VII. *On the Theory of Local Probability, applied to Straight Lines drawn at random in a plane; the methods used being also extended to the proof of certain new Theorems in the Integral Calculus.*

By MORGAN W. CROFTON, *B.A., of the Royal Military Academy, Woolwich; late Professor of Natural Philosophy in the Queen's University, Ireland. Communicated by* J. J. SYLVESTER, *F.R.S.*

Received February 5,—Read February 27, 1868.

1. THE new Theory of Local or Geometrical Probability, so far as it is known, seems to present, in a remarkable degree, the same distinguishing features which characterize those portions of the general Theory of Probability which we owe to the great philosophers of the past generation. The rigorous precision, as well as the extreme beauty of the methods and results, the extent of the demands made on our mathematical resources, even by cases apparently of the simplest kind, the subtlety and delicacy of the reasoning, which seem peculiar to that wonderful application of modern analysis—*ce calcul délicat*, as it has been aptly described by LAPLACE—reappear, under new forms, in this, its latest development. The first trace which we can discover of the Theory of Local Probability seems to be the celebrated problem of BUFFON, the great naturalist *—a given rod being placed at random on a space ruled with equidistant parallel lines, to find the chance of its crossing one of the lines. Although the subject was noticed so early, and though BUFFON'S and one or two similar questions have been considered by LAPLACE, no real attention seems to have been bestowed upon it till within the last few years, when this new field of research has been entered upon by several English mathematicians, among whom the names of SYLVESTER and WOOLHOUSE† are particularly

* The mathematical ability evinced by BUFFON may well excite surprise; that one whose life was devoted to other branches of science should have had the sagacity to discern the true mathematical principles involved in a question of so entirely novel a character, and to reduce them correctly to calculation by means of the integral calculus, thereby opening up a new region of inquiry to his successors, must move us to admiration for a mind so rarely gifted.

† Many remarkable propositions on the subject, by these eminent mathematicians, have appeared in the mathematical columns of the 'Educational Times' and other periodicals. A very important principle has been introduced by Professor SYLVESTER, which may be termed *decomposition of probabilities*. For instance, he has shown that the probability of a group of three points, taken at random within a given triangle, fulfilling a given *intrinsic* condition (*i. e.* one depending solely on the internal relations of the points among each other), may be expressed as a linear function of two simpler probabilities; viz. that of the same condition being fulfilled (1) when one of the points is fixed at a vertex of the triangle, and a second restricted to the opposite side; (2) when all three points are restricted, one to each side of the triangle. The order of the integrations required

distinguished. It is true that in a few cases differences of opinion have arisen as to the principles, and discordant results have been arrived at, as in the now celebrated *three-point* problem, by Mr. WOOLHOUSE, and the *four-point* problem of Professor SYLVESTER; but all feel that this arises, not from any inherent ambiguity in the subject matter, but from the weakness of the instrument employed; our undisciplined conceptions of a novel subject requiring to be repeatedly and patiently reviewed, tested, and corrected by the light of experience and comparison, before they are purged from all latent error.

The object of the present paper is, principally, the application of the Theory of Probability to straight lines drawn at random in a plane; a branch of the subject which has not yet been investigated. It will be necessary to begin by some remarks on the general principles of Local Probability. Some portion of what follows I have already given elsewhere*.

2. The expression "*at random*" has in common language a very clear and definite meaning; one which cannot be better conveyed than by Mr. WILSON's expression "*according to no law.*" It is thus of very wide application, being often used in cases altogether beyond the province of mathematical measurement or calculation.

In Mathematical Probability, which consists essentially in arithmetical calculation, when we speak of a thing of any kind taken at random, there is always a direct reference to the *assemblage of things* to which it belongs and from which it is taken, at random,—which here comes to the same thing as saying that any one is as likely to be taken as any other. When we have a clear conception of what the assemblage is, from which we take, and not till then, we can proceed to sum up the favourable cases.

In many problems on probability there is no difficulty in forming a clear conception of the total number of cases. Thus if balls are drawn from an urn, the number of cases is the number of balls, or of certain combinations of them; and if the number of balls be supposed infinite, no obscurity arises from this. But there are several classes of questions in which the totality of cases is not merely infinite, but of an inconceivable nature. Thus if we try to imagine how to determine completely by experiment the probability of a hemisphere thrown into the air falling on its base, we may suppose an infinite number of persons to make one trial each; afterwards we may suppose each person to make two, three, or an infinite number of trials; again, we may suppose for every trial that has taken place an infinite number of others, varying, for instance, in the substance, size, &c. of the body employed; and so on. We can thus continually suppose variations of the experiment, each variation giving a new infinity of cases. Now problems of this nature are treated by means of the following principle:—

In any question of probability regarding an infinite number of cases, all equally pro-

---

is thus reduced by *three.* The same method applies to any polygon, and also to the points taken in space within a tetrahedron. It is to be hoped that Professor SYLVESTER will give these remarkable results to the public in a detailed form: a general account of them was given to the British Association at Birmingham in 1865.

* Educational Times, May 1867.

bable, the result will be unaltered if we take, instead of these cases, *any lesser infinity of cases, chosen at random from among them**.

3. The case of a point or straight line taken at random in a plane or in space is a problem of the above description. Thus, if a point be taken at random in a plane, the total number of cases is of an inconceivable nature, inasmuch as a plane cannot be *filled* with mathematical points, any infinitesimal element of the plane containing an unlimited number of points. We see, however, by means of the above principle, that we may consider the assemblage we are dealing with, as *an infinity of points all taken at random in the plane.*

Let us examine the nature of this assemblage. As the points continue to be scattered at random over the plane, their density tends to become uniform. It is evident, in fact, that a random point is as likely to be in any element dS of the surface, as in any equal element dS'; and therefore by continuing to multiply points, the number in dS will be equal (or *subequal*, to use a term of Professor DE MORGAN'S) to that in dS'. Of course, though the density tends to become uniform, the disposition of the points does not tend to become symmetrical; those within any element dS will be dispersed in the most irregular manner over that element†. However, it is important to remark that, *for all purposes of calculation*, the ultimate disposition may be supposed symmetrical; for as the position of any point is determined by that of the element dS, within which it falls, it matters not what arbitrary arrangement we assume for the points within the element.

* This proposition, of which, in a somewhat different form, a mathematical demonstration is given by LAPLACE (Théorie Analytique des Probabilités, chap. 3), may be regarded as almost axiomatic. Thus, suppose an urn to contain an infinite number of black and white balls, in the proportion of 2 to 3; if any lesser infinite number of balls be drawn from it, the black ones among them will be to the white as 2 to 3. For, imagine all the balls ranged in a row ACB, the black from A to C, the white from C to B; if we now select an infinite number at random from among them, it appears self-evident that, if the line be divided into five equal parts, the numbers of balls taken from each part will be the same, or rather, will *tend* to equality on being increased indefinitely. Hence the black balls selected will be to the white as AC to CB, or as 2 to 3. When the numbers are *large*, but not infinite, this principle is approximately true, and forms, as is well known, the basis of most of the practical applications of Probability. Thus the chance of an infant living to the age of twenty is as truly found from, say, 1,000,000 of observed cases, as it would be from the total number.

In its strict mathematical form, the proposition may be thus stated:—In any unlimited number of cases, divided into favourable and unfavourable, if p be the ratio of the favourable to the whole number of cases, and if we select any infinite number of cases at random from among them, *the probability is infinitely small, that the same ratio, as determined from the selected cases, shall differ from* p *by a finite quantity.*

† Order thus results from disorder, the uniform density of the aggregate being unaffected by the disorder and irregularity of arrangement of its ultimate constituents; much as a nebula of uniform brightness is related to the stars which compose it. This remarkable law is to be traced, under one form or another, in most of the applications of the Theory of Probability.

"Elle mérite l'attention des philosophes, en faisant voir comment la regularité finit par s'établir dans les choses même qui nous paraissent entièrement livrées au hasard."—*Laplace.*

A familiar illustration of the tendency to uniform density in the random points may be derived by observing the drops of rain on a pavement at the commencement of a shower: as the drops multiply, it will be evident to the eye that their density tends more and more to uniformity.

Hence we may, if we please, assume that, when a point is taken at random in a plane, those from which it is taken are an infinite number symmetrically disposed over the plane.

Likewise, points taken at random in a line may be supposed equidistant. And if random values be taken for any *quantity*, they may be supposed to form an arithmetical series, with an infinitesimal difference.

Let us now consider the case of a straight line drawn at random in an infinite plane: the assemblage from which we select it is, as before, *an infinity of lines drawn at random in the plane.* What is the nature of this aggregate? First, since any direction is as likely as any other, as many of the lines are parallel to any given direction as to any other. Consider one of these systems of parallels; let them be cut by any infinite perpendicular. As this infinite system of parallels is drawn at random, they are as thickly disposed along any part of the perpendicular as along any other; the intersections being in fact random points on the perpendicular. Hence it is easily seen that, for all purposes of calculation, the assemblage of lines may be thus conceived. Divide the angular space round any point into a number of equal angles $\delta\theta$, and for every direction let the plane be ruled with an infinity of equidistant parallel lines, the common infinitesimal distance being the same for every set of parallels. Or we may suppose one such system of parallels drawn, and then turned through an angle $\delta\theta$, then through another equal angle, and so on, till they have returned to their former direction.

If we take any fixed axes in the plane, a random line may be represented by the equation

$$x\cos\theta+y\sin\theta=p,$$

where p and θ are constants taken at random.

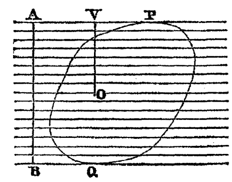
There is no difficulty in extending now our conceptions to points, straight lines, and planes, taken at random in *space.*

4. We may take any plane area as the *measure* of the number of random points within it: in the case of random lines, I proceed to prove the following important principle:—

The measure of the number of random lines which meet a given closed convex plane boundary, is the length of the boundary.

Draw any system of parallels meeting the boundary, their common infinitesimal distance being δp. *If we take this distance as unity*, the number of these parallels is AB, a line cutting them at right angles. Let $AB=\mathfrak{C}$, and let θ be its inclination to any fixed direction in the plane; conceive now a consecutive system of parallels inclined to the former at an angle $\delta\theta$, then a third, and so on, till the parallels return to the direction in the figure; then the total number of lines will be

Fig. 1.



$$\frac{1}{\delta\theta}\int_0^\pi \mathfrak{C}\,d\theta;$$

or, if O be any fixed pole inside the boundary, and $OV=p$, the perpendicular on the

tangent to the boundary, θ its inclination to a fixed axis, the measure of the number of lines* is

$$N=\int_0^{2\pi} pd\theta.$$

Now the integral $\int pd\theta$ extended through four right angles gives the *whole length of the boundary*, whatever be its nature, provided it be convex†.

Hence if L be the length of the boundary,

$$N=L.$$

This result may be obtained also as follows. It may be shown very simply by the above principles that the measure of the number of random lines which meet any finite straight line of length a, is $2a$ (it may indeed be assumed as self-evident that the number is proportional to a). Conceiving now the boundary L as consisting of straight elements, the number of lines meeting any element ds, is $2ds$; so that the whole number which meet the boundary would be 2L; but as each line cuts the boundary in *two* points, we should thus count each line twice over; hence the true number is L.

Hence if L be the length of any convex boundary, and l that of another, lying wholly inside the former, the probability that a line drawn at random across L shall also intersect l, is

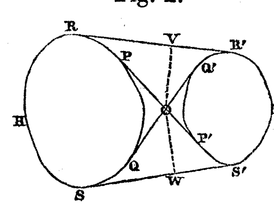
$$p=\frac{l}{L}.$$

It is important to observe that the measure of the number of lines which meet any *non-convex* boundary is *the length of a string drawn tightly round it*; as is obvious on consideration. The same is true for a boundary which is not closed.

5. Let there be any two boundaries external to each other: let X be the length of an endless band passing round both, and crossing between them, and Y the length of another endless band also enveloping both, but not crossing; then *the measure of the number of random lines which meet both boundaries is* X−Y.

It will be easily found from the principles explained above, that the number required will be the integral $\int pd\theta$ (referred to O as pole), taken for the left-hand curve from the position RR′ of its tangent, to the position PO; then for the right-hand one from the position P′O of its tangent, to the position S′S; then for the left-hand one, from SS′ to QO; then for the right-hand one, from Q′O to R′R. Now the values of these integrals are, drawing the perpendiculars OV, OW to RR′, SS′,

Fig. 2.



* It will be well to remember that this *measure* of the number of lines, N, means *the actual number multiplied by the constant factor* $\delta\theta$. Our notation is thus simplified, and no confusion need arise from sometimes saying "the number of lines," for shortness, instead of "the measure of the number of lines." As $\delta\theta$ remains constant throughout our investigations, henceforth we will denote it by δ.

† As $L=\int_0^{\pi}\mathfrak{C}d\theta$, we see that *the mean breadth of any convex area is equal to the diameter of a circle whose circumference equals the length of the boundary*. By *breadth* is meant the distance between two parallel tangents, whose direction is supposed to alter by uniform increments.

1. the mixed line RPO − RV,
2. „ „ S′P′O − S′W,
3. „ „ SQO − SW,
4. „ „ R′Q′O − R′V,

and the sum of these is evidently equal to X−Y.

I will add a different proof of this proposition, deduced from art. 4, as it is interesting to see our results verified.

For shortness, I will use the symbol N(S) for "the number of random lines meeting the space S;" and N(S, S′) for the number meeting both S and S′.

The number of lines meeting both boundaries is evidently identical with the number meeting both the mixtilinear figures OPHQ, OP′H′Q′. These two figures together form the mixtilinear reentrant figure HPP′H′Q′Q, and by art. 4, N(HPP′H′Q′Q)=Y.

Now N(OPHQ)+N(OP′H′Q′)=N(HPP′H′Q′Q)+N(OPHQ, OP′H′Q′). But OPHQ, OP′H′Q′ being convex figures, the number of lines meeting each is represented by its length; therefore

$$X=Y+N(HPQ, H'P'Q').$$

The probability that a line drawn at random across a given convex boundary of length L shall also meet a given *external* boundary is therefore

$$p=\frac{X-Y}{L}.$$

6. If two convex boundaries L, L′ intersect each other, in two or more points, it may be proved in a similar manner that the number of random lines which meet both is represented by L+L′−Y, where Y is the length of an endless band passing round both. Hence the probability that a line which meets L shall also meet L′, is

$$p=\frac{L+L'-Y}{L}.$$

7. It may easily be proved that the measure of *the number of random lines which pass between two given convex boundaries is*

$$N=PP'+QQ'-\text{arc } PQ-\text{arc } P'Q',$$

where PP′, QQ′ are the two common tangents which cross each other.

Thus the number of random lines which pass between the two branches of an hyperbola is represented by Δ, the difference between the whole length of the hyperbola and that of its asymptotes. This difference, as is known, is given by the definite integral

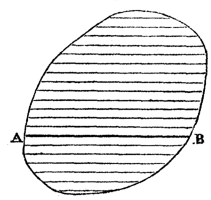
$$\Delta=4a\int_0^\alpha\sqrt{1-e^2\sin^2\theta}\,.\,d\theta,$$

where $\sin\alpha=\frac{1}{e}$.

8. Two lines are drawn at random across a given convex area: to find the probability of their intersection lying within the area.

Let AB be the internal portion of any random line crossing the area: the number of its intersections with all the random lines in the area is the number of those lines which meet it. Now this number is $\frac{2AB}{\delta}$ (art. 4); hence the number of intersections of the system of parallels to AB with all the random lines in the area, is twice the sum of the lengths of all these parallel chords divided by δ. But this sum is the area of the figure (we have taken the common distance δp of the chords as unity).

Fig. 3.



Let Ω be the area, L the length of the boundary. As, then, $\frac{2\Omega}{\delta}$ is the number of intersections for any system of parallels, and the number of those systems is $\frac{\pi}{\delta}$, the total number of intersections is $\frac{2\pi\Omega}{\delta^2}$. But we have thus counted each intersection twice; so that the real number of intersections which fall inside the area Ω is $\frac{\pi\Omega}{\delta^2}$.

Hence the required probability is

$$p=\frac{2\pi\Omega}{L^2},$$

since the whole number of intersections is $\frac{1}{2}\left(\frac{L}{\delta}\right)^2$.

Thus it is an even chance that two random chords of a circle intersect within the circle; for any other figure the chance is less than $\frac{1}{2}$.

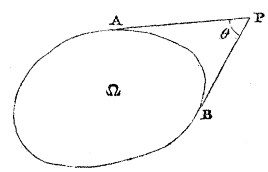
If an infinity of lines are drawn at random in an infinite plane, *the density of their intersections* (*i. e.* the measure of the number* of intersections in any given space, divided by the space) is uniform, and equal to π.

9. If an infinity of random lines meet a given area, *the density of their intersections, at any external point* P, is

$$\varrho=\theta-\sin\theta,$$

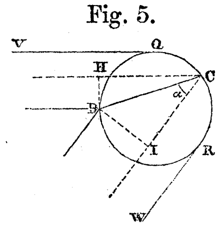
where θ is the apparent angular magnitude of the area from that point.

Fig. 4.



Conceive an infinitely small circle, or other figure (whose dimensions, however, infinitely exceed δp), at P, and let us calculate the number of the said intersections which fall inside this circle. Let the figure represent this circle, *magnified* as it were; QV, RW being the tangents PA, PB. Draw one of the random lines CD, which meet both the circle and the area Ω, the actual number of intersections which lie on CD will be $\frac{1}{\delta}N(\Omega, CD)$, which is found from art. 5 to be

Fig. 5.



$$\frac{1}{\delta}(2CD-CH-CI),$$

* We take for this measure the actual number multiplied by $\delta\theta^2$, or δ^2 (see note, art. 4).

or

$$\frac{\mathrm{CD}}{\delta}\Big(2-\cos\alpha-\cos(\theta-\alpha)\Big).$$

Hence the actual number of intersections on all the chords parallel to CD is

$$\frac{1}{\delta}(\text{area of circle})\Big(2-\cos\alpha-\cos(\theta-\alpha)\Big).$$

Therefore the measure* of the whole number of intersections lying within the circle is

$$\tfrac{1}{2}(\text{area})\int_0^\theta\Big(2-\cos\alpha-\cos(\theta-\alpha)\Big)d\alpha=(\text{area})(\theta-\sin\theta),$$

which proves the theorem.

10. The number of the intersections external to the given area is, then, measured by the integral

$$\iint(\theta-\sin\theta)d\mathrm{S}$$

extended over the whole plane outside Ω; $d\mathrm{S}$ being the element of the area. Now the number of internal intersections is $\pi\Omega$ (art. 8), and the sum of both is $\frac{1}{2}\mathrm{L}^2$. We obtain thus, in a singular manner, the following remarkable theorem in Definite Integrals:—

If θ be the angle between the tangents drawn from any external point (x, y) *to any given convex boundary, of length* L, *enclosing an area* Ω, *then*

$$\iint(\theta-\sin\theta)dxdy=\tfrac{1}{2}\mathrm{L}^2-\pi\Omega,$$

the integration extending over the whole space outside Ω.

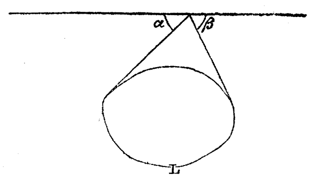
It does not seem easy to deduce this integral, in its generality, by any other method. It may be verified by direct integration for the cases of a circle, and of a finite straight line. It forms a striking example of what will doubtless be found, as the study of Local Probability advances, to be one of its most remarkable applications, viz. the evaluation of Definite Integrals. All who have studied the subject must have remarked the variety of ways in which almost every problem may be considered; now it often happens that a question in which we are baffled by the difficulties of the integration, when we attempt it in a particular way, may be solved with comparative ease by other considerations: we can then return to the integrals which we were unable to solve, and assign their values. I proceed to give some further applications of the above theory to Integration.

11. Given any infinite straight line outside a given convex boundary of length L, let dx be any element of this line; α, β the inclinations of dx to the two tangents drawn from it to the boundary, then

$$\int_{-\infty}^{\infty}(\cos\alpha+\cos\beta)dx=\mathrm{L}.$$

* We take for this measure the actual number multiplied by $\delta\theta^2$, or δ^2 (see note, art. 4).

Fig. 6.



It is easy to see from art. 5 that the number of random lines cutting L, which also meet dx, is $dx(\cos\alpha+\cos\beta)$; now the sum of all such elements gives the number of lines cutting both L and the given infinite straight line; that is, L (art. 4). This integral may be otherwise verified.

If the boundary L be enclosed within any outer convex boundary, let ds be the differential of the length of the latter, α, β the inclinations of ds to the tangents from it to L, then we find in the same manner,

$$\int(\cos\alpha+\cos\beta)ds=2\text{L},$$

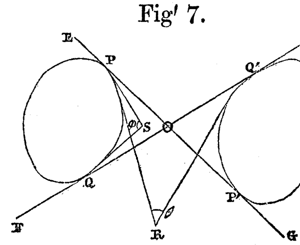
the integral extending all round the outer curve.

I mention this merely as an illustration; it is in fact easy to show independently that

$$\text{L}=\int\cos\alpha ds=\int\cos\beta ds.$$

12. If an infinite number of random lines pass between two convex areas, the density of their intersections will be (as in art. 9) at any point R in the angle FOG, or in EOH,

Fig' 7.



$$\varrho=\theta-\sin\theta;$$

and at any point S in the spaces POQ, P′OQ′,

$$\varrho=\pi-\phi-\sin\phi;$$

now the whole number of intersections is (art. 7) measured by

$$\tfrac{1}{2}(\text{PP}'+\text{QQ}'-\text{PQ}-\text{P}'\text{Q}')^2.$$

Hence

$$\iint(\theta-\sin\theta)d\text{S}+\iint(\pi-\phi-\sin\phi)d\text{S}=\tfrac{1}{2}(\text{PP}'+\text{QQ}'-\text{PQ}-\text{P}'\text{Q}')^2,$$

the first integral extending over the infinite spaces FOG, EOH, and the second over the spaces POQ, P′OQ′.

Thus if θ be the angle between the tangents drawn from any external point to an hyperbola,

$$\iint(\theta-\sin\theta)dxdy=\tfrac{1}{2}\Delta^2,$$

where Δ is the difference between the hyperbola and its asymptotes, and θ means the *external* angle of the tangents, in the cases where they touch the same branch of the curve, the integral extending over the whole space outside the hyperbola.

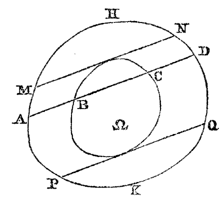
Received February 13, 1868.

13. If we consider a system of random lines disposed over the whole surface of an infinite plane, and a second system all of which meet a given convex area Ω within the plane, and then fix our attention on the infinite system of points in which the latter system cuts the former, it will be seen that *the density of these intersections, at any point* (x, y) *exterior to* Ω, *is equal to* 2θ, θ being the angle which Ω subtends at the point

(x, y); hence $2\iint\theta dxdy$ represents the number of these intersections which lie on any given portion of the plane outside Ω.

Take now an arbitrary convex boundary surrounding Ω; we will calculate in a different way the number of intersections which lie on the annular space between the two boundaries, and thus arrive at a value for the above definite integral, extended over the same annulus.

Fig. 8.



Let AD be a random line of the second system, meeting Ω; the number (within the annulus) of its intersections with the first system will be measured by (art. 4) 2AB+2CD; and hence the total number of intersections of all parallels to AD (between the tangents MN, PQ), with the first system, will be measured by double the area, cut from the annulus, between MN and PQ. Hence if Θ represent the annulus, the actual number of intersections which lie on those random lines of the second system which are parallel to those in the figure, is

$$n_0=\frac{1}{\delta}(2\Theta-2 \text{ segment MHN}-2 \text{ segment PKQ}).$$

Making now the parallel tangents MN, PQ revolve by constant changes of inclination, δ, through two right angles, we have for the *measure* of the total number of intersections, if ϕ be the inclination of MN to a fixed line,

$$\mathrm{N}=\int_0^\pi(2\Theta-2\mathrm{MHN}-2\mathrm{PKQ})d\phi.$$

But if we make the tangent MN revolve through 4 right angles instead of 2, it will occupy all the positions of PQ; denoting then the segment MHN by Σ, we have

$$\mathrm{N}=2\pi\Theta-2\int_0^{2\pi}\Sigma d\phi;$$

therefore

$$\iint\theta dxdy=\pi\Theta-\int_0^{2\pi}\Sigma d\phi.$$

The mean or average value of the segment Σ, as the tangent alters by uniform changes of inclination, is

$$\mathrm{A}=\frac{1}{2\pi}\int_0^{2\pi}\Sigma d\phi;$$

we have, then, the following theorem:—

If θ be the angle subtended at any point (x, y) *by a given convex area Ω, then*

$$\iint\theta dxdy=\pi(\Theta-2\mathrm{A}),$$

the integration extending over the annulus between Ω and any given exterior convex boundary; Θ standing for the area of that annulus, and A *denoting the average area of the segments cut from the annulus by the tangents to the boundary of Ω.*

This theorem gives the value of the integral in those cases where we are able to calculate the value of A: if Σ is constant, we have the theorem:—

Let there be any two convex boundaries so related that a tangent to the inner cuts off a constant area from the outer. Let θ be the angle subtended by the inner boundary at any external point (x, y); *and let Δ be the difference of the parts into which the annular space between the two is divided by any tangent to the inner, then*

$$\iint \theta dxdy = \pi\Delta,$$

the integration extending over the whole of the annulus.

For instance, we may apply the theorem to two similar coaxial ellipses. We may deduce thus the following definite integral,

$$\iint \tan^{-1}\left(\frac{2\sqrt{a^2y^2+b^2x^2-a^2b^2}}{x^2+y^2-a^2-b^2}\right)dxdy = \pi abk^2(\pi\sin^2\tfrac{1}{2}\alpha - \alpha + \sin\alpha),$$

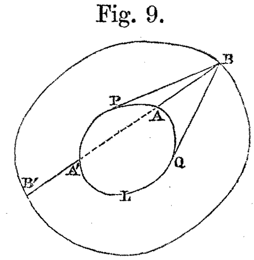
the limits being given by $1<\frac{x^2}{a^2}+\frac{y^2}{b^2}<k^2$; putting $\cos\frac{1}{2}\alpha=\frac{1}{k}$.

In the case where $k^2=2$, the value of the integral is $2\pi ab$; that is, the area of the outer ellipse.

14. If we suppose an infinite plane covered with random lines, and then imagine these divided into two systems, the first comprising all those lines which meet a given convex boundary, the second all those which do not meet it, and if we now consider the assemblage of points in which the first system intersects the second, we shall find (as in art. 9) that *the density of these intersections, at any point outside the boundary*, is $2\sin\theta$, θ being, as before, the apparent angular magnitude of the boundary.

Hence the number of intersections which lie on any given space is represented by the integral $2\iint \sin\theta dS$.

Fig. 9.



If we now suppose an endless string (of length Y) passed round the given boundary (whose perimeter we call L), and if this string be kept stretched by a moving point which thus traces out a new contour enclosing the given one (as the outer of any two confocal ellipses may be generated from the inner), we may estimate in a different manner the number of intersections which lie on the intermediate annular space, and thus obtain the following value for the above integral extended over that space,

$$\iint \sin\theta dS = L(Y-L).$$

Let AB be a line of the first system meeting the two boundaries in A, B; the number of points in which AB is cut by a system of random lines *covering the whole plane* is (art. 4)

$$\frac{2AB}{\delta}.$$

If we subtract from this the number of intersections of AB with those lines which

meet the boundary L, the remainder will be the number of intersections of AB with the *second* system of lines above, viz. (art. 6)

$$\frac{2\mathrm{AB}}{\delta}-\frac{1}{\delta}(2\mathrm{AB}+\mathrm{L}-\mathrm{Y}),$$

that is,

$$\frac{1}{\delta}(\mathrm{Y}-\mathrm{L}).$$

This is constant for every position of AB; hence the number of intersections lying on the annulus will be the above constant, multiplied by the number of positions of AB; now this number is $\frac{2\mathrm{L}}{\delta}$ (art. 4) (remembering that for every line AB, there is also one A′B′, forming a portion of the same straight line). Hence the total number of intersections is

$$\frac{2}{\delta^2}\mathrm{L}(\mathrm{Y}-\mathrm{L}).$$

If, then, the integration extend over the annulus,

$$\iint \sin\theta\,.\,d\mathrm{S}=\mathrm{L}(\mathrm{Y}-\mathrm{L}).$$

This theorem will apply to an ellipse, the outer boundary being a confocal ellipse. A particular case, which admits of verification by using elliptic coordinates, will be:—

If θ be the angle which two fixed points F, F′ subtend at the element dS,

$$\iint \sin\theta d\mathrm{S}=8c(a-c)\,;$$

the integration extending over an ellipse whose foci are F, F′, $2a$ being the axis of the ellipse, and $2c$=FF′.

The above method will also show that in this case the integral remains unchanged in value, if it extend over any Cartesian oval whose *internal* foci are FF′, and whose axis is $2a$. An instance of such a Cartesian is a circle from F as centre with a as radius, provided $a>2c$. The same will appear by means of elliptic coordinates*.

15. I will mention the following integral here, as, though strictly not derived from the theory which forms the subject of this paper, the principle used in obtaining it is, as in the cases which precede, the calculation of the number of intersections lying on a given space, of a given reticulation of straight lines.

Given a closed convex boundary without salient points; if we draw an infinity of tangents to it, each making an infinitesimal angle (δ) with the preceding, and consider the *intersections* of all these tangents with each other, it will not be difficult to show (as in art. 9) that the number of intersections lying on any element dS will be

$$\frac{1}{\delta^2}\cdot\frac{\sin\theta}{\mathrm{TT}'}\,d\mathrm{S},$$

* The general integral above admits also of being established by means of a certain generalization of elliptic coordinates, which defines the position of a point by the sum and difference of two strings, each of which is attached to a fixed point on a given oval curve; they are then wrapped round the curve in opposite directions, and leave it as two tangents, meeting and terminating at the proposed point.

where T, T′ *are the tangents from* dS *to the boundary, and* θ *their mutual inclination.*

Now the whole number of tangents is $\frac{2\pi}{\delta}$, and that of intersections $\frac{2\pi^2}{\delta^2}$. We infer therefore that

$$\iint \frac{\sin\theta}{\mathrm{TT}'} d\mathrm{S} = 2\pi^2,$$

the integral extending over the whole external surface.

If the integral extend over *the annulus between the given boundary and an outer line along which* θ *has a constant value* (α), then

$$\iint \frac{\sin\theta}{\mathrm{TT}'} d\mathrm{S} = 2\pi(\pi-\alpha).$$

If the same integral extend over *the space between the given boundary and two fixed tangents, including an angle* α, its value will be $\frac{1}{2}(\pi-\alpha)^2$. If it extend over *the infinite angle formed by those tangents produced*, its value will be $\frac{1}{2}\alpha^2$.

If the given boundary contain salient points, then for every such point, where the bounding line changes direction abruptly through an angle A, a number of the tangents, equal to $\frac{\mathrm{A}}{\delta}$, meet at that point; hence a number $\left(\frac{1}{2}\frac{\mathrm{A}^2}{\delta^2}\right)$ of intersections coincide there, and consequently we must subtract $\frac{1}{2}\mathrm{A}^2$ from each of the above integrals. Hence if there are any number of salient points A A′ A″, &c. in the boundary, the first integral becomes

$$\iint \frac{\sin\theta}{\mathrm{TT}'} d\mathrm{S} = 2\pi^2 - \frac{1}{2}\Sigma \mathrm{A}^2,$$

and likewise for the second.

Thus for a regular polygon of (n) sides, the value is

$$2\pi^2\left(1-\frac{1}{n}\right).$$

If instead of drawing tangents to the given boundary at uniform angular intervals, we draw a system of tangents whose points of contact are distant from each other by a common infinitesimal interval, we shall find that the density of the intersections in this case varies as

$$\frac{\varrho\varrho'}{\mathrm{TT}'}\sin\theta,$$

where $\varrho\varrho'$ *are the radii of curvature of the boundary at the points of contact of* TT′: this gives us the integral

$$\iint \frac{\varrho\varrho'}{\mathrm{TT}'}\sin\theta\, d\mathrm{S} = \frac{1}{2}\mathrm{L}^2,$$

L being the whole perimeter of the boundary, the integral extending over the whole plane.

Many analytical definite integrals may be deduced by expressing the general theorems now given, in the language of different systems of coordinates, for various particular

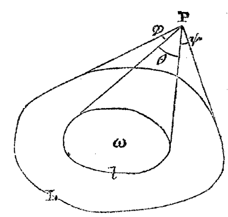
cases. Thus the first theorem in this article, applied to the ellipse, gives

$$\iint \frac{a^2y^2+b^2x^2}{\{(x^2+y^2+c^2)^2-4c^2x^2\}\sqrt{a^2y^2+b^2x^2-a^2b^2}}\,dxdy=\pi^2;$$

the equation of limits being $\frac{x^2}{a^2}+\frac{y^2}{b^2}>1$.

16. Let there be a closed convex area ω, length of boundary l, enclosed within another of length L; let θ be the apparent magnitude of ω at any external point; by considering two systems of random lines, one crossing the boundary L, and the other l, and examining the law of the density of the intersections of the former with the latter, we arrive at the theorem:—if we put for shortness

Fig. 10.



$$\alpha-\sin\alpha=u_\alpha,$$

$$\iint(u_{\theta+\phi}+u_{\theta+\psi}-u_\phi-u_\psi)d\mathrm{S}+2\iint\theta d\mathrm{S}=\mathrm{L}l-2\pi\omega;$$

the first integral extending over the whole space outside L, the second over the space between L and l.

17. But few problems on random straight lines admit of such simple results and of such generality as those we have been discussing. In general they can only be solved for particular forms of the boundaries. However, the above principles, applied to each particular question, generally suffice to reduce it at least to a problem of the Integral Calculus. I will give one or two examples.

If two random lines cross a given convex area, the chance of their intersection falling on any *internal* portion of the area ω, is evidently (art. 8)

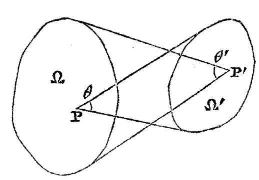
$$p=\frac{2\pi\omega}{\mathrm{L}^2}.$$

But the chance of the intersection falling on any *external* area is less easy to find; it depends on the integral $\iint(\theta-\sin\theta)d\mathrm{S}$ extended over that area. Could we succeed in finding the required probability by any different method, we could give the value of this integral for any external area.

A line is drawn at random across each of two given convex areas Ω, Ω', external to each other, lengths of boundaries L, L′; to find the chance of their intersection being outside both areas.

The density of the intersections of the system of random lines crossing Ω with those crossing Ω', at any point P within Ω, is 2θ, where θ means the apparent magnitude of Ω' at P. Within Ω', the density is $2\theta'$. Hence it is easy to see that, as the whole number of intersections is LL′, the required probability is

Fig. 11.



$$p=1-\frac{2}{\mathrm{LL}'}\left(\iint\theta d\mathrm{S}+\iint\theta' d\mathrm{S}'\right),$$

the integrals extending over Ω and Ω' respectively. It is evident that these integrals, however, can only be evaluated for particular forms of the areas.

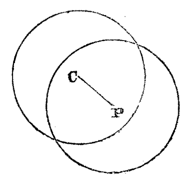
18. The following problems relate to a circular boundary:—

1. A random point falls within a given circle and a random straight line is drawn across the circle; to find the chance of the line passing within a given distance of the point.

As the general solution is somewhat complicated, I will take the particular case where the given distance is the radius of the circle, which will serve equally well as an example of the application of the foregoing principles.

Let C be the centre of the given circle, P any position of the random point, r the radius of the given circle; draw an equal circle with P as centre; then the number of random lines meeting the given circle and passing within a distance r of the point P, is the same as the number of random lines cutting *both* circles; this number is measured (art. 6) by the excess of the two circumferences over an endless band wrapped round them; that is, putting $\mathrm{CP}=\varrho$,

Fig. 12.



$$2\pi r-2\varrho.$$

If $d\mathrm{S}$ be an element of the surface at P, the sum of the favourable cases will be

$$\mathrm{F}=\iint(2\pi r-2\varrho)d\mathrm{S}=2\int_0^r(\pi r-\varrho).2\pi\varrho d\varrho\,;$$

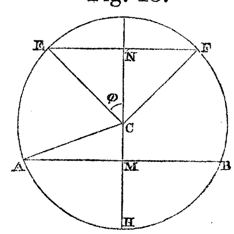
$$\therefore\ \mathrm{F}=(\pi-\tfrac{2}{3})2\pi r^3.$$

But the whole number of cases is $2\pi r\times\pi r^2$; hence the required chance is

$$p=1-\frac{2}{3\pi}.$$

I will give another solution of this problem:—Let AB be a position of the random line; take $\mathrm{MN}=r$, then all the favourable positions of the random point are within the segment EHF; the number of favourable points is therefore

Fig. 13.



$$r^2(\pi-\phi+\sin\phi\,\cos\phi).$$

We have to multiply this by the differential of CM, and integrate from $\mathrm{CM}=r$ to $\mathrm{CM}=0$, which will give the favourable combinations for all random lines parallel to AB, passing between C and H; doubling this, we have the result for *all* lines parallel to AB; that is,

$$\mathrm{F}_0=2r^2\int_r^0(\pi-\phi+\sin\phi\,\cos\phi)d.r\cos\phi$$

$$=2r^3\int_0^{\frac{\pi}{2}}(\pi-\phi+\sin\phi\,\cos\phi)\sin\phi\,d\phi\,;$$

$$\therefore\ \mathrm{F}_0=2r^3(\pi-\tfrac{2}{3}).$$

Now if the system of lines parallel to AB revolve through two right angles, we have for the *measure* of the whole number of favourable cases

$$\mathrm{F}=2\pi r^3(\pi-\tfrac{2}{3})\,;$$

hence, as before,

$$p=1-\frac{2}{3\pi}.$$

The problem in its general form can be solved without any great difficulty by the same methods. The result may be expressed in this form:—Let D be the given maximum distance; draw a circle of radius D with its centre on the given circumference; let Y be a band enveloping both circles, and 2θ the inclination of the two straight portions of this band; then the probability of the line passing within a distance D of the point will be

$$p=\frac{2\pi r+2\pi \mathrm{D}-\mathrm{Y}}{2\pi r}+\frac{\cos^3\theta}{3\pi};$$

or, if p_0 be the probability when the point is taken anywhere on the *circumference* of the given circle, then the general value of the probability is

$$p=p_0+\frac{\cos^3\theta}{3\pi}.$$

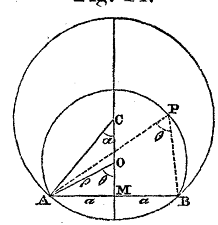
If a random point and a random straight line be taken within any convex boundary of length L, the chance that the line shall pass within a distance D of the point, D being small, is approximately,

$$p=\frac{2\pi \mathrm{D}}{\mathrm{L}}.$$

2. If three lines are drawn at random across a given circle, to determine the probability that their three intersections shall lie within the circle.

Let AB be one of the random lines. The total number of favourable triads of random lines, each triad of which includes AB, is the same as the *number of intersections, which fall within the circle*, of all random lines which cross AB. For every such intersection which lies within the circle, gives a pair of lines meeting AB, forming a triad whose intersections all lie within the circle. Now if θ be the angle which AB subtends at any internal point P, the number of these intersections will be measured by (art. 9)

Fig. 14.



$$\mathrm{N}=\iint(\theta-\sin\theta)d\mathrm{S},$$

extended over the whole circle.

To integrate this, conceive the circle divided into an infinite number of elementary crescents, by segments of circles on AB; let O be the centre of the segment APB, ϱ its radius; then the area of the segment APB is, putting $\mathrm{AB}=2a$,

$$\text{segment }=(\pi-\theta)\varrho^2+a\varrho\cos\theta,\ \text{ or as } \varrho\sin\theta=a$$

$$=a^2\left(\frac{\pi-\theta}{\sin^2\theta}+\cot\theta\right).$$

Differentiating this for θ, we obtain for the area of the crescent between APB and the consecutive arc on AB,

$$\text{crescent} = \frac{2a^2}{\sin^2\theta}\left(1+(\pi-\theta)\cot\theta\right)d\theta.$$

Hence the number of intersections *above* AB will be

$$N=2a^2\int_\alpha^\pi \frac{\theta-\sin\theta}{\sin^2\theta}\left(1+(\pi-\theta)\cot\theta\right)d\theta;$$

$$\therefore\quad \frac{N}{2a^2}=\int_\alpha^\pi\left\{\frac{\theta d\theta}{\sin^2\theta}+\pi\frac{\theta\cos\theta d\theta}{\sin^3\theta}-\frac{\theta^2\cos\theta}{\sin^3\theta}d\theta-\frac{d\theta}{\sin\theta}-\pi\frac{\cos\theta d\theta}{\sin^2\theta}+\frac{\theta\cos\theta}{\sin^2\theta}d\theta\right\}.$$

All these are elementary integrals, and give (reducing the indeterminate forms by the usual methods)

$$\frac{N}{2a^2}=\frac{3}{2}-\frac{\alpha^2}{\sin^2\alpha}-\frac{\pi-\alpha}{\sin\alpha}+\frac{\pi}{2}\cot\alpha+\frac{\pi\alpha}{2\sin^2\alpha}.$$

To find the number of intersections *below* AB, change α into $\pi-\alpha$; this gives for the whole number of favourable triads (including AB),

$$N=2a^2\left(3-\frac{\pi}{\sin\alpha}+\frac{\alpha\pi-\alpha^2}{\sin^2\alpha}\right);$$

or if c be the radius of the given circle, $a=c\sin\alpha$;

$$\therefore\quad N=2c^2(3\sin^2\alpha-\pi\sin\alpha+\alpha\pi-\alpha^2).$$

Multiply this by the differential of CM, and integrate from c to $-c$, and we have the sum of all favourable triads, each of which includes *any one* of the random lines parallel to AB,

$$F=2c^3\int_0^\pi(3\sin^2\alpha-\pi\sin\alpha+\alpha\pi-\alpha^2)\sin\alpha d\alpha$$

$$=2c^3\left(8-\frac{\pi^2}{2}\right).$$

Multiply this by π, and we have the measure* of the total number of favourable triads: however, this must be divided by 3, as it is clear we should thus count each triad thrice; hence total value of

$$F=\frac{\pi c^3}{3}(16-\pi^2);$$

and the whole number of cases being $\frac{1}{6}(2\pi c)^3$, we find for the probability sought,

$$P=\frac{4}{\pi^2}-\frac{1}{4}\dagger.$$

19. An interesting inquiry, though of a more difficult nature than that which has occupied us in this Paper, would be the extension of the foregoing principles to straight lines and planes drawn at random in space. It involves several intricate and curious points relating to the general theory of surfaces. With regard to the *measure of the number of random straight lines which meet a given closed convex surface*, it is easy to show that this measure is *the surface itself*.

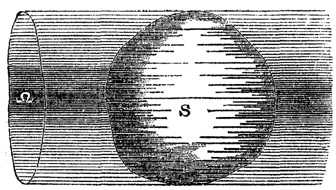
* *i. e.* the actual number multiplied by δ^3 (as art. 8).

† It is not unlikely that this result may be obtained in some simpler manner.

It may be assumed as self-evident that if space be filled with an infinity of random straight lines, and they be cut by any infinite plane, the points in which it cuts them are distributed with uniform density over the plane; and this density will be the same for any other plane. Hence the number of the random lines which meet any plane area is proportional to that area. Hence the number meeting any plane element dS of the surface is proportional to dS; the same is true for every other element; and each random line cuts two elements and only two; hence the whole number of lines is proportional to S.

We might view the question as follows. The entire body of random lines may be considered (as in art. 3) as a system of parallels disposed uniformly and symmetrically in space, which is afterwards turned round by infinitely small angular displacements, into every possible position. Let the figure represent one of these systems of parallels meeting the surface S, and of course bounded by the cylinder, enveloping S, whose generatrix is parallel to these lines. Let Ω be the area of the perpendicular section of this cylinder, then Ω is the measure of the number of these parallels. Let θ, ϕ be the angular coordinates of the direction of these parallels, and let them now pass into every possible angular position; the whole number of lines which meet S will be proportional to

Fig. 15.



$$\iint \Omega \sin\theta d\theta d\phi,$$

extended through half the solid angular space round a point. We infer from this that

$$\int_0^{2\pi}\int_0^{\frac{1}{2}\pi} \Omega \sin\theta d\theta d\phi = kS.$$

To determine the constant k, we may apply the theorem to any particular case, as a sphere; this gives $k=\frac{\pi}{2}$. We may accept this manner of viewing a system of random lines, then, as a proof of the theorem in surfaces:—

If Ω be the area of the section of a cylinder enveloping a convex surface S; θ, ϕ the angular coordinates of the generatrix of the cylinder,

$$\int_0^{2\pi}\int_0^{\frac{\pi}{2}} \Omega \sin\theta d\theta d\phi = \frac{\pi}{2}S.$$

The measure of *the number of random planes* which meet a given surface is easily seen to be (as in art. 4)

$$N = \int_0^{2\pi}\int_0^{\pi} p \sin\theta d\theta d\phi,$$

where p is the perpendicular from any internal point on the tangent plane, and θ, ϕ the angular coordinates of p. I am not aware that this integral has ever been considered. It is probable that it admits of some simple geometrical representation, which possibly will be found to be the length of some closed curve, traced upon the given surface, and bearing some remarkable relation to the general curvature of the surface.

It is well to notice, with regard to the applications to integration of the theory laid down in this Paper, that the theorems thereby deduced in no way depend for their truth upon the doctrine of Probability, although it has been the occasion which has led to them. The apparatus of a system of equidistant parallels, revolving through constant angular displacements, which has been used in establishing their truth, is a strictly geometrical conception, and which, as here employed, may be viewed as a method in the Integral Calculus. A simpler species of reticulation, consisting of *two* systems of parallels, crossing at a finite angle, has already been used by EISENSTEIN and others in the Theory of Numbers and in Elliptic Functions.

It will be borne in mind also that this apparatus of lines is used only as a correct and convenient *representation* of an infinite system of random lines, for the purposes of calculation. Of course it is not asserted that all those random lines which are parallel to a given direction will be equidistant, or that there will be none of the random lines intermediate in direction between θ and $\theta+\delta\theta$. Just as an infinity of points arranged in horizontal rows and vertical columns will faithfully represent, for the purposes of calculation, an infinity of random points, so will the above apparatus represent the lines. Other arrangements, in either case, may easily be conceived which will represent them equally correctly, and which possibly will be found, in certain cases, more convenient. Thus if an infinite plane be covered with points arranged symmetrically, the system of lines obtained by joining each pair of points will, undoubtedly, truly represent a system of random lines.

It is unnecessary to point out, that if we can succeed in the difficult inquiry involved in extending the above methods to space, not only will the theory of probability be advanced, but various remarkable results in the Integral Calculus may be expected.