II. *A Supplementary Memoir on Caustics.* By A. CAYLEY, *F.R.S.*

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IT is near the conclusion of my "Memoir on Caustics," Philosophical Transactions, vol. cxlvii. (1857), pp. 273–312, remarked that for the case of parallel rays refracted at a circle, the ordinary construction for the secondary caustic cannot be made use of (the entire curve would in fact pass off to an infinite distance), and that the simplest course is to measure off the distance GQ from a line through the centre of the refracting circle perpendicular to the direction of the incident rays. The particular secondary caustic, or orthogonal trajectory of the refracted rays, obtained on the above supposition was shown to be a curve of the order 8; and it was further shown (by consideration of the case wherein the distance GQ is measured off from an arbitrary line perpendicular to the incident rays) that the general secondary caustic or orthogonal trajectory of the refracted rays was a curve of the same order 8. The last-mentioned curve in the case of reflexion, or for $\mu=-1$, degenerates into a curve of the order 6; and I propose in the present supplementary memoir to discuss this sextic curve, viz. the sextic curve which is the general secondary caustic or orthogonal trajectory of parallel rays reflected at a circle.

1. For parallel rays refracted at a circle, taking the equation of the circle to be $x^2+y^2=1$, and the incident rays to be parallel to the axis of x, then if $x=m$ be an arbitrary line perpendicular to the direction of the incident rays, the secondary caustic is the envelope of the circle,

$$\mu^2\{(x-\alpha)^2+(y-\beta)^2\}-(x-m)^2=0,$$

where (α, β) are the coordinates of a variable point on the refracting circle, and as such satisfy the equation $\alpha^2+\beta^2=1$. Or, what is the same thing, writing $\alpha=\cos\theta$, $\beta=\sin\theta$, the secondary caustic is the envelope of the circle

$$\mu^2\{(x-\cos\theta)^2+(y-\sin\theta)^2\}-(x-m)^2=0,$$

where θ is a variable parameter.

2. The equation may be written

$$\mathrm{A}\cos 2\theta+\mathrm{B}\sin 2\theta+\mathrm{C}\cos\theta+\mathrm{D}\sin\theta+\mathrm{E}=0,$$

where

$$\begin{aligned}
\mathrm{A}&=1,\\
\mathrm{B}&=0,\\
\mathrm{C}&=4\mu^2x-4m,\\
\mathrm{D}&=4\mu^2y,\\
\mathrm{E}&=-2\mu^2(x^2+y^2)-2\mu^2+1+2m^2,
\end{aligned}$$

and which in the case of reflexion, or for $\mu=-1$, become

$$\begin{aligned}
&A=1,\\
&B=0,\\
&C=4x-4m,\\
&D=4y,\\
&E=-2(x^2+y^2)-1+2m^2,
\end{aligned}$$

viz. the equation of the variable circle is in this case

$$\cos 2\theta+4(x-m)\cos\theta+4y\sin\theta+2m^2-1-2(x^2+y^2)=0.$$

3. Now in general for the equation

$$A\cos 2\theta+B\sin 2\theta+C\cos\theta+D\sin\theta+E=0,$$

where the coefficients are any functions whatever of the coordinates (x, y), the equation of the envelope is

$$S^3-T^2=0,$$

where

$$\begin{aligned}
S&=12(A^2+B^2)-3(C^2+D^2)+4E^2,\\
-T&=27\,A(C^2-D^2)+54BCD-\big(72(A^2+B^2)+9(C^2+D^2)\big)E+8E^3.
\end{aligned}$$

4. Hence, substituting for A, B, C, D, E the above reflexion values, we find

$$\begin{aligned}
S&=12-48\big((x-m)^2+y^2\big)+4(2m^2-1-2x^2-2y^2)^2,\\
-T&=432\big((x-m)^2-y^2\big)\\
&\quad-72\Big(12+144\big((x-m)^2+y^2\big)\Big)(2m^2-1-2x^2-2y^2)\\
&\quad+\ 8(2m^2-1-2x^2-2y^2)^3.
\end{aligned}$$

Writing in these equations

$$\begin{aligned}
(x-m)^2+y^2&=x^2+y^2-2mx+m^2,\\
(x-m)^2-y^2&=2x^2-2mx+m^2-(x^2+y^2),
\end{aligned}$$

then after some simple reductions, we find

$$\begin{aligned}
S&=16\{(x^2+y^2-m^2-1)^2+6m(x-m)\},\\
T&=32\{2(x^2+y^2-m^2-1)^3+18m(x-m)(x^2+y^2-m^2-1)-27(x-m)^2\},
\end{aligned}$$

and thence

$$S^3-T^2=1024(x-m)^2U,$$

where

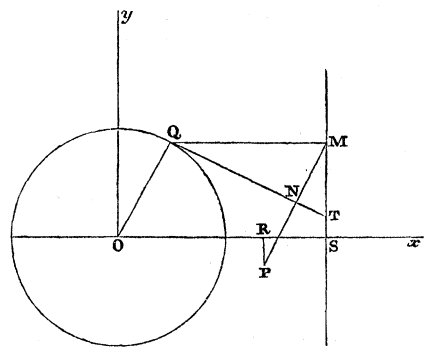
$$\begin{aligned}
U=&\quad\ \ 4(x^2+y^2-m^2-1)^3\\
&+\ 4m^2(x^2+y^2-m^2-1)^2\\
&+36m\,(x^2+y^2-m^2-1)(x-m)\\
&-27\ \ (x-m)^2\\
&+32m^3(x-m),
\end{aligned}$$

or, what is the same thing,

$$\begin{aligned} U= \quad & \quad 4(x^2+y^2)^3 \\ & -(8m^2+12)\ (x^2+y^2)^2 \\ & +(36mx+4m^4-20m^2+12)\ (x^2+y^2) \\ & -\ 27x^2+(-4m^2+18)mx+m^2-4\,; \end{aligned}$$

so that the equation of the secondary caustic is $U=0$, or the secondary caustic is, as stated above, a sextic curve.

5. It is easy to see that the foregoing envelope may be geometrically constructed as follows: viz. if from the point Q (coordinates $\cos\theta$, $\sin\theta$) on the reflecting circle we draw QM perpendicular to the line $x-m=0$, and then from the point M draw MN perpendicular to QT, the tangent at T, and produce MN to a point P such that PN=NM, then P is a point of the envelope; and we thence obtain for the coordinates (x, y) of a point P of the envelope the values



$$x= \quad m-2(m-\cos\theta)\cos^2\theta,$$
$$y= \sin\theta-2(m-\cos\theta)\cos\theta\sin\theta,$$

or, what is the same thing,

$$x=2\cos^3\theta-m(2\cos^2\theta-1),$$
$$y=\sin\theta(2\cos^2\theta+1)-2m\sin\theta\cos\theta,$$

or, as these equations may also be written,

$$x=\tfrac{3}{2}\cos\theta-m\cos 2\theta+\tfrac{1}{2}\cos 3\theta,$$
$$y=\tfrac{3}{2}\sin\theta-m\sin 2\theta+\tfrac{1}{2}\sin 3\theta.$$

6. This result may be verified by showing that these values satisfy the equation

$$\cos 2\theta+4(x-m)\cos\theta+4y\sin\theta+2m^2-1-2(x^2+y^2)=0,$$

and also the derived equation

$$\sin 2\theta+2(x-m)\sin\theta-2y\cos\theta=0.$$

We in fact have

$$x\sin\theta-y\cos\theta= \quad m\sin\theta-\tfrac{1}{2}\sin 2\theta,$$
$$x\cos\theta+y\sin\theta=\tfrac{3}{2}-m\cos\theta+\tfrac{1}{2}\cos 2\theta,$$

and thence

$$(x-m)\sin\theta-y\cos\theta= \quad -\tfrac{1}{2}\sin 2\theta,$$

which is one of the equations to be verified; and also

$$(x-m)\cos\theta+y\sin\theta=\tfrac{3}{2}-2m\cos\theta+\tfrac{1}{2}\cos 2\theta.$$

We have moreover

$$x^2+y^2= \quad \tfrac{5}{2}+m^2-4m\cos\theta+\tfrac{3}{2}\cos 2\theta\,;$$

and, by means of these last equations, the other equation

$$\cos 2\theta+4(x-m)\cos\theta+4y\sin\theta+2m^2-1-2(x^2+y^2)=0,$$

is also verified.

7. The foregoing values of (x, y) give

$$dx=(-\tfrac{3}{2}\sin\theta+2m\sin 2\theta-\tfrac{3}{2}\sin 3\theta)d\theta=-\sin 2\theta(3\cos\theta-2m)d\theta,$$
$$dy=(\ \tfrac{3}{2}\cos\theta-2m\cos 2\theta+\tfrac{3}{2}\cos 3\theta)d\theta=\ \ \cos 2\theta(3\cos\theta-2m)d\theta,$$

or, what is the same thing,

$$dx : dy=-\sin 2\theta : \cos 2\theta.$$

Hence taking for a moment (X, Y) as the current coordinates of a point in the tangent of the envelope, the equation of the tangent of the envelope is

$$\mathrm{X}dy-\mathrm{Y}dx=xdy-ydx,$$

or, substituting for x, y, dx, dy their values, this equation takes the very simple form

$$\mathrm{X}\cos 2\theta-\mathrm{Y}\sin 2\theta-2\cos\theta+m=0,$$

or writing (x, y) in place of (X, Y), that is taking now (x, y) as the current coordinates of a point in the tangent, the equation of the tangent is

$$x\cos 2\theta-y\sin 2\theta-2\cos\theta+m=0;$$

whence observing that this equation may be expressed as a rational equation of the fourth order in terms of the parameter $\tan\frac{1}{2}\theta$ (or $\cos\theta+\sqrt{-1}\sin\theta$), it appears that the class of the secondary caustic is$=4$.

8. The secondary caustic may be considered as the envelope of the tangent, and the equation be obtained in this manner. Comparing with the general equation

$$\mathrm{A}\cos 2\theta+\mathrm{D}\sin 2\theta+\mathrm{C}\cos\theta+\mathrm{D}\sin\theta+\mathrm{E}=0,$$

we have

$$\mathrm{A}=x,$$
$$\mathrm{B}=-y,$$
$$\mathrm{C}=-2,$$
$$\mathrm{D}=0,$$
$$\mathrm{E}=m,$$

and thence

$$\mathrm{S}=4\{3(x^2+y^2)+m^2-3\},$$
$$\mathrm{T}=4\{18m(x^2+y^2)-27x-2m^3+9m\},$$

giving

$$\mathrm{S}^3-\mathrm{T}^2=16\mathrm{V},$$

if for a moment

$$\mathrm{V}=4\{3(x^2+y^2)+m^2-3\}^3$$
$$-\{18m(x^2+y^2)-27x-2m^3+9m\}^2.$$

The equation of the curve is thus obtained in the form $\mathrm{V}=0$; this should of course be equivalent to the before-mentioned equation $\mathrm{U}=0$; and by developing V, and com-

paring with the second of the two expressions of U, it appears that we in fact have $V=27U$.

9. Taking as parameter $\tan\frac{1}{2}\theta$, or if we please $\cos\theta+\sqrt{-1}\sin\theta$, the foregoing values of (x, y) in terms of θ give $(x, y, 1)$ proportional to rational and integral functions of the parameter of the degree 6; so that not only the curve is a sextic curve, but it is a unicursal sextic, or curve of the order 6 with the maximum number, $=10$, of nodes and cusps; that is, if δ be the number of nodes and $\varkappa$ the number of cusps, we have $\delta+\varkappa=10$. Moreover, introducing the same parameter into the equation of the tangent, this equation is seen to be of the degree 4 in the parameter; that is, the class of the curve is $=4$: this implies $2\delta+3\varkappa=26$, and we have therefore $\delta=4$, $\varkappa=6$. To verify these numbers, it is to be remarked that it appears by the equation of the curve that there is at each of the circular points at infinity a triple point in the nature of the point $x=0$, $y=0$ on the curve $y^3=x^4$; such a point is in fact equivalent to two cusps and a node, and we have thus the two circular points at infinity counting together as 2 nodes and 4 cusps; there should therefore besides be 2 nodes and 2 cusps, and I proceed to establish the existence of these by means of the expressions for (x, y) in terms of θ.

10. To find the cusps, we have

$$\frac{dx}{d\theta}=-\sin 2\theta(3\cos\theta-2m)=0,$$

$$\frac{dy}{d\theta}=\quad\cos 2\theta(3\cos\theta-2m)=0,$$

which are each of them satisfied if only $3\cos\theta-2m=0$, or $\cos\theta=\frac{2m}{3}$; the corresponding values of (x, y) are found to be

$$x=m-\frac{8m^3}{27},\quad y=\pm\left(1-\frac{4m^2}{9}\right)^{\frac{3}{2}},$$

and we have thus two cusps situate symmetrically in regard to the axis of x; the cusps are real if $m<\frac{3}{2}$, imaginary if $m>\frac{3}{2}$; for $m=\frac{3}{2}$, the two cusps unite together at the point $x=\frac{1}{2}$ on the axis of x, giving rise to a higher singularity, which will be further examined, *post*, No. 12.

11. The curve is symmetrical in regard to the axis of x, and hence any intersection with the axis of x, not being a point where the curve cuts the axis at right angles, will be a node. Hence, in order to find the nodes, writing $y=0$, this is

$$\sin\theta(1-2m\cos\theta+2\cos^2\theta)=0,$$

giving $\sin\theta=0$, that is,

$$\theta=0,\quad x=\quad 2-m;$$

or

$$\theta=\pi,\quad x=-2-m;$$

but these are each of them ordinary points on the axis of x; or else giving

$$1-2m\cos\theta+2\cos^2\theta=0,$$

that is,

$$\cos\theta=\tfrac{1}{2}(m\pm\sqrt{m^2-2}).$$

The corresponding values of x are

$$x=\cos\theta(2\cos^2\theta-2m\cos\theta)+m,\ =m-\cos\theta,\ =\tfrac{1}{2}(m\mp\sqrt{m^2-2});$$

each of the points in question, viz. the points

$$x=\tfrac{1}{2}(m\mp\sqrt{m^2-2}),\quad y=0,$$

is a node on the axis of x.

12. It is to be observed that for $m<\sqrt{2}$ the nodes are both imaginary; for $m=\sqrt{2}$ they coincide together at the point $x=\frac{1}{\sqrt{2}}$; for $m>\sqrt{2}$ they are both real: it is to be further noticed that

$$\text{node } x=\tfrac{1}{2}(m+\sqrt{m^2-2}) \text{ corresponds to } \cos\theta=\tfrac{1}{2}(m-\sqrt{m^2-2}),$$

where (m being $>\sqrt{2}$) the point ($\cos\theta$, $\sin\theta$) is a real point on the circle $x^2+y^2=1$; in fact for $m<\frac{3}{2}$ (that is, $m=\sqrt{2}$ to $m=\frac{3}{2}$) we have $\frac{1}{2}(m-\sqrt{m^2-2})\ <\frac{1}{2}m$, that is, $\cos\theta<\frac{3}{4}$; but $m=$ or $>\frac{3}{2}$, then $\cos\theta=\frac{1}{2}(m-\sqrt{m^2-2})=\frac{1}{m+\sqrt{m^2-2}}$ is $=$ or $<\frac{1}{2}$, and

$$\text{node } x=\tfrac{1}{2}(m-\sqrt{m^2-2}) \text{ corresponds to } \cos\theta=\tfrac{1}{2}(m+\sqrt{m^2-2}),$$

where (m being $>\sqrt{2}$) the point ($\cos\theta$, $\sin\theta$) is a real point on the circle $x^2+y^2=1$ so long as m is not $>\frac{3}{2}$, that is, from $m=\sqrt{2}$ to $m=\frac{3}{2}$; but if $m>\frac{3}{2}$, then the point in question is an imaginary point on the circle—whence also the node $x=\frac{1}{2}(m-\sqrt{m^2-2})$ is an acnode or isolated point.

In the case $m=\frac{3}{2}$ we have

$$\begin{array}{lll}\text{node } x=1 & \text{corresponding to} & \cos\theta=\tfrac{1}{2} \text{ or } \theta=60^\circ,\\ \quad ,,\quad x=\tfrac{1}{2} & \quad ,, & \cos\theta=1 \text{ or } \theta=0^\circ,\end{array}$$

the last-mentioned point $x=\frac{1}{2}$ being in fact the point of union of two cusps in the case $m=\frac{3}{2}$ now in question. Hence in this case we have at ($x=\frac{1}{2}$, $y=0$) a triple point equivalent to two cusps and a node; visibly, there is only a single branch cutting the axis of x at right angles.

In the case $m=\sqrt{2}$, the nodes coincide as above mentioned at the point $x=\frac{1}{\sqrt{2}}$ on the axis; for this value of m the coordinates of the cusps are

$$x=\tfrac{11}{27}\sqrt{2}\left(=\tfrac{22}{27}\frac{1}{\sqrt{2}},\text{ which is }<\frac{1}{\sqrt{2}}\right);\quad y=\pm\tfrac{1}{27}.$$

13. Starting from the equation $1024\,(x-m)^2\mathrm{U}=\mathrm{S}^3-\mathrm{T}^2=0$, it is clear that the cusps are included among the intersections of the curves $\mathrm{S}=0$, $\mathrm{T}=0$; these two curves intersect in 24 points which lie $9+9$ at the circular points at infinity, $2+2$ at the points $x=m$, $y^2-1=0$, and $1+1$ are the cusps, or points $x=m-\frac{8m^3}{27}$, $y^2=\left(1-\frac{4m^2}{9}\right)^3$. To verify this, writing for a moment

$$\begin{aligned}\mathrm{S}'&=\ (x^2+y^2-m^2-1)^2+\ 6m(x-m),\\ \mathrm{T}'&=2(x^2+y^2-m^2-1)^3+18m(x-m)(x^2+y^2-m^2-1)-27(x-m)^2,\end{aligned}$$

then we have

$$\begin{aligned} \mathrm{T}'-2(x^2+y^2-m^2-1)\mathrm{S}' &= 6m(x-m)(x^2+y^2-m^2-1)-27(x-m)^2 \\ &= 3(x-m)\{2m(x^2+y^2-m^2-1)-9(x-m)\}; \end{aligned}$$

so that the equations $\mathrm{S}=0$, $\mathrm{T}=0$, or, what is the same thing, $\mathrm{S}'=0$, $\mathrm{T}'=0$ give

$$(x-m)\{2m(x^2+y^2-m^2-1)-9(x-m)\}=0,$$

that is, $x-m=0$, or else $x^2+y^2-m^2-1=\frac{9}{2m}(x-m)$. And combining herewith the equation $\mathrm{S}'=(x^2+y^2-m^2-1)^2+6m(x-m)=0$, we have $x-m=0$, $(y^2-1)^2=0$, or else

$$(x^2+y^2-m^2-1)^2=\frac{81}{4m^2}(x-m)^2=6m(x-m),$$

and therefore

$$(x-m)\frac{3}{4m^2}\{27(x-m)-8m^3\}=0,$$

the second factor of which gives $x=m-\frac{8}{27}m^3$, and thence $x^2+y^2-m^2-1=-\frac{4}{3}m^2$, that is, $x^2+y^2=1-\frac{1}{3}m^2$, and therefore $y^2=(1-\frac{1}{3}m^2)-(m-\frac{8}{27}m^3)^2$, $=(1-\frac{4}{9}m^2)^3$, that is, we have

$$x=m-\tfrac{8}{27}m^3,\quad y^2=(1-\tfrac{4}{9}m^2)^3,$$

which, as appears above, gives the two cusps.

14. Similarly, in the equation $16\mathrm{V}=\mathrm{S}^3-\mathrm{T}^2=0$, the intersections of the curves $\mathrm{S}=0$, $\mathrm{T}=0$ must include the cusps; the curves in question are the two circles

$$\begin{gathered} 3\ (x^2+y^2)+m^2-3=0, \\ 18m(x^2+y^2)-27x-2m^3+9m=0, \end{gathered}$$

meeting in the circular points at infinity, and in the two cusps. It is to be added that the tangent at the cusp coincides with the tangent of the last-mentioned circle,

$$18m(x^2+y^2)-27x-2m^3+9m=0,$$

or, as this may also be written,

$$\left(x-\frac{3}{4m}\right)^2+y^2=\left(\frac{4m^2-9}{12m}\right)^2.$$

15. The axis of x meets the secondary caustic in the two nodes counting as 4 intersections, and besides in 2 points, viz. the points $x=2-m$, $x=-2-m$; these correspond to the values $\theta=0$ and $\theta=\pi$ respectively. But to verify them by means of the equation

$$16\mathrm{V}=\mathrm{S}^3-\mathrm{T}^2=0$$

of the curve, it may be remarked that for $y=0$ we have

$$\mathrm{S}=4(3x^2+m^2-3),\ \mathrm{T}=4(18mx^2-27x-2m^3+9m);$$

and writing herein $x=\pm 2-m$, we find

$$\mathrm{S}=4(2m\mp 3)^2,\quad \mathrm{T}=8(2m\mp 3)^3,$$

values which satisfy the equation $\mathrm{S}^3-\mathrm{T}^2=0$.

16. In the equation $U=0$ of the curve, writing $x-m=0$, the equation becomes

$$4(y^2-1)^3+4m^2(y^2-1)^2=0,$$

that is,

$$4(y^2-1)^2(y^2-1+m^2)=0,$$

and the line $(x-m)=0$ is thus a double tangent to the curve touching it at the points $x=m$, $y=\pm 1$, and besides meeting it at the points $x=m$, $y=\pm\sqrt{1-m^2}$, that is, at the intersections of the line $x-m=0$, with the circle $x^2+y^2=1$.

17. The maximum or minimum values of y correspond to the values $\theta=\frac{\pi}{4}$, $\theta=\frac{3\pi}{4}$, $\theta=\frac{5\pi}{4}$, $\theta=\frac{7\pi}{4}$ of θ; and we have for

$$\begin{array}{lll} \theta=\frac{\pi}{4}, & x=\frac{1}{2}\sqrt{2}, & y=\sqrt{2}-m, \\ \theta=\frac{3\pi}{4}, & x=-\frac{1}{2}\sqrt{2}, & y=\sqrt{2}+m, \\ \theta=\frac{5\pi}{4}, & x=-\frac{1}{2}\sqrt{2}, & y=-\sqrt{2}-m, \\ \theta=\frac{7\pi}{4}, & x=\frac{1}{2}\sqrt{2}, & y=-\sqrt{2}+m. \end{array}$$

18. It is now easy to trace the secondary caustic; we may without loss of generality assume that m is positive, and the values to be considered are

$$m=0, \quad m=1, \quad m=\sqrt{2}, \quad m=\tfrac{3}{2},$$

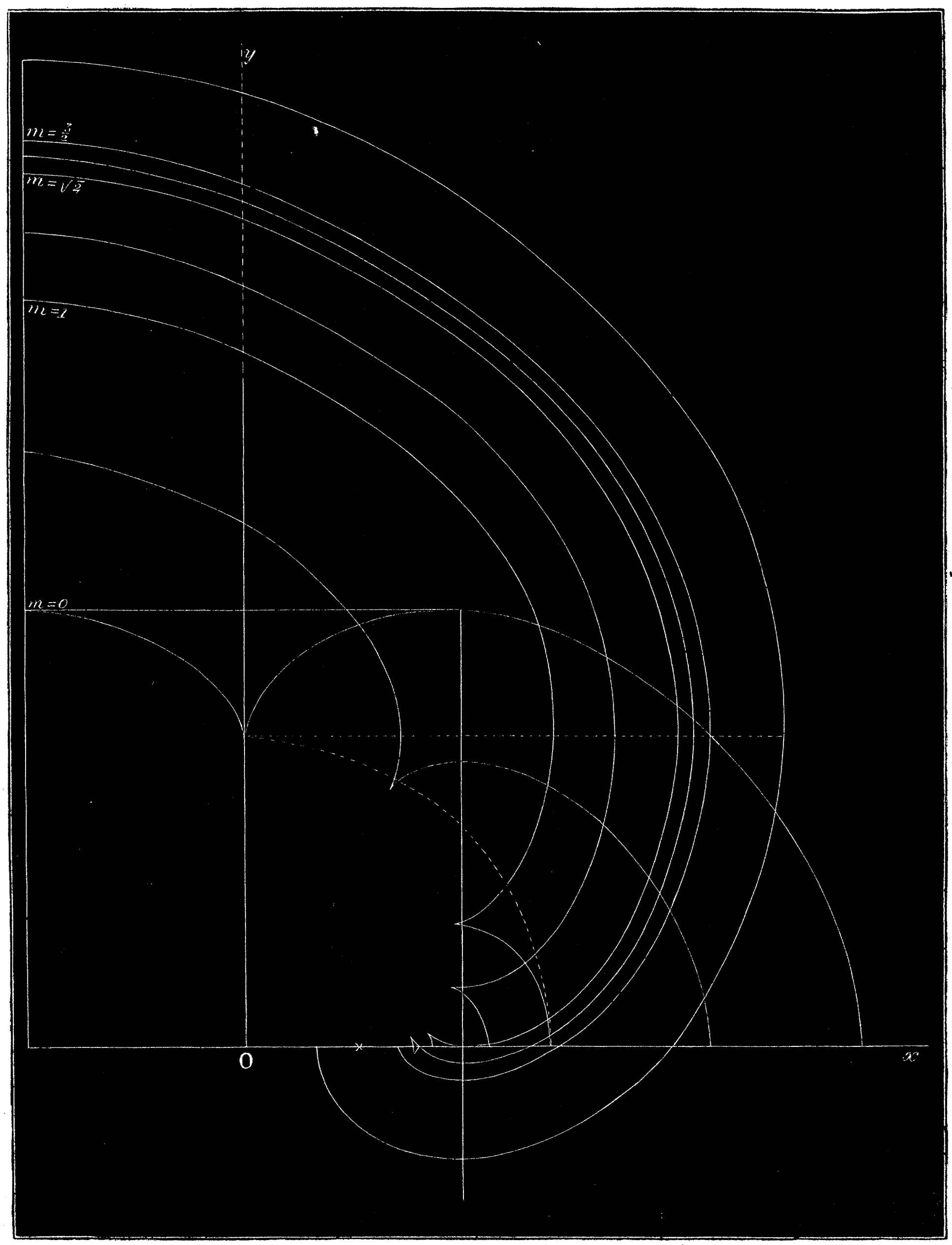
with the intermediate values $m>0<1$, &c. ... and $m>\frac{3}{2}$. I have for convenience delineated in the figure only a portion of each curve, viz. the figure is terminated at the negative value $x=-\frac{1}{2}\sqrt{2}$, which corresponds to the maximum value $y=\sqrt{2}+m$; as x increases negatively, the value of the ordinate y diminishes continuously from this maximum value, becoming $=0$ for the value $x=-2-m$, and the curve at this point cutting the axis of x at right angles; this is a sufficient explanation of the form of the curves beyond the limits of the figure. Moreover the curve is symmetrical in regard to the axis of x, and I have within the limits of the figure delineated only one of the two halves of the curve.

19. For $m>\frac{3}{2}$ the cusps are both imaginary, the nodes both real, but one of them is an isolated point or acnode (shown in the figure by a small cross). The curve has an interior loop, as shown in the figure, and there is also the acnode lying within the loop.

For $m=\frac{3}{2}$, there is still an interior loop, but the acnode has united itself to the loop, the point of union, although presenting no visible singularity, being really a triple point equivalent to a node and two cusps. And in all the cases which follow there are two real cusps.

For $m=<\frac{3}{2}>\sqrt{2}$, the loop has altered its form in such wise as to exhibit the node and two cusps, the curve has therefore two real nodes.

For $m=\sqrt{2}$, the two nodes unite together into a tacnode, so that the loop is on the point of disappearing; and for $m<\sqrt{2}>1$ the nodes are imaginary, and there is thus no longer any loop.



In all the above forms the double tangent $x=m$ touches the curve at the points $y=\pm 1$, but the other two intersections of the double tangent with the curve are imaginary.

For $m=1$, the double tangent has the two coincident real intersections $y=0$, or it is in fact a triple tangent.

For $m<1>0$, the double tangent has with the curve two real intersections, viz. they are the points where the double tangent meets the circle $x^2+y^2=1$.

And finally, for $m=0$, the points in question unite themselves with the points of contact, the double tangent $x=0$ being in this case the common tangent at the two cusps $x=0$, $y=\pm 1$.

Added May 13, 1867.—A. C.

20. As remarked in the original memoir, p. 312, the secondary caustic, in the last-mentioned case $m=0$, is a curve similar to and double the magnitude of the caustic itself (viz. the caustic for parallel rays reflected at a circle), the position of the two curves differing by a right angle.

The secondary caustics corresponding to the different values of m form, it is clear, a system of parallel curves; and, by the remark just referred to, it appears that this system is similar to the system of curves parallel to the caustic for parallel rays reflected at a circle.