, w conjugate, $a>0$, and therefore α positive. The two nodes $\frac{\partial}{\partial z}$ and $\frac{\partial}{\partial w}$ are conjugate, the nodal cone at the real node $\frac{\partial}{\partial y}$ is imaginary. The real lines are the axis joining the conjugate nodes, and its transversal. The real planes are the plane of the three nodes, the singular tangent plane through the real axis, two other planes which pass through the real axis, the plane passing through the real transversal and the real node, and the transversal plane. VIII. 3.

4. z, w conjugate, $a<-4$, and therefore α negative. The nodal cone at the real node is real, but its two (disengaged) rays are imaginary and conjugate. The rest as before. VIII. 4.

5. z, w conjugate, $-4<a<0$. The nodal cone at the real node is real. The real lines are the axis joining the conjugate nodes, its transversal, and the two (disengaged) rays of the real node. The real planes are that of the three nodes, the singular tangent plane through the real axis, the plane passing through the real transversal and the real node, and the transversal plane. VIII. 5.

IX. *Cubic surface of the sixth class with two biplanar nodes.*

Art. 29. From art. 20 it appears that the reduced equation of this sort of surface is

$$xzw+(y+\alpha x)(y+\beta x)(y+\gamma x)=0,$$

where $\frac{\partial}{\partial z}$, $\frac{\partial}{\partial w}$ are the biplanar nodes. These have in common the nodal plane $x=0$,

which osculates the surface along the axis ($x=0$, $y=0$). The other nodal planes are $z=0$, $w=0$, each of which intersects the surface in three nodal rays. In order however to find the reciprocal equation it is more convenient to write

$$12xzw+ax^3+3bx^2y+3cxy^2+dy^3=0$$

as the equation in point-coordinates. Then the discriminant of the binary cubic

$$rs(ax^3+3bx^2y+3cxy^2+dy^3)+3x(px+qy)^2,$$

divided by rs and equated to zero, will furnish the equation in plane-coordinates as follows; viz. this is

$$\begin{aligned}&[a^2d^2-6abcd-3b^2c^2+4ac^3+4b^3d]r^3s^3\\&+6[(ad^2-3bcd+2c^3)p^2+(4b^2d-2acd-2bc^2)pq+(2ac^2-abd-b^2c)q^2]r^2s^2\\&+3[3d^2p^4-12cdp^3q+(10bd+8c^2)p^2q^2-(4ad+8bc)pq^3+(4ac-b^2)q^4]rs\\&-4q^3(dp^3-3cp^2q+3bpq^2-aq^3)=0.\end{aligned}$$

The Hessian of the cubic $12xzw+ax^3+3bx^2y+3cxy^2+dy^3$ is

$$64\cdot 81x\{4zw(cx+dy)-(ac-b^2)x^3-(ad-bc)x^2y-(bd-c^2)xy^2\}.$$

The system of the two expressions equated to zero breaks up into four times the axis ($x=0$, $y=0$) and four conics which lie in the planes

$$(4ac-3b^2)x^4+4adx^3y+6bdx^2y^2+4cdxy^3+d^2y^4=0,$$

and touch the nodal planes $z=0$, $w=0$ at the corresponding nodes. Two of these four planes are always imaginary and conjugate, the two remaining ones are real. For let

$$\mathrm{K}=a^2d^2-6abcd-3b^2c^2+4ac^3+4b^3d,\ k^3=\tfrac{1}{4}d^2\mathrm{K},$$

and take for k the single real value; again, let $l^2=k+c^2-bd$, which is positive, since

$$k^3+(c^2-bd)^3=\tfrac{1}{4}(ad^2-3bcd+2c^3)^2,$$

and determine the value of l by the condition that $l(ad^2-3bcd+2c^3)$ shall become positive; then the constants m^2, n^2 determined by

$$l(m^2+n^2)=2(ad^2-3bcd+2c^3),\quad n^2-m^2=2k+4(bd-c^2)$$

will be positive, because this system implies $mn=\sqrt{3}\,.\,k$; and the equation of the four planes breaks up into

$$\{(dy+(c+l)x)^2-m^2x^2\}\{(dy+(c-l)x)^2+n^2x^2\}=0.$$

The section made by the real plane $dy+(c+l+m)x=0$ is represented by

$$24d^2zw-m\big((2l+m)^2+n^2\big)x^2=0.$$

In the case therefore when both z and w are real, the two real planes contain also real conics; but when z and w are conjugate, one only of the two real planes intersects the surface also in a real conic, the other real plane has, besides the axis, no real point in common with the surface.

Art. 30. We now suppose $xzw+(y+\alpha x)(y+\beta x)(y+\gamma x)$ to be the equation of the surface. As this form results from that of art. 21, by changing z, w respectively into kz,

$\frac{w}{k}$, and letting k vanish, we may readily thence get a knowledge of the disposition of the twenty-seven lines and forty-five triangle-planes, and we shall in particular see that the axis here unites all the nine lines immediately afforded by a trihedral-pair. Changing then the notation for the sake of greater symmetry, we can regard the surface as though the six planes of $uvw+xyz=0$ coincided with the singular osculating plane ($x=0$), while the nine lines $\overline{ux}$, &c. coincided with the axis. One of the two remaining nodal planes will then unite all the six planes such as (lmn), and contain the three nodal rays (l, m, n), (l', m', n'), (l'', m'', n''); the other nodal plane will unite all the six planes such as (pqr), and contain the three nodal rays (p, q, r), (p', q', r'), (p'', q'', r''). One of the remaining triangle-planes passing through the axis, for instance the plane which combines the nodal rays (l, m, n) and (p, q, r), would then unite the nine triangle-planes

$$(ux),\ (uy),\ (uz),\ (vx),\ (vy),\ (vz),\ (wx),\ (wy),\ (wz),$$

and the other two like planes would answer to the two remaining accents. In the whole $1\cdot 9+6\cdot 3=27$ lines and $3\cdot 9+3\cdot 6=45$ triangle-planes.

The singular osculating plane $x=0$, and one at least of the three other planes passing through the axis $y+dx=0$, for instance, must be real. But z, w can be either real or conjugate, and so also the constants β, γ. From this double reason of partition we get four species, IX. 1; IX. 2; IX. 3; IX. 4.

X. *Cubic surface of the sixth class with a biplanar node and a proper node.*

Art. 31. The cubic surface with a biplanar node which lowers the class by *four* can only in this way have a second node distinct from the first, when the two (disengaged) rays of one nodal plane unite themselves together apart from the nodal edge. The equation then takes the form

$$xyw+(x+y)(z^2-ax^2)=0.$$

Changing z, w respectively into $\sqrt{a}\,.\,z$, aw, we might reduce this equation to

$$xyw+(x+y)(z^2-x^2)=0.$$

But since a may be either positive or negative, in the latter case we should get z as the product of the numerical factor i $(=\sqrt{-1})$ by a real function; and to avoid this we shall retain the constant a.

If we let $a=1$ and denote the discriminant of the binary cubic

$$s^2x(x+y)^2+sy(x+y)(px+qy)+\tfrac{1}{4}r^2xy^2$$

by $\frac{1}{27}s^2\Theta$, then

$$\Theta=[r^2-(p-q)^2]s^4+[(2p-5q)r^2-2(p-2q)(p-q)^2]s^3+$$
$$[\tfrac{1}{2}r^4+\tfrac{1}{2}(p^2-pq+6q^2)r^2-p^2(p-q)^2]s^2+[\tfrac{1}{4}(2p+3q)r^4-\tfrac{1}{2}p^2(p+q)r^2]s+\tfrac{1}{16}r^4(r^2-p^2)=0$$

is the reciprocal equation of the surface.

The quartic function, the Hessian of $xyw+(x+y)(z^2-x^2)$, is

$$x(x+y)(yw+3x^2-xy)+z^2(x-y)^2.$$

Hence the spinode curve breaks up into four times the axis ($x=0$, $z=0$) joining both nodes, twice the nodal edge, or also axis, ($x=0$, $y=0$), and the complete curve

$$(yw+z^2=0, \quad x^2y+x^3-yz^2=0)$$

of the sixth degree. It has passing through the biplanar node three branches, represented in the lowest approximation by

$$yw+z^2=0, \quad x^3w+z^4=0,$$

where only w is finite, and through the proper node two branches, the tangents whereof are the two disengaged rays of this proper node, represented by

$$w=0, \quad z^2-x^2=0.$$

Art. 32. Let

$$\begin{array}{lll} \mathrm{U}=-w+2h(x+y), & \mathrm{V}=\ \ z+x+hy, & \mathrm{W}=\ \ z-x-hy, \\ \mathrm{X}=\ \ w+2h(x+y), & \mathrm{Y}=-z-x-hy, & \mathrm{Z}=-z+x-hy, \end{array}$$

where h denotes a constant which ultimately vanishes; then

$$\mathrm{UVW}+\mathrm{XYZ}=4h\{xyw+(x+y)(z^2-x^2-h^2y^2)\}=0$$

represents the surface in question, and

$$\mathrm{V}+\mathrm{W}+\mathrm{Y}+\mathrm{Z}=0, \quad \mathrm{U}-(1+h)\mathrm{V}+(1+h)\mathrm{W}+\mathrm{X}-(1-h)\mathrm{Y}+(1-h)\mathrm{Z}=0$$

are identical relations, the former of which, in virtue of the latter, stands for the three equations which correspond to the roots of the condition ABC=DEF. Hence we get the following survey of the manner of coincidence of some of the twenty-seven lines and forty-five triangle-planes (accents and permutations are omitted).

The axis joining both nodes ($x=0$, $z=0$) unites $\overline{vz}$, $\overline{wy}$, m, n, eight lines. The nodal edge, also an axis, ($x=0$, $y=0$) unites l, p, six lines. The two disengaged rays of the biplanar node count each of them four times, viz. ($y=0$, $x+z=0$) unites $\overline{vy}$, r, and ($y=0$, $x-z=0$) unites $\overline{wz}$, q. The two disengaged rays of the proper node count each of them twice, viz. ($w=0$, $x+z=0$) unites $\overline{uy}$, $\overline{vx}$, and ($w=0$, $x-z=0$) unites $\overline{uz}$, $\overline{wx}$. Lastly, the transversal of the nodal edge ($w=0$, $x+y=0$) is the only simple line $\overline{ux}$. Together $1\cdot8+1\cdot6+2\cdot4+2\cdot2+1\cdot1=27$ lines.

The planes of the biplanar node count twelve times, viz. $x=0$ (a singular tangent plane) unites (vz), (wy), (lmn), and $y=0$ unites (vy), (wz), (pqr). The two planes combining the double ray of the biplanar node with each of its two simple rays count eight times, viz. $z+x=0$ unites v, (vx), y, (uy); and $x-z=0$ unites w, (wx), z, (uz). The plane $x+y=0$ touching the surface along the nodal edge unites (ux) three triangle-planes, and the plane $w=0$ combining the two simple rays of the proper node, unites u, x, two triangle-planes of the general surface. In all $2\cdot12+2\cdot8+1\cdot3+1\cdot2=45$ planes.

Because no two of the four linear functions x, y, z enter in a similar manner into the form $xyw+(x+y)(z^2-ax^2)=0$, all of them must be real. Only the constant a, according as it is positive or negative, gives rise to a distinction between *two* species. X. 1; X. 2.

XI. *Cubic surface of the sixth class with a biplanar node.*

Art. 33. From art. 7 (see art. 23) we know that one of the nodal planes must osculate the surface along the nodal edge, in order that the node $\frac{\partial}{\partial w}$ may lower the class by *six*, and since in the first term xyw of the equation of the surface all other terms divisible by xy may be included, we write the equation immediately in the form

$$xyw+xz^2+2ax^2z+bx^3+dy^3=0,$$

or, what is the same thing,

$$x.dy.dw+x(dz+adx)^2+d^2(b-a^2)x^3+(dy)^3=0,$$

or, to save constants,

$$xyw+xz^2+ax^3+y^3=0,$$

which is the assumed form for the equation of the cubic. It is well to observe that here all the letters are necessarily real, provided that the surface be real. Putting $a=-\mu^6$ and changing y, z, w respectively into μ^2y, μ^3z, μ^4w, we might get

$$xyw+xz^2-x^3+y^3=0,$$

where no explicit constant remains; but then z would cease to be necessarily real.

If we denote the discriminant of the binary cubic

$$(3as^2,\ -ps,\ -(qs+\tfrac{1}{4}r^2),\ 3s^2 \between x,\ y)^3$$

by $\frac{3}{16}s^2\Theta$, then

$$\Theta=-64s^3p^3-(4qs+r^2)^2p-72as^3(4qs+r^2)p-a(4qs+r^2)^3+432a^2s^6=0$$

is the reciprocal equation. It is obvious that

$$27a^2\Theta=\{8p^3+9a(4qs+r^2)p-108a^2s^3\}^2-\{4p^2+3a(4qs+r^2)\}^3.$$

The Hessian of the cubic is

$$4x(xyw+xz^2-3ax^3-3y^3).$$

The spinode curve then breaks up into six times the axis ($x=0$, $y=0$), and the three distinct conics

$$(ax^3+y^3=0,\quad yw+z^2=0).$$

Art. 34. The trihedral-pair form can only be obtained by the help of two constants which ultimately vanish. Let them be h and ω, the finite constant a be $=-\varrho^2$, and

$$\begin{aligned}
\mathrm{U}&=(1+\omega+\omega h^3\varrho)y+h(1+2\omega)z+hx-\omega h^2w,\\
\mathrm{V}&=(1+\omega)y+\omega hz+\omega h\varrho x,\\
\mathrm{W}&=y+hz-h\varrho x,\\
\mathrm{X}&=-(1+\omega-\omega h^3\varrho)y-h(1+2\omega)z+h\varrho x+\omega h^2w,\\
\mathrm{Y}&=-(1+\omega)y-\omega hz+\omega h\varrho x,\\
\mathrm{Z}&=-y-hz-h\varrho x;
\end{aligned}$$

then

$$\mathrm{UVW}+\mathrm{XYZ}=2\omega h^3\varrho\{xyw+(x+\omega h^2y)(z^2-\varrho^2x^2)+(1+\omega)y^3\}$$

becomes the cubic of the surface as soon as h and ω vanish. Of the two identical relations

$$V+\omega W+Y+\omega Z=0,$$
$$U-\omega h^3\varrho V+(1+\omega^2 h^3\varrho)W+X+\omega h^3\varrho Y+(1-\omega^2 h^3\varrho)Z=0,$$

the former, in virtue of the latter, stands for the three equations which correspond to the condition ABC=DEF. We may therefore, in the following survey of lines and triangle-planes, omit accents and permutations.

The nodal edge ($x=0$, $y=0$) unites $\overline{ux}$, $\overline{vy}$, $\overline{wz}$, l, p, q, r, fifteen lines. The two nodal rays count six times, viz.

$$(y=0,\ z+\varrho x=0) \text{ unites } \overline{uy},\ \overline{vz},\ \overline{wx},\ n,$$
$$(y=0,\ z-\varrho x=0) \text{ unites } \overline{vx},\ \overline{wy},\ \overline{uz},\ m;$$

in the same order as they here are written, they form a double six. Together $1\cdot15+2\cdot6=27$ lines.

Each line of the one six, combined successively with the five not corresponding lines of the other six, gives rise to five triangle-planes; all the thirty planes so obtained coincide with the nodal plane $y=0$, viz. u, v, w, x, y, z, (uy), (uz), (vx), (vz), (wx), (wy), (lmn). Again, as to the fifteen lines first mentioned, which, as we know, form fifteen triangles, all their planes here coincide with the osculating nodal plane $x=0$, viz. (ux), (vy), (wz), (pqr). Together $1\cdot30+1\cdot15=45$ triangle-planes.

We can distinguish only two species, according as the constant a is negative or positive (1, the two disengaged nodal rays are real; 2, they are conjugate). XI. 1; XI. 2. The case where $a=0$ is not considered, because it would imply a proper node at the point $\frac{\partial}{\partial x}$ with the cone $yw+z^2=0$.

XII. *Cubic surface of the sixth class with a uniplanar node.*

Art. 35. The simplest form of the equation is

$$(x+y+z)^2 w+xyz=0.$$

If all four letters are real, the three nodal rays ($x+y+z=0$, $xyz=0$) are all of them real, and imply the applanished proximity of the node into six angular spaces alternatively full and empty, so that there appear three flat thorns having the node for their common point*. (The surface here considered arises from III. 4, if there all the conjugate values be allowed to coincide by pairs.)

Let $sw=t$, $x+y+z=u$, and $\frac{1}{27}\Theta$ be the discriminant of the binary cubic

$$(t-pu)(t-qu)(t-ru)+\tfrac{1}{4}st^2u$$

in respect to t, u; then $\Theta=0$ will be the equation reciprocal to $u^2w+xyz=0$. Putting

$$\alpha=p+q+r,\quad \beta=qr+rp+pq,\quad \gamma=pqr,$$

* A notion of the form of the surface may be most readily acquired by taking the equation to be

$$z^2+xy(z-mx-ny)=0.\text{—A. C.}$$

we have

$$\Theta=-(q-r)^2(r-p)^2(p-q)^2+\tfrac{1}{2}(\alpha\beta^2+9\beta\gamma-6\alpha^2\gamma)s+\tfrac{1}{16}(12\alpha\gamma-\beta^2)s^2-\tfrac{1}{16}\gamma s^3.$$

The Hessian of the original cubic is

$$4(x+y+z)^2(x^2+y^2+z^2-2yz-2zx-2xy).$$

The spinode curve therefore breaks up into twice the nodal rays (or *axes*)

$$(x+y+z=0,\ xyz=0)$$

and a complete curve of the sixth degree, arising from the intersection of a quadratic cone, which cone is inscribed in the trihedral $(xyz=0)$ of the singular tangent planes in such manner that the lines of contact are harmonical with the nodal rays in respect to the edges of the trihedral. The nodal plane does not really intersect this cone when all three planes of the trihedral are real; but it does so when one of them is real and the two others are conjugate. The node is a quadruple point on the curve of the sixth degree, and the two intersection-lines last mentioned are here a kind of cuspidal tangents.

Art. 36. In order to get a trihedral-pair form, let a, b, c be finite numbers, h a number which ultimately vanishes, and put $(b-c)(c-a)(a-b)=m$, and moreover

$$\begin{aligned}
U&=(b-c)(1+ah)x+mh^2w,\\
V&=(c-a)(1+bh)y+mh^2w,\\
W&=(a-b)(1+ch)z+mh^2w,\\
X&=-mh^2w,
\end{aligned}$$

$$\begin{aligned}
hY&=-mh^3w+(1+ah)(1+ch)x+(1+bh)(1+ah)y+(1+ch)(1+bh)z,\\
-hZ&=\quad mh^3w+(1+ah)(1+bh)x+(1+bh)(1+ch)y+(1+ch)(1+ah)z;
\end{aligned}$$

then the equation

$$UVW+XYZ=m(1+ah)(1+bh)(1+ch)\{w(x+y+z)[x+y+z+h(ax+by+cz)]+xyz\}$$

is identically true, and the six functions U, V, W, X, Y, Z satisfy the identical relations

$$U+V+W+X+Y+Z=0,\quad AU+BV+CW+DX+EY+FZ=0,$$

where the numbers

$$A=(c-a)(a-b)(1+ch),\quad B=(a-b)(b-c)(1+ah),\quad C=(b-c)(c-a)(1+bh),$$
$$D=A-(b-c)^2(1+ah),\quad E=-mh,\quad F=0$$

satisfy the conditions

$$A+B+C=D+E+F,\quad BC+CA+AB=EF+FD+DE,$$

without ABC$-$DEF vanishing. As long therefore as h is finite, the surface UVW+XYZ=0 has a biplanar node at the point $\frac{\partial}{\partial w}$, and this becomes uniplanar when h vanishes. Omitting then accents and permutations, because the three roots of the auxiliary cubic condition are equal, we get the following survey.

The three nodal rays count eight times; for

$$(u=0,\ x=0) \text{ unites } \overline{uy},\ \overline{uz},\ l\ ,\ p,$$
$$(u=0,\ y=0) \text{ unites } \overline{vy},\ \overline{vz},\ n\ ,\ r,$$
$$(u=0,\ z=0) \text{ unites } \overline{wy},\ \overline{wz},\ m,\ q.$$

The sides of the triangle ($w=0$, $xyz=0$) are simple, because they do not pass through the node; they are $\overline{ux}$, $\overline{vx}$, $\overline{wx}$ of the old notation. Together $3{\cdot}8+3{\cdot}1=27$ lines.

The nodal plane $u=0$ unites y, z, (uy), (uz), (vy), (vz), (wy), (wz), (lmn), (pqr), thirty-two triangle-planes. The three singular tangent planes count four times; for $x=0$ unites u, (ux), and so on. The transversal plane $w=0$ is the only simple triangle-plane x of the old notation. In the whole $1{\cdot}32+3{\cdot}4+1{\cdot}1=45$ triangle-planes.

All this might have been foreseen by the help of easy geometrical considerations.

As to reality, the function w must be real, and so must also one at least of the three functions x, y, z, for instance x. We then have only *two* species, according as y, z are real or conjugate. XII. 1; and XII. 2.

XIII. *Cubic surface of the fifth class with a biplanar and two proper nodes.*

Art. 37. Such surface arises from art. 21, when there the binary cubic

$$(y+\alpha x)(y+\beta x)(y+\gamma x)$$

has two equal roots. We are then at liberty to put $\beta=\gamma=0$, $\alpha=1$, and permuting x and y we get

$$yzw+x^2(x+y+z)=0$$

as the equation of the surface, where $\frac{\partial}{\partial w}$ is the biplanar and $\frac{\partial}{\partial y}$, $\frac{\partial}{\partial z}$ are the proper nodes. And the survey given in the same article changes into the following:—

Lines unite.			*Planes* unite.		
$(x=0,\ y=0)$	$m,\ n,$	6	$x=0$	$(uz),\ (uy),\ (vx),\ (wx),$	12
$(x=0,\ z=0)$	$q,\ r,$	6	$y=0$	$(lmn),$	6
$(x=0,\ w=0)$	$\overline{uy},\ \overline{uz},\ \overline{vx},\ \overline{wx},$	4	$z=0$	$(pqr),$	6
$(y=0,\ x+z=0)$	$l,$	3	$x+y=0$	$(vz),\ (wy),$	6
$(z=0,\ x+y=0)$	$p,$	3	$x+z=0$	$(vy),\ (wz),$	6
$(x=-z=w)$	$\overline{vy},\ \overline{wz},$	2	$x-w=0$	$v,\ w,\ y,\ z,$	4
$(x=-y=w)$	$\overline{vz},\ \overline{wy},$	2	$x+y+z=0$	$(ux),$	3
$(w=0,\ x+y+z=0)$	$\overline{ux},$	1	$w=0$	$u,\ x,$	2
		27			45

The discriminant of the ternary cubic $sx^2(x+y+z)-yz(px+qy+vz)$ divided by $q^2r^2s^3$ and then equated to zero is the reciprocal equation of the surface. But this may also

be derived from art. 20, and will be found to be

$$\begin{aligned}\Theta&=(s+p-q-r)\{4(q+r)s+p^2\}^2-8qr(2s+p)^3+9qr(2s+p)\{4(q+r)s+p^2\}-27q^2r^2s\\&=16(q-r)^2s^3+8\{p^2(q+r)+2p(q^2-4qr+r^2)-(q+r)(2q-r)(q-2r)\}s^2\\&\quad+\{p^4+8p^3(q+r)-2p^2(4q^2+23qr+4r^2)+36pqr(q+r)-27q^2r^2\}s\\&\quad+p^3(p-q)(p-r)=0.\end{aligned}$$

The Hessian of $yzw+x^2(x+y+z)$ is $4\{yzw(3x+y+z)+x^2(y-z)^2\}$. Hence the spinode curve breaks up into three times the axes joining the biplanar to the two proper nodes, twice the third axis, and a complete curve of the fourth degree formed by the intersection of the cones

$$(3x+4y+4z)^2-(6x+5y+5z)^2+9(y-z)^2=0,$$
$$(3x+4y+4z)(9w+4x+4y+4z)-16(y-z)^2=0,$$

the latter of which passes through the vertex of the former, *i. e.* through the biplanar node. This is therefore a double point of the curve, and the tangents are

$$(3x+4y+4z=0,\ yz=0).$$

There are but two species; for x, w must be real, and only y, z can either be real or conjugate.

1. All is real. XIII. 1.

2. y and z are conjugate. The two proper nodes are conjugate, and so are also the two planes of the biplanar node. The axis joining the two proper nodes, and the transversal of this axis are the only real lines. XIII. 2.

XIV. *Cubic surface of the fifth class with a biplanar node and a proper node.*

Art. 38. As we have seen above (art. 23), the presence of a biplanar node such as lowers the class by *five* reduces the equation of the surface to the form

$$xyw+xz^2+y^2z-ax^3=0.$$

Because the nodal plane $x=0$ contains but *one* disengaged ray ($x=0$, $z=0$), only the union of the *two* disengaged rays ($y=0$, $z^2-ax^2=0$) in the other nodal plane can give rise to a proper node. Hence the constant a must vanish. The surface in question is therefore represented by

$$xyw+xz^2+y^2z=0$$

in point-coordinates, and consequently by

$$\Theta=27p^2s^3+(36pqr+16q^3)s^2+(pr^3+8q^2r^2)s+qr^4=0$$

in plane-coordinates; $48s^3\Theta$ is the discriminant of the binary cubic

$$12sx^2(px+qy)+3y(rx-sy)^2;$$

and

$$108s\Theta=(54ps^2+36qrs+r^3)^2+(12qs-r^2)^3.$$

The Hessian of the original cubic is $4\{x^2yw+x^2z^2-3xy^2z+y^4\}$. The spinode curve

therefore breaks up into five times the axis ($y=0$, $z=0$) joining the two nodes, four times the nodal edge ($x=0$, $y=0$), and a partial curve of the third degree, which may be represented by

$$\left\| \begin{matrix} y, & z, & w \\ 4x, & y, & -5z \end{matrix} \right\| =0,$$

or, which is the same thing, by $y=2\lambda x$, $z=\lambda^2 x$, $w=-\frac{5}{2}\lambda^3 x$, where λ denotes a variable number. Since the plane touching the original surface at this current point has the equation

$$-8\lambda^3 x+3\lambda^2 y+12\lambda z+4w=0,$$

the spinode develope is represented by the vanishing of the discriminant of the binary cubic $(-8x, y, 4z, 4w)(\lambda, 1)^3$, that is to say, by

$$V=64x^2w^2+(48xyz+y^3)w-128xz^3-3y^2z^2=0;$$

and we have in fact

$$xy^2V=(64x^2yw-64x^2z^2-16xy^2z+y^4)(xyw+xz^2+y^2z)+z(4xz-y^2)^3,$$

which shows that the curve is contained three times in the intersection of the original surface and the developable $V=0$. The cuspidal line of this developable is represented by

$$y=8\lambda x,\ z=-2\lambda^2 x,\ w=2\lambda^3 x,$$

and is therefore a partial curve of the third degree. The equation in plane-coordinates of the spinode curve is

$$675p^2s^2+16pr^3+360pqrs-320q^3s-16q^2r^2=0.$$

In point-coordinates the developable formed by the tangents of the spinode curve is represented by the vanishing of the discriminant of the binary cubic

$$(10x,\ -5y,\ 10z,\ 4w)(\lambda,\ 1)^3.$$

Art. 39. On putting

$$\begin{aligned} U&=\ \ z-hw+k(x+hy), & V&=-y-hz-hkx, & W&=-z+kx, \\ X&=-z+hw+k(x+hy), & Y&=\ \ y+hz-hkx, & Z&=\ \ z+kx, \end{aligned}$$

whence arise the identical relations

$$V+hW+Y+hZ=0 \text{ (holding three times)},$$
$$U+hkV-(1+h^2k)W+X-hkY-(1-h^2k)Z=0 \text{ (accidental)},$$

the identical equation

$$UVW+XYZ=2hk\{xyw+(x+hy)z^2+y^2z-k^2x^3-hk^2x^2y\},$$

when the constants h, k are made to vanish, enables us to perceive what arrangement is here undergone by the 27 lines and 45 triangle-planes of the general surface.

The edge	$(x=0,\ y=0)$	unites $\overline{vy}$, l, p, r,	10
the axis	$(y=0,\ z=0)$	unites $\overline{uy}$, $\overline{vx}$, $\overline{vz}$, $\overline{wy}$, m, n,	10
the ray	$(x=0,\ z=0)$	unites $\overline{ux}$, $\overline{wz}$, q,	5
the ray	$(z=0,\ w=0)$	unites $\overline{uz}$, $\overline{wx}$,	2
			27 lines.

The axis joining the two nodes thus unites five rays of the proper node.

Of the nodal planes, $y=0$ unites v, y, (uy), (vx), (vz), (wy), (lmn), 20 triangle-planes; $x=0$ unites (ux), (vy), (wz), (pqr), 15 triangle-planes; and the only plane containing an actual triangle, $z=0$, unites u, w, x, z, (uz), (wx), 10 triangle-planes; $20+15+10=45$.

The only disengaged ray of the proper node unites two independent lines of the surface. The five lines intersecting both of these coincide in the disengaged ray of the biplanar node. The ten lines meeting but one of the two original lines coincide in the axis joining both nodes. And the ten remaining lines coincide in the edge of the biplanar node.

There is but *one* species, because all four linear functions x, y, z, w must be real. XIV. 1.

XV. *Cubic surface of the fifth class with a uniplanar node.*

Art. 40. We have seen above that the cubic surface with a uniplanar node can always be represented by an equation of the form $x^2w+\mathrm{P}+\mathrm{Q}x$, where $\mathrm{P}=(y,\ z)^3$, $\mathrm{Q}=(y,\ z)^2$, and that, whenever P has no two equal factors, the uniplanar node $\frac{\partial}{\partial w}$ for itself lowers the class by six; but upon considering the case where P has two equal factors, it appears that there is a further reduction of one, making the whole reduction of class to be equal *seven*. We are here allowed to write $\mathrm{P}=y^2z$; and the equation of the surface accordingly is

$$x^2w+y^2z+x(ay^2+2byz+cz^2)=0$$

or, what is the same thing,

$$x^2[cw-ac(ac+b^2)x-2abcy-c(2ac+b^2)z]+(y+bx)^2(cz+acx)+x(cz+acx)^2=0$$

or simply

$$x^2w+y^2z+xz^2=0,$$

where all the variables are necessarily real. The equation in plane-coordinates arises when the discriminant of $(-\frac{3}{2}q^2,\ qr,\ 2ps,\ 3qs\between x,\ y)^3$ is cleared of the factor $\frac{3}{4}q^2s$; hence it is

$$-64p^3s^2-16p^2r^2s+72pq^2rs+27q^4s+16q^2r^3=0.$$

The Hessian of $x^2w+y^2z+xz^2$ is $16x^2(xz-y^2)$. Hence the spinode curve breaks up into six times the double nodal ray $(x=0,\ y=0)$, twice the simple nodal ray $(x=0,\ z=0)$, and once the complete curve $(xz-y^2=0,\ xw+2z^2=0)$, which has the node for a

double point, where the double nodal ray is a tangent common to both branches of the curve.

Art. 41. Denoting by h a number which ultimately vanishes, the surface in question may also be represented by the equation

$$x^2(z+h^2w)-z(x+hy)(x-hy-h^2z)=0.$$

Hence we can see that there is but one simple line ($z=0$, $w=0$), and that all the ten lines intersecting it coincide in the simple nodal ray ($x=0$, $z=0$), while the double nodal ray ($x=0$, $y=0$) unites all the sixteen remaining lines. Again, the plane $z=0$ unites all the five triangle-planes that pass through the only simple line ($z=0$, $w=0$), and the nodal plane $x=0$ unites alone all the forty remaining triangle-planes.

There is but one species. XV. 1.

XVI. *Cubic surface of the fourth class with four proper nodes.*

Art. 42. If we choose the four nodes as points of reference, the equation of the surface necessarily takes the form $ayzw+bxzw+cxyw+dxyz=0$. None of the four constants can vanish, unless the surface break up into a plane and a quadratic surface. We are therefore at liberty to change x, y, z, w respectively into ax, by, cz, dw, when the equation of the surface becomes

$$yzw+xzw+xyw+xyz=0.$$

Since

$$p:q:r:s=\frac{1}{x^2}:\frac{1}{y^2}:\frac{1}{z^2}:\frac{1}{w^2},$$

we have

$$\sqrt{p}+\sqrt{q}+\sqrt{r}+\sqrt{s}=0,$$

or in a rational form

$$(\Sigma p^2-2\Sigma pq)^2-64pqrs=\Sigma p^4-4\Sigma p^3q+6\Sigma p^2q^2+4\Sigma p^2qr-40pqrs=0,$$

as an equation in plane-coordinates.

The Hessian of Σyzw is

$$-4\Sigma(xz+yw)(xw+yz)=-4\Sigma x^2yz=4\{4xyzw-\Sigma x\,.\,\Sigma yzw\}.$$

The spinode curve consequently breaks up into twice each axis (or edge of the tetrahedron of reference).

Art. 43. Trihedral-pair forms are, for instance,

$$(x+y)zw+(z+w)xy=0,$$
$$(x+y)(x+z)(x+w)-x^2(x+y+z+w)=0.$$

The latter shows a transversal triangle-plane $x+y+z+w=0$, which is simple as containing none of the four nodes. Its sides are the transversals of the axes; each of them belongs to two opposite axes, as for instance ($x+y=0$, $z+w=0$), being the transversal common to both singular tangent planes $x+y=0$ and $z+w=0$. The six singular tangent planes lie harmonically in regard to the point $x=y=z=w$. Let any plane $px+qy+rz+sw=0$ pass through this point, whence $p+q+r+s=0$; then to this plane

will harmonically answer the point $px=qy=rz=sw$, and this will describe the surface

$$\frac{1}{x}+\frac{1}{y}+\frac{1}{z}+\frac{1}{w}=0,$$

while the plane turns about that fixed point.

The six axes count four times; the three transversals are simple; $6\cdot4+3\cdot1=27$ lines.

The four planes each of which contains three nodes count eight times, the six singular tangent planes count twice, the transversal plane is simple; together $4\cdot8+6\cdot2+1\cdot1=45$ triangle-planes.

There are three species; the transversal plane is always real.

1. All is real. XVI. 1.

2. x, y are real, z, w conjugate. Two nodes are real, and two are conjugate. Two axes and but one transversal are real. XVI. 2.

3. x, y are conjugate, and so also z, w. All four nodes are imaginary and conjugate by pairs. Two axes and the three transversals are real. XVI. 3.

XVII. *Cubic surface of the fourth class with two biplanar nodes and one proper node.*

Art. 44. Such surface arises from the kind IX. when there $\beta=\gamma$. With a change of letters

$$xyz+xw^2+w^3=0$$

(implying only fourteen constants) is a form to which the equation of such a surface can always be reduced.

Let

$$\mathrm{R}^2=s^2-12qr,$$

then

$$p:(2s-\mathrm{R}):(\mathrm{R}-s)=(yz+w^2):3w^2:2xw,$$

whence

$$9p(\mathrm{R}-s)+2(\mathrm{R}-2s)^2=0,$$

or in a rational form

$$(s^2+4qr)^2-ps^3-36pqrs+27p^2qr=0$$

is the reciprocal equation of the surface.

The Hessian of the original cubic is $4x(xyz+3yzw+xw^2)$; consequently the spinode curve breaks up into four times the axis joining both biplanar nodes, three times the two other axes, and once the conic ($4x+3w=0$, $3yz-w^2=0$). Along this last conic the cone $(8x+9w)^2-27yz=0$ osculates the surface.

The axis ($x=0$, $w=0$) joining both biplanar nodes counts nine times, the two other axes ($w=0$, $yz=0$) count six times, and the two disengaged rays of the biplanar nodes ($y=0$, $x+w=0$) and ($z=0$, $x+w=0$) count three times; $1\cdot9+2\cdot6+2\cdot3=27$ lines.

The plane $w=0$ passing through the three nodes counts eighteen times, the plane $x+w=0$ counts nine times, and the three nodal planes count six times; in the whole $1\cdot18+1\cdot9+3\cdot6=45$ triangle-planes.

Three species may be distinguished.

1. All is real. XVII. 1.

2. y, z are conjugate. The two biplanar nodes are conjugate, and the cone of the proper node is imaginary (it has but one real point). XVII. 2.

3. y and $-z$ are conjugate. The two biplanar nodes are conjugate, and the cone of the proper node is real. XVII. 3.

XVIII. *Cubic surface of the fourth class with a biplanar node and two proper nodes.*

Art. 45. The equation of the surface in point-coordinates is

$$xyw+(x+y)z^2=0,$$

and in plane-coordinates it is

$$(p-q)^2s^2+\tfrac{1}{2}(p+q)r^2s+\tfrac{1}{16}r^4=0,$$

this last equation arising from the discriminant of the binary quadric

$$2s(x+y)(px+qy)+\tfrac{1}{2}r^2xy.$$

The Hessian of the original cubic is

$$4\{(x+y)xyw+(x-y)^2z^2\}.$$

The spinode curve therefore breaks up into four times each of the lines joining the biplanar node to the two proper nodes, twice the line joining both the proper nodes, and twice the nodal edge.

Let h, k be numbers which ultimately vanish, and write

$$\begin{aligned} \mathrm{U}&=-w+2hk(x+y), & \mathrm{V}&=\ \ z+hx+ky, & \mathrm{W}&=\ \ z-hx-ky, \\ \mathrm{X}&=\ \ w+2hk(x+y), & \mathrm{Y}&=-z-hx+ky, & \mathrm{Z}&=-z+hx-ky; \end{aligned}$$

then

$$\mathrm{UVW}+\mathrm{XYZ}=4hk\{xyw+(x+y)(z^2-h^2x^2-k^2y^2)\},$$

$$\mathrm{V}+\mathrm{W}+\mathrm{Y}+\mathrm{Z}=0 \text{ (holding three times)},$$

$$\mathrm{U}-(h+k)\mathrm{V}+(h+k)\mathrm{W}+\mathrm{X}-(h-k)\mathrm{Y}+(h-k)\mathrm{Z}=0 \text{ (accidental)}.$$

Then the lines are as follows, viz.

The axis $(x=0,\ z=0)$ unites	$\overline{vz}$, $\overline{wy}$, m, n,	8
the axis $(y=0,\ z=0)$ unites	$\overline{vy}$, $\overline{wz}$, q, r,	8
the edge $(x=0,\ y=0)$ unites	l, p,	6
the axis $(z=0,\ w=0)$ unites	$\overline{uy}$, $\overline{uz}$, $\overline{vx}$, $\overline{wx}$,	4
the line $(x+y=0,\ w=0)$ is	$\overline{ux}$,	1
		27

the last-mentioned line $\overline{ux}$ being the transversal common to the nodal edge and the axis joining the two proper nodes.

The plane of the three nodes $z=0$ unites v, w, y, z, (uy), (uz), (vx), (wx), sixteen triangle-planes; the nodal planes count each of them twelve times, since $x=0$ unites (vz), (wy), (lmn), and $y=0$ unites (vy), (wz), (pqr). Of the singular tangent planes, that

along the nodal edge, $x+y=0$, unites (ux), three triangle-planes, and that through the two proper nodes, $w=0$, unites u, x, two triangle-planes. In all $16+12+12+3+2=45$ triangle planes.

There are two species, according as x, y are real or conjugate. As an example of the latter species, I may notice the surface generated by a variable circle the diameter whereof is parallel to the axis of a fixed parabola and intercepted between this curve and its tangent at the vertex, while the plane of the circle is perpendicular to that of the parabola.

XIX. *Cubic surface of the fourth class with a biplanar and a proper node.*

Art. 46. Such a surface is represented $xyw+xz^2+y^3=0$ in point-coordinates, and by $64ps^3+(4qs+r^2)^2=0$ in plane-coordinates. The Hessian of the original cubic is $4x(xyw+xz^2-3y^3)$, whence the spinode curve breaks up into six times the edge $(x=0, y=0)$ of the biplanar node and six times the axis $(y=0, z=0)$ joining the two nodes.

From art. 34 it appears that the axis $(y=0, z=0)$ joining the two nodes unites the twelve lines of a double six, and that the edge $(x=0, y=0)$ unites the fifteen remaining lines, $12+15=27$ lines. Moreover it is plain that the axis unites all six rays of the proper node. The nodal plane $y=0$ containing the proper node unites the thirty triangle-planes immediately arising from the double six, and the osculating nodal plane $x=0$ unites all the fifteen remaining triangle-planes, $30+15=45$ triangle-planes.

The plane $z=0$ is not fixed, for we may also write

$$xy(w-2\lambda z-\lambda^2y)+x(z+\lambda y)^2+y^3=0.$$

The equation of the surface therefore implies but thirteen disposable constants.

There is but one species, because everything must be real. XIX. 1.

XX. *Cubic surface of the fourth class with a uniplanar node.*

Art. 47. When in the form $x^2w+P+Qx=0$ of art. 40, P is a perfect cube, which we may denote by y^3, this equation can be reduced to $x^2w+y^3+xz^2=0$. The equation reciprocal to this is $27(4ps+r^2)^2-64q^3s=0$. Since we may also write the equation in the form

$$x^2(w+2\lambda z-\lambda^2x)+y^3+x(z-\lambda x)^2=0,$$

there is nothing to fix the positions of the planes $z=0$ and $w=0$; and the equation of the surface implies only thirteen disposable constants.

The Hessian is $48x^3y$, and the spinode curve breaks up into ten times the line $(x=0, y=0)$ and once the conic section $(y=0, xw+z^2=0)$, along which the cone $xw+z^2=0$ osculates the surface.

By the help of a constant h, which ultimately vanishes, we may represent the surface here considered in the form of art. 40,

$$x^2(-2x-h^2y-3h^3z+h^6w)+(x-h^2y)^2(x+h^2y+h^3z)+x(x+h^2y+h^3z)^2=0,$$

whence we see that here all the twenty-seven lines of the general surface coincide in the line ($x=0$, $y=0$), and all the forty-five triangle-planes in the plane $x=0$.

There is but one species. XX. 1.

XXI. *Cubic surface of the third class with three biplanar nodes.*

Art. 48. The equation is $xyz+w^3=0$ in point-coordinates, and $27pqr-s^3=0$ in plane-coordinates. The Hessian is $12xyzw$; hence the spinode curve breaks up into four times the three axes.

The three axes, as counting each for nine lines, unite all the twenty-seven lines of the surface, and this distribution of them into three groups of nine lines answers to a triad of trihedral-pairs*. The plane $w=0$ of the three nodes counts for twenty-seven triangle-planes, and each of the singular osculating planes $x=0$, $y=0$, $z=0$ counts for six triangle-planes, $27+6+6+6=45$ triangle-planes.

There are two species (if x, w be supposed to be real), according as y, z are real or conjugate. XXI. 1; XXI. 2.

XXII. *Ruled surface of the third order and third class.*

Art. 49. Let us imagine a continuous system of straight lines forming a surface of the nth order, and take at pleasure any one of these lines as an axis about which we turn an intersecting plane. The section will then consist of the axis itself and a plane curve of the $(n-1)$th order, which, of course, intersects the axis in $n-1$ points. But of these one alone can move, while the $n-2$ remaining intersections must be fixed. For the plane cuts an indefinitely near (or *consecutive*) straight line of the system in only one point, and this alone moves. Should any one of the other intersections also move, the axis would be a double line of the surface, whereas it was taken at hazard. Because then the $n-2$ remaining intersections on the axis are fixed, they must arise from a double line of the surface, such double line being met by every generating line in $n-2$ points. Again, to investigate the class of this surface we take an arbitrary line in space; it will intersect the surface in n points, and therefore meet the same number of generating lines. Each plane passing through the arbitrary line and one of these n generating lines will be a tangent plane to the surface. And since there are no other tangent planes than such as pass through a generating line, therefore the class of the surface is equal to its order.

Art. 50. For $n=3$ the double line cannot be a curve; for else an arbitrary plane section of the surface would have two double points at least, and would therefore consist of a straight line and a conic section; but this cannot be the case, unless the surface break up into a plane and a quadratic surface. The double line must therefore be a straight line. Again, since through any point of it there pass (in general) two distinct generating lines, the plane of these two lines must besides cut the surface in a third line (not belonging to the system of generating lines), and this will meet all the gene-

* See Quart. Math. Journal, vol. ii. p. 114.

rating lines. For if we turn a plane about it, the section will always break up into this line itself and a quadratic curve having a double point on the double line of the surface; in other words, the section will be a triangle whereof the vertex moves along the double line, while the two sides are current generating lines, and the base rests on a straight line fixed in position*, which we shall term the *transversal.* It plays the part of the node-couple-develope, since every plane passing through it touches the surface in two points away from the double line, whereas every plane passing through a generating line touches the surface in only one point away from the double line.

Suppose now that there pass through the double line the planes $x=0$, $y=0$, and through the transversal the planes $z=0$, $w=0$. Then the equation of the surface will assume the form $Mz+Nw=0$, and for indefinitely small values of x, y this cubic must become of the second order. Therefore M, N cannot contain z, w, but must be of the form $(x, y)^2$, whence the equation may also be presented in the form $Ax^2+2Bxy+Cy^2=0$, where A, B, C mean homogeneous linear functions of z, w. If then we inquire for what value of the ratio $z:w$ this equation gives two equal values to the ratio $x:y$, the corresponding condition $AC-B^2$ is of the second degree in respect to the ratio required. Hence there lie on the double line only two uniplanar nodes†. We are allowed to let pass through them respectively the planes $z=0$, $w=0$. But then M, N are perfect squares, and we are also at liberty to represent them by $-y^2$, x^2, so that now the equation of the surface becomes

$$x^2w-y^2z=0.$$

Since it obviously implies only thirteen constants, the existence of a double line counts in the cubic surface for *six* conditions. The system $y=\lambda x$, $w=\lambda^2 z$, where λ is an arbitrary parameter, shows the generating line in movement, and affords an easy geometrical construction of the surface, which I think it is not necessary to explain.

The equation reciprocal to $x^2w-y^2z=0$ is $p^2s+q^2r=0$; hence the surface keeps its properties, though point and plane be interchanged.

The Hessian is $-16x^2y^2$. The spinode curve therefore breaks up into eight times the double line and twice the generating lines which pass through the uniplanar nodes and along which the surface is touched by the two singular tangent planes $z=0$ and $w=0$.

There are two species, according as the two uniplanar nodes are real or conjugate. In the first species x, y, z, w are real, and whenever the ratio $z:w$ is negative, the ratio $x:y$ becomes *lateral.* In other words, when the double line between the two uniplanar

* The same thing might also be thus proved. Take any four distinct generating lines; they will in general not lie on a quadratic surface, and, because they are already intersected by the straight double line of the surface, there will be a second straight line intersecting all of them. But since this now has four points in common with the cubic surface, it must lie wholly in the surface.

The problem of drawing through a given generating line a triangle-plane is of the fifth degree, and it may be foreseen that the plane passing through it and the transversal is a single solution; the four remaining solutions must all coincide in the plane passing through the given line and the double line.

† In the language of Dr. SALMON and myself, cuspidal points.—A. C.

nodes is contiguous with the rest of the surface, then it is isolated without them; and when isolated within, then it is contiguous without. The two planes through the transversal and one of the uniplanar nodes are singular tangent planes, and both real. XXII. 1.

In the second species we may assume w conjugate to $-z$ and y to x, and write

$$(x+iy)^2(z+iw)+(x-iy)^2(z-iw)=0,$$

or, what is the same thing,

$$(x^2-y^2)z-2xyw=0,$$

whence arises the system

$$(w=\lambda z, \quad x^2-2\lambda xy-y^2=0),$$

which for all real values of λ gives also real values to the ratio $x:y$. The double line is therefore throughout contiguous to the rest of the surface, and the two singular tangent planes are conjugate. XXII. 2.

[Dr. SCHLÄFLI has omitted to notice a special form of the ruled surface of the third order which presented itself to me, and which I communicated to M. CREMONA and Dr. SALMON, and which is in fact that in which the transversal coincides with the double line. For this species, say XXII. 3, the equation may be taken to be

$$y^3+x(zx+wy)=0:$$

see SALMON'S 'Geometry of Three Dimensions,' pp. 378, 379, where however in the construction of the surface a necessary condition was (by an oversight of mine) omitted. The correct construction is as follows, viz., Given a cubic curve having a double point, and a line meeting the curve in this point (the double line of the surface); if on the line we have a series of points, and through the line a series of planes, corresponding anharmonically to each other, *and such that to the double point considered as a point of the line, there corresponds the plane through one of the tangents at the double point*, then the line drawn through a point (of the double line), and in the corresponding plane, to meet the cubic, generates the surface. The special form in question must, however, have been familiar to M. CHASLES, as I find it alluded to in the foot-note, p. 188, to a paper by him, "Description des Courbes, &c.," Comptes Rendus, 18 November 1861.—A. C.]