XII. *On the Orders and Genera of Ternary Quadratic Forms.* By Henry J. Stephen Smith, *M.A., F.R.S., Savilian Professor of Geometry in the University of Oxford.*

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Eisenstein, in a Memoir entitled "Neue Theoreme der höheren Arithmetik"*, has defined the ordinal and generic characters of ternary quadratic forms of an uneven determinant; and, in the case of definite forms, has assigned the weight of any given order or genus. But he has not considered forms of an even determinant, neither has he given any demonstrations of his results. To supply these omissions, and so far to complete the work of Eisenstein, is the object of the present memoir.

Art. 2. We represent by f the ternary quadratic form

$$ax^2+a'y^2+a''z^2+2byz+2b'xz+2b''xy\,; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

we suppose that f is *primitive* (*i. e.* that the six integral numbers a, a', a'', b, b', b'' admit of no common divisor other than unity), and that its discriminant is different from zero; this discriminant, or the determinant of the matrix

$$\begin{vmatrix} a\,, & b'', & b' \\ b'', & a', & b \\ b', & b\,, & a'' \end{vmatrix}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

we represent by D; by Ω we denote the greatest common divisor of the minor determinants of the matrix (2); by ΩF the contravariant of f, or the form

$$\left.\begin{array}{l}(a'a''-b^2)x^2+(a''a-b'^2)y^2+(aa'-b''^2)z^2 \\ \quad +2(b'b''-ab)yz+2(b''b-a'b')zx+2(bb'-a''b'')xy\,;\end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

we shall term F the *primitive contravariant of* f, and we shall write

$$\mathrm{F}=\mathrm{A}x^2+\mathrm{A}'y^2+\mathrm{A}''z^2+2\mathrm{B}yz+2\mathrm{B}'xz+2\mathrm{B}''xy. \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

If $\mathrm{D}=\Delta\Omega^2$, Δ is an integral number, and the discriminant, contravariant, and primitive contravariant of F are respectively $\Omega\Delta^2$, Δf, and f. The numbers Ω and Δ are arithmetical invariants of f; *i. e.* they remain unaltered when f is transformed by any substitution of which the determinant is unity and the coefficients integral numbers. We shall accordingly describe the primitive form f, and the class of forms containing f, as a form, and class, of the invariants $[\Omega, \Delta]$. Similarly, F is a form of the invariants $[\Delta, \Omega]$, and the class containing F is a class of those invariants. The relation between the forms f and F is reciprocal; and this reciprocity extends throughout the whole

* Crelle's Journal, vol. xxxv. p. 117.

theory, the contravariants f and F, and the invariants Ω and Δ, being everywhere simultaneously interchangeable.

Of definite forms we shall consider only those which are positive; and in the case of such forms we shall suppose Ω, as well as Δ, to be positive, in order that F as well as f may be positive. In the case of indefinite forms we shall always attribute opposite signs to Ω and Δ; so that in this case the discriminants of f and F will be of opposite signs. Thus the definiteness, or indefiniteness, of a form is indicated by the signs of its invariants; if, for example, p and q are positive numbers, the forms $x^2+py^2+pqz^2$, $x^2-py^2-pqz^2$, $-x^2+py^2+pqz^2$ are respectively of the invariants $[p, q]$, $[-p, q]$, $[p, -q]$; and their primitive contravariants, $pqx^2+qy^2+z^2$, $-pqx^2+qy^2+z^2$, $pqx^2-qy^2-z^2$, are respectively of the invariants $[q, p]$, $[q, -p]$, $[-q, p]$.

Art. 3. A primitive form f is properly primitive when one at least of its three *principal* coefficients a, a', a'' is uneven; it is improperly primitive when those coefficients are all even. In an improperly primitive form, one at least of the three coefficients b, b', b'' is uneven (or the form would not be primitive); if, therefore, f is improperly primitive, Ω is uneven and F properly primitive; and, reciprocally, if F is improperly primitive, Δ is uneven and f properly primitive. Again, the discriminant of an improperly primitive form is always even. Whenever, therefore, Ω and Δ are both even, or both uneven, neither f nor F is improperly primitive. Primitive forms of the same invariants $[\Omega, \Delta]$ are said to belong to the same order when they and their primitive contravariants are alike properly or alike improperly primitive. An order of properly primitive forms of the invariants $[\Omega, \Delta]$ always exists, for the form

$$x^2+\Omega y^2+\Omega\Delta z^2$$

is a form of that order. And we shall show hereafter that, when Ω is uneven and Δ even, there is always an improperly primitive order of forms of the invariants $[\Omega, \Delta]$, in which f is improperly and F properly, primitive except when Ω is an uneven square, and $\frac{1}{2}\Delta$ an even or uneven square. And, reciprocally, when Δ is uneven and Ω even, there is always an improperly primitive order of forms of the invariants $[\Omega, \Delta]$, in which f is properly and F improperly primitive, except when Δ is an uneven square, and $\frac{1}{2}\Omega$ an even or uneven square. These exceptions cannot occur if the forms are indefinite.

For example, there are two orders of forms of the invariants $[1, 12]$. The properly primitive order contains three classes, represented by the forms

$$\begin{pmatrix}1, & 1, & 12\\ 0, & 0, & 0\end{pmatrix}, \quad \begin{pmatrix}1, & 3, & 4\\ 0, & 0, & 0\end{pmatrix}, \quad \begin{pmatrix}2, & 3, & 3\\ 1, & 1, & 1\end{pmatrix}.$$

The improperly primitive order, in which the forms are improperly primitive, but their contravariants properly primitive, contains two classes, represented by the forms

$$\begin{pmatrix}2, & 2, & 4\\ -1, & -1, & 0\end{pmatrix}, \quad \begin{pmatrix}2, & 2, & 4\\ 0, & 0, & -1\end{pmatrix}.$$

Art. 4. From the identical equations

$$\left.\begin{aligned}&f(x_1, y_1, z_1)\times f(x_2, y_2, z_2)-\tfrac{1}{4}\left[x_1\frac{df}{dx_2}+y_1\frac{df}{dy_2}+z_1\frac{df}{dz_2}\right]^2\\&\quad=\Omega\mathrm{F}(y_1z_2-z_1y_2,\ z_1x_2-x_1z_2,\ x_1y_2-y_1x_2),\end{aligned}\right\} \quad . \ . \ . \ . \ . \quad (5)$$

$$\left.\begin{aligned}&\mathrm{F}(x_1, y_1, z_1)\times \mathrm{F}(x_2, y_2, z_2)-\tfrac{1}{4}\left[x_1\frac{d\mathrm{F}}{dx_2}+y_1\frac{d\mathrm{F}}{dy_2}+z_1\frac{d\mathrm{F}}{dz_2}\right]^2\\&\quad=\Delta f(y_1z_2-z_1y_2,\ z_1x_2-x_1z_2,\ x_1y_2-y_1x_2),\end{aligned}\right\} \quad . \ . \ . \ . \ . \quad (6)$$

we obtain the subdivision of the Orders into Genera. If ω represent any uneven prime dividing Ω, δ any uneven prime dividing Δ, these equations imply the theorems—

I. "The numbers, prime to ω, which are represented by f, are either all quadratic residues of ω, or all non-quadratic residues of ω." In the first case we attribute to f the *particular generic character* $\left(\frac{f}{\omega}\right)=+1$, in the second we attribute to f the particular generic character $\left(\frac{f}{\omega}\right)=-1$.

II. "The numbers, prime to δ, which are represented by F, are either all quadratic residues of δ, or all non-quadratic residues of δ." We attribute to F the particular generic character $\left(\frac{\mathrm{F}}{\delta}\right)=+1$ in the first case, $\left(\frac{\mathrm{F}}{\delta}\right)=-1$ in the second.

If Ω and Δ are both divisible by any uneven prime, f and F will both have particular characters with respect to that prime. These theorems are due to EISENSTEIN.

Besides its particular characters with respect to uneven primes dividing Ω, f, if properly primitive, will have in certain cases particular characters (which we shall call *supplementary*) with respect to 4 and 8. If the uneven numbers represented by f are all $\equiv 1$, mod 4, we attribute to f the particular character $(-1)^{\frac{f-1}{2}}=+1$; if they are all $\equiv 3$, mod 4, we attribute to f the particular character $(-1)^{\frac{f-1}{2}}=-1$. If they are all either $\equiv 1$, or $\equiv 7$, mod 8, we attribute to f the particular character $(-1)^{\frac{f^2-1}{8}}=+1$; if they are all either $\equiv 3$, or $\equiv 5$, mod 8, we attribute to f the particular character $(-1)^{\frac{f^2-1}{8}}=-1$. Lastly, if they are all either $\equiv 1$, or $\equiv 3$, mod 8, f has the character $(-1)^{\frac{f-1}{2}+\frac{f^2-1}{8}}=+1$; if they are all either $\equiv 5$, or $\equiv 7$, mod 8, it has the character $(-1)^{\frac{f-1}{2}+\frac{f^2-1}{8}}=-1$. Similarly, if F is properly primitive, it will, in certain cases, acquire the characters $(-1)^{\frac{\mathrm{F}-1}{2}}=+1$, or $=-1$; $(-1)^{\frac{\mathrm{F}^2-1}{8}}=+1$, or $=-1$; $(-1)^{\frac{\mathrm{F}-1}{2}+\frac{\mathrm{F}^2-1}{8}}=+1$, or $=-1$.

The following Table is useful for ascertaining the supplementary characters of any proposed form.

TABLE I.

A.—f and F properly primitive.

	$\Omega\equiv1$, mod 2.	$\Omega\equiv2$, mod 4.	$\Omega\equiv4$, mod 8.	$\Omega\equiv0$, mod 8.
$\Delta\equiv1$, mod 2.	Ψ	$(-1)^{\frac{f^2-1}{8}}\Psi$	$(-1)^{\frac{f-1}{2}}$, $*(-1)^{\frac{F-1}{2}}$	$(-1)^{\frac{f-1}{2}}$, $*(-1)^{\frac{F-1}{2}}$ $(-1)^{\frac{f^2-1}{8}}$
$\Delta\equiv2$, mod 4.	$(-1)^{\frac{F^2-1}{8}}\Psi$	$(-1)^{\frac{f^2-1}{8}+\frac{F^2-1}{8}}\Psi$	$(-1)^{\frac{f-1}{2}}$, $\dagger(-1)^{\frac{F^2-1}{8}}$ $*(-1)^{\frac{F-1}{2}+\frac{F^2-1}{8}}$	$(-1)^{\frac{f-1}{2}}$, $*(-1)^{\frac{F-1}{2}+\frac{F^2-1}{8}}$ $(-1)^{\frac{f^2-1}{8}}$, $\dagger(-1)^{\frac{F^2-1}{8}}$
$\Delta\equiv4$, mod 8.	$*(-1)^{\frac{f-1}{2}}$, $(-1)^{\frac{F-1}{2}}$	$\dagger(-1)^{\frac{f^2-1}{8}}$, $(-1)^{\frac{F-1}{2}}$ $*(-1)^{\frac{f-1}{2}+\frac{f^2-1}{8}}$	$(-1)^{\frac{f-1}{2}}$, $(-1)^{\frac{F-1}{2}}$	$(-1)^{\frac{f-1}{2}}$, $(-1)^{\frac{F-1}{2}}$ $(-1)^{\frac{f^2-1}{8}}$
$\Delta\equiv0$, mod 8.	$*(-1)^{\frac{f-1}{2}}$, $(-1)^{\frac{F-1}{2}}$ $(-1)^{\frac{F^2-1}{8}}$	$*(-1)^{\frac{f-1}{2}+\frac{f^2-1}{8}}$, $(-1)^{\frac{F-1}{2}}$ $\dagger(-1)^{\frac{f^2-1}{8}}$, $(-1)^{\frac{F^2-1}{8}}$	$(-1)^{\frac{f-1}{2}}$, $(-1)^{\frac{F-1}{2}}$ $(-1)^{\frac{F^2-1}{8}}$	$(-1)^{\frac{f-1}{2}}$, $(-1)^{\frac{F-1}{2}}$ $(-1)^{\frac{f^2-1}{8}}$, $(-1)^{\frac{F^2-1}{8}}$

B.—f improperly, F properly primitive.

$\Omega\equiv1$, mod 2; $(-1)^{\frac{F-1}{2}}=-(-1)^{\frac{\Omega-1}{2}}$.

$\Delta\equiv2$, mod 4.	$(-1)^{\frac{F-1}{2}}$
$\Delta\equiv0$, mod 4.	$(-1)^{\frac{F-1}{2}}$, $(-1)^{\frac{F^2-1}{8}}$

C.—f properly, F improperly primitive.

$\Delta\equiv1$, mod 2; $(-1)^{\frac{f-1}{2}}=-(-1)^{\frac{\Delta-1}{2}}$.

$\Omega\equiv2$, mod 4.	$(-1)^{\frac{f-1}{2}}$
$\Omega\equiv0$, mod 4.	$(-1)^{\frac{f-1}{2}}$, $(-1)^{\frac{f^2-1}{8}}$

In this Table the asterisk, prefixed to a supplementary character of f, indicates that that character is attributed to f only when $(-1)^{\frac{F-1}{2}}=(-1)^{\frac{\Omega'-1}{2}}$; prefixed to a supplementary character of F, it indicates that that character is attributed to F only when $(-1)^{\frac{f-1}{2}}=(-1)^{\frac{\Delta'-1}{2}}$, Ω' and Δ' denoting the greatest uneven divisors of Ω and Δ, taken

with the same signs as Ω and Δ. Similarly, the obelisk prefixed to a character of f or F indicates that that character is attributable to f or F only when $(-1)^{\frac{F-1}{2}}=-(-1)^{\frac{\Omega'-1}{2}}$ in the first case, and $(-1)^{\frac{f-1}{2}}=-(-1)^{\frac{\Delta'-1}{2}}$ in the second.

The use of the Table is most easily explained by an example. Let the proposed form be

$$f=2x^2+7y^2+7z^2-2yz;$$

its invariants are [2, 24], and its primitive contravariant is

$$F=24x^2+7y^2+7z^2+2yz.$$

Since $\Omega\equiv 2$, mod 4, $\Delta\equiv 0$, mod 8, F has the supplementary characters $(-1)^{\frac{F-1}{2}}$ and $(-1)^{\frac{F^2-1}{8}}$; the values of these characters are found by an inspection of the coefficients, and are -1 and $+1$ respectively. Again, since $\Omega'=1$, $(-1)^{\frac{F-1}{2}}=-(-1)^{\frac{\Omega'-1}{2}}$; the character $(-1)^{\frac{f^2-1}{8}}$ is therefore attributable to f, and an inspection of its coefficients shows that $(-1)^{\frac{f^2-1}{8}}=+1$.

The demonstration of the assertions implied in the Table (so far as they relate to supplementary characters) is obtained without difficulty from the equations (5) and (6). It will suffice to consider one case as an example of the rest. Let f and F be both properly primitive, and let $\Omega=2\Omega'\equiv 2$, mod 4; $\Delta\equiv 0$, mod 8. If $M_1=F(x_1, y_1, z_1)$, $M_2=F(x_2, y_2, z_2)$ are two uneven numbers represented by F, we infer from equation (6) that $\frac{1}{2}\left(x_1\frac{dF}{dx_2}+y_1\frac{dF}{dy_2}+z_1\frac{dF}{dz_2}\right)$ is an uneven number, and consequently that $M_1\times M_2\equiv 1$, mod 8; M_1 and M_2 are therefore congruous to one another, mod 8; *i. e.* all uneven numbers represented by F are congruous, mod 8, or F has the characters $(-1)^{\frac{F-1}{2}}$ and $(-1)^{\frac{F^2-1}{8}}$. To prove that f has the supplementary character attributed to it in the Table, we observe first of all that F cannot represent unevenly even numbers; for, if possible, let $F(x_1, y_1, z_1)$ be unevenly even, and let $F(x_2, y_2, z_2)$ be any uneven number represented by F; then in the equation (6) we have a square congruous, mod 8, to an unevenly even number, which is impossible. Now let $m_1=f(x_1, y_1, z_1)$, $m_2=f(x_2, y_2, z_2)$ be any two uneven numbers represented by f; the number $\frac{1}{2}\left(x_1\frac{df}{dx_2}+y_1\frac{df}{dy_2}+z_1\frac{df}{dz_2}\right)$ is uneven in equation (5); and considering that equation as a congruence for the modulus 8, we find $m_1\times m_2\equiv 1$, or $m_1\times m_2\equiv 1+2\times(-1)^{\frac{\Omega'-1}{2}+\frac{F-1}{2}}$, according as

$$F(y_1z_2-y_2z_1,\ z_1x_2-z_2x_1,\ x_1y_2-x_2y_1)$$

is evenly even, or uneven. If, then, $(-1)^{\frac{F-1}{2}}=(-1)^{\frac{\Omega'-1}{2}}$, $m_1\times m_2$ is $\equiv 1$, or $\equiv 3$, mod 8; *i. e.* the uneven numbers represented by f are either all of one or other of the linear forms $8k+1$, $8k+3$, or else all of one or other of the linear forms $8k+5$, $8k+7$; so that f has the supplementary character $(-1)^{\frac{f-1}{2}+\frac{f^2-1}{8}}$. But if $(-1)^{\frac{F-1}{2}}=-(-1)^{\frac{\Omega'-1}{2}}$,

$m_1 \times m_2 \equiv +1$, or -1, mod 8, and the uneven numbers represented by f are either all of the linear forms $8k \pm 1$, or else all of the linear forms $8k \pm 3$, so that f has the character $(-1)^{\frac{f^2-1}{8}}$.

The signification of the symbols Ψ, $(-1)^{\frac{f^2-1}{8}}\Psi$, $(-1)^{\frac{F^2-1}{8}}\Psi$, $(-1)^{\frac{f^2-1}{8}+\frac{F^2-1}{8}}\Psi$, which occur in the Table, is explained in Arts. 6 and 7. In the next article we shall establish an auxiliary proposition which is frequently useful.

Art. 5. "There exist pairs of forms ϕ and Φ, equivalent to f and F, and satisfying the congruences

$$\left.\begin{aligned} \phi &\equiv \alpha x^2+\beta\Omega y^2+\gamma\Omega\Delta z^2, \\ \Phi &\equiv \beta\gamma\Omega\Delta x^2+\alpha\gamma\Delta z^2+\alpha\beta z^2, \\ \alpha\beta\gamma &\equiv 1, \end{aligned}\right] \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

for any proposed modulus ∇; but this modulus must be uneven, if either f or F is improperly primitive."

In the proof of this proposition we shall employ two lemmas of a very elementary character.

(i) A properly primitive form f represents numbers prime to any given number ∇; and an improperly primitive form f represents the doubles of numbers prime to any given number ∇.

Let p be any prime divisor of ∇, and if f is improperly primitive, let p be an uneven prime. If one of the numbers a, a', a'' is prime to p, let a be prime to p; then if x is prime to p and y and z are divisible by p, f will acquire a value prime to p. If a, a', a'' are all divisible by p, one of the three numbers b, b', b'' must be prime to p; let b be prime to p; then if x is divisible by p, and y and z are prime to p, f will acquire a value prime to p.

If f is improperly primitive and $p=2$, we may consider $\frac{1}{2}f$ instead of f and $\frac{1}{2}a$, $\frac{1}{2}a'$, $\frac{1}{2}a''$ instead of a, a', a''; and we may prove in the same way that $\frac{1}{2}f$ represents uneven numbers.

Thus, among the p^3 systems of values which can be attributed to x, y, z for the modulus p, there are always some which render f (or $\frac{1}{2}f$) prime to p; there are, therefore, among the ∇^3 systems of values which can be attributed to x, y, z for the modulus ∇, some which render f (or $\frac{1}{2}f$) simultaneously prime to every prime dividing ∇.

(ii) If $\Omega\Delta$ is uneven, f represents numbers of both the linear forms $4k+1$ and $4k+3$.

One at least of the principal coefficients of f is uneven, because its discriminant is uneven: let then a be uneven, and let $a' \equiv \lambda$, mod 2, $a'' \equiv \mu$, mod 2; the substitution $x = x + \lambda y + \mu z$ will transform f into a form f_1, in which a_1, a'_1, a''_1 are all uneven, and in which, because the discriminant is uneven, either only one, or else all three, of the coefficients b_1, b'_1, b''_1 are even. The four numbers a_1, a'_1, a''_1, $a_1+a'_1+a''_1+2b_1+2b'_1+2b''_1$ are then all uneven; they are all represented by f_1, that is by f; but they are not all congruous to one another for the modulus 4; therefore f represents numbers of both the linear forms $4k+1$ and $4k+3$.

It follows from these lemmas (i) that if f is an improperly primitive form, we can find

a form equivalent to f, and having one of its principal coefficients unevenly even and prime to any uneven number; (ii) that if f is properly primitive, we can find a form equivalent to f, and having one of its principal coefficients prime to any given number; (iii) that if $\Omega\Delta$ is uneven, we may suppose this principal coefficient of either of the two linear forms $4k+1$, or $4k+3$, at our option.

We shall first suppose that the forms f and F, which it is proposed to transform into forms ϕ and Φ satisfying the congruences (7), are properly primitive. Let $\nabla'=\nabla\Omega^2\Delta$, and let us assume that in the form F, A'' is prime to ∇', and also that $A''\equiv\Omega$, mod. 4, if $\Omega\Delta$ is uneven. Let $\gamma\equiv\frac{1}{A''}$, mod ∇'; the redundant system of congruences

$$\left.\begin{aligned} a\,x+b''y+b' &\equiv 0, \\ b''x+a'y+b &\equiv 0 \\ b'x+by+a'' &\equiv \gamma\Omega\Delta, \end{aligned}\right] \text{mod } \nabla',$$

is resoluble, admitting Ω incongruous solutions*. Let

$$\left.\begin{aligned} x &\equiv \lambda, \\ y &\equiv \mu, \end{aligned}\right] \text{mod } \nabla',$$

be any one of these solutions, and let us transform f by the substitution

$$\left.\begin{aligned} x &= x+\lambda z, \\ y &= y+\mu z, \end{aligned}\right]$$

into an equivalent form f_1. The coefficients a_1, b''_1, a'_1 are the same as a, b'', a'; the coefficients a''_1, b_1, b'_1 are respectively congruous for the modulus ∇' to $\gamma\Omega\Delta$, 0, 0; so that f_1 satisfies the congruence

$$f_1\equiv ax^2+2b''xy+a'y^2+\gamma\Omega\Delta z^2, \text{ mod } \nabla'.$$

The binary form (a, b'', a') is primitive; for if d is a prime dividing a, b'', a', it divides $-\Omega A''$, the determinant of (a, b'', a'), and $\Omega^2\Delta$, the discriminant of f; it therefore divides Ω (because A'' and Δ are relatively prime), and is a common divisor of the coefficients of the primitive form f_1, i. e. $d=1$. Again, (a, b'', a') is not improperly primitive; if $\Omega\Delta$ is even, this is manifest, for f_1 is not improperly primitive; if $\Omega\Delta$ is uneven, $\Omega A''$ is by hypothesis of the form $4k+1$, and there are no improperly primitive binary forms of the determinant $-\Omega A''$. We may now suppose that, in the properly primitive binary form (a, b'', a'), a is uneven and prime to ∇'; let $\beta\equiv\frac{A''}{a}$, mod ∇'; then the congruences

$$\left.\begin{aligned} a\,x+b'' &\equiv 0, \\ b''x+a' &\equiv \beta\Omega, \end{aligned}\right] \text{mod } \nabla',$$

are resoluble and admit of one solution. Let $x\equiv\lambda$, mod ∇', be that solution; if f_1 be transformed by the substitution $x=x+\lambda y$, the resulting form ϕ will satisfy the congruence

$$\phi\equiv ax^2+\beta\Omega y^2+\gamma\Omega\Delta z^2, \text{ mod } \nabla',$$

and the forms ϕ and Φ will satisfy the congruences (7) for the modulus ∇.

* Philosophical Transactions, vol. cli. p. 323.

Attributing to ∇' the value $\nabla\Omega'^2\Delta'$, we may apply the same demonstration to the case in which either f or F is improperly primitive, the modulus ∇ being supposed uneven.

Art. 6. When neither f nor F is improperly primitive, and neither Ω nor Δ evenly even, f and F, considered separately, have no particular characters, properly so called, with respect to 4 or 8. But, considered jointly, they have a certain character with respect to 4 or 8, which we shall term their *simultaneous generic character*, and which it is important to consider here.

If $m=f(x, y, z)$, $\mathrm{M}=\mathrm{F}(\mathrm{X}, \mathrm{Y}, \mathrm{Z})$, the representations of m by f and of M by F are said to be *simultaneous* when x, y, z, X, Y, Z satisfy the equation

$$x\mathrm{X}+y\mathrm{Y}+z\mathrm{Z}=0. \qquad (8)$$

This definition of simultaneous representation suffices for our immediate purpose; we add, however, that if the two representations are *primitive* * as well as simultaneous, the equations

$$\left\| \begin{matrix} x, & y, & z \\ x', & y', & z' \end{matrix} \right\| = \mathrm{X}, \mathrm{Y}, \mathrm{Z}, \qquad (9)$$

$$\left\| \begin{matrix} \mathrm{X}, & \mathrm{Y}, & \mathrm{Z} \\ \mathrm{X}', & \mathrm{Y}', & \mathrm{Z}' \end{matrix} \right\| = x, y, z \qquad (10)$$

are resoluble in integral numbers x', y', z', $\mathrm{X}', \mathrm{Y}', \mathrm{Z}'$. For, considering the first of these equations, we observe that $[x, y, z]$ is a given solution, in relatively prime numbers, of the equation (8); let $[x', y', z']$ be a solution of the same equation which with $[x, y, z]$ forms a fundamental set; the equation (9) is then satisfied by virtue of the characteristic property of the fundamental set (Philosophical Transactions, vol. cli. p. 297). Thus if the representations of m by f, and of M by F, are primitive and simultaneous, m is primitively represented by a binary form of determinant $-\Omega\mathrm{M}$, which is itself primitively represented by f †; and, reciprocally, M is primitively represented by a binary form of determinant $-\Delta m$, which is itself primitively represented by F.

In the four cases in which neither Ω nor Δ is evenly even, the simultaneous character of f and F is given in the Table of Art. 4. The symbol Ψ in that Table represents the unit $(-1)^{\frac{\Delta'f+1}{2}\cdot\frac{\Omega'\mathrm{F}+1}{2}}$, *i.e.* the unit $(-1)^{\frac{\Delta'm+1}{2}\cdot\frac{\Omega'\mathrm{M}+1}{2}}$, Ω' and Δ' denoting the same numbers as in Art. 4, and m, M being any two uneven numbers simultaneously represented by f and F. Thus, if $\Omega\equiv\Delta\equiv1$, mod 2, the simultaneous character Ψ is attributed in the Table to f and F; *i. e.* the uneven numbers simultaneously represented by f and F either all satisfy the equation $\Psi=1$, or else all satisfy the equation $\Psi=-1$.

To demonstrate these simultaneous characters we consider the four cases separately, and for the forms f and F we substitute forms ϕ and Φ equivalent to them, and satisfying

* The representation of a number by a form is said to be *primitive* when the values of the indeterminates do not admit of any common divisor besides unity.

† See Art. 10.

certain congruences for the modulus 4 or 8. The existence of the equivalent forms thus assumed results, in each case, from the theorem of the last article.

Case (i) Let $\Omega \equiv \Delta \equiv 1$, and let φ and Φ satisfy the congruences

$$\left.\begin{array}{l} \Delta\varphi \equiv \alpha x^2+\beta y^2+\gamma z^2, \\ \Omega\Phi \equiv \beta\gamma X^2+\alpha\beta Y^2+\alpha\beta Z^2, \\ \text{or} \quad \Omega\Phi \equiv \alpha X^2+\beta Y^2+\gamma Z^2, \\ \alpha\beta\gamma \equiv 1, \end{array}\right\} \text{mod } 4.$$

Attributing in succession to the indeterminates

$$\begin{array}{c} x,\ y,\ z \\ X,\ Y,\ Z \end{array}$$

all systems of values, mod 2, which satisfy the congruence

$$xX+yY+zZ \equiv 0, \text{ mod } 2,$$

and which render m and M simultaneously uneven, we find that in every case Δm is congruous, for the modulus 4, to one of the numbers α, β, γ, and ΩM to one of the remaining two. Thus $\frac{\Delta m+1}{2}\times\frac{\Omega M+1}{2}$ is necessarily congruous, for the modulus 2, to one of the three numbers

$$\frac{(\beta+1)(\gamma+1)}{4},\ \frac{(\gamma+1)(\alpha+1)}{4},\ \frac{(\alpha+1)(\beta+1)}{4}.$$

But these numbers are all congruous to one another for the modulus 2, because the congruence $\alpha\beta\gamma=1$, mod 4, implies the congruence $\alpha+\beta+\gamma+1=0$, mod 4. Therefore the unit Ψ has always the same value for every pair of uneven numbers simultaneously represented by f and F.

It will be seen that $\Psi=-1$, or $\Psi=+1$, according as the congruences $\alpha\equiv\beta\equiv\gamma\equiv 1$, mod 4, are or are not satisfied.

Case (ii) Let $\Omega\equiv 2$, mod 4, $\Delta\equiv 1$, mod 2, and let

$$\Delta\varphi \equiv \alpha x^2+2\beta y^2+2\gamma z^2, \text{ mod } 8,$$
$$\Omega'\Phi \equiv 2\alpha X^2+\beta Y^2+\gamma Z^2, \text{ mod } 4,$$
$$\alpha\beta\gamma \equiv 1, \text{ mod } 4.$$

The admissible combinations of the values of x, y, z, X, Y, Z, mod 2, give rise to eight cases,

$$\begin{array}{lll} \Delta m \equiv \alpha, & \text{mod } 8; & \Omega'M \equiv \ \ \beta, \text{ or } \ \ \gamma, \text{ mod } 4, \\ \Delta m \equiv \alpha+2\beta, & \text{mod } 8; & \Omega'M \equiv -\beta, \text{ or } +\gamma, \text{ mod } 4, \\ \Delta m \equiv \alpha+2\gamma, & \text{mod } 8; & \Omega'M \equiv \ \ \beta, \text{ or } -\gamma, \text{ mod } 4, \\ \Delta m \equiv \alpha+2\beta+2\gamma, & \text{mod } 8; & \Omega'M \equiv -\beta, \text{ or } -\gamma, \text{ mod } 4, \end{array}$$

and, in all of them, the value of the unit $(-1)^{\frac{\Delta^2m^2-1}{8}}\Psi$, and therefore of the unit $(-1)^{\frac{f^2-1}{8}}\Psi$, is the same, because, by virtue of the congruence $\alpha+\beta+\gamma+1\equiv 0$, mod 4,

the four numbers

$$\frac{(\alpha+1)(\alpha+2\beta+1)}{8},\ \frac{(\alpha+1)(\alpha+2\gamma+1)}{8},$$

$$\frac{(\alpha+2\beta+2\gamma+1)(\alpha+2\beta+1)}{8},\ \frac{(\alpha+2\beta+2\gamma+1)(\alpha+2\gamma+1)}{8}$$

are all congruous to one another for the modulus 2.

Case (iii) $\Omega\equiv1$, mod 2, $\Delta\equiv2$, mod 4. In this case the simultaneous character of the forms f and F may be demonstrated as in case (ii), or may be inferred by reciprocation from the result in that case.

Case (iv) $\Omega\equiv\Delta\equiv2$, mod 4. Let

$$\Delta'\phi\equiv\alpha x^2+2\beta y^2+4\gamma z^2,\ \text{mod } 8,$$
$$\Omega'\Phi\equiv4\alpha X^2+2\beta Y^2+\gamma Z^2,\ \text{mod } 8,$$
$$\alpha\beta\gamma\equiv1,\ \text{mod } 4.$$

Here again there are eight cases,

$$\begin{array}{lll}
\Delta'm\equiv\alpha & ;\ \Omega'M\equiv\gamma, & \text{or } \gamma+2\beta\quad, \text{mod } 8,\\
\Delta'm\equiv\alpha+2\beta & ;\ \Omega'M\equiv\gamma, & \text{or } \gamma+2\beta+4, \text{mod } 8,\\
\Delta'm\equiv\alpha+4 & ;\ \Omega'M\equiv\gamma+4, & \text{or } \gamma+2\beta+4, \text{mod } 8,\\
\Delta'm\equiv\alpha+2\beta+4 & ;\ \Omega'M\equiv\gamma+4, & \text{or } \gamma+2\beta\quad, \text{mod } 8;
\end{array}$$

and in all eight the value of the unit $(-1)^{\frac{\Delta'^2m^2-1}{8}+\frac{\Omega'^2M^2-1}{8}}\Psi$, and therefore of the unit $(-1)^{\frac{f^2-1}{8}+\frac{F^2-1}{8}}\Psi$, is the same, because by virtue of the congruence $\alpha+\beta+\gamma+1\equiv0$, mod 4, the two numbers

$$\frac{(\alpha+\gamma)(\alpha+\gamma+2)}{8},\ \frac{(\alpha+\gamma+2\beta)(\alpha+\gamma+2\beta+2)}{8}$$

are congruous to one another for the modulus 2.

Art. 7. The following observations will serve to show more clearly the import of the simultaneous character in each of the four cases.

Case (i) Let $\Psi=-1$; then, if m and M are any two uneven numbers simultaneously represented by f and F, $m\equiv\Delta$, mod 4, and $M\equiv\Omega$, mod 4. Also f cannot represent numbers congruous to 7Δ, mod 8, nor F numbers congruous to 7Ω, mod 8; for the congruences

$$\frac{(\beta+1)(\gamma+1)}{4}\equiv\frac{(\gamma+1)(\alpha+1)}{4}\equiv\frac{(\alpha+1)(\beta+1)}{4}\equiv1,\ \text{mod } 2,$$

imply that $\alpha\equiv\beta\equiv\gamma\equiv1$, mod 4; *i. e.* that ϕ, or, which is the same thing, f can only represent uneven numbers congruous to Δ, 3Δ, 5Δ. And similarly of uneven numbers F can only represent those which are congruous to Ω, 3Ω, 5Ω. Numbers congruous to 3Δ are represented by f, and numbers congruous to 3Ω are represented by F; but these representations are not simultaneous with the representation of any uneven number by F in the first case, and by f in the second.

Let $\Psi=+1$; then if m and M are uneven numbers simultaneously represented by f and F, one at least of the two congruences $m\equiv-\Delta$, mod 4, $\mathrm{M}\equiv-\Omega$, mod 4, must be satisfied. Subject to this restriction, m and M may have any of the four linear forms $8k+1$, 3, 5, 7.

Case (ii) The restrictions imposed on the numbers m and M by the simultaneous characters are exhibited in the annexed Table.

If	$(-1)^{\frac{f^2-1}{8}}\Psi=(-1)^{\frac{\Delta^2-1}{8}}$.	$(-1)^{\frac{f^2-1}{8}}\Psi=-(-1)^{\frac{\Delta^2-1}{8}}$.
$\mathrm{M}\equiv\ \Omega'$, mod 4.	$m\equiv5\Delta,\ 7\Delta$, mod 8.	$m\equiv\ \Delta,\ 3\Delta$, mod 8.
$\mathrm{M}\equiv3\Omega'$, mod 4.	$m\equiv\ \Delta,\ 7\Delta$, mod 8.	$m\equiv3\Delta,\ 5\Delta$, mod 8.

Except when Ω and Δ are both uneven it will be found that, in the case of any two properly primitive forms f and F, every representation of an uneven number by either of the two is simultaneous with the representation of uneven numbers by the other. If therefore $(-1)^{\frac{f^2-1}{8}}\Psi=(-1)^{\frac{\Delta^2-1}{8}}$, f cannot represent numbers congruous to 3Δ, mod 8, because it cannot represent them simultaneously with uneven numbers, and if $(-1)^{\frac{f^2-1}{8}}\Psi=-(-1)^{\frac{\Delta^2-1}{8}}$, f cannot represent numbers congruous to 7Δ, mod 8.

Case (iii) In this case, which is the reciprocal of the last, we have the Table,

If	$(-1)^{\frac{\mathrm{F}^2-1}{8}}\Psi=(-1)^{\frac{\Omega^2-1}{8}}$.	$(-1)^{\frac{\mathrm{F}^2-1}{8}}\Psi=-(-1)^{\frac{\Omega^2-1}{8}}$.
$m\equiv\ \Delta'$, mod 4.	$\mathrm{M}\equiv5\Omega,\ 7\Omega$, mod 8.	$\mathrm{M}\equiv\ \Omega,\ 3\Omega$, mod 8.
$m\equiv3\Delta'$, mod 4.	$\mathrm{M}\equiv\ \Omega,\ 7\Omega$, mod 8.	$\mathrm{M}\equiv3\Omega,\ 5\Omega$, mod 8.

And F cannot represent numbers congruous to 3Ω, or cannot represent numbers congruous to 7Ω, according as $(-1)^{\frac{\mathrm{F}^2-1}{8}}\Psi=(-1)^{\frac{\Omega^2-1}{8}}$, or $-(-1)^{\frac{\Omega^2-1}{8}}$.

Case (iv) In this case both f and F represent numbers of all the four linear forms $8k+1$, 3, 5, 7. The Table, in which the modulus is everywhere 8, exhibits the restrictions imposed by the simultaneous character.

If	$(-1)^{\frac{f^2-1}{8}+\frac{\mathrm{F}^2-1}{8}}\Psi=(-1)^{\frac{\Delta'^2-1}{8}+\frac{\Omega'^2-1}{8}}$	$(-1)^{\frac{f^2-1}{8}+\frac{\mathrm{F}^2-1}{8}}\Psi=-(-1)^{\frac{\Delta'^2-1}{8}+\frac{\Omega'^2-1}{8}}$
$m\equiv\ \Delta'$	$\mathrm{M}\equiv5\Omega',\ 7\Omega'$	$\mathrm{M}\equiv\ \Omega',\ 3\Omega'$
$m\equiv3\Delta'$	$\mathrm{M}\equiv3\Omega',\ 5\Omega'$	$\mathrm{M}\equiv\ \Omega',\ 7\Omega'$
$m\equiv5\Delta'$	$\mathrm{M}\equiv\ \Omega',\ 3\Omega'$	$\mathrm{M}\equiv5\Omega',\ 7\Omega'$
$m\equiv7\Delta'$	$\mathrm{M}\equiv\ \Omega',\ 7\Omega'$	$\mathrm{M}\equiv3\Omega',\ 5\Omega'$

Art. 8. The complete generic character of a form or class is the complex of all the particular characters attributable to the form or class, and to its primitive contravariant, including their simultaneous character, if any. And two forms, or classes, which have the same complete generic character are said to belong to the same genus. But not every complete generic character that can be assigned *à priori*, is the character of any really existing genus of forms. The annexed Table will serve, in the case of any given order, to distinguish those complete generic characters, which are possible, *i. e.* to which actually existing genera correspond, from those which are impossible.

In this Table Ω_2^2 and Δ_2^2 are the greatest squares dividing Ω and Δ; the quotients $\Omega\div\Omega_2^2$, $\Delta\div\Delta_2^2$ are respectively represented, if uneven, by Ω_1 and Δ_1, if even by $2\Omega_1$ and $2\Delta_1$, so that Ω_1 and Δ_1 are always uneven and not divisible by any square; ω_1 and δ_1 are any primes dividing Ω_1 and Δ_1, ω_2 and δ_2 are any uneven primes dividing Ω_2 and Δ_2, but ω_2 must not divide Ω_1, nor must δ_2 divide Δ_1; lastly, Ψ is still the unit $(-1)^{\frac{\Omega'F+1}{2}\cdot\frac{\Delta'F+1}{2}}$, or, which is the same thing, the unit $(-1)^{\frac{\Omega_1F+1}{2}\cdot\frac{\Delta_1F+1}{2}}$, f and F in the exponents of these units denoting uneven numbers simultaneously represented by the forms f and F.

The Table A of properly primitive generic characters contains twenty-five compartments corresponding to the twenty-five cases indicated in its margin; the Tables B and C of improperly primitive genera contain three such compartments each. In each compartment are inscribed all the particular characters which make up the complete generic character of a form coming under the case to which the compartment corresponds; the symbols $\left(\frac{f}{\omega_1}\right)$, $\left(\frac{f}{\omega_2}\right)$, $\left(\frac{F}{\delta_1}\right)$, $\left(\frac{F}{\delta_2}\right)$ implying that f has a particular character with respect to every prime ω_1 or ω_2, F a particular character with respect to every prime δ_1 or δ_2. Each compartment is divided into two parts by a vertical line, and the particular characters (one of which in Table A either is or contains Ψ) placed to the left of this line are subject to the condition that their product is equal in Table A to the unit $(-1)^{\frac{\Omega_1+1}{2}\cdot\frac{\Delta_1+1}{2}}$, in Table B to the unit $(-1)^{\frac{\Omega_1^2-1}{8}}\times(-1)^{\frac{\Omega_1+1}{2}\cdot\frac{\Delta_1+1}{2}}$, in Table C to the unit $(-1)^{\frac{\Delta_1^2-1}{8}}\times(-1)^{\frac{\Omega_1+1}{2}\cdot\frac{\Delta_1+1}{2}}$. If $\alpha=+1$, or -1, according as Ω is of the form $\Omega_1\Omega_2^2$, or $2\Omega_1\Omega_2^2$, and if, similarly $\beta=+1$, or -1, according as Δ is of the form $\Delta_1\Delta_2^2$, or $2\Delta_1\Delta_2^2$, we may express this condition in Table A by the equation

$$\Psi\times\alpha^{\frac{f^2-1}{8}}\times\beta^{\frac{F^2-1}{8}}\times\left(\frac{f}{\Omega_1}\right)\times\left(\frac{F}{\Delta_1}\right)=(-1)^{\frac{\Omega_1+1}{2}\cdot\frac{\Delta_1+1}{2}}; \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

and in Tables B and C respectively by the equations

$$(-\beta)^{\frac{F^2-1}{8}}\times\left(\frac{f}{\Omega_1}\right)\times\left(\frac{F}{\Delta_1}\right)=(-1)^{\frac{\Omega_1^2-1}{8}+\frac{\Omega_1+1}{2}\cdot\frac{\Delta_1+1}{2}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

$$(-\alpha)^{\frac{f^2-1}{8}}\times\left(\frac{f}{\Omega_1}\right)\times\left(\frac{F}{\Delta_1}\right)=(-1)^{\frac{\Delta_1^2-1}{8}+\frac{\Omega_1+1}{2}\cdot\frac{\Delta_1+1}{2}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

The condition distinguishes the possible and impossible genera, every generic character which satisfies it being the character of an actually existing genus, and every

A.—f

	$\Omega=\Omega_1\Omega_2^2.$	$\Omega_2\equiv1$, mod 2.	$\Omega=\Omega_1\Omega_2^2.$	$\Omega_2\equiv2$, mod 4.
$\Delta=\Delta_1\Delta_2^2.$ $\Delta_2\equiv1$, mod 2.	Ψ $\left(\frac{f}{\omega_1}\right)$, $\left(\frac{F}{\delta_1}\right)$ S	$\left(\frac{f}{\omega_2}\right)$, $\left(\frac{F}{\delta_2}\right)$ $\Gamma=2^\gamma$	Ψ $\left(\frac{f}{\omega_1}\right)$, $\left(\frac{F}{\delta_1}\right)$ Q	$(-1)^{\frac{f-1}{2}}$ $\left(\frac{f}{\omega_2}\right)$, $\left(\frac{F}{\delta_2}\right)$ $\Gamma=3\times2^{\gamma-1}$
$\Delta=\Delta_1\Delta_2^2.$ $\Delta_2\equiv2$, mod 4.	Ψ $\left(\frac{f}{\omega_1}\right)$, $\left(\frac{F}{\delta_1}\right)$ Q	$(-1)^{\frac{F-1}{2}}$ $\left(\frac{f}{\omega_2}\right)$, $\left(\frac{F}{\delta_2}\right)$ $\Gamma=3\times2^{\gamma-1}$	Ψ $\left(\frac{f}{\omega_1}\right)$, $\left(\frac{F}{\delta_1}\right)$ P	$(-1)^{\frac{f-1}{2}}$, $(-1)^{\frac{F-1}{2}}$ $\left(\frac{f}{\omega_2}\right)$, $\left(\frac{F}{\delta_2}\right)$ $\Gamma=2^{\gamma+1}$
$\Delta=\Delta_1\Delta_2^2.$ $\Delta_2\equiv0$, mod 4.	Ψ $\left(\frac{f}{\omega_1}\right)$, $\left(\frac{F}{\delta_1}\right)$ Q	$(-1)^{\frac{F-1}{2}}$ $(-1)^{\frac{F^2-1}{8}}$ $\left(\frac{f}{\omega_2}\right)$, $\left(\frac{F}{\delta_2}\right)$ $\Gamma=3\times2^\gamma$	Ψ $\left(\frac{f}{\omega_1}\right)$, $\left(\frac{F}{\delta_1}\right)$ P	$(-1)^{\frac{f-1}{2}}$, $(-1)^{\frac{F-1}{2}}$ $(-1)^{\frac{F^2-1}{8}}$ $\left(\frac{f}{\omega_2}\right)$, $\left(\frac{F}{\delta_2}\right)$ $\Gamma=2^{\gamma+2}$
$\Delta=2\Delta_1\Delta_2^2.$ $\Delta_2\equiv1$, mod 2.	$(-1)^{\frac{F^2-1}{8}}\Psi$ $\left(\frac{f}{\omega_1}\right)$, $\left(\frac{F}{\delta_1}\right)$ S	$\left(\frac{f}{\omega_2}\right)$, $\left(\frac{F}{\delta_2}\right)$ $\Gamma=2^\gamma$	$(-1)^{\frac{F^2-1}{8}}\Psi$ $\left(\frac{f}{\omega_1}\right)$, $\left(\frac{F}{\delta_1}\right)$ R	$(-1)^{\frac{f-1}{2}}$ $\left(\frac{f}{\omega_2}\right)$, $\left(\frac{F}{\delta_2}\right)$ $\Gamma=2^{\gamma+1}$
$\Delta=2\Delta_1\Delta_2^2.$ $\Delta_2\equiv0$, mod 2.	Ψ $(-1)^{\frac{F^2-1}{8}}$ $\left(\frac{f}{\omega_1}\right)$, $\left(\frac{F}{\delta_1}\right)$ Q	$(-1)^{\frac{F-1}{2}}$ $\left(\frac{f}{\omega_2}\right)$, $\left(\frac{F}{\delta_2}\right)$ $\Gamma=3\times2^\gamma$	Ψ $(-1)^{\frac{F^2-1}{8}}$ $\left(\frac{f}{\omega_1}\right)$, $\left(\frac{F}{\delta_1}\right)$ P	$(-1)^{\frac{f-1}{2}}$, $(-1)^{\frac{F-1}{2}}$ $\left(\frac{f}{\omega_2}\right)$, $\left(\frac{F}{\delta_2}\right)$ $\Gamma=2^{\gamma+2}$

TABLE II. OF COMPLETE GENERIC CHARACTERS.

A.—f and F properly primitive.

od 4.	$\Omega=\Omega_1\Omega_2^2.$	$\Omega_2\equiv 0$, mod 4.	$\Omega=2\Omega_1\Omega_2^2.$	$\Omega_2\equiv 1$, mod 2.	$\Omega=2\Omega_1\Omega_2^2$
$\left(\frac{F}{\delta_2}\right)$ $3\times 2^{\gamma-1}$	Ψ $\left(\frac{f}{\omega_1}\right), \left(\frac{F}{\delta_1}\right)$ Q	$(-1)^{\frac{f-1}{2}}$ $(-1)^{\frac{f^2-1}{8}}$ $\left(\frac{f}{\omega_2}\right), \left(\frac{F}{\delta_2}\right)$ $\Gamma=3\times 2^{\gamma}$	$(-1)^{\frac{f^2-1}{8}}\Psi$ $\left(\frac{f}{\omega_1}\right), \left(\frac{F}{\delta_1}\right)$ S	$\left(\frac{f}{\omega_2}\right), \left(\frac{F}{\delta_2}\right)$ $\Gamma=2^{\gamma}$	Ψ $(-1)^{\frac{f^2-1}{8}}$ $\left(\frac{f}{\omega_1}\right), \left(\frac{F}{\delta_1}\right)$ Q
$-1)^{\frac{F-1}{2}}$ $\left(\frac{F}{\delta_2}\right)$ $2^{\gamma+1}$	Ψ $\left(\frac{f}{\omega_1}\right), \left(\frac{F}{\delta_1}\right)$ P	$(-1)^{\frac{f-1}{2}}, (-1)^{\frac{F-1}{2}}$ $(-1)^{\frac{f^2-1}{8}}$ $\left(\frac{f}{\omega_2}\right), \left(\frac{F}{\delta_2}\right)$ $\Gamma=2^{\gamma+2}$	$(-1)^{\frac{f^2-1}{8}}\Psi$ $\left(\frac{f}{\omega_1}\right), \left(\frac{F}{\delta_1}\right)$ R	$(-1)^{\frac{F-1}{2}}$ $\left(\frac{f}{\omega_2}\right), \left(\frac{F}{\delta_2}\right)$ $\Gamma=2^{\gamma+1}$	Ψ $(-1)^{\frac{f^2-1}{8}}$ $\left(\frac{f}{\omega_1}\right), \left(\frac{F}{\delta_1}\right)$ P
$-1)^{\frac{F-1}{2}}$ $-1)^{\frac{F^2-1}{8}}$ $\frac{F}{\delta_2}\right)$ $2^{\gamma+2}$	Ψ $\left(\frac{f}{\omega_1}\right), \left(\frac{F}{\delta_1}\right)$ P	$(-1)^{\frac{f-1}{2}}, (-1)^{\frac{F-1}{2}}$ $(-1)^{\frac{f^2-1}{8}}, (-1)^{\frac{F^2-1}{8}}$ $\left(\frac{f}{\omega_2}\right), \left(\frac{F}{\delta_2}\right)$ $\Gamma=2^{\gamma+3}$	$(-1)^{\frac{f^2-1}{8}}\Psi$ $\left(\frac{f}{\omega_1}\right), \left(\frac{F}{\delta_1}\right)$ R	$(-1)^{\frac{F-1}{2}}$ $(-1)^{\frac{F^2-1}{8}}$ $\left(\frac{f}{\omega_2}\right), \left(\frac{F}{\delta_2}\right)$ $\Gamma=2^{\gamma+2}$	Ψ $(-1)^{\frac{f^2-1}{8}}$ $\left(\frac{f}{\omega_1}\right), \left(\frac{F}{\delta_1}\right)$ P
$\left(\frac{F}{\delta_2}\right)$ $2^{\gamma+1}$	$(-1)^{\frac{F^2-1}{8}}\Psi$ $\left(\frac{f}{\omega_1}\right), \left(\frac{F}{\delta_1}\right)$ R	$(-1)^{\frac{f-1}{2}}$ $(-1)^{\frac{f^2-1}{8}}$ $\left(\frac{f}{\omega_2}\right), \left(\frac{F}{\delta_2}\right)$ $\Gamma=2^{\gamma+2}$	$(-1)^{\frac{f^2-1}{8}+\frac{F^2-1}{8}}\Psi$ $\left(\frac{f}{\omega_1}\right), \left(\frac{F}{\delta_1}\right)$ S	$\left(\frac{f}{\omega_2}\right), \left(\frac{F}{\delta_2}\right)$ $\Gamma=2^{\gamma}$	$(-1)^{\frac{F^2-1}{8}}\Psi$ $(-1)^{\frac{f^2-1}{8}}$ $\left(\frac{f}{\omega_1}\right), \left(\frac{F}{\delta_1}\right)$ R
$-1)^{\frac{F-1}{2}}$ $\left(\frac{F}{\delta_2}\right)$ $2^{\gamma+2}$	Ψ $(-1)^{\frac{F^2-1}{8}}$ $\left(\frac{f}{\omega_1}\right), \left(\frac{F}{\delta_1}\right)$ P	$(-1)^{\frac{f-1}{2}}, (-1)^{\frac{F-1}{2}}$ $(-1)^{\frac{f^2-1}{8}}$ $\left(\frac{f}{\omega_2}\right), \left(\frac{F}{\delta_2}\right)$ $\Gamma=2^{\gamma+3}$	$(-1)^{\frac{f^2-1}{8}}\Psi$ $(-1)^{\frac{F^2-1}{8}}$ $\left(\frac{f}{\omega_1}\right), \left(\frac{F}{\delta_1}\right)$ R	$(-1)^{\frac{F-1}{2}}$ $\left(\frac{f}{\omega_2}\right), \left(\frac{F}{\delta_2}\right)$ $\Gamma=2^{\gamma+2}$	Ψ $(-1)^{\frac{f^2-1}{8}}, (-1$ $\left(\frac{f}{\omega_1}\right), \left(\frac{F}{\delta_1}\right)$ P

$=2\Omega_1\Omega_2^2.$	$\Omega_2\equiv 0$, mod 2.
Ψ $\frac{-1}{8}$ $, \left(\frac{F}{\delta_1}\right)$	$(-1)^{\frac{f-1}{2}}$ $\left(\frac{f}{\omega_2}\right), \left(\frac{F}{\delta_2}\right)$ $\Gamma=3\times 2^\gamma$
Ψ $\frac{-1}{8}$ $, \left(\frac{F}{\delta_1}\right)$	$(-1)^{\frac{f-1}{2}}, (-1)^{\frac{F-1}{2}}$ $\left(\frac{f}{\omega_2}\right), \left(\frac{F}{\delta_2}\right)$ $\Gamma=2^{\gamma+2}$
Ψ $\frac{-1}{8}$ $, \left(\frac{F}{\delta_1}\right)$	$(-1)^{\frac{f-1}{2}}, (-1)^{\frac{F-1}{2}}$ $(-1)^{\frac{F^2-1}{8}}$ $\left(\frac{f}{\omega_2}\right), \left(\frac{F}{\delta_2}\right)$ $\Gamma=2^{\gamma+3}$
$\frac{-1}{8}\Psi$ $\frac{-1}{8}$ $, \left(\frac{F}{\delta_1}\right)$	$(-1)^{\frac{f-1}{2}}$ $\left(\frac{f}{\omega_2}\right), \left(\frac{F}{\delta_2}\right)$ $\Gamma=2^{\gamma+2}$
Ψ $\frac{-1}{8}, (-1)^{\frac{F^2-1}{8}}$ $, \left(\frac{F}{\delta_1}\right)$	$(-1)^{\frac{f-1}{2}}, (-1)^{\frac{F-1}{2}}$ $\left(\frac{f}{\omega_2}\right), \left(\frac{F}{\delta_2}\right)$ $\Gamma=2^{\gamma+3}$

B.—f improperly, F properly primitive.

$$(-1)^{\frac{F-1}{2}}=-(-1)^{\frac{\Omega-1}{2}};\ \Omega\equiv 1,\ \text{mod } 2.$$

	$\Omega=\Omega_1\Omega_2^2.$	$\Omega_2\equiv 1$, mod 2.
$\Delta=\Delta_1\Delta_2^2.$ $\Delta_2\equiv 0$, mod 2.	$(-1)^{\frac{F^2-1}{8}}$ $\left(\frac{f}{\omega_1}\right), \left(\frac{F}{\delta_1}\right)$	$\left(\frac{f}{\omega_2}\right), \left(\frac{F}{\delta_2}\right)$ $\Gamma=2^\gamma$
$\Delta=2\Delta_1\Delta_2^2.$ $\Delta_2\equiv 1$, mod 2.	$\left(\frac{f}{\omega_1}\right), \left(\frac{F}{\delta_1}\right)$	$\left(\frac{f}{\omega_2}\right), \left(\frac{F}{\delta_2}\right)$ $\Gamma=2^{\gamma-1}$
$\Delta=2\Delta_1\Delta_2^2.$ $\Delta_2\equiv 0$, mod 2.	$\left(\frac{f}{\omega_1}\right), \left(\frac{F}{\delta_1}\right)$	$(-1)^{\frac{F^2-1}{8}}$ $\left(\frac{f}{\omega_2}\right), \left(\frac{F}{\delta_2}\right)$ $\Gamma=2^\gamma$

C.—f properly, F improperly primitive.

$$(-1)^{\frac{f-1}{2}}=-(-1)^{\frac{\Delta-1}{2}};\ \Delta\equiv 1,\ \text{mod } 2.$$

	$\Delta=\Delta_1\Delta_2^2.$	$\Delta_2\equiv 1$, mod 2.
$\Omega=\Omega_1\Omega_2^2.$ $\Omega_2\equiv 0$, mod 2.	$(-1)^{\frac{f^2-1}{8}}$ $\left(\frac{f}{\omega_1}\right), \left(\frac{F}{\delta_1}\right)$	$\left(\frac{f}{\omega_2}\right), \left(\frac{F}{\delta_2}\right)$ $\Gamma=2^\gamma$
$\Omega=2\Omega_1\Omega_2^2.$ $\Omega_1\equiv 1$, mod 2.	$\left(\frac{f}{\omega_1}\right), \left(\frac{F}{\delta_1}\right)$	$\left(\frac{f}{\omega_2}\right), \left(\frac{F}{\delta_2}\right)$ $\Gamma=2^{\gamma-1}$
$\Omega=2\Omega_1\Omega_2^2.$ $\Omega_1\equiv 0$, mod 2.	$\left(\frac{f}{\omega_1}\right), \left(\frac{F}{\delta_1}\right)$	$(-1)^{\frac{f^2-1}{8}}$ $\left(\frac{f}{\omega_2}\right), \left(\frac{F}{\delta_2}\right)$ $\Gamma=2^\gamma$

generic character which does not satisfy it belonging to no forms whatever. The demonstration of this important theorem will occupy the next articles; it is, however, requisite to show in the first place that the enumeration of the supplementary characters in Table II. is in accordance with the Table I. of Art. 4. For the Tables B and C this is evident; in Table A it is necessary to attend to the signification of the symbol Ψ, which serves to represent the simultaneous character of f and F (as has been already explained in Arts. 6 and 7) in those cases (marked S in the Table) in which neither $(-1)^{\frac{f-1}{2}}$ nor $(-1)^{\frac{F-1}{2}}$ is a character, but which also appears in every compartment of the Table without exception.

(1) When $(-1)^{\frac{f-1}{2}}$ and $(-1)^{\frac{F-1}{2}}$ are both characters (cases P in the Table), Ψ is not an independent character, because its value is determined by the values of $(-1)^{\frac{f-1}{2}}$ and $(-1)^{\frac{F-1}{2}}$. It is retained in the Table only because it serves to express the criterion of possibility.

(2) When $(-1)^{\frac{f-1}{2}}$ and Ψ, but not $(-1)^{\frac{F-1}{2}}$, are incribed as characters, Ψ represents the character $(-1)^{\frac{F-1}{2}}$, if $(-1)^{\frac{f-1}{2}}=(-1)^{\frac{\Delta_1-1}{2}}$, and is simply $+1$ (*i. e.* not a character at all), if $(-1)^{\frac{f-1}{2}}=-(-1)^{\frac{\Delta_1-1}{2}}$. This is in accordance with Table I., according to which, in the cases under consideration, $(-1)^{\frac{F-1}{2}}$ is or is not a character, according as $(-1)^{\frac{f-1}{2}}=(-1)^{\frac{\Delta_1-1}{2}}$, or $=-(-1)^{\frac{\Delta_1-1}{2}}$. Similarly, if $(-1)^{\frac{F-1}{2}}$ and Ψ, but not $(-1)^{\frac{f-1}{2}}$, are inscribed as characters, Ψ represents the character $(-1)^{\frac{f-1}{2}}$, or is not a character at all, according as $(-1)^{\frac{F-1}{2}}=(-1)^{\frac{\Omega_1-1}{2}}$, or $=-(-1)^{\frac{\Omega_1-1}{2}}$; which again agrees with Table I.

In these cases, marked Q in the Table, the symbol Ψ supersedes the asterisks and obelisks of Table I., and also serves to express the criterion of possibility.

(3) When $(-1)^{\frac{f-1}{2}}$ and $(-1)^{\frac{F^2-1}{8}}\times\Psi$, but not $(-1)^{\frac{F-1}{2}}$, are characters in the Table, $(-1)^{\frac{F^2-1}{8}}\times\Psi$ represents the character $(-1)^{\frac{F-1}{2}+\frac{F^2-1}{8}}$, or $(-1)^{\frac{F^2-1}{8}}$, according as $(-1)^{\frac{f-1}{2}}=(-1)^{\frac{\Delta_1-1}{2}}$, or $-(-1)^{\frac{\Delta_1-1}{2}}$. And again, when $(-1)^{\frac{F-1}{2}}$ and $(-1)^{\frac{f^2-1}{8}}\times\Psi$, but not $(-1)^{\frac{f-1}{2}}$, are characters in the Table, $(-1)^{\frac{f^2-1}{8}}\times\Psi$ represents the character $(-1)^{\frac{f-1}{2}+\frac{f^2-1}{8}}$, or $(-1)^{\frac{f^2-1}{8}}$, according as $(-1)^{\frac{F-1}{2}}=(-1)^{\frac{\Omega_1-1}{2}}$, or $-(-1)^{\frac{\Omega_1-1}{2}}$.

The result in these cases (marked R in the Table) is again in accordance with Table I.; and the use of the symbol Ψ is the same as in the cases Q.

Thus the units Ψ, $(-1)^{\frac{f^2-1}{8}}\times\Psi$, $(-1)^{\frac{F^2-1}{8}}\times\Psi$, which properly represent simultaneous characters of the forms f and F, are employed, in the cases Q and R of the Table, to represent supplementary characters. This use of these symbols is admissible, because, when $\Omega\Delta$ is even (as it is in the cases Q and R), every representation of an uneven number by f or F is simultaneous with the representation of uneven numbers by F or f.

In the lower right-hand corner of each compartment in the Table, the number of possible genera contained in the order to which the compartment refers is represented by Γ; γ is the number of uneven primes dividing Ω, together with the number of uneven primes dividing Δ, so that if the same prime divide both Ω and Δ, it is to be counted twice. But it is to be observed that, when Ω and Δ are both perfect squares (a case which can only arise when the forms are definite), the number of possible genera is two-thirds of the number stated in the Table in the cases (Q), and one half in the cases (P). And again (as has been already stated in Art. 3), in Table B, when Ω is an uneven square, and Δ the double of a square, there are no possible genera; and when Δ is an uneven square, and Ω the double of an uneven square, there are none in Table C.

Art. 9. It results from the theorem of Art. 5 that if f and F are properly primitive, they simultaneously and primitively represent uneven numbers prime to $\Omega\Delta$. We may therefore suppose that in f and F, a and A'' are uneven and prime to $\Omega\Delta$; we may also suppose that these numbers are prime to one another, because A'' being prime to $\Omega\Delta$, and a being uneven, the binary form (a, b'', a') is properly primitive (Art. 5), and so represents numbers prime to its determinant. Lastly, we may assume that a and A'' are positive. If the forms f and F are definite, a and A'' are certainly positive; if they are indefinite, Δ and Ω are of opposite signs; supposing, for example, that Δ is positive and Ω negative, let m be any positive number primitively represented by f, and M any number simultaneously and primitively represented by F, then M is positive as well as m; otherwise mMf, which is of the type $MX^2+\Omega Y^2+m\Omega\Delta Z^2$, would be a definite form. Positive numbers are therefore simultaneously and primitively represented by f and F; *i. e.* we may suppose a and A'' simultaneously positive. The complete generic character of f is then determined by the characters of a and A''. But

$$aa'-b^2=\Omega A'', \quad A'A''-B^2=\Delta a,$$

whence it follows that

$$\left(\frac{-\Omega}{a}\right)\left(\frac{A''}{a}\right)=1, \quad \left(\frac{-\Delta}{A''}\right)\left(\frac{a}{A''}\right)=1;$$

multiplying these equations together and observing that, by the laws of quadratic residues,

$$\left(\frac{A''}{a}\right)\left(\frac{a}{A''}\right)=(-1)^{\frac{a-1}{2}\cdot\frac{A''-1}{2}},$$

we find

$$(-1)^{\frac{a-1}{2}\cdot\frac{A''-1}{2}}\left(\frac{-\Omega}{a}\right)\left(\frac{-\Delta}{A''}\right)=1. \quad \quad (14)$$

Let α and β retain the significations assigned to them in equation (11), Art. 8; transforming $\left(\frac{-\Omega}{a}\right)$ and $\left(\frac{-\Delta}{A''}\right)$ by the law of quadratic reciprocity, we find

$$\left(\frac{-\Omega}{a}\right)=(-1)^{\frac{a-1}{2}\cdot\frac{\Omega_1+1}{2}}\alpha^{\frac{a^2-1}{8}}\left(\frac{a}{\Omega_1}\right),$$

$$\left(\frac{-\Delta}{A''}\right)=(-1)^{\frac{A''-1}{2}\cdot\frac{\Delta_1+1}{2}}\beta^{\frac{A''^2-1}{8}}\left(\frac{A''}{\Delta_1}\right),$$

and equation (14) becomes

$$(-1)^{\frac{\Omega_1+1}{2}\cdot\frac{\Delta_1+1}{2}+\frac{\Omega_1+A''}{2}\cdot\frac{\Delta_1+a}{2}}\alpha^{\frac{a^2-1}{8}}\beta^{\frac{A''^2-1}{8}}\times\left(\frac{a}{\Omega_1}\right)\left(\frac{A''}{\Delta_1}\right)=1;$$

or observing that

$$(-1)^{\frac{\Omega_1+A''}{2}\cdot\frac{\Delta_1+a}{2}}=(-1)^{\frac{\Omega_1 A''+1}{2}\cdot\frac{\Delta_1 a+1}{2}}=\Psi,$$

and writing f and F for a and A″,

$$\Psi\times\alpha^{\frac{f^2-1}{8}}\times\beta^{\frac{F^2-1}{8}}\times\left(\frac{f}{\Omega_1}\right)\times\left(\frac{F}{\Delta_1}\right)=(-1)^{\frac{\Omega_1+1}{2}\cdot\frac{\Delta_1+1}{2}},$$

i. e. the generic character of f satisfies the condition of possibility (11).

Again, if f is improperly and F properly primitive, let A″ be prime to $2\Omega\Delta$; then the binary form (a, b'', a') is primitive, because A″ is prime to $\Delta\Omega$, and improperly primitive, because f is improperly primitive. We may therefore suppose that $\frac{1}{2}a$ is uneven and prime to $\Omega A''$, and, as before, that a and A″ are positive. Multiplying together the equations

$$\left(\frac{-\Omega}{\frac{1}{2}a}\right)\times\left(\frac{A''}{\frac{1}{2}a}\right)=1,\quad\left(\frac{-2\Delta}{A''}\right)\left(\frac{\frac{1}{2}a}{A''}\right)=1,$$

and transforming the result by the law of reciprocity, we find

$$(-\beta)^{\frac{A''^2-1}{8}}\times\left(\frac{a}{\Omega_1}\right)\times\left(\frac{A''}{\Delta_1}\right)=(-1)^{\frac{\Omega_1^2-1}{8}+\frac{\Omega_1+1}{2}\cdot\frac{\Delta_1+1}{2}},$$

i. e. the condition (12) is satisfied by the generic character of f.

The case in which f is properly and F improperly primitive, is the reciprocal of the preceding.

To show that the conditions (11), (12), (13) are sufficient as well as necessary, other principles are required. These principles relate to the representation of binary by ternary quadratic forms, and will be found in the 'Disquisitiones Arithmeticæ,' arts. 282–284; it will, however, be convenient briefly to restate them here, in a form suited for our present purpose.

Art. 10. A binary quadratic form (p, q'', p') or ϕ is said to be represented by a ternary form f when f is transformed into ϕ by a substitution of the type

$$x=\alpha_1 x+\beta_1 y,$$
$$y=\alpha_2 x+\beta_2 y,$$
$$z=\alpha_3 x+\beta_3 y.$$

The representation is said to be *primitive* when the determinants of the matrix

$$\left|\begin{matrix}\alpha_1, & \beta_1\\ \alpha_2, & \beta_2\\ \alpha_3, & \beta_3\end{matrix}\right|$$

are relatively prime. If ϕ is primitively represented by f, f is equivalent to a form containing ϕ as a part, *i. e.* to a form f' of the type

$$f'=px^2+p'y^2+p''z^2+2qyz+2q'xz+2q''xy,$$

for f is transformed into such a form by a substitution of which the matrix is

$$\begin{vmatrix} \alpha_1, & \beta_1, & \gamma_1 \\ \alpha_2, & \beta_2, & \gamma_2 \\ \alpha_3, & \beta_3, & \gamma_3, \end{vmatrix}$$

$\gamma_1, \gamma_2, \gamma_3$ denoting any three numbers which render the determinant of that matrix equal to $+1$.

Let

$$F' = Px^2 + P'y^2 + P''z^2 + 2Qyz + 2Q'xz + 2Q''xy$$

be the primitive contravariant of f', so that, in particular,

$$\Omega P'' = q''^2 - pp'; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

multiplying the equations

$$\left.\begin{aligned} P'P'' - \quad Q^2 &= \Delta p, \\ QQ' - P'Q'' &= \Delta q'', \\ PP'' - \quad Q'^2 &= \Delta p' \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

(which result from the contravariance of f' and F') by x^2, $2xy$, y^2 respectively, we obtain

$$-\Delta \times (px^2 + 2q''xy + p'y^2) = (Q^2 - P'P'')x^2 - 2(QQ' - P''Q'')xy + (Q'^2 - PP'')y^2;$$

and this equation, considered as a congruence for the modulus P'', becomes

$$\Delta \times \phi + (Qx - Q'y)^2 \equiv 0, \text{ mod } P'', \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

the coefficients of x^2, $2xy$, y^2 in the left-hand member being all divisible by P''. If therefore ϕ is a binary quadratic form of determinant $-\Omega P''$, admitting of primitive representation by a ternary form of order $[\Omega, \Delta]$, $-\Delta\phi$ is a quadratic residue of P''. And we shall now show that if ϕ is a primitive (and not negative) binary form of determinant $-\Omega P''$, P'' being of the same sign as Δ and prime to Δ, ϕ admits of primitive representation by ternary forms of the invariants $[\Omega, \Delta]$, whenever $-\Delta\phi$ is a quadratic residue of P''.

Because $-\Delta\phi$ is a quadratic residue of P'', the congruence (17) admits of solution in integral numbers Q, Q'. Any solution of this congruence supplies a system of five numbers, P, P', Q, Q', Q'', satisfying the equations (16). The greatest common divisor of these five numbers divides Δ, because p, q'', p' are relatively prime; but P'' is prime to Δ; therefore the six numbers P, P', P'', Q, Q', Q'' are relatively prime. Let q and q' be determined by the equations

$$\left.\begin{aligned} qq'' - q'p' &= \quad \Omega Q', \\ qp - q'q'' &= -\Omega Q, \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

which are always resoluble because their determinant $q''^2 - pp' = \Omega P''$ is different from zero. Also let p'' be determined by the equation

$$q'Q' + qQ + p''P'' = \Omega\Delta. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

The values of q, q', p'' are rational; and, if they are fractions, their denominators, when they are expressed in their lowest terms, are divisors of P''. Substituting in (19) for

P'', Q', Q their values derived from the equations (15) and (18), we find that $\Omega^2\Delta$ is the determinant of the matrix

$$\begin{vmatrix} p\,, & q'', & q' \\ q'', & p', & q \\ q', & q\,, & p'' \end{vmatrix} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

Let

$$\begin{vmatrix} \Omega[p], & \Omega[q''], & \Omega[q'] \\ \Omega[q''], & \Omega[p'], & \Omega[q] \\ \Omega[q'], & \Omega[q], & \Omega[p''] \end{vmatrix} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

be the matrix reciprocal to the matrix (20); we know, from the equations (15) and (18), that $[p'']=P''$, $[q']=Q'$, $[q]=Q$. Again, in the reciprocal matrices (20) and (21), we must have

$$[p'][p'']-[q]^2=\Delta p,$$
$$[q][q']\ -[p''][q'']=\Delta q'',$$
$$[p][p'']-[q'^2]=\Delta p',$$

or substituting for $[p'']$, $[q]$, $[q']$ their values,

$$[p']P''-Q^2=\Delta p,$$
$$QQ'-P''[q'']=\Delta q'',$$
$$[p]P''-Q'^2=\Delta p''.$$

Comparing these equations with the equations (16), and observing that P'' is not zero, we infer that

$$[p']=P',\quad [q'']=Q'',\quad [p]=P.$$

The matrix reciprocal to the matrix (20) is therefore

$$\begin{vmatrix} \Omega P\,, & \Omega Q'', & \Omega Q' \\ \Omega Q'', & \Omega P'\,, & \Omega Q \\ \Omega Q'\,, & \Omega Q\,, & \Omega P'', \end{vmatrix} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (22)$$

and, consequently,

$$\Delta q=Q'Q''-PQ,$$
$$\Delta q'=QQ''-P'Q',$$
$$\Delta p''=PP'-Q''^2.$$

These equations prove that the denominators of q, q', p'' are divisors of Δ; *i. e.* that q, q', p'' are integral numbers, because P'' is prime to Δ. The coefficients of the ternary form

$$f'=px^2+p'y^2+p''z^2+2qyz+2q'xz+2''qxy$$

are therefore integral; this form is primitive, and represents primitively the form (p, q'', p'); it is also a form of the given invariants $[\Omega, \Delta]$; for its discriminant is $\Delta\Omega^2$, and the greatest common divisor of the first minors of its matrix is Ω; hence its second invariant is Δ, and its first invariant either $+\Omega$, or $-\Omega$. But when the given invariants Ω and Δ are both positive, ϕ is a positive binary form of the negative determinant

$-\Omega P''$; and such a form cannot be represented by an indefinite ternary form of a positive discriminant; f' is therefore definite, and its first invariant is $+\Omega$. When the given invariants Ω and Δ are of opposite signs, ϕ is a binary form of the positive determinant $-\Omega P''$; such a form cannot be represented by a definite ternary form; f' is therefore indefinite, and, as its invariants must be of opposite signs, in this case also its first invariant is $+\Omega$.

Also, if ϕ is properly primitive and P'' uneven, the forms f' and F' are both properly primitive, one of the principal coefficients of each being uneven. In this case, therefore, ϕ is represented by a form of the properly primitive order $[\Omega, \Delta]$. If ϕ is improperly primitive (a supposition which implies that $\Omega P''\equiv 3$, mod 4), and if Δ is even, f' is improperly primitive. For no properly primitive ternary form of even discriminant can represent primitively an improperly primitive binary form, the supposition that (p, q'', p') is improperly primitive and p'' uneven implying that the discriminant is uneven. And the same thing follows from the preceding analysis; for, considering the equations (18) as congruences for the modulus 2, we find on the supposition that ϕ is improperly primitive, $q\equiv Q'$, mod 2, $q'\equiv Q$, mod 2, and substituting in (19), $p''\equiv 0$, mod 2, so that f' is improperly primitive.

Art. 11. We can now assign a properly primitive form of any given invariants $[\Omega, \Delta]$, and of any given generic character satisfying the condition of possibility. Let M be any number prime to 2Δ, of the same sign as Δ, and possessing all the particular characters (except the simultaneous character, if any) which are attributed to F in the given generic character; also if Ω is uneven, and Δ uneven or unevenly even, we shall suppose that $M\equiv\Omega$, mod 4. Let ϕ be any properly primitive, and not negative binary quadratic form of determinant $-\Omega M$; and let m be any number prime to $2\Omega M$ which is represented by ϕ. By the theory of binary quadratic forms, the generic characters which are attributable to ϕ, are (i) its characters with respect to primes dividing M, (ii) its characters with respect to primes dividing Ω, (iii) its supplementary characters. These last are exhibited in the following Table,

If $-\Omega M\equiv$	Supplementary characters.
1, mod 4.	None.
3, mod 4.	$(-1)^{\frac{\phi-1}{2}}$
2, mod 8.	$(-1)^{\frac{\phi^2-1}{8}}$
6, mod 8.	$(-1)^{\frac{\phi-1}{2}+\frac{\phi^2-1}{8}}$
4, mod 8.	$(-1)^{\frac{\phi-1}{2}}$
0, mod 8.	$(-1)^{\frac{\phi-1}{2}}$, $(-1)^{\frac{\phi^2-1}{8}}$

Let μ be any prime divisor of M, and let us determine the first set of characters by the equations

$$\left(\frac{\phi}{\mu}\right)=\left(\frac{-\Delta}{\mu}\right), \quad \ldots \quad (23)$$

the second set by the equations

$$\left(\frac{\phi}{\omega}\right)=\left(\frac{f}{\omega}\right), \quad \ldots \quad (24)$$

$\left(\frac{f}{\omega}\right)$ being a particular character of f, of which the value is assigned in the proposed generic character. With respect to the supplementary characters of ϕ, it will be found on a comparison of the above Table with Table II. A, that, when the proposed generic character includes no simultaneous character, the supplementary characters attributable to ϕ are the same as those attributable to f; we then assign to the supplementary characters of ϕ the same values which are assigned to the supplementary characters of f in the proposed generic character. But when the proposed generic character includes a simultaneous character, there is always a supplementary character (and only one) attributable to ϕ, and not to f; this character of ϕ we determine so that the value of the simultaneous character of f and F, and the value of the unit similarly formed with m and M, may be coincident. This determination is always possible, as will be seen on a comparison of the cases (S) of Table II. A, with the above Table of supplementary characters of binary forms. As we have now assigned a value to every particular character attributable to ϕ, it is necessary to inquire whether a form ϕ, possessing such a complete character, actually exists; *i. e.* whether the character that we have assigned to ϕ satisfies the condition of possibility for binary forms of determinant $-\Omega$M.

If, as in art. 8, $\alpha=+1$, or -1, according as Ω is of the form $\Omega_1\Omega_2^2$, or $2\Omega_1\Omega_2^2$, that condition is

$$(-1)^{\frac{\Omega_1 M+1}{2}\cdot\frac{\phi-1}{2}}\alpha^{\frac{\phi^2-1}{8}}\left(\frac{\phi}{\Omega_1 M}\right)=1, \quad \ldots \quad (25)$$

or, since

$$(-1)^{\frac{\Delta_1+1}{2}+\frac{\Delta_1\phi+1}{2}}=(-1)^{\frac{\phi-1}{2}},$$

$$(-1)^{\frac{\Omega_1 M+1}{2}\cdot\frac{\Delta_1+1}{2}+\frac{\Omega_1 M+1}{2}\cdot\frac{\Delta_1\phi+1}{2}}\alpha^{\frac{\phi^2-1}{8}}\left(\frac{\phi}{\Omega_1 M}\right)=1, \quad \ldots \quad (26)$$

But $\left(\frac{\phi}{\Omega_1}\right)=\left(\frac{f}{\Omega_1}\right)$, by the equations (24), and if (again as in art. 8) $\beta=+1$, or -1, according as Δ is of the form $\Delta_1\Delta_2^2$, or $2\Delta_1\Delta_2^2$,

$$\left(\frac{\phi}{M}\right)=\left(\frac{-\Delta}{M}\right)=(-1)^{\frac{M-1}{2}\cdot\frac{\Delta_1+1}{2}}\times\beta^{\frac{M^2-1}{8}}\times\left(\frac{M}{\Delta_1}\right).$$

Substituting these values in (26), and observing that in every case

$$(-1)^{\frac{\Omega_1 M+1}{2}\cdot\frac{\Delta_1\phi+1}{2}}\alpha^{\frac{\phi^2-1}{8}}\beta^{\frac{M^2-1}{8}}=\Psi\times\alpha^{\frac{f^2-1}{8}}\beta^{\frac{F^2-1}{8}},$$

we obtain

$$\Psi\times\alpha^{\frac{f^2-1}{8}}\beta^{\frac{F^2-1}{8}}\left(\frac{f}{\Omega_1}\right)\left(\frac{F}{\Delta_1}\right)=(-1)^{\frac{\Omega_1+1}{2}\cdot\frac{\Delta_1+1}{2}}.$$

But this equation is the equation (11) of art. 8, which is by hypothesis satisfied by the proposed generic character; therefore the equation (25) is also satisfied; *i. e.* a properly primitive binary form ϕ exists, of determinant $-\Omega M$, possessing the generic character which we have assigned to it. This form, multiplied by $-\Delta$, is a quadratic residue of M; for the equation

$$\left(\frac{-\Delta\phi}{\mu}\right)=1$$

is satisfied for every prime dividing M, by virtue of the equations (23). Let, then, a ternary form f, of the properly primitive order, and of the invariants $[\Omega, \Delta]$, be determined, representing ϕ primitively. The generic character of this form is completely determined by the numbers m and M, which are uneven numbers simultaneously represented by f and F; it is therefore a form of the proposed generic character.

Of the two improperly primitive orders, it will suffice to consider that in which f is improperly and F properly primitive; so that Ω is uneven and Δ even. Let M be a number prime to $2\Omega\Delta$, of the same sign as Δ, and satisfying the generic characters of F, including the congruence $M\equiv-\Omega$, mod 4; also let ϕ be an improperly primitive binary form of determinant $-\Omega M$; the generic characters attributable to ϕ are (i) its characters $\left(\frac{\phi}{\mu}\right)$, (ii) its characters $\left(\frac{\phi}{\omega}\right)$. These characters we determine, as before, by the equations (23) and (24). The complete generic character thus assigned to ϕ is possible; for the condition that it should be possible is

$$\left(\frac{\phi}{\Omega_1 M}\right)=(-1)^{\frac{\Omega_1^2-1}{8}+\frac{M^2-1}{8}},$$

or

$$\left(\frac{f}{\Omega_1}\right)\left(\frac{-2\Delta}{M}\right)=(-1)^{\frac{\Omega_1^2-1}{8}}.$$

Transforming $\left(\frac{-2\Delta}{M}\right)$ by the law of reciprocity, we find

$$(-\beta)^{\frac{F^2-1}{8}}\times\left(\frac{f}{\Omega_1}\right)\left(\frac{F}{\Delta_1}\right)=(-1)^{\frac{\Omega_1+1}{2}\cdot\frac{\Delta_1+1}{2}+\frac{\Omega_1^2-1}{8}},$$

an equation which the proposed generic character satisfies by hypothesis (equation (12) Art. 8). An improperly primitive form ϕ of determinant $-\Omega M$ therefore actually exists, having the generic character which we have assigned to it; *i. e.* ternary forms exist having the proposed generic character.

It is evident from the demonstration that if M is of the same sign as Δ, prime to 2Δ, and also (when Ω is uneven and Δ uneven or unevenly even) congruous to Ω, mod 4, there is always one genus of properly primitive binary forms of determinant $-\Omega M$ capable of primitive representation by a given genus of ternary forms of the properly primitive order $[\Omega, \Delta]$, of which the contravariant characters coincide with the characters of M. And similarly, if Δ is even, Ω uneven, M prime to Δ, and $\equiv-\Omega$, mod 4, there is always one genus of improperly primitive binary forms of determinant $-\Omega M$

capable of primitive representation by a given genus of ternary forms of the improperly primitive order $[\Omega, \Delta]$, of which the contravariant characters coincide with the characters of M. And in both cases no other primitive form (if M is prime to Ω, no other form, primitive or derived) of determinant $-\Omega M$ is capable of such representation.

Art. 12. By a rational substitution we shall understand in this article a substitution of which the determinant is unity, and of which the coefficients are rational. If the common denominator of the coefficients is prime to any number m, we shall say that the substitution is prime to m.

If f_1 and f_2 are ternary forms, having integral coefficients, of which f_1 is a form of the invariants (Ω, Δ), and is transformed by a rational substitution, prime to $2\Omega\Delta$, into f_2, f_2 is a form of the same invariants, of the same order, and of the same genus as f_1. This may be proved nearly in the same way in which it is proved that equivalent forms have the same invariants and are of the same order and genus; it is only necessary to observe that F_1 and F_2, as well as f_1 and f_2, are transformable into one another by rational substitutions, prime to $2\Omega\Delta$. The converse proposition,

"If f_1 and f_2 are two forms of the same invariants (Ω, Δ), of the same order, and of the same genus, they are transformable, each into the other, by rational substitutions prime to $2\Omega\Delta$," is also true, and is of importance in the present theory, because it establishes the completeness of the enumeration of the generic characters of ternary forms. To avoid the introduction, in this place, of principles relating to quaternary quadratic forms, we shall give an indirect demonstration of it, depending on the following lemma which relates to binary quadratic forms.

"If ϕ_1, ϕ_2 are two primitive binary quadratic forms of the same determinant, and of the same genus, the resolubility of the equation $\phi_1(x, y)=M$ implies the resolubility of the equation $\phi_2(x, y)=Mz^2$; and in the solution of this equation the value of z may be supposed prime to any given number k."

Because ϕ_1 and ϕ_2 are of the same genus, ϕ_2 is transformable, by a bipartite linear substitution, into the product $\chi\times\phi_1$, χ representing a properly primitive form of the principal genus (Disq. Arith. art. 251). But χ is transformable, by a quadratic substitution, into the square of a properly primitive form ψ (ibid. art. 287). Therefore, by a mixed quadratic and linear substitution, ϕ_2 is transformed into the product $\psi^2\times\phi_1$. Attributing, in this mixed substitution, to the indeterminates of ϕ_1 the values which satisfy the equation $\phi_1=M$, and to the indeterminates of ψ any values whatever for which ψ acquires a value z prime to k, we obtain a solution of the equation $\phi_2=Mz^2$.

Let us first suppose that the given ternary forms f_1 and f_2 belong to the properly primitive order of the invariants (Ω, Δ); let M_1, M_2 be two numbers of the same sign as Δ, prime to $2\Omega\Delta$, and primitively represented by F_1, F_2 respectively; we may suppose that $M_1\equiv M_2$, mod 8; and that the representations of M_1 and M_2 are simultaneous with the representations of uneven numbers by f_1 and f_2. Let ϕ_1, ϕ_2 be two binary quadratic forms, of the determinants $-\Omega M_1$, $-\Omega M_2$ respectively, represented by f_1 and f_2 simul-

taneously with the representations of M_1 and M_2 by F_1 and F_2*. Then ϕ_1 and ϕ_2 are properly primitive; their generic characters with respect to uneven primes dividing Ω will coincide, because

$$\left(\frac{\phi}{\omega}\right)=\left(\frac{f_1}{\omega}\right)=\left(\frac{f_2}{}\right)=\left(\frac{\phi_2}{\omega}\right);$$

their supplementary characters will also coincide; for the same supplementary characters are attributable to ϕ_1 and ϕ_2, and these supplementary characters are determined for ϕ_1, in accordance with the supplementary characters of f_1, or the simultaneous character of f_1 and F_1, and for ϕ_2 in accordance with characters which are the same with these; lastly, if μ is any prime dividing both M_1 and M_2, the characters of ϕ_1 and ϕ_2 with respect to μ will also coincide; for

$$\left(\frac{\phi_1}{\mu}\right)=\left(\frac{-\Delta}{\mu}\right)=\left(\frac{\phi_2}{\mu}\right).$$

The remaining characters of ϕ_1 and ϕ_2 (*i. e.* their characters with respect to primes dividing only one of the two numbers M_1 and M_2), being characters with respect to different primes, cannot be incompatible. The complete generic characters of ϕ_1 and ϕ_2 are therefore compatible, and are satisfied by the numbers contained in certain arithmetical progressions. Each of these progressions contains (by the theorem of LEJEUNE DIRICHLET) an infinite number of positive and negative primes. Let p be one of these primes of the same sign as Ω, and not dividing $2\Omega\Delta$; p will satisfy the generic characters both of ϕ_1 and ϕ_2, and will be represented by some form of determinant $-\Omega M_1$, and of the same genus as ϕ_1, and by some form of determinant $-\Omega M_2$, and of the same genus as ϕ_2. Therefore, by the lemma of this article, $p\theta_1^2$ will be primitively represented by ϕ_1, and $p\theta_2^2$ by ϕ_2, θ_1 and θ_2 denoting numbers prime to $2\Omega\Delta$. Let Φ_1, Φ_2 be two properly primitive binary forms represented by F_1, F_2, simultaneously with the representations of $p\theta_1^2$, $p\theta_2^2$, by f_1, f_2. The determinants of Φ_1, Φ_2 are $-\Delta p\theta_1^2$, $-\Delta p\theta_2^2$; and it will be found (as in the case of the forms ϕ_1, ϕ_2) that the generic characters of Φ_1, Φ_2 are compatible; and that a prime P of the same sign as Δ, and not dividing $2\Omega\Delta$, is assignable, such that $P\Theta_1^2$, $P\Theta_2^2$ are primitively represented by Φ_1, Φ_2 respectively, Θ_1 and Θ_2 denoting numbers prime to $2\Omega\Delta$. Thus the numbers $p\theta_1^2$, $P\Theta_1^2$ are simultaneously and primitively represented by f_1 and F_1; the numbers $p\theta_2^2$, $P\Theta_2^2$ are simultaneously and primitively represented by f_2 and F_2. We may therefore suppose that ψ_1 is a form equivalent to f_1, in which $a_1=p\theta_1^2$, $A_1''=P\Theta_1^2$, and that ψ_2 is a form equivalent to

* If

$$M=F(\alpha'\beta''-\alpha''\beta',\ \alpha''\beta-\alpha\beta'',\ \alpha\beta'-\alpha'\beta),$$

and if f is transformed into a binary form ϕ by the substitution

$$x=\alpha x+\beta y,$$
$$y=\alpha' x+\beta' y,$$
$$z=\alpha'' x+\beta'' y,$$

the representations of M by F, and of ϕ by f, are said to be simultaneous, or to appertain to one another (GAUSS, Disq. Arith. Art. 280).

f_2, in which $a_2 = p\theta_2^2$, $A_2'' = P\Theta_2^2$. The fractional form

$$\frac{1}{Pp}[PX^2 + \Omega Y^2 + p\Omega\Delta Z^2]$$

is then transformed into ψ_1 by the substitution

$$\left|\begin{array}{ccc} \frac{a_1}{\theta_1}, & \frac{b_1''}{\theta_1}, & \frac{b_1'}{\theta_1} \\ 0, & \frac{A_1''}{\Theta_1\theta_1}, & -\frac{B_1}{\Theta_1\theta_1} \\ 0, & 0, & \frac{1}{\Theta_1} \end{array}\right|$$

of which the determinant is Pp, and into ψ_2 by a similar substitution of the same determinant. Either of the two forms ψ_1, ψ_2 (and consequently either of the two f_1, f_2) is therefore transformable into the other, by a rational substitution prime to $2\Omega\Delta$. It will be found that if the signs of Θ_1, Θ_2, θ_1, θ_2 are properly determined, the primes P, p will not appear in the denominators of these substitutions.

If f_1 and f_2 belong to an improperly primitive order, the preceding proof requires very little modification. It will suffice to consider the case in which f_1 and f_2 are improperly, F_1 and F_2 properly primitive. We take $M_1 \equiv M_2 \equiv -\Omega$, mod 4; ϕ_1 and ϕ_2 are then improperly primitive and have compatible generic characters; let $2p\theta_1^2$ be represented by ϕ_1, and $2p\theta_2^2$ by ϕ_2; Φ_1 and Φ_2 are properly primitive and of the determinants $-2\Delta p\theta_1^2$, $-2\Delta p\theta_2^2$; these forms have compatible generic characters (their supplementary characters, in particular, being determined by those of F_1 and F_2); let, then, $P\Theta_1^2$ be represented by Φ_1 and $P\Theta_2^2$ by Φ_2, and let us suppose that ψ_1, ψ_2 are forms equivalent to f_1, f_2, in which $a_1 = 2p\theta_1^2$, $A_1'' = P\Theta_1^2$, $a_2 = 2p\theta_2^2$, $A_2'' = P\Theta_2^2$; the fractional form

$$\frac{1}{Pp}[\tfrac{1}{2}(P+\Omega)X^2 + (\Omega - P)XY + \tfrac{1}{2}(P+\Omega)Y^2 + p\Omega\Delta Z^2]$$

is transformed into ψ_1 by the substitution

$$\left|\begin{array}{ccc} \frac{1}{2}\frac{a_1}{\theta_1}, & \frac{1}{2}\frac{b''\Theta_1 + A_1''}{\Theta_1\theta_1}, & \frac{1}{2}\frac{b_1'\Theta_1 - B_1}{\Theta_1\theta_1} \\ -\frac{1}{2}\frac{a_1}{\theta_1}, & -\frac{1}{2}\frac{b''\Theta_1 - A_1''}{\Theta_1\theta_1}, & -\frac{1}{2}\frac{b_1'\Theta_1 + B_1}{\Theta_1\theta_1} \\ 0, & 0\ \ , & \frac{1}{\Theta_1} \end{array}\right|$$

and into ψ_2 by a similar substitution. The determinant of each of these substitutions is Pp, and the denominators of their coefficients do not contain the prime 2, because b_1'', b_2'', A_1'', A_2'', Θ_1, Θ_2 are all uneven, and because $B_1 \equiv b_1'$, mod 2, $B_2 \equiv b_2'$, mod 2. Each of the forms f_1, f_2 is therefore transformable into the other by a rational substitution prime to $2\Omega\Delta$.

Art. 13. We have hitherto considered ternary forms of a negative determinant, definite or indefinite; we shall now confine our attention to definite forms. By a binary

form we shall henceforward understand a positive form of negative determinant, by a ternary form a positive and definite form; and we shall occupy ourselves in the remainder of this memoir with the determination of the weight of a given genus or order of such ternary forms.

A ternary form has always 1, 2, 4, 6, 8, 12, or 24 *positive* automorphics, *i. e.* automorphics of which the determinant is a positive unit. The weight of a form is the reciprocal of the number of its positive automorphics; so that a form and its contravariant have the same weight; the weight of a class is the weight of any form contained in the class; the weight of a genus or of an order is the sum of the weights of the non-equivalent classes contained in the genus or order. When a number is represented by a ternary form, the weight of the representation is the weight of the ternary form. The weight of a binary form, or class, is also the reciproeal of the number of its positive automorphics; thus the weight of a binary form is always $\frac{1}{2}$, except when the form either is, or is derived from, a form of determinant -1, or an improperly primitive form of determinant -3; in these excepted cases the weight of the binary form is $\frac{1}{4}$ and $\frac{1}{6}$ respectively. When a binary form is represented by a ternary form, the weight of the representation is the product of the weights of the two forms.

To determine the weight of a given genus of ternary forms, we avail ourselves of the principles introduced into arithmetic by GAUSS and DIRICHLET, and employed by them to determine the number of binary forms of any given determinant. Let (f, F) represent a given genus of ternary forms of the invariants $[\Omega, \Delta]$, and either of the properly primitive order, or of that improperly primitive order in which f is improperly and F properly primitive. Let $f_1, f_2, \ldots$ or (f) denote a system of forms representing the classes of the given genus; $F_1, F_2, \ldots$ or (F), the primitive contravariants of those forms. Let M represent any positive number, prime to $2\Omega\Delta$ and satisfying the generic characters of F; when (f, F) is a properly primitive genus, Ω being uneven, and Δ uneven or unevenly even, we shall also suppose that M satisfies the congruence $\Omega M \equiv 1$, mod 4: the numbers designated by M will be subject to the restrictions here stated throughout the whole investigation. Lastly, let L be a positive quantity which we shall afterwards suppose to increase without limit; and let T be the sum of the weights of the representations by the forms (F) of all the numbers M which do not surpass L. The quotient $T \div L^{\frac{3}{2}}$ approximates to a finite limit, when L is increased without limit. Of this limit, we shall obtain two distinct expressions, the one containing as a factor the weight W of the genus (f, F), the other not containing that factor, and depending on the arithmetical relation which subsists between the sum of the weights of the representations of a given number M by the forms (F), and the sum of the weights of the properly or improperly primitive binary classes of determinant $-\Omega M$. A comparison of the two expressions will then give the required weight of the genus (f, F).

Art. 14. The first determination of the limit of the quotient $T \div L^{\frac{3}{2}}$ depends on the following auxiliary propositions, in which F represents any form of the system (F).

(1) If δ is an uneven prime dividing Δ, F acquires a value prime to δ for $\delta^2(\delta-1)$ systems of values of x, y, z, mod δ.

As, instead of f and F, we may consider any forms equivalent to f and F, we may suppose that f and F satisfy, for any assigned powers of the uneven primes dividing $\Omega\Delta$, the congruences of Art. 5,

$$f\equiv\alpha x^2+\beta\Omega y^2+\gamma\Omega\Delta z^2,$$
$$F\equiv\beta\gamma\Omega\Delta x^2+\alpha\gamma\Delta y^2+\alpha\beta z^2,$$
$$\alpha\beta\gamma\equiv1.$$

The congruence $F\equiv0$, mod δ, is then satisfied by δ^2 systems of values of x, y, z, mod δ; for z must be divisible by δ, but x and y may have any values, mod δ; F is therefore prime to δ for the remaining $\delta^2(\delta-1)$ systems of values of x, y, z, mod δ.

(2) If ω is an uneven prime dividing Ω, but not Δ, F is prime to Ω, for $\omega(\omega-1)\left(\omega-\left(\frac{-\Delta f}{\omega}\right)\right)$ systems of values of x, y, z, mod ω.

For if $F\equiv0$, mod ω, x may have any value, mod ω, but y and z must have values satisfying the congruence $\gamma\Delta y^2+\beta z^2\equiv0$, mod ω. If $\left(\frac{-\Delta\beta\gamma}{\omega}\right)=-1$, the only values of y and z that satisfy this congruence are $y\equiv0$, $z\equiv0$, mod ω; and the congruence $F\equiv0$, mod ω, is satisfied by ω systems of values of x, y, z, mod ω. If $\left(\frac{-\Delta\beta\gamma}{\omega}\right)=+1$, the congruence $\gamma\Delta y^2+\beta z^2\equiv0$, mod ω, is satisfied by $2\omega-1$ systems of values of y and z; in this case therefore the congruence $F\equiv0$, mod ω, admits of $\omega(2\omega-1)$ solutions. And, observing that $\left(\frac{-\Delta\beta\gamma}{\omega}\right)=\left(\frac{-\Delta\alpha}{\omega}\right)=\left(\frac{-\Delta f}{\omega}\right)$, we find in both cases alike that F is prime to ω for $\omega(\omega-1)\left(\omega-\left(\frac{-\Delta f}{\omega}\right)\right)$ systems of values of x, y, z, mod ω.

(3) It is evident from the congruence

$$F\equiv Ax^2+A'y^2+A''z^2,\ \text{mod}\ 2,$$

in which one at least of the numbers A, A′, A″ is uneven, that F acquires an uneven value for 4 out of the 8 systems of values, mod 2, which can be attributed to x, y, z.

(4) If $\Omega\Delta$ is uneven, the number of solutions of the congruence $\Omega F\equiv1$, mod 4, is $8(2-\Psi)$.

For this congruence may be written in the form (art. 6)

$$\alpha x^2+\beta y^2+\gamma z^2\equiv1,\ \text{mod}\ 4,$$

α, β, γ representing uneven numbers which satisfy the congruence $\alpha+\beta+\gamma+1\equiv0$, mod 4. Of the three numbers x, y, z one must be uneven, the other two even. The number of solutions in which x is uneven, y and z even, is 8 or 0, according as $\alpha\equiv+1$, or $\equiv-1$, mod 4. The whole number of solutions is therefore

$$12+4\left[(-1)^{\frac{\alpha-1}{2}}+(-1)^{\frac{\beta-1}{2}}+(-1)^{\frac{\gamma-1}{2}}\right],$$

i. e. 24, or 8, according as the congruences $\alpha\equiv\beta\equiv\gamma\equiv1$, mod 4, are, or are not satisfied; or again (Art. 6), according as $\Psi=-1$, or $\Psi=+1$. The congruence $\Omega F\equiv1$, mod 4, admits therefore of $8(2-\Psi)$ solutions.

(5) If Ω is uneven, and Δ unevenly even, f as well as F being properly primitive, there are 16 solutions of the congruence $\Omega F \equiv 1$, mod 4; for this congruence may be written in the form (Art. 6)

$$2\alpha x^2+2\beta y^2+\gamma z^2 \equiv 1, \text{ mod } 4.$$

For clearness, we shall henceforward represent by r any uneven prime dividing both Ω and Δ, by δ any uneven prime dividing Δ, but not Ω; by ω any uneven prime dividing Ω, but not Δ. Let $\theta=2-\Psi$, if $\Omega \equiv \Delta \equiv 1$, mod 2; $\theta=2$, if, f and F being properly primitive, Ω is uneven and Δ unevenly even; $\theta=4$ in every other case; also let

$$\nabla=4\Pi r \times \Pi\omega \times \Pi\delta,$$

$$\psi(\nabla)=\frac{\theta}{8}\nabla^3\Pi\left[1-\frac{1}{r}\right]\Pi.\left[1-\frac{1}{\omega}\right]\Pi\left[1-\frac{1}{\delta}\right]\Pi\left[1-\left(\frac{-\Delta f}{\omega}\right)\frac{1}{\omega}\right].$$

Combining the lemmas (1)...(5) we obtain the theorem—

"The form F represents numbers of the series M for $\psi(\nabla)$ of the ∇^3 systems of values, mod ∇, that can be attributed to x, y, z."

Let x_i, y_i, z_i represent one of these $\psi(\nabla)$ systems of values; it is evident that F represents a number of the series M for every system of values of x, y, z included in the formulæ

$$\left.\begin{aligned} x&=\nabla X+x_i, \\ y&=\nabla Y+y_i, \\ z&=\nabla Z+z_i, \end{aligned}\right\} \quad \text{. . . . .} \quad (27)$$

in which X, Y, Z represent any integral numbers whatever. It is also evident that there are as many systems of values of x, y, z included in the formulæ (27), for which F acquires a value not surpassing L, as there are points having their rectangular coordinates of the form

$$x=\frac{\nabla X+x_i}{\sqrt{L}},$$

$$y=\frac{\nabla Y+y_i}{\sqrt{L}},$$

$$z=\frac{\nabla Z+z_i}{\sqrt{L}},$$

and lying inside, or on the surface of, the ellipsoid,

$$F(x, y, z)=1. \quad \text{. . . . .} \quad (28)$$

Let ν_i be the number of these points, and let L be increased without limit; the limit of the fraction $\frac{\nabla^3\nu_i}{L^{\frac{3}{2}}}$ is the volume of the ellipsoid (28), or $\frac{4}{3}\frac{\pi}{\Delta\sqrt{\Omega}}$. Extending this result to all the $\psi(\nabla)$ values of i, we find

$$\lim\frac{\Sigma\nu_i}{L^{\frac{3}{2}}}=\frac{4}{3}\frac{\psi(\nabla)}{\nabla^3}\cdot\frac{\pi}{\Delta\sqrt{\Omega}}. \quad \text{. . . . .} \quad (29)$$

Let τ be the sum of the weights of the representations of the numbers M which do not surpass L by the form F, and let w be the weight of f or F, so that $\tau = w\Sigma\nu_i$; the equation (29) becomes

$$\lim \frac{\tau}{L^{\frac{3}{2}}} = \frac{4}{3}\frac{\psi(\nabla)}{\nabla^3}\cdot\frac{\pi}{\Delta\sqrt{\Omega}}\cdot w; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (30)$$

or considering in succession all the forms of (F), and observing that $T = \Sigma\tau$, $W = \Sigma w$,

$$\lim \frac{T}{L^{\frac{3}{2}}} = \frac{\theta W}{6} \times \Pi\left(1-\frac{1}{r}\right)\Pi\left(1-\frac{1}{\delta}\right)\Pi\left(1-\frac{1}{\omega}\right)\times\Pi\left[1-\left(\frac{-\Delta f}{\omega}\right)\frac{1}{\omega}\right], \quad . \quad . \quad (31)$$

which is the first determination of the limit of the quotient $\dfrac{T}{L^{\frac{3}{2}}}$.

Art. 15. The second determination of the limit of the quotient $T \div L^{\frac{3}{2}}$ depends on the following theorem:—

"The sum of the weights of the primitive representations by the forms (F) of a given number M divisible by μ unequal primes, is 2^μ times the weight of a genus of binary forms, of determinant $-\Omega M$, and properly or improperly primitive, according as the forms (f) are properly or improperly primitive."

The principles which give the demonstration of this theorem are contained in Arts. 280–284 of the 'Disquisitiones Arithmeticæ,' and have been in part already employed in Art. 10 of this memoir. We have shown in Art. 11 that one genus and only one of binary forms of determinant $-\Omega M$ admits of primitive representation by the forms (f) of the ternary genus (f, F). Let $\phi_1, \phi_2, \ldots$ or (ϕ) be a system of forms representing the classes of that binary genus; these forms are properly or improperly primitive, according as the forms (f) are properly or improperly primitive: let n be their number and ν the sum of their weights; as their weights are all equal, the weight of each of them is $\frac{\nu}{n}$; so that each has $\frac{n}{\nu}$ positive automorphics, and is transformed into any equivalent form by $\frac{n}{\nu}$ positive substitutions. We shall first show that the sum of the weights of the primitive representations of the forms (ϕ) by the forms (f) is equal to $2^\mu \times \nu$; and secondly, that the sum of the weights of the primitive representations of the numbers M by the forms (F) is equal to the sum of the weights of the primitive representations of the forms (ϕ) by the forms (f).

(i) Each of the n congruences

$$-\Delta\phi \equiv (Qx - Q'y)^2, \bmod M, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (32)$$

in which Q, Q′ are the numbers to be determined, is resoluble, and admits of 2^μ incongruous solutions. From each such solution we deduce, by the method of GAUSS employed in Art. 10, a ternary form f' of the given genus, containing one of the forms (ϕ) as a part, and having Q, Q′, M for the coefficients of $2yz$, $2xz$, z^2 in its primitive contravariant. There are $2^\mu \times n$ of these forms (f'); none of them is the same as any other, and none of them can be transformed into any other by a substitution of the type

$$\begin{vmatrix} 1, & 0, & \varkappa' \\ 0, & 1, & \varkappa \\ 0, & 0, & 1; \end{vmatrix} \quad \ldots \ldots \ldots \ldots \quad (33)$$

for if one of them could be so transformed into another, these two would contain as a part the same form ϕ, and the values of Q, Q' in the primitive contravariant of the one would be congruous, for the modulus M, to the values of Q, Q' in the primitive contravariant of the other; the two forms would thus be derived from the same solution of the same congruence (32). Again, the primitive representations of the forms (ϕ) by the forms (f) are equal in number to the positive transformations of the forms (f) into the forms (f'). For every positive transformation of a form of (f) into a form of (f') supplies a primitive representation of some form of (ϕ) by that form of (f); and these representations are all different, because the same form f cannot be transformed into two of the forms (f'), or twice into one of them, by positive substitutions of which the first two columns are the same; otherwise one of the forms (f') could be transformed into another by a substitution of the type (33), or else one of those forms would have an automorphic of that type, whereas no substitution of the type (33), in which $\varkappa$ and $\varkappa'$ are different from zero, can be an automorphic of any ternary form. There are therefore at least as many different primitive representations of the forms (ϕ) by the forms (f), as there are positive transformations of the forms (f) into the forms (f'). And there are no more; for if

$$\begin{vmatrix} \alpha\ , & \beta \\ \alpha', & \beta' \\ \alpha'', & \beta'' \end{vmatrix}$$

is a given primitive representation of ϕ by f, let γ, γ', γ'' be numbers which render the determinant of the substitution

$$\begin{vmatrix} \alpha\ , & \beta\ , & \gamma \\ \alpha', & \beta', & \gamma' \\ \alpha'', & \beta'', & \gamma'' \end{vmatrix} \quad \ldots \ldots \ldots \ldots \quad (34)$$

equal to $+1$; and let f_1 be the form, containing ϕ as a part, into which f is transformed by the substitution (34). The coefficient of z^2 in the primitive contravariant of f_1 is M, and if the coefficients of $2yz$, $2xz$ in that contravariant are Q_1, Q'_1, these numbers supply a solution of the congruence (32). Let f' be that form of (f') which is deduced from this solution; then f_1 is equivalent to f', and is transformed into it by a substitution of the type (33), in which $\varkappa=\frac{Q_1-Q}{M}$, $\varkappa'=\frac{Q'_1-Q'}{M}$. Therefore f is transformed into f' by the substitution

$$\begin{vmatrix} \alpha\ , & \beta\ , & \gamma+\varkappa'\alpha+\varkappa\beta \\ \alpha', & \beta', & \gamma'+\varkappa'\alpha'+\varkappa\beta' \\ \alpha'', & \beta'', & \gamma''+\varkappa'\alpha''+\varkappa\beta'', \end{vmatrix},$$

i. e. the given primitive representation of ϕ by f is included among those supplied by the positive transformations of the forms (f) into the forms (f'). Thus the number of the primitive representations of the forms (ϕ) by the forms (f) is equal to the number of the positive transformations of the forms (f) into the forms (f'): to obtain the sum of the weights of these representations, we consider, in particular, f one of the forms of (f); let d be the number of its positive automorphics, so that $\frac{1}{d}$ is its weight, and let s be the number of the forms (f') which are equivalent to it. Then there are ds primitive representations of the forms (ϕ) by f; but the weight of each of these representations is $\frac{1}{d}\times\frac{\nu}{n}$; the sum of the weights of the primitive representations of the forms (ϕ) by f is therefore $s\times\frac{\nu}{n}$. Extending this conclusion to all the forms of (f), and observing that Σs is equal to the number of the forms (f'), *i. e.* to $2^{\mu}\times n$, we find that the sum of the weights of the primitive representations of the forms (ϕ) by the forms (f) is $2^{\mu}\times\nu$.

(ii) Let $M=F(\Gamma, \Gamma', \Gamma'')$ be a given primitive representation of M by F; and let

$$\left|\begin{matrix}\alpha\,, & \beta\\ \alpha', & \beta'\\ \alpha'', & \beta''\end{matrix}\right| \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (35)$$

be a matrix, of which the constituents satisfy the equations

$$\alpha'\beta''-\alpha''\beta'=\Gamma, \quad \alpha''\beta-\alpha\beta''=\Gamma', \quad \alpha\beta'-\alpha'\beta=\Gamma''. \quad . \quad . \quad . \quad . \quad (36)$$

All the matrices, of which the constituents satisfy these equations, are then included in the formula

$$\left|\begin{matrix}\alpha\,, & \beta\\ \alpha', & \beta'\\ \alpha'', & \beta''\end{matrix}\right| \times |v| \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (37)$$

in which $|v|$ is a square binary matrix of which the determinant is $+1$. Thus the binary forms, which are represented by f simultaneously with the given representation of M by F, are all equivalent to one another, and to some form of (ϕ); let ϕ be that form of (ϕ) to which they are equivalent, and let us suppose (as we may do) that f is transformed into ϕ by the substitution (35). Substituting successively for $|v|$ in the formula (37), the $\frac{n}{\nu}$ positive automorphics of ϕ, we obtain $\frac{n}{\nu}$ representations of ϕ by f: these representations are all different, and they include every representation of ϕ by f which is simultaneous with the given representation of M by F: the weight of each of them is $\frac{1}{d}\times\frac{\nu}{n}$; the sum of their weights is therefore equal to $\frac{1}{d}$, or to the weight of the given representation of M by F. Hence the sum of the weights of all the primitive representations of M by the forms (F) is equal to the sum of the weights of the simultaneous representations of the forms (ϕ) by the forms (f), or, which is the same thing,

to the sum of the weights of all the primitive representations of the forms (φ) by the forms (f); because every primitive representation of a form (φ) by a form (f) is simultaneous with one and only one primitive representation of M by a form (F).

Combining the conclusions (i) and (ii), we obtain the result enunciated at the beginning of this article.

Art. 16. Let σ represent the number of uneven primes dividing Ω, counting those which also divide Δ; let $\sigma'=-1$, when $\Omega M\equiv -1$, mod 4*; $\sigma'=+1$, when $\Omega\equiv 0$, mod 8; and $\sigma'=0$ in all other cases. Let also $h(\Omega M)$ and $h'(\Omega M)$ be the weights of the properly and improperly primitive orders of binary forms of determinant $-\Omega M$; then $2^{\mu}\times\nu=\frac{h(\Omega M)}{2^{\sigma+\sigma'}}$, or $\frac{h'(\Omega M)}{2^{\sigma+\sigma'}}$, according as the forms (f) are properly or improperly primitive. If λ^2 is any square divisor of M, the sum of the weights of those representations of M by the forms (F), which are derived from primitive representations of $\frac{M}{\lambda^2}$ by the same forms, is $\frac{h\left(\frac{\Omega M}{\lambda^2}\right)}{2^{\sigma+\sigma'}}$, or $\frac{h'\left(\frac{\Omega M}{\lambda^2}\right)}{2^{\sigma+\sigma'}}$. Therefore the sum of the weights of all the representations of M by the forms (F) is $\frac{\Sigma\,.\,h\left(\frac{\Omega M}{\lambda^2}\right)}{2^{\sigma+\sigma'}}$, or $\frac{\Sigma\,.\,h'\left(\frac{\Omega M}{\lambda^2}\right)}{2^{\sigma+\sigma'}}$, the signs of summation extending to every square divisor of M. Or, if we represent by $H(\Omega M)$ the sum of the weights of those uneven binary classes of determinant $-\Omega M$ which are prime to Ω, and by $H'(\Omega M)$ the sum of the weights of those even classes of determinant $-\Omega M$ which are prime to Ω, the sum of the weights of all the representations of M by the forms (F) is

$$\frac{H(\Omega M)}{2^{\sigma+\sigma'}},\ \text{or}\ \frac{H'(\Omega M}{2^{\sigma+\sigma'}},$$

according as the forms (f) are properly or improperly primitive.

Art. 17. We now consider the sums

$$\Sigma[xz-y^2=\Omega M], \quad \ldots\ldots \quad (38)$$

$$\Sigma'[xz-y^2=\Omega M]. \quad \ldots\ldots \quad (39)$$

In both the sign of summation extends to every solution in integral numbers of the equation

$$xz-y^2=\Omega M,$$

in which the greatest common divisor of x, y, z is prime to Ω, and in which x, y, z satisfy the inequalities

$$\left.\begin{array}{lll} x>0, & y\geqq 0, & z>0, \\ x\geqq 2y\leqq z, & & x\leqq z \end{array}\right\} \quad \ldots\ldots \quad (40)$$

But, in the first sum, one at least of the two numbers x and z is uneven; in the second, x and z are even, and y is uneven. The symbol $[xz-y^2=\Omega M]$ is 1, or $\frac{1}{2}$, or $\frac{1}{4}$, or $\frac{1}{6}$,

* If this congruence is satisfied by any one number of the series M, it is satisfied by every number of that series.

according as the inequalities (40) are satisfied, excluding all signs of equality, or admitting one, or two, or three such signs. Again, representing by $(2y)$ the absolute value of $2y$, we observe that a *reduced* binary form is a form (x, y, z) of which the coefficients satisfy the inequalities,

$$\left.\begin{array}{lll} \text{(i)} & x>0, & z>0, \\ & x \geqq (2y) \leqq z, & x \leqq z. \\ \text{(ii)} & \text{If } x=(2y), & y>0, \\ & \text{If } x=z, & y \geqq 0. \end{array}\right\} \quad \ldots \ldots \quad (41)$$

and that, by a fundamental proposition in the theory of binary forms, every class contains one and only one reduced form. Attending only to those uneven classes of determinant $-\Omega\mathrm{M}$ which are prime to Ω, and comparing the inequalities (40) and (41), we find that the sum (38) contains (i) an unit corresponding to every pair of reduced forms (x, y, z), $(x, -y, z)$ of which the coefficients satisfy none of the equalities $y=0$, $x=2y$, $x=z$; (ii) one-half of an unit corresponding to every reduced form of which the coefficients satisfy one of them; and (iii) one-fourth of an unit corresponding to a reduced form (if there be such a form of determinant $-\Omega\mathrm{M}$ prime to Ω) of which the coefficients satisfy the two equalities, $y=0$, $x=z$, and of which the weight is consequently $\frac{1}{4}$. We thus obtain the equation

$$\mathrm{H}(\Omega\mathrm{M})=\Sigma[xz-y^2=\Omega\mathrm{M}].$$

Again, attending only to those even classes of the uneven determinant $-\Omega\mathrm{M}$ which are prime to Ω, we find that the sum (39) contains units corresponding to pairs of reduced forms, and half units corresponding to single reduced forms; it also contains one-sixth of an unit corresponding to a reduced form (if there be such a reduced form of determinant $-\Omega\mathrm{M}$ prime to Ω) of which the coefficients satisfy the three equalities $x=2y$, $2y=z$, $x=z$, and of which the weight is consequently $\frac{1}{6}$. We therefore have the equation

$$\mathrm{H}'(\Omega\mathrm{M})=\Sigma'[xz-y^2=\Omega\mathrm{M}].$$

Art. 18. According as the forms (f) are properly or improperly primitive, let

$$\Upsilon=\Sigma\,.\,\Sigma[xz-y^2=\Omega\mathrm{M}],$$

or

$$\Upsilon=\Sigma\,.\,\Sigma'[xz-y^2=\Omega\mathrm{M}],$$

the first sign of summation extending to all values of M not surpassing L; so that, in both cases alike,

$$\mathrm{T}=\frac{\Upsilon}{2^{\sigma+\sigma'}}.$$

To determine the limit of the quotient $\frac{\Upsilon}{\mathrm{L}^{\frac{3}{2}}}$, when L is increased without limit, we shall again employ the geometric method of Gauss. For its application here the following preliminary lemmas are requisite, relating to the arithmetical properties of the function $xz-y^2$.

(1) If p is any uneven prime, and m any given number, the congruence

$$xz-y^2\equiv m, \bmod p, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (42)$$

admits of $p\left[p+\left(\frac{-m}{p}\right)\right]$ solutions.

For if $\left(\frac{-m}{p}\right)=+1$, y^2+m is prime to p for $p-2$ values of y, and is divisible by p for 2 values of y. When y^2+m is prime to p, we may assign to z any value prime to p, determining x by the congruence $xz\equiv y^2+m$; we thus obtain $(p-1)(p-2)$ solutions of (42). When y^2+m is divisible by p, the congruence $xz\equiv 0$, mod p, admits of $2p-1$ solutions; we thus obtain in all $(p-1)(p-2)+2(2p-1)=p(p+1)$ solutions of (42).

If $\left(\frac{-m}{p}\right)=-1$, y^2+m is prime to p for every value of y; there are thus $p(p-1)$ solutions of (42).

Lastly, if $\left(\frac{-m}{p}\right)=0$, *i. e.* if $m\equiv 0$, mod p, y^2+m is prime to p for $p-1$ values of y, and divisible by p for one value of y. There are thus $(p-1)^2+2p-1=p^2$ solutions of (42).

We shall have to use the following corollary of this lemma,

If m is prime to p, and if we successively attribute to x, y, z the p^3 systems of values, mod p, of which they are susceptible, $xz-y^2$ will have the same quadratic character as m for $\frac{1}{2}p(p-1)\left[p+\left(\frac{-m}{p}\right)\right]$ of these systems.

(2) The congruences $xz-y^2\equiv 1, 3, 5, 7$, mod 8, each admit 48 solutions in which x and z are not simultaneously even; of the congruences, $xz-y^2\equiv 3, \equiv 7$, mod 8, the first admits 16, the second 48 solutions in which x and z are simultaneously even.

For example, let the proposed congruence be $xz-y^2\equiv 3$, mod 8. If y has one of its four even values, mod 8, we may give to z any one of its four uneven values, mod 8, and determine the value of x in the resulting congruence; we thus obtain 4×4 solutions in which x and z are uneven. If y has one of its four uneven values, mod 8, the congruence becomes $xz\equiv 4$, mod 8, which admits of 8 solutions in which x and z are not simultaneously even, and 4 in which they are simultaneously even. There are thus $(4\times 4)+(4\times 8)=48$ solutions of the congruence $xz-y^2\equiv 3$, mod 8, in which x and z are not simultaneously even, and $4\times 4=16$ in which x and z are simultaneously even.

(3) If p is any prime, even or uneven, i and i' integral exponents, of which $i>0$, $i'\geqq 0$, and m any given number, prime to p, or divisible by any power of p, the congruence

$$xz-y^2\equiv mp^i, \bmod p^{i+i'} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (43)$$

admits of $p^{2i+2i'}\left(1-\frac{1}{p^2}\right)$ *primitive* solutions, *i. e* .solutions in which x, y, z, or, which is the same thing, x, z are not simultaneously divisible by p.

(i) If the assertion is true for i, i', and if $j\leqq i'$, it is true for $i+j$, $i'-j$. For, on writing mp^j for m in (43), it becomes

$$xz-y^2\equiv mp^{i+j}, \bmod p^{(i+j)+(i'-j)};$$

if therefore the former congruence admits

$$p^{2i+2i'}\left(1-\frac{1}{p^2}\right)=p^{2(i+j)+2(i'-j)}\left(1-\frac{1}{p^2}\right)$$

primitive solutions, the latter does so too.

(ii) If the assertion is true for i, 0, it is also true for i, i', where $i'\leqq i$.

For if x, y, z is a given primitive solution of

$$xz-y^2\equiv mp^i, \bmod p^i, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (44)$$

Xp^i+x, Yp^i+y, Zp^i+z is a primitive solution of (43), whenever X, Y, Z satisfy the congruence

$$Xz-2Yy+Zx+\frac{xz-y^2}{p^i}\equiv m, \bmod p^{i'}.$$

This congruence admits of $p^{2i'}$ solutions; for the given numbers x and z are not simultaneously divisible by p. Thus from each primitive solution of (44) we obtain $p^{2i'}$ primitive solutions of (43). These solutions are all different, and exhaust all the solutions of (43); if therefore (44) admits of $p^{2i}\left(1-\frac{1}{p^2}\right)$ solutions, (43) admits of $p^{2i+2i'}\left(1-\frac{1}{p^2}\right)$ solutions.

(iii) The assertion is true if $i=1$, $i'=0$. For (lemma 1) there are p^2 solutions of the congruence $xz-y^2\equiv 0$, $\bmod p$, and of these one is not primitive.

The proposition is, therefore, true universally. We shall have to employ the following corollaries from it.

(1) The function $xz-y^2$ is divisible by p^i, but not by p^{i+1}, for $p^{2i}(p-1)^2(p+1)$ systems of values of $x, y, z, \bmod p^{i+1}$; the values of x, y, z not being simultaneously divisible by p.

(2) If p is an uneven prime, the quotients obtained by dividing these $p^{2i}(p-1)^2(p+1)$ values of $xz-y^2$ by p^i, are half quadratic residues, and half non-quadratic residues of p.

(3) If $p=2$, the function $xz-y^2$ is divisible by 2^i, but not by 2^{i+1}, for $3\times 2^{2i+6}$ systems of values of x, y, z, mod 2^{i+3}, the values of x, y, z not being simultaneously even. And if these $3\times 2^{2i+6}$ values of $xz-y^2$ be divided by 2^i, one-fourth part of the quotients is contained in each of the linear forms $8k+1$, 3, 5, 7.

Art. 19. Let $\nabla=8\Omega\times\Pi r\times\Pi\omega\times\Pi\delta$, and let us successively attribute to x, y, z in the function $xz-y^2$ the ∇^3 systems of values, mod ∇, of which they are susceptible; let $\phi(\nabla)$ represent the number of those systems, in which the greatest common divisor of x, y, z is prime to ∇, and which give to $xz-y^2$ a value divisible by Ω, and such that the quotient $\frac{xz-y^2}{\Omega}$ is a number of the series M; if the forms (f) are properly primitive, x and z are not to be simultaneously even; if those forms are improperly primitive, x and z are to be simultaneously even. We shall now show that $\phi(\nabla)$ is determined by the equation

$$\left.\begin{aligned}\phi(\nabla)=\tfrac{3}{8}\times\eta\times\nabla^3\times\frac{1}{\Omega}\times\Pi\left(1-\frac{1}{r}\right)\times\Pi\left(1-\frac{1}{\delta}\right)\times\Pi\left(1-\frac{1}{\omega}\right)\\ \times\Pi\tfrac{1}{2}\left(1-\frac{1}{r^2}\right)\times\Pi\left(1-\frac{1}{\omega^2}\right)\times\Pi\tfrac{1}{2}\left[1+\left(\frac{-\Omega F}{\delta}\right)\frac{1}{\delta}\right],\end{aligned}\right\} \quad . \quad . \quad . \quad (45)$$

η being a coefficient of which the value is $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}$, or $\frac{1}{12}$*, as shown in the following Table.

(i) (f) properly primitive.

	$\Omega\equiv 1$, mod 2.	$\Omega\equiv 2$, mod 4.	$\Omega\equiv 4$, mod 8.	$\Omega\equiv 0$, mod 8.
$\Delta\equiv 1$, mod 2.	$\frac{1}{2}$	1	$\frac{1}{4}\left[3+(-1)^{\frac{\Delta f+1}{2}}\right]$	$\frac{1}{4}\left[3+(-1)^{\frac{\Delta f+1}{2}}\right]$
$\Delta\equiv 2$, mod 4.	$\frac{1}{2}$	1	$\frac{1}{2}$	$\frac{1}{2}$
$\Delta\equiv 4$, mod 8.	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
$\Delta\equiv 0$, mod 8.	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$

(ii) (f) improperly primitive.

$\Delta\equiv 2$, mod 4.	$\frac{1}{3}$
$\Delta\equiv 0$, mod 4.	$\frac{1}{12}\left[2+(-1)^{\frac{\Omega^2-1}{8}+\frac{F^2-1}{8}}\right]$

To establish the equation (45), we consider separately the different primes dividing ∇. And first let us take an uneven prime δ, dividing Δ but not Ω. Of the δ^3 systems of values of x, y, z, mod δ,

$$\delta^3\times\left(1-\frac{1}{\delta}\right)\times\frac{1}{2}\left[1+\left(\frac{-\Omega F}{\delta}\right)\frac{1}{\delta}\right] \text{ systems}$$

give to $xz-y^2$ a value prime to δ, and satisfying the equation (Lemma i, Cor.)

$$\left(\frac{xz-y^2}{\delta}\right)=\left(\frac{\Omega F}{\delta}\right).$$

Secondly, let us consider an uneven prime ω dividing Ω but not Δ; and let ω^i be the highest power of ω dividing Ω. Of the ω^{3i+3} systems of values of x, y, z, mod ω^{i+1},

$$\omega^{3i+3}\times\frac{1}{\omega^i}\times\left(1-\frac{1}{\omega}\right)\times\left(1-\frac{1}{\omega^2}\right) \text{ systems,}$$

in which x, y, z are not simultaneously divisible by ω, render $xz-y^2$ divisible by ω^i, and also render the quotient $\frac{xz-y^2}{\omega^i}$ prime to ω (Lemma iii. Cor. 1).

Thirdly, let us consider an uneven prime r dividing both Δ and Ω, and let r^i be the highest power of r dividing Ω. Of the r^{3i+3} systems of values of x, y, z, mod r^{i+1},

$$r^{3i+3}\times\frac{1}{r}\times\left(1-\frac{1}{r}\right)\times\frac{1}{2}\left(1-\frac{1}{r^2}\right) \text{ systems,}$$

in which x, y, z are not simultaneously divisible by r, render $xz-y^2$ divisible by

* It will be seen that 4η in the Table (i), is in every case the number of the linear forms $8k+1, 3, 5, 7$, in which the numbers M are contained.

r^i, but not by r^{i+1}, and also render the quotients $\frac{xz-y^2}{r^i}$ all quadratic residues, or all non-quadratic residues of r (Lemma iii. Cor. 2.).

Lastly, let us consider the even prime 2, and let 2^i be the highest power of 2 dividing Ω. Considering separately the eighteen cases of the Tables (i) and (ii), we find that of the 2^{3i+9} systems of values of x, y, z, mod 2^{i+3},

$$\tfrac{3}{8}\times\eta\times 2^{3i+9}\times\frac{1}{2^i}\text{ systems}$$

(in which x, y, z are not simultaneously even, but x and z are or are not simultaneously even, according as the forms (f) are improperly or properly primitive) give to $\frac{xz-y^2}{2^i}$ an integral and uneven value, satisfying the supplementary character (if any) of $\frac{\Omega}{2^i}$ F, and, if the forms (f) are properly primitive, satisfying the congruence $xz-y^2\equiv 1$, mod 4, when Ω is uneven, and Δ uneven or unevenly even.

For example, let $i\geqq 1$, $\Delta\equiv 0$, mod 8. Here F, or $\frac{\Omega}{2^i}$ F, has two supplementary characters, and of the 2^{3i+9} systems of values of x, y, z, mod 2^{i+3},

$$\tfrac{3}{8}\times\tfrac{1}{4}\times 2^{3i+9}\times\frac{1}{2^i}\text{ systems,}$$

in which x and z are not simultaneously even, give to $\frac{xz-y^2}{2^i}$ an integral and uneven value, satisfying the supplementary characters of $\frac{\Omega}{2^i}$ F (Lemma iii. Cor. 3).

Again, let $i\geqq 2$, $\Delta\equiv 1$, mod 2. Here F has or has not a supplementary character, according as $(-1)^{\frac{\Delta f+1}{2}}=-1$, or $=+1$. In the former case, of the 2^{3i+9} systems of values of x, y, z, mod 2^{i+3},

$$\tfrac{3}{8}\times\tfrac{1}{2}\times 2^{3i+9}\times\frac{1}{2^i}\text{ systems}$$

(in which x and z are not simultaneously even) give to $\frac{xz-y^2}{2^i}$ an integral and uneven value satisfying the supplementary character of $\frac{\Omega}{2^i}$ F. In the latter case, of the same 2^{3i+9} systems of values,

$$\tfrac{3}{8}\times 1\times 2^{3i+9}\times\frac{1}{2^i}\text{ systems,}$$

in which x and z are not simultaneously even, give to $\frac{xz-y^2}{2^i}$ an integral and uneven value. Both results are comprised in the formula

$$\tfrac{3}{8}\times\tfrac{1}{4}\left[3+(-1)^{\frac{\Delta f+1}{2}}\right]\times 2^{3i+9}\times\frac{1}{2^i}.$$

As a third example, let $i=0$, $\Delta\equiv 0$, mod 4, and let the forms considered be of an improperly primitive order. Then $\Omega F\equiv 3$, mod 4; and either $\Omega F\equiv 3$, mod 8, or $\Omega F\equiv 7$,

mod 8. The congruence $xz-y^2\equiv\Omega\mathrm{F}$, mod 8, in which only even values of x and z are to be admitted, is satisfied in the former case by 16, in the latter by 48 systems; *i. e.* in either case by

$$\tfrac{3}{8}\times\tfrac{1}{12}\left[2+(-1)^{\frac{\Omega^2-1}{8}+\frac{\mathrm{F}^2-1}{8}}\right]\times 2^9 \text{ systems.}$$

The formula (45) results immediately from the combination of these determinations relative to the primes δ, ω, r, and 2.

Art. 20. Let x_i, y_i, z_i be one of the $\varphi(\nabla)$ systems of values of x, y, z, mod ∇, defined in the last article; and let us decompose the sum Υ of art. 18 into $\varphi(\nabla)$ partial sums Υ_1, Υ_2, ..., comprising in the sum Υ_i all those terms of Υ in which x, y, z are of the linear forms

$$x=\nabla\mathrm{X}+x_i,$$
$$y=\nabla\mathrm{Y}+y_i,$$
$$z=\nabla\mathrm{Z}+z_i,$$

X, Y, Z denoting any integral numbers whatever. The sum Υ_i is equal to the number of points having their positive rectangular coordinates of the forms

$$x=\frac{\nabla\mathrm{X}+x_i}{\sqrt{\Omega\mathrm{L}}},$$
$$y=\frac{\nabla\mathrm{Y}+y_i}{\sqrt{\Omega\mathrm{L}}},$$
$$z=\frac{\nabla\mathrm{Z}+z_i}{\sqrt{\Omega\mathrm{L}}},$$

and lying within the hyperboloidal cuneus, bounded by the planes $y=0$, $x=z$, $x=2y$, and the hyperboloid $xz-y^2=1$; points lying on the hyperboloidal boundary are counted as lying within the cuneus; points lying on its plane boundaries are counted as $\frac{1}{2}$ each, and points lying on the intersection of $y=0$ with $x=z$, and with $x=2y$ respectively as $\frac{1}{4}$ and $\frac{1}{6}$. Let V be the volume of the cuneus, and let L be increased without limit; we have

$$\lim\frac{\Upsilon_i}{\mathrm{L}^{\frac{3}{2}}}=\Omega^{\frac{3}{2}}\times\frac{\mathrm{V}}{\nabla^3};$$

and since this limit is thus ascertained to be the same for all the partial sums Υ_1, Υ_2, . .,

$$\lim\frac{\Upsilon}{\mathrm{L}^{\frac{3}{2}}}=\Omega^{\frac{3}{2}}\times\frac{\varphi(\nabla)}{\nabla^3}\times\mathrm{V},$$

or, which is the same thing,

$$\lim\frac{\mathrm{T}}{\mathrm{L}^{\frac{3}{2}}}=\frac{1}{2^{\sigma+\sigma'}}\times\Omega^{\frac{3}{2}}\times\frac{\Phi(\nabla)}{\nabla^3}\times\mathrm{V}.$$

The value of V may be determined by dividing the cuneus into laminæ parallel to the plane of xz; if A be the area of a section at a distance y from that plane, we find

$$\mathrm{A}=(1+y^2)[\tfrac{1}{2}\log(1+y^2)-\log 2y]-\tfrac{1}{2}(1-3y^2);$$

whence

$$\mathrm{V}=\int_0^{\sqrt{\frac{1}{3}}}\mathrm{A}dy=\frac{\pi}{9}. \quad \ldots\ldots\ldots\ldots\ldots \quad (46)$$

Substituting for $\frac{\phi(\nabla)}{\nabla^3}$ and for V their values, given by the equations (45) and (46), we find

$$\left.\begin{aligned}\lim \frac{T}{L^{\frac{3}{2}}}=\frac{\pi}{24}\times\frac{\eta}{2^{\sigma'}}\times\sqrt{\Omega}\times\Pi\left(1-\frac{1}{r}\right)\times\Pi\left(1-\frac{1}{\delta}\right)\times\Pi\left(1-\frac{1}{\omega}\right)\\ \times\Pi\tfrac{1}{4}\left(1-\frac{1}{r^2}\right)\times\Pi\tfrac{1}{2}\left(1-\frac{1}{\omega^2}\right)\times\Pi\tfrac{1}{2}\left[1+\left(\frac{-\Omega F}{\delta}\right)\frac{1}{\delta}\right]\end{aligned}\right\}, \quad . \quad . \quad . \quad . \quad (47)$$

which is the second determination of the limit of the quotient $\frac{T}{L^{\frac{3}{2}}}$.

Finally, equating the two values of this limit, and denoting the coefficient $\frac{1}{2^{\sigma'}}\times\frac{\eta}{\theta}$ by $\frac{1}{2}\zeta$, we obtain the following determination of the weight of the proposed genus,

$$W=\frac{\Delta\Omega}{8}\times\zeta\times\Pi\tfrac{1}{4}\left(1-\frac{1}{r^2}\right)\times\Pi\tfrac{1}{2}\left[1+\left(\frac{-\Delta f}{\omega}\right)\frac{1}{\omega}\right]\times\Pi\tfrac{1}{2}\left[1+\left(\frac{-\Omega F}{\delta}\right)\frac{1}{\delta}\right], \quad . \quad . \quad (48)$$

the values of ζ (which are computed from those of σ', η, θ) being as follows:—

(A).—(f) and (F) properly primitive.

	$\Omega\equiv1$, mod 2.	$\Omega\equiv2$, mod 4.	$\Omega\equiv4$, mod 8.	$\Omega\equiv0$, mod 8.
$\Delta\equiv1$, mod 2.	$\frac{1}{3}[2+\Psi]$	$\frac{1}{2}$	$\frac{1}{8}\left[3+(-1)^{\frac{\Delta f+1}{2}}\right]$	$\frac{1}{16}\left[3+(-1)^{\frac{\Delta f+1}{2}}\right]$
$\Delta\equiv2$, mod 4.	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$
$\Delta\equiv4$, mod 8.	$\frac{1}{8}\left[3+(-1)^{\frac{\Omega F+1}{2}}\right]$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{8}$
$\Delta\equiv0$, mod 8.	$\frac{1}{16}\left[3+(-1)^{\frac{\Omega F+1}{2}}\right]$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{16}$

(B).—(f) improperly, (F) properly primitive.
$\Omega\equiv1$, mod 2; $\Omega F\equiv3$, mod 4.

$\Delta\equiv2$, mod 4.	$\frac{1}{3}$
$\Delta\equiv0$, mod 4.	$\frac{1}{12}\left[2+(-1)^{\frac{\Omega^2F^2-1}{8}}\right]$

(C).—(f) properly, (F) improperly primitive.
$\Delta\equiv1$, mod 2; $\Delta f\equiv3$, mod 4.

$\Omega\equiv2$, mod 4.	$\frac{1}{3}$
$\Omega\equiv0$, mod 4.	$\frac{1}{12}\left[2+(-1)^{\frac{\Delta^2f^2-1}{8}}\right]$

The last of these Tables is obtained by reciprocation from the second.

The result in the case $\Omega\equiv\Delta\equiv 1$, mod 2, is given in the memoir of EISENSTEIN (Crelle, vol. xxxv. p. 128).

Art. 21. The equation (47) may also be deduced from the theorem of Art. 15 by another method. We consider first and principally the case in which the forms (f) and (F) are both properly primitive.

From Art. 16 we obtain the equation

$$T=\frac{1}{2^{\sigma+\sigma'}}\sum_{M\geq 1}^{M\leq L}\left[\Sigma h\left(\frac{\Omega M}{\lambda^2}\right)\right],$$

the interior sign of summation extending to every square divisor of M. Inverting the order of the summations, and designating by m any number prime to $2\Omega\Delta$, we may write this equation in the form

$$T=\frac{1}{2^{\sigma+\sigma'}}\sum_{m=1}^{m\leq\sqrt{L}}\ \sum_{M\geq 1}^{M\leq\frac{L}{m^2}} h(\Omega M).$$

But, by a theorem of LEJEUNE DIRICHLET,

$$h(\Omega M)=\frac{1}{\pi}\sqrt{\Omega M}\,\Sigma\left(\frac{-\Omega M}{n}\right)\frac{1}{n},$$

the sign of summation extending to all uneven numbers prime to ΩM. The limit of $\frac{T}{L}$ is therefore the limit of the expression

$$\frac{1}{2^{\sigma+\sigma'}}\frac{\sqrt{\Omega}}{\pi}\times\frac{1}{L^{\frac{3}{2}}}\times\sum_{m=1}^{m\leq\sqrt{L}}\ \sum_{M\geq 1}^{M\leq\frac{L}{m^2}}\sqrt{M}\,\Sigma\left(\frac{-\Omega M}{n}\right)\frac{1}{n},$$

or, leaving the summation with respect to n to be effected last, of the expression

$$\frac{1}{2^{\sigma+\sigma'}}\frac{\sqrt{\Omega}}{\pi}\sum_{n=1}^{n=\infty}\frac{1}{n}\sum_{m=1}^{m\leq\sqrt{L}}\frac{1}{L^{\frac{3}{2}}}\sum_{M>1}^{M\leq\frac{L}{m^2}}\left(\frac{-\Omega M}{n}\right)\sqrt{M}. \quad \text{(49)}$$

In this expression n is uneven and prime to Ω; but n is not necessarily prime to Δ. Let $n=n_1^2 n_2$, n_1^2 denoting the greatest square dividing n, so that n_2 is a product of unequal primes; also let ν represent any prime dividing n, other than one of the primes δ; and let η represent $\frac{1}{4}$, $\frac{1}{2}$, or 1, according as the numbers M are contained in one, two, or all four of the linear forms $8k+1$, 3, 5, 7; so that η has the same value as in Art. 19. The limit of the sum

$$\frac{1}{L^{\frac{3}{2}}}\sum_{M\geq 1}^{M\leq\frac{L}{m^2}}\left(\frac{-\Omega M}{n}\right)\sqrt{M} \quad \text{(50)}$$

is zero, or

$$\frac{\eta}{3}\times\left(\frac{-\Omega F}{n_2}\right)\times\Pi\left(1-\frac{1}{\omega}\right)\times\Pi\tfrac{1}{2}\left(1-\frac{1}{r}\right)\times\Pi\tfrac{1}{2}\left(1-\frac{1}{\delta}\right)\times\Pi\left(1-\frac{1}{\nu}\right)\times\frac{1}{m^3}, \quad \text{(51)}$$

according as n_2 does or does not contain any primes other than the primes δ. For, in

the sum (50), it is only necessary to consider those numbers M which are are prime to n; because $\left(\frac{-\Omega\mathrm{M}}{n}\right)=0$, if M is not prime to n; and if

$$\nabla=8\Pi\omega\times\Pi r\times\Pi\delta\times\Pi\nu,$$

$$\chi(\nabla)=\nabla\times\tfrac{1}{2}\eta\times\Pi\left(1-\frac{1}{\omega}\right)\times\Pi\tfrac{1}{2}\left(1-\frac{1}{r}\right)\times\Pi\tfrac{1}{2}\left(1-\frac{1}{\delta}\right)\times\Pi\left(1-\frac{1}{\nu}\right),$$

the sum (50) contains $\chi(\nabla)$ numbers M inferior to ∇; let these be represented by $x_1, x_2, \ldots x_i$; then all the numbers M, which enter into that sum, are contained in the $\chi(\nabla)$ linear forms $x\nabla+x_i$; and the sum (50) may be decomposed into $\chi(\nabla)$ partial sums, of which the sum

$$\left(\frac{-\Omega x_i}{n_2}\right)\frac{1}{\mathrm{L}^{\frac{3}{2}}}\sum_{x=0}^{x\nabla+x_i\leq\frac{\mathrm{L}}{m^2}}\sqrt{x\nabla+x_i}$$

is one. The limit of this sum is

$$\tfrac{2}{3}\times\frac{1}{\nabla}\times\frac{1}{m^3}\times\left(\frac{-\Omega x_i}{n_2}\right),$$

so that the limit of the sum (50) is

$$\tfrac{2}{3}\times\frac{1}{\nabla}\times\frac{1}{m^3}\Sigma_i\left(\frac{-\Omega x_i}{n_2}\right).$$

If n_2 is divisible by any prime other than the primes δ, the symbols $\left(\frac{-\Omega x_i}{n_2}\right)$ are one half equal to $+1$, and one half equal to -1; in this case, therefore, the limit of the sum (50) is zero. But if n_2 contain no prime other than the primes δ, the symbols $\left(\frac{-\Omega x_i}{n_2}\right)$ are all equal to one another and to $\left(\frac{-\Omega\mathrm{F}}{n_2}\right)$; and the limit of the sum (50) is

$$\tfrac{2}{3}\times\frac{\chi(\nabla)}{\nabla}\times\frac{1}{m^3}\times\left(\frac{-\Omega\mathrm{F}}{n_2}\right),$$

in accordance with the formula (51). Substituting in the expression (49) for the sum (50) its limiting value, we find

$$\left.\begin{aligned}\lim\frac{\mathrm{T}}{\mathrm{L}^{\frac{3}{2}}}=\frac{1}{2^{\sigma+\sigma'}}\times\frac{\sqrt{\Omega}}{\pi}\times\frac{\eta}{3}\times\Pi\left(1-\frac{1}{\omega}\right)\times\Pi\tfrac{1}{2}\left(1-\frac{1}{r}\right)\times\Pi\tfrac{1}{2}\left(1-\frac{1}{\delta}\right)\\ \times\sum_{m=1}^{m=\infty}\frac{1}{m^3}\times\Sigma\Sigma\left[\frac{\left(\frac{-\Omega\mathrm{F}}{n_2}\right)\Pi\left(1-\frac{1}{\nu}\right)}{n_1^2n_2}\right]\end{aligned}\right\}\quad\ldots\ldots\quad(52)$$

In the sum $\Sigma\Sigma\left[\frac{\left(\frac{-\Omega\mathrm{F}}{n_2}\right)\Pi\left(1-\frac{1}{\nu}\right)}{n_1^2n_2}\right]$ the summations extend to all values of n_2 composed of unequal primes δ, and to all values of n_1 prime to 2Ω; ν is any prime divisor of n_1, other than one of the primes δ. Thus the two summations are independent, and

$$\Sigma\Sigma\left[\left(\frac{-\Omega\mathrm{F}}{n_2}\right)\frac{\Pi\left(1-\frac{1}{\nu}\right)}{n_1^2n_2}\right]=\Sigma\left(\frac{-\Omega\mathrm{F}}{n_2}\right)\frac{1}{n_2}\times\Sigma\frac{\left(1-\frac{1}{\nu}\right)}{n_1^2}.$$

But

$$\Sigma\left(\frac{-\Omega F}{n_2}\right)\frac{1}{n_2}=\Pi\left[1+\left(\frac{-\Omega F}{\delta}\right)\frac{1}{\delta}\right]$$

and

$$\Sigma.\frac{\Pi\left(1-\frac{1}{\nu}\right)}{n_1^2}=\Pi\left[\frac{1}{1-\frac{1}{\delta^2}}\right]\times\Pi\left[1+\frac{1-\frac{1}{\nu}}{\nu^2}+\frac{1-\frac{1}{\nu}}{\nu^4}+\frac{1-\frac{1}{\nu}}{\nu^6}+\ldots\right]=\Pi\left[\frac{1}{1-\frac{1}{\delta^2}}\right]\times\Pi\frac{1+\frac{1}{\nu}+\frac{1}{\nu^2}}{1+\frac{1}{\nu}},$$

the last sign of multiplication extending to all primes ν which do not divide $2\Omega\Delta$. Also

$$\sum_{m=1}^{m=\infty}\frac{1}{m^3}=\Pi\frac{1}{1-\frac{1}{\nu^3}},=\Pi\frac{1}{1-\frac{1}{\nu}}\times\Pi\frac{1}{1+\frac{1}{\nu}+\frac{1}{\nu^2}},$$

so that the product

$$\Sigma\frac{1}{m^3}\times\Sigma\Sigma\left(\frac{-\Omega F}{n^2}\right)\frac{\Pi\left(1-\frac{1}{\nu}\right)}{n_1^2 n_2}$$

is equal to

$$\Pi\left[1+\left(\frac{-\Omega F}{\delta}\right)\frac{1}{\delta}\right]\times\Pi\frac{1}{1-\frac{1}{\delta^2}}\times\Pi\frac{1}{1-\frac{1}{\nu^2}},$$

or to

$$\frac{\pi^2}{8}\Pi\left[1-\frac{1}{\omega^2}\right]\times\Pi\left[1-\frac{1}{r^2}\right]\times\Pi\left[1+\left(\frac{-\Omega F}{\delta}\right)\frac{1}{\delta}\right], \quad \ldots \ldots \quad (53)$$

because

$$\Pi\frac{1}{1-\frac{1}{\omega^2}}.\Pi\frac{1}{1-\frac{1}{r^2}}.\Pi\frac{1}{1-\frac{1}{\delta^2}}.\Pi\frac{1}{1-\frac{1}{\nu^2}}$$

is equal to the sum of the squares of the reciprocals of the uneven numbers, that is to $\frac{\varpi^2}{8}$. Substituting for the product (52) its equivalent (53) in the equation (51), we obtain the formula (47).

If the forms (f) are improperly primitive, we have to employ the equation

$$h'(\Omega M)=\tfrac{1}{3}\left[2+(-1)^{\frac{\Omega^2-1}{8}+\frac{M^2-1}{8}}\right]\frac{\sqrt{\Omega M}}{\pi}\,\Sigma\left(\frac{-\Omega M}{n}\right)\frac{1}{n};$$

and the proof is the same as in the former case. Only, if $\Delta\equiv2$, mod 4, it is convenient, on account of the factor $2+(-1)^{\frac{\Omega^2-1}{8}+\frac{M^2-1}{8}}$, separately to determine the limit $T\div L^{\frac{3}{2}}$ for the numbers M which satisfy the congruences $M\equiv3\Omega$, $M\equiv7\Omega$, mod 8; and then to add the results.

Art. 22. The weight of an order (Art. 13) is the sum of the weights of the genera contained in the order. The determination of this sum may in every case be effected by means of the formulæ

$$R=\Sigma\left\{\Pi\tfrac{1}{4}\left[1-\tfrac{1}{r^2}\right]\times\Pi\tfrac{1}{2}\left[1+\left(\tfrac{-\Delta f}{\omega}\right)\tfrac{1}{\omega}\right]\times\Pi\tfrac{1}{2}\left[1+\left(\tfrac{-\Omega F}{\delta}\right)\tfrac{1}{\delta}\right]\right\}$$

$$=\Pi\left(1-\tfrac{1}{r^2}\right),$$

$$R'=\Sigma\left\{\left(\tfrac{f}{\Omega_1}\right)\left(\tfrac{F}{\Delta_1}\right)\Pi\tfrac{1}{4}\left[1-\tfrac{1}{r^2}\right]\Pi\left[1+\left(\tfrac{-\Delta f}{\omega}\right)\tfrac{1}{\omega}\right]\times\Pi\tfrac{1}{2}\left[1+\left(\tfrac{-\Omega F}{\delta}\right)\tfrac{1}{\delta}\right]\right\}$$

$$=0,\text{ or }=-\frac{(-1)^{\frac{\Omega_1+1}{2}\cdot\frac{\Delta_1+1}{2}}}{\Omega_1\Delta_1}\times\alpha^{\frac{\Delta^2_1-1}{8}}\times\beta^{\frac{\Omega^2_1-1}{8}}\times\Pi\left(1-\tfrac{1}{r^2}\right),$$

according as $\Omega_1\Delta_1$ is or is not divisible by any of the primes r; *i. e.* according as $\Omega_1\Delta_1$ is not, or is prime to the greatest common divisor of Ω and Δ. In the expressions of R and R′ the signs of summation extend to every combination of the equations

$$\left(\tfrac{f}{r}\right)=+1,\text{ or }-1;\quad\left(\tfrac{F}{r}\right)=+1,\text{ or }-1;\quad\left(\tfrac{f}{\omega}\right)=+1,\text{ or }-1;\quad\left(\tfrac{F}{\delta}\right)=+1,\text{ or }-1;$$

i. e. the value of the continued product is to be determined on each of these suppositions, and the sum of these values is to be taken. From this definition it is evident that in the sum R, we may substitute for any factor of the form

$$\tfrac{1}{2}\left[1+\left(\tfrac{-\Delta f}{\omega}\right)\tfrac{1}{\omega}\right],$$

or

$$\tfrac{1}{2}\left[1+\left(\tfrac{-\Omega F}{\delta}\right)\tfrac{1}{\delta}\right],$$

a factor of the form

$$\tfrac{1}{2}\left\{\left[1+\left(\tfrac{-\Delta}{\omega}\right)\tfrac{1}{\omega}\right]+\left[1-\left(\tfrac{-\Delta}{\omega}\right)\tfrac{1}{\omega}\right]\right\},\quad\ldots\ldots\quad(54)$$

or

$$\tfrac{1}{2}\left\{\left[1+\left(\tfrac{-\Omega}{\delta}\right)\tfrac{1}{\delta}\right]+\left[1-\left(\tfrac{-\Omega}{\delta}\right)\tfrac{1}{\delta}\right]\right\}\quad\ldots\ldots\quad(55)$$

outside the sign of summation. And similarly for any factor $\tfrac{1}{4}\left(1-\tfrac{1}{r^2}\right)$ we may substitute the factor

$$1-\tfrac{1}{r^2}$$

outside the sign of summation. Observing that the factors (54) and (55) are all positive units, we obtain immediately

$$R=\Pi\left(1-\tfrac{1}{r^2}\right).$$

Again, if a prime r divide Ω_1 or Δ_1, the sum R′ vanishes, being composed of pairs of terms equal in absolute magnitude and opposite in sign; if, for example, r divide Ω_1, the two terms in one of which $\left(\tfrac{f}{r}\right)$, contained in $\left(\tfrac{f}{\Omega_1}\right)$, is $+1$, and in the other -1, but which are in other respects identical, will destroy one another. But if none of the primes r divide Ω_1 or Δ_1, we replace those factors of the general term of R′, which

contain primes not dividing Ω_1 and Δ_1, by factors placed outside the sign of summation; we thus find

$$R'=\Pi\left(1-\tfrac{1}{r^2}\right)\Sigma\left\{\Pi\tfrac{1}{2}\left[\left(\tfrac{f}{\omega_1}\right)+\left(\tfrac{-\Delta}{\omega_1}\right)\tfrac{1}{\omega_1}\right]\times\Pi\tfrac{1}{2}\left[\left(\tfrac{F}{\delta_1}\right)+\left(\tfrac{-\Omega}{\delta_1}\right)\tfrac{1}{\delta_1}\right]\right\},$$

where only primes ω_1 which divide Ω_1 and primes δ_1 which divide Δ_1 occur after the sign of summation. We then substitute for each factor containing ω_1 or δ_1 a factor of the form

$$\tfrac{1}{2}\left\{\left[1+\left(\tfrac{-\Delta}{\omega_1}\right)\tfrac{1}{\omega_1}\right]+\left[-1+\left(\tfrac{-\Delta}{\omega_1}\right)\tfrac{1}{\omega_1}\right]\right\}=\left(\tfrac{-\Delta}{\omega_1}\right)\tfrac{1}{\omega_1},$$

or

$$\tfrac{1}{2}\left\{\left[1+\left(\tfrac{-\Omega}{\delta_1}\right)\tfrac{1}{\delta_1}\right]+\left[-1+\left(\tfrac{-\Omega}{\delta_1}\right)\tfrac{1}{\delta_1}\right]\right\}=\left(\tfrac{-\Omega}{\delta_1}\right)\tfrac{1}{\delta_1}$$

outside the sign of summation; and observing that by the law of reciprocity

$$\left(\tfrac{-\Delta}{\Omega_1}\right)\cdot\left(\tfrac{-\Omega}{\Delta_1}\right)=-(-1)^{\frac{\Omega_1+1}{2}\cdot\frac{\Delta_1+1}{2}}\times\alpha^{\frac{\Delta_1^2-1}{8}}\times\beta^{\frac{\Omega_1^2-1}{8}},$$

we find

$$R'=-\frac{(-1)^{\frac{\Omega_1+1}{2}\cdot\frac{\Delta_1+1}{2}}}{\Omega_1\Delta_1}\times\alpha^{\frac{\Delta_1^2-1}{8}}\times\beta^{\frac{\Omega_1^2-1}{8}}\times\Pi\left(1-\tfrac{1}{r^2}\right).$$

As an example of the application of these formulæ, let us consider the properly primitive order in the case in which $\Delta\equiv1$, mod 2, $\Omega\equiv4$, mod 8. We may determine separately the weights of those genera for which $(-1)^{\frac{\Delta f+1}{2}}=-1$, and of those for which $(-1)^{\frac{\Delta f+1}{2}}=+1$. In a genus of the former kind the characters

$$\left(\tfrac{f}{r}\right),\ \left(\tfrac{F}{r}\right),\ \left(\tfrac{f}{\omega}\right),\ \left(\tfrac{F}{\delta}\right)\quad\ldots\ldots\ldots\quad(56)$$

may have any assigued values because the condition of possibility is

$$(-1)^{\frac{\Omega_1F+1}{2}}\left(\tfrac{f}{\Omega_1}\right)\left(\tfrac{F}{\Delta_1}\right)=(-1)^{\frac{\Omega_1+1}{2}\cdot\frac{\Delta_1+1}{2}}.$$

Therefore the sum of the weights of these genera is $\frac{\Omega\Delta}{8}\times\zeta\times R$, or $\frac{\Omega\Delta}{32}\times R$, because $\zeta=\frac{1}{4}$. But in a genus of the latter kind the characters (56), or some of them, are subject to the condition

$$\left(\tfrac{f}{\Delta_1}\right)\left(\tfrac{F}{\Delta_1}\right)=(-1)^{\frac{\Omega_1+1}{2}\cdot\frac{\Delta_1+1}{2}};\quad\ldots\ldots\ldots\quad(57)$$

we have therefore to consider a sum of which the general term is the same as that of R, but into which only those terms are admitted which are formed with values of $\left(\frac{f}{\omega_1}\right)$ and $\left(\frac{F}{\delta_1}\right)$ satisfying the condition (57). This sum is expressed by the formula

$$\tfrac{1}{2}\left(R+(-1)^{\frac{\Omega_1+1}{2}\cdot\frac{\Delta_1+1}{2}}R'\right);$$

so that the sum of the weights of the genera of the latter kind is

$$\frac{\Omega\Delta}{16}\times\tfrac{1}{2}\times\left(\mathrm{R}+(-1)^{\frac{\Omega_1+1}{2}\cdot\frac{\Delta_1+1}{2}}\mathrm{R}'\right).$$

Adding the two sums together, and substituting for R and R′ their values, we find for the weight of the proposed order the expression

$$\frac{\Omega\Delta}{16}\Pi\left(1-\frac{1}{r^2}\right),\text{ or }\frac{\Omega\Delta}{32}\left(2-\frac{1}{\Omega_1\Delta_1}\right)\Pi\left(1-\frac{1}{r^2}\right)$$

according as $\Omega_1\Delta_1$ is not or is prime to the greatest common divisor of Ω and Δ.

If, in general, we represent the weight of any proposed order of the invariants $[\Omega, \Delta]$ by the expression

$$\frac{\Omega\Delta}{8}\times\mathrm{Z}\times\Pi\left(1-\frac{1}{r^2}\right),$$

the following Table (with which we shall conclude this memoir) will assign the value of the coefficient Z, and will thus serve to ascertain the weight of the order*. The determinations contained in it have been obtained by the method just described; λ is $\frac{1}{\Omega_1\Delta_1}$, or 0, according as $\Omega_1\Delta_1$ is or is not prime to the greatest common divisor of Ω and Δ; I_1, I_2 are the exponents of the highest powers of 2 dividing Ω and Δ respectively.

(A).—(f) and (F) properly primitive.

	$\mathrm{I}_1=0.$	I_1 even.	I_1 uneven.
$\mathrm{I}_2=0.$	$\frac{1}{3}(2-\lambda)$	$\frac{1}{4}(2-\lambda)$	$\frac{1}{2}$
I_2 even.	$\frac{1}{4}(2-\lambda)$	$\frac{1}{4}(2-\lambda)$	$\frac{1}{2}$
I_2 uneven.	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$

(B).—(f) improperly, (F) properly primitive.

	$\mathrm{I}_1=0,\ \mathrm{I}_2>0.$
I_2 even.	$\frac{1}{12}(2-\lambda)$
I_2 uneven.	$\frac{1}{6}(1-\lambda)$

* For the case of uneven invariants, the result has been given by EISENSTEIN (Crelle, vol. xxxv. p. 123); there is, however, a slight discrepancy. According to EISENSTEIN, λ is not zero, when the greatest common divisor of Δ and Ω is a square; according to the definition in the text, λ is always zero, except when the exponent of every uneven prime common to Δ and Ω is even both in Δ and Ω. For the invariants (p^2, p^3) the weight assigned by the formula of EISENSTEIN is $\frac{p^5}{24}\left(2-\frac{1}{p}\right)\left(1-\frac{1}{p^2}\right)$, p denoting an uneven prime; a result which can hardly be right, because the weight of each genus separately is $\equiv 0$, mod p^3.

(C).—(f) properly, (F) improperly primitive.

	$I_2=0, I_1>0$.
I_1 even.	$\frac{1}{12}(2-\lambda)$
I_1 uneven.	$\frac{1}{6}(1-\lambda)$

TABLE II. OF COMPLETE GENERIC CHARACTERS.

A.—f and F properly primitive.

	$\Omega=\Omega_1\Omega_2^2.$	$\Omega_2\equiv 1$, mod 2.	$\Omega=\Omega_1\Omega_2^2.$	$\Omega_2\equiv 2$, mod 4.	$\Omega=\Omega_1\Omega_2^2.$	$\Omega_2\equiv 0$, mod 4.	$\Omega=2\Omega_1\Omega_2^2.$	$\Omega_2\equiv 1$, mod 2.	$\Omega=2\Omega_1\Omega_2^2.$	$\Omega_2\equiv 0$, mod 2.
$\Delta=\Delta_1\Delta_2^2.$ $\Delta_2\equiv 1$, mod 2.	Ψ $\left(\frac{f}{\omega_1}\right), \left(\frac{F}{\delta_1}\right)$ S	$\left(\frac{f}{\omega_2}\right), \left(\frac{F}{\delta_2}\right)$ $\Gamma=2^\gamma$	Ψ $\left(\frac{f}{\omega_1}\right), \left(\frac{F}{\delta_1}\right)$ Q	$(-1)^{\frac{f-1}{2}}$ $\left(\frac{f}{\omega_2}\right), \left(\frac{F}{\delta_2}\right)$ $\Gamma=3\times 2^{\gamma-1}$	Ψ $\left(\frac{f}{\omega_1}\right), \left(\frac{F}{\delta_1}\right)$ Q	$(-1)^{\frac{f-1}{2}}$ $(-1)^{\frac{f^2-1}{8}}$ $\left(\frac{f}{\omega_2}\right), \left(\frac{F}{\delta_2}\right)$ $\Gamma=3\times 2^\gamma$	$(-1)^{\frac{f^2-1}{8}}\Psi$ $\left(\frac{f}{\omega_1}\right), \left(\frac{F}{\delta_1}\right)$ S	$\left(\frac{f}{\omega_2}\right), \left(\frac{F}{\delta_2}\right)$ $\Gamma=2^\gamma$	Ψ $(-1)^{\frac{f^2-1}{8}}$ $\left(\frac{f}{\omega_1}\right), \left(\frac{F}{\delta_1}\right)$ Q	$(-1)^{\frac{f-1}{2}}$ $\left(\frac{f}{\omega_2}\right), \left(\frac{F}{\delta_2}\right)$ $\Gamma=3\times 2^\gamma$
$\Delta=\Delta_1\Delta_2^2.$ $\Delta_2\equiv 2$, mod 4.	Ψ $\left(\frac{f}{\omega_1}\right), \left(\frac{F}{\delta_1}\right)$ Q	$(-1)^{\frac{F-1}{2}}$ $\left(\frac{f}{\omega_2}\right), \left(\frac{F}{\delta_2}\right)$ $\Gamma=3\times 2^{\gamma-1}$	Ψ $\left(\frac{f}{\omega_1}\right), \left(\frac{F}{\delta_1}\right)$ P	$(-1)^{\frac{f-1}{2}}, (-1)^{\frac{F-1}{2}}$ $\left(\frac{f}{\omega_2}\right), \left(\frac{F}{\delta_2}\right)$ $\Gamma=2^{\gamma+1}$	Ψ $\left(\frac{f}{\omega_1}\right), \left(\frac{F}{\delta_1}\right)$ P	$(-1)^{\frac{f-1}{2}}, (-1)^{\frac{F-1}{2}}$ $(-1)^{\frac{f^2-1}{8}}$ $\left(\frac{f}{\omega_2}\right), \left(\frac{F}{\delta_2}\right)$ $\Gamma=2^{\gamma+2}$	$(-1)^{\frac{f^2-1}{8}}\Psi$ $\left(\frac{f}{\omega_1}\right), \left(\frac{F}{\delta_1}\right)$ R	$(-1)^{\frac{F-1}{2}}$ $\left(\frac{f}{\omega_2}\right), \left(\frac{F}{\delta_2}\right)$ $\Gamma=2^{\gamma+1}$	Ψ $(-1)^{\frac{f^2-1}{8}}$ $\left(\frac{f}{\omega_1}\right), \left(\frac{F}{\delta_1}\right)$ P	$(-1)^{\frac{f-1}{2}}, (-1)^{\frac{F-1}{2}}$ $\left(\frac{f}{\omega_2}\right), \left(\frac{F}{\delta_2}\right)$ $\Gamma=2^{\gamma+2}$
$\Delta=\Delta_1\Delta_2^2.$ $\Delta_2\equiv 0$, mod 4.	Ψ $\left(\frac{f}{\omega_1}\right), \left(\frac{F}{\delta_1}\right)$ Q	$(-1)^{\frac{F-1}{2}}$ $(-1)^{\frac{F^2-1}{8}}$ $\left(\frac{f}{\omega_2}\right), \left(\frac{F}{\delta_2}\right)$ $\Gamma=3\times 2^\gamma$	Ψ $\left(\frac{f}{\omega_1}\right), \left(\frac{F}{\delta_1}\right)$ P	$(-1)^{\frac{f-1}{2}}, (-1)^{\frac{F-1}{2}}$ $(-1)^{\frac{F^2-1}{8}}$ $\left(\frac{f}{\omega_2}\right), \left(\frac{F}{\delta_2}\right)$ $\Gamma=2^{\gamma+2}$	Ψ $\left(\frac{f}{\omega_1}\right), \left(\frac{F}{\delta_1}\right)$ P	$(-1)^{\frac{f-1}{2}}, (-1)^{\frac{F-1}{2}}$ $(-1)^{\frac{f^2-1}{8}}, (-1)^{\frac{F^2-1}{8}}$ $\left(\frac{f}{\omega_2}\right), \left(\frac{F}{\delta_2}\right)$ $\Gamma=2^{\gamma+3}$	$(-1)^{\frac{f^2-1}{8}}\Psi$ $\left(\frac{f}{\omega_1}\right), \left(\frac{F}{\delta_1}\right)$ R	$(-1)^{\frac{F-1}{2}}$ $(-1)^{\frac{F^2-1}{8}}$ $\left(\frac{f}{\omega_2}\right), \left(\frac{F}{\delta_2}\right)$ $\Gamma=2^{\gamma+2}$	Ψ $(-1)^{\frac{f^2-1}{8}}$ $\left(\frac{f}{\omega_1}\right), \left(\frac{F}{\delta_1}\right)$ P	$(-1)^{\frac{f-1}{2}}, (-1)^{\frac{F-1}{2}}$ $(-1)^{\frac{F^2-1}{8}}$ $\left(\frac{f}{\omega_2}\right), \left(\frac{F}{\delta_2}\right)$ $\Gamma=2^{\gamma+3}$
$\Delta=2\Delta_1\Delta_2^2.$ $\Delta_2\equiv 1$, mod 2.	$(-1)^{\frac{F^2-1}{8}}\Psi$ $\left(\frac{f}{\omega_1}\right), \left(\frac{F}{\delta_1}\right)$ S	$\left(\frac{f}{\omega_2}\right), \left(\frac{F}{\delta_2}\right)$ $\Gamma=2^\gamma$	$(-1)^{\frac{F^2-1}{8}}\Psi$ $\left(\frac{f}{\omega_1}\right), \left(\frac{F}{\delta_1}\right)$ R	$(-1)^{\frac{f-1}{2}}$ $\left(\frac{f}{\omega_2}\right), \left(\frac{F}{\delta_2}\right)$ $\Gamma=2^{\gamma+1}$	$(-1)^{\frac{F^2-1}{8}}\Psi$ $\left(\frac{f}{\omega_1}\right), \left(\frac{F}{\delta_1}\right)$ R	$(-1)^{\frac{f-1}{2}}$ $(-1)^{\frac{f^2-1}{8}}$ $\left(\frac{f}{\omega_2}\right), \left(\frac{F}{\delta_2}\right)$ $\Gamma=2^{\gamma+2}$	$(-1)^{\frac{f^2-1}{8}+\frac{F^2-1}{8}}\Psi$ $\left(\frac{f}{\omega_1}\right), \left(\frac{F}{\delta_1}\right)$ S	$\left(\frac{f}{\omega_2}\right), \left(\frac{F}{\delta_2}\right)$ $\Gamma=2^\gamma$	$(-1)^{\frac{F^2-1}{8}}\Psi$ $(-1)^{\frac{f^2-1}{8}}$ $\left(\frac{f}{\omega_1}\right), \left(\frac{F}{\delta_1}\right)$ R	$(-1)^{\frac{f-1}{2}}$ $\left(\frac{f}{\omega_2}\right), \left(\frac{F}{\delta_2}\right)$ $\Gamma=2^{\gamma+2}$
$\Delta=2\Delta_1\Delta_2^2.$ $\Delta_2\equiv 0$, mod 2.	Ψ $(-1)^{\frac{F^2-1}{8}}$ $\left(\frac{f}{\omega_1}\right), \left(\frac{F}{\delta_1}\right)$ Q	$(-1)^{\frac{F-1}{2}}$ $\left(\frac{f}{\omega_2}\right), \left(\frac{F}{\delta_2}\right)$ $\Gamma=3\times 2^\gamma$	Ψ $(-1)^{\frac{F^2-1}{8}}$ $\left(\frac{f}{\omega_1}\right), \left(\frac{F}{\delta_1}\right)$ P	$(-1)^{\frac{f-1}{2}}, (-1)^{\frac{F-1}{2}}$ $\left(\frac{f}{\omega_2}\right), \left(\frac{F}{\delta_2}\right)$ $\Gamma=2^{\gamma+2}$	Ψ $(-1)^{\frac{F^2-1}{8}}$ $\left(\frac{f}{\omega_1}\right), \left(\frac{F}{\delta_1}\right)$ P	$(-1)^{\frac{f-1}{2}}, (-1)^{\frac{F-1}{2}}$ $(-1)^{\frac{f^2-1}{8}}$ $\left(\frac{f}{\omega_2}\right), \left(\frac{F}{\delta_2}\right)$ $\Gamma=2^{\gamma+3}$	$(-1)^{\frac{f^2-1}{8}}\Psi$ $(-1)^{\frac{F^2-1}{8}}$ $\left(\frac{f}{\omega_1}\right), \left(\frac{F}{\delta_1}\right)$ R	$(-1)^{\frac{F-1}{2}}$ $\left(\frac{f}{\omega_2}\right), \left(\frac{F}{\delta_2}\right)$ $\Gamma=2^{\gamma+2}$	Ψ $(-1)^{\frac{f^2-1}{8}}, (-1)^{\frac{F^2-1}{8}}$ $\left(\frac{f}{\omega_1}\right), \left(\frac{F}{\delta_1}\right)$ P	$(-1)^{\frac{f-1}{2}}, (-1)^{\frac{F-1}{2}}$ $\left(\frac{f}{\omega_2}\right), \left(\frac{F}{\delta_2}\right)$ $\Gamma=2^{\gamma+3}$

B.—f improperly, F properly primitive.

$(-1)^{\frac{F-1}{2}}=-(-1)^{\frac{\Omega-1}{2}}$; $\Omega\equiv 1$, mod 2.

	$\Omega=\Omega_1\Omega_2^2.$	$\Omega_2\equiv 1$, mod 2.
$\Delta=\Delta_1\Delta_2^2.$ $\Delta_2\equiv 0$, mod 2.	$(-1)^{\frac{F^2-1}{8}}$ $\left(\frac{f}{\omega_1}\right), \left(\frac{F}{\delta_1}\right)$	$\left(\frac{f}{\omega_2}\right), \left(\frac{F}{\delta_2}\right)$ $\Gamma=2^\gamma$
$\Delta=2\Delta_1\Delta_2^2.$ $\Delta_2\equiv 1$, mod 2.	$\left(\frac{f}{\omega_1}\right), \left(\frac{F}{\delta_1}\right)$	$\left(\frac{f}{\omega_2}\right), \left(\frac{F}{\delta_2}\right)$ $\Gamma=2^{\gamma-1}$
$\Delta=2\Delta_1\Delta_2^2.$ $\Delta_2\equiv 0$, mod 2.	$\left(\frac{f}{\omega_1}\right), \left(\frac{F}{\delta_1}\right)$	$(-1)^{\frac{F^2-1}{8}}$ $\left(\frac{f}{\omega_2}\right), \left(\frac{F}{\delta_2}\right)$ $\Gamma=2^\gamma$

C.—f properly, F improperly primitive.

$(-1)^{\frac{f-1}{2}}=-(-1)^{\frac{\Delta-1}{2}}$; $\Delta\equiv 1$, mod 2.

	$\Delta=\Delta_1\Delta_2^2.$	$\Delta_2\equiv 1$, mod 2.
$\Omega=\Omega_1\Omega_2^2.$ $\Omega_2\equiv 0$, mod 2.	$(-1)^{\frac{f^2-1}{8}}$ $\left(\frac{f}{\omega_1}\right), \left(\frac{F}{\delta_1}\right)$	$\left(\frac{f}{\omega_2}\right), \left(\frac{F}{\delta_2}\right)$ $\Gamma=2^\gamma$
$\Omega=2\Omega_1\Omega_2^2.$ $\Omega_1\equiv 1$, mod 2.	$\left(\frac{f}{\omega_1}\right), \left(\frac{F}{\delta_1}\right)$	$\left(\frac{f}{\omega_2}\right), \left(\frac{F}{\delta_2}\right)$ $\Gamma=2^{\gamma-1}$
$\Omega=2\Omega_1\Omega_2^2.$ $\Omega_1\equiv 0$, mod 2.	$\left(\frac{f}{\omega_1}\right), \left(\frac{F}{\delta_1}\right)$	$(-1)^{\frac{f^2-1}{8}}$ $\left(\frac{f}{\omega_2}\right), \left(\frac{F}{\delta_2}\right)$ $\Gamma=2^\gamma$