XXII. *A Memoir on Prepotentials.* *By Professor* Cayley, *F.R.S.*

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The present Memoir relates to multiple integrals expressed in terms of the $(s+1)$ ultimately disappearing variables $(x\,..\,z, w)$, and the same number of parameters $(a\,..\,c, e)$, and being of the form

$$\int \frac{\varrho\, d\varpi}{\{(a-x)^2\,..+(c-z)^2+(e-w)^2\}^{\frac{1}{2}s+q}},$$

where ϱ and $d\varpi$ depend only on the variables $(x\,..\,z, w)$. Such an integral, in regard to the index $\frac{1}{2}s+q$, is said to be "prepotential," and in the particular case $q=-\frac{1}{2}$ to be "potential."

I use throughout the language of hyper-tridimensional geometry: $(x\,..\,z, w)$ and $(a\,..\,c, e)$ are regarded as coordinates of points in $(s+1)$dimensional space, the former of them determining the position of an element $\varrho d\varpi$ of attracting matter, the latter being the attracted point; viz. we have a mass of matter $=\int \varrho d\varpi$ distributed in such manner that, $d\varpi$ being the element of $(s+1)$- or lower-dimensional volume at the point $(x\,..\,z, w)$, the corresponding density is ϱ, a given function of $(x\,..\,z, w)$, and that the element of mass $\varrho d\varpi$ exerts on the attracted point $(a\,..\,c, e)$ a force inversely proportional to the $(s+2q+1)$th power of the distance $\{(a-x)^2\,..+(c-z)^2+(e-w)^2\}^{\frac{1}{2}}$. The integration is extended so as to include the whole attracting mass $\int \varrho d\varpi$; and the integral is then said to represent the Prepotential of the mass in regard to the point $(a\,..\,c, e)$. In the particular case $s=2$, $q=-\frac{1}{2}$, the force is as the inverse square of the distance, and the integral represents the Potential in the ordinary sense of the word.

The element of volume $d\varpi$ is usually either the element of solid (spatial or $(s+1)$-dimensional) volume $dx\,..\,dz\,dw$, or else the element of superficial (s-dimensional) volume $d\mathrm{S}$. In particular, when the surface (s-dimensional locus) is the (s-dimensional) plane $w=0$, the superficial element $d\mathrm{S}$ is $=dx\,...\,dz$. The cases of a less-than-s-dimensional volume are in the present memoir considered only incidentally. It is scarcely necessary to remark that the notion of density is dependent on the dimensionality of the element of volume $d\varpi$: in passing from a spatial distribution, $\varrho\, dx\,...\,dz\,dw$, to a superficial distribution, $\varrho\, d\mathrm{S}$, we alter the signification of ϱ. In fact if, in order to connect the two, we imagine the spatial distribution as made over an indefinitely thin layer or stratum bounded by the surface, so that at any element $d\mathrm{S}$ of the surface the normal thickness is $d\nu$, where $d\nu$ is a function of the coordinates $(x\,...\,z, w)$ of the element $d\mathrm{S}$, the spatial element is $=d\nu\, d\mathrm{S}$, and the element of mass $\varrho\, dx\,...\,dz\,dw$ is $=\varrho\, d\nu\, d\mathrm{S}$; and

then changing the signification of ϱ, so as to denote by it the product $\varrho\, d\nu$, the expression for the element of mass becomes $\varrho\, d\mathrm{S}$, which is the formula in the case of the superficial distribution.

The space or surface over which the distribution extends may be spoken of as the material space or surface; so that the density ϱ is not $=0$ for any finite portion of the material space or surface; and if the distribution be such that the density becomes $=0$ for any point or locus of the material space or surface, then such point or locus, considered as an infinitesimal portion of space or surface, may be excluded from and regarded as not belonging to the material space or surface. It is allowable, and frequently convenient, to regard ϱ as a discontinuous function, having its proper value within the material space or surface, and having its value $=0$ beyond these limits; and this being so, the integrations may be regarded as extending as far as we please beyond the material space or surface (but so always as to include the whole of the material space or surface)—for instance, in the case of a spatial distribution, over the whole $(s+1)$dimensional space; and in the case of a superficial distribution, over the whole of the s-dimensional surface of which the material surface is a part.

In all cases of surface-integrals it is, unless the contrary is expressly stated, assumed that the attracted point does not lie on the material surface; to make it do so is, in fact, a particular supposition. As to solid integrals, the cases where the attracted point is not, and is, in the material space may be regarded as cases of coordinate generality; or we may regard the latter one as the general case, deducing the former one from it by supposing the density at the attracted point to become $=0$.

The present memoir has chiefly reference to three principal cases, which I call A, C, D, and a special case, B, included both under A and C: viz. these are:—

A. The prepotential-plane case; q general, but the surface is here the plane $w=0$, so that the integral is

$$\int \frac{\varrho\, dx \ldots dz}{\{(a-x)^2 \ldots + (c-z)^2 + e^2\}^{\frac{1}{2}s+q}}.$$

B. The potential-plane case; $q=-\frac{1}{2}$, and the surface the plane $w=0$, so that the integral is

$$\int \frac{\varrho\, dx \ldots dz}{\{(a-x)^2 \ldots + (c-z)^2 + e^2\}^{\frac{1}{2}s-\frac{1}{2}}}.$$

C. The potential-surface case; $q=-\frac{1}{2}$, the surface arbitrary, so that the integral is

$$\int \frac{\varrho\, d\mathrm{S}}{\{(a-x)^2 \ldots + (c-z)^2 + (e-w)^2\}^{\frac{1}{2}s-\frac{1}{2}}}.$$

D. The potential-solid case; $q=-\frac{1}{2}$, and the integral is

$$\int \frac{\varrho\, dx \ldots dz\, dw}{(a-x)^2 \ldots + (c-z)^2 + (e-w)^2\}^{\frac{1}{2}s-\frac{1}{2}}}.$$

It is, in fact, only the prepotential-plane case which is connected with the partial differential equation

$$\left(\frac{d^2}{da^2}\ldots+\frac{d^2}{dc^2}+\frac{d^2}{de^2}+\frac{2q+1}{e}\,\frac{d}{de}\right)\mathrm{V}=0,$$

considered in GREEN'S memoir 'On the Attractions of Ellipsoids' (1835), and called here "the prepotential equation." For this equation is satisfied by the function

$$\frac{1}{\{a^2\ldots+c^2+e^2\}^{\frac{1}{2}s+q}},$$

and therefore also by

$$\frac{1}{\{(a-x)^2\ldots+(c-z)^2+e^2\}^{\frac{1}{2}s+q}},$$

and consequently by the integral

$$\int\frac{\varrho\,dx\ldots dz}{\{(a-x)^2\ldots+(c-z)^2+e^2\}^{\frac{1}{2}s+q}}, \quad \ldots\ldots\ldots\ldots \quad \text{(A)}$$

that is by the prepotential-plane integral; but the equation is *not* satisfied by the value

$$\frac{1}{\{(a-x)^2\ldots+(c-z)^2+(e-w)^2\}^{\frac{1}{2}s+q}},$$

nor, therefore, by the prepotential-solid, or general superficial, integral.

But if $q=-\frac{1}{2}$, then, instead of the prepotential equation, we have "the potential equation"

$$\left(\frac{d^2}{da^2}\ldots+\frac{d^2}{dc^2}+\frac{d^2}{de^2}\right)\mathrm{V}=0\,;$$

and this is satisfied by

$$\frac{1}{\{a^2\ldots+c^2+e^2\}^{\frac{1}{2}s-\frac{1}{2}}},$$

and therefore also by

$$\frac{1}{\{(a-x)^2\ldots+(c-z)^2+(e-w)^2\}^{\frac{1}{2}s-\frac{1}{2}}}.$$

Hence it is satisfied by

$$\int\frac{\varrho\,dx\ldots dz\,dw}{\{(a-x)^2\ldots+(c-z)^2+(e-w)^2\}^{\frac{1}{2}s-\frac{1}{2}}}, \quad \ldots\ldots\ldots \quad \text{(D)}$$

the potential-solid integral, *provided that the point* $(a\ldots c,\ e)$ *does not lie within the material space*: I would rather say that the integral does *not* satisfy the equation, but of this more hereafter; and it is satisfied by

$$\int\frac{\varrho\,d\mathrm{S}}{\{(a-x)^2\ldots+(c-z)^2+(e-w)^2\}^{\frac{1}{2}s-\frac{1}{2}}}, \quad \ldots\ldots\ldots \quad \text{(C)}$$

the potential-surface integral. The potential-plane integral (B), as a particular case of (C), of course also satisfies the equation.

Each of the four cases give rise to what may be called a distribution-theorem; viz. given V a function of $(a\ldots c,\ e)$ satisfying certain prescribed conditions, but otherwise arbitrary, then the form of the theorem is that there exists and that we can find an expres-

sion for ρ, the density or distribution of matter over the space or surface to which the theorem relates, such that the corresponding integral V has its given value, viz. in A and B there exists such a distribution over the plane $w=0$, in C such a distribution over a given surface, and in D such a distribution in space. The establishment, and exhibition in connexion with each other, of these four distribution-theorems is the principal object of the present memoir; but the memoir contains other investigations which have presented themselves to me in treating the question. It is to be noticed that the theorem A belongs to GREEN, being in fact the fundamental theorem of his memoir of 1835, already referred to. Theorem C, in the particular case of tridimensional space, belongs also to him, being given in his 'Essay on the Application of Mathematical Analysis to the theories of Electricity and Magnetism' (Nottingham, 1828), being partially rediscovered by GAUSS in the year 1840; and theorem D, in the same case of tridimensional space, to LEJEUNE-DIRICHLET: see his memoir "Sur un moyen général de vérifier l'expression du potentiel relatif à une masse quelconque homogène ou hétérogène," *Crelle*, t. xxxii. pp. 80–84 (1840). I refer more particularly to these and other researches by GAUSS, JACOBI, and others in an Annex to the present memoir.

On the Prepotential Surface-integral.—Art. Nos. 1 to 18.

1. In what immediately follows we require

$$V=\int\frac{dx\ldots dz}{(x^2\ldots+z^2+e^2)^{\frac{1}{2}s+q}},$$

limiting condition $x^2\ldots+z^2=R^2$, the prepotential of a uniform (s-coordinal) circular disk*, radius R, in regard to a point $(0\ldots0, e)$ on the axis; and in particular the value is required in the case where the distance e (taken to be always positive) is indefinitely small in regard to the radius R.

Writing $x=r\xi\ldots z=r\zeta$, where the s new variables $\xi\ldots\zeta$ are such that $\xi^2\ldots+\zeta^2=1$, the integral becomes

$$\int\frac{r^{s-1}dr\,dS}{(r^2+e^2)^{\frac{1}{2}s+q}},\quad =\int dS\int_0^R\frac{r^{s-1}dr}{(r^2+e^2)^{\frac{1}{2}s+q}},$$

where dS is the element of surface of the s-dimensional unit-sphere $\xi^2\ldots+\zeta^2=1$; the integral $\int dS$ denotes the entire surface of this sphere, which (see Annex I.) is $=\frac{2(\Gamma\frac{1}{2})^s}{\Gamma\frac{1}{2}s}$. The other factor,

$$\int_0^R\frac{r^{s-1}dr}{(r^2+e^2)^{\frac{1}{2}s+q}},$$

is the r-integral of Annex II.

* It is to be throughout borne in mind that $x\ldots z$ denotes a set of s coordinates, $x\ldots z, w$ a set of $s+1$ coordinates; the adjective coordinal refers to the number of coordinates which enter into the equation; thus, $x^2\ldots+z^2+w^2=f^2$ is an $(s+1)$coordinal sphere (observe that the surface of such a sphere is s-dimensional); $x^2\ldots+z^2=f^2$, according as we tacitly associate with it the condition $w=0$, or w arbitrary, is an s-coordinal circle, or cylinder, the surface of such circle or cylinder being s-dimensional, but the circumference of the circle $(s-1)$dimensional; or if we attend only to the s-dimensional space constituted by the plane $w=0$, the locus may be considered as an s-coordinal sphere, its surface being $(s-1)$dimensional.

2. We now consider the prepotential-surface integral

$$V=\int\frac{\varrho\,dS}{\{(a-x)^2\ldots+(c-z)^2+(e-w)^2\}^{\frac{1}{2}s+q}}.$$

As already mentioned, it is only a particular case of this, the prepotential-plane integral, which is specially discussed; but at present I consider the general case, for the purpose of establishing a theorem in relation thereto. The surface (s-dimensional surface) S is any given surface whatever.

Let the attracted point P be situate indefinitely near to the surface, on the normal thereto at a point N, say the normal distance NP is $=\vartheta$*; and let this point N be taken as the centre of an indefinitely small circular (s-dimensional) disk or segment (of the surface), the radius of which R, although indefinitely small, is indefinitely large in comparison with the normal distance ϑ. I proceed to determine the prepotential of the disk; for this purpose, transforming to new axes, the origin being at N and the axes of $x\ldots z$ in the tangent-plane at N, then the coordinates of the attracted point P will be $(0\ldots0,\ \vartheta)$, and the expression for the prepotential of the disk will be

$$V=\int\frac{\varrho\,dx\ldots dz}{\{x^2\ldots+z^2+\ ^2\}^{\frac{1}{2}s+q}},$$

where the limits are given by $x^2\ldots+z^2<R^2$.

Suppose for a moment that the density at the point N is $=\varrho'$, then the density throughout the disk may be taken $=\varrho'$, and the integral becomes

$$V=\varrho'\int\frac{dx\ldots dz}{\{x^2\ldots+z^2+\vartheta^2\}^{\frac{1}{2}s+q}},$$

where instead of ϱ' I write ϱ; viz. ϱ now denotes the density at the point N. Making this change, then (by what precedes) the value is

$$=\varrho\cdot\frac{2(\Gamma\frac{1}{2})^s}{\Gamma(\frac{1}{2}s)}\cdot\int_0^R\frac{r^{s-1}dr}{\{r^2+\vartheta^2\}^{\frac{1}{2}s+q}}.$$

$q=$*Positive*.—Nos. 3 to 7.

3. I consider first the case where q is positive. The value is here

$$=\varrho\frac{2(\Gamma\frac{1}{2})^s}{\Gamma(\frac{1}{2}s)}\frac{1}{2\vartheta^{2q}}\left\{\frac{\Gamma\frac{1}{2}s\Gamma q}{\Gamma(\frac{1}{2}s+q)}-\int_0^{\frac{\vartheta^2}{R^2}}\frac{x^{q-1}dx}{(1+x)^{\frac{1}{2}s+q}}\right\};$$

or since $\frac{\vartheta}{R}$ is indefinitely small, the x-integral may be neglected, and the value is

$$=\frac{1}{\vartheta^{2q}}\varrho\frac{(\Gamma\frac{1}{2})^s\Gamma q}{\Gamma(\frac{1}{2}s+q)}.$$

Observe that this value is independent of R, and that the expression is thus the same as if (instead of the disk) we had taken the whole of the infinite tangent-plane, the

* ϑ is positive; in afterwards writing $\vartheta=0$, we mean by 0 the limit of an indefinitely small positive quantity

density at every point thereof being $=\varrho$. It is proper to remark that the neglected terms are of the orders

$$\frac{1}{s^{2q}}\left\{\left(\frac{s}{\mathrm{R}}\right)^{2q}, \left(\frac{s}{\mathrm{R}}\right)^{2q+2}, \&c.\right\};$$

so that the complete value multiplied by s^{2q} is equal to the constant $\varrho\frac{(\Gamma\frac{1}{2})^s\Gamma q}{\Gamma(\frac{1}{2}s+q)}+$terms of the orders $\left(\frac{s}{\mathrm{R}}\right)^{2q}, \left(\frac{s}{\mathrm{R}}\right)^{2q+2}$, &c.

4. Let us now consider the prepotential of the remaining portion of the surface; every part thereof is at a distance from P exceeding, in fact far exceeding, R; so that imagining the whole mass $\int\varrho\, d\mathrm{S}$ to be collected at the distance R, the prepotential of the remaining portion of the surface is less than

$$\frac{\int\varrho\, d\mathrm{S}}{\mathrm{R}^{s+2q}};$$

viz. we have thus, in the case where the mass $\int\varrho\, d\mathrm{S}$ is finite, a superior limit to the prepotential of the remaining portion of the surface. This will be indefinitely small in comparison with the prepotential of the disk, provided only s^{2q} is indefinitely small compared with R^{s+2q}, that is s indefinitely small in comparison with $\mathrm{R}^{1+\frac{s}{2q}}$. The proof assumes that the mass $\int\varrho\, d\mathrm{S}$ is finite; but considering the very rough manner in which the limit $\frac{\int\varrho\, d\mathrm{S}}{\mathrm{R}^{s+2q}}$ was obtained, it can scarcely be doubted that, if not universally, at least for very general laws of distribution, even when $\int\varrho\, d\mathrm{S}$ is infinite, the same thing is true; viz. that by taking s sufficiently small in regard to R, we can make the prepotential of the remaining portion of the surface vanish in comparison with that of the disk. But without entering into the question I assume that the prepotential of the remaining portion does thus vanish; the prepotential of the whole surface in regard to the indefinitely near point P is thus equal to the prepotential of the disk; viz. its value is

$$=\frac{1}{s^{2b}}\varrho\frac{(\Gamma\frac{1}{2})^s\Gamma q}{\Gamma(\frac{1}{2}s+q)},$$

which, observe, is infinite for a point P on the surface.

5. Considering the prepotential V of an arbitrary point $(a\ldots c, e)$ as a given function of $(a\ldots c, e)$ the coordinates of this point, and taking $(x\ldots z, w)$ for the coordinates of the point N, which is, in fact, an arbitrary point on the surface, then the value of V at the point P indefinitely near to N will be $=\mathrm{W}$, if W denote the same function of $(x\ldots z, w)$ that V is of $(a\ldots c, e)$. The result just obtained is therefore

$$\mathrm{W}=\frac{1}{s^{2q}}\varrho\frac{(\Gamma\frac{1}{2})^s\Gamma q}{\Gamma(\frac{1}{2}s+q)},\ (s=0),$$

or, what is the same thing,

$$\varrho=\frac{\Gamma(\frac{1}{2}s+q)}{(\Gamma\frac{1}{2})^s\Gamma q}(s^{2q}\mathrm{W})_{s=0}.$$

As to this, remark that V is not an arbitrary function of $(a \ldots c, e)$: *non constat* that there is any distribution of matter, and still less that there is any distribution of matter on the surface, which will produce at the point $(a \ldots c, e)$, that is at every point whatever, a prepotential the value of which shall be a function assumed at pleasure of the coordinates $(a \ldots c, e)$. But suppose that V, the given function of $(a \ldots c, e)$, is such that there does exist a corresponding distribution of matter on the surface (viz. that V satisfies the conditions, whatever they are, required in order that this may be the case), then the foregoing formula determines the distribution, viz. it gives the expression of ϱ, that is, the density at any point of the surface.

6. The theorem may be presented in a somewhat different form; regarding the prepotential as a function of the normal distance $\mathfrak{s}$, its derived function in regard to $\mathfrak{s}$ is

$$=-\frac{2q}{\mathfrak{s}^{2q+1}}\varrho\frac{(\Gamma\frac{1}{2})^s\Gamma q}{\Gamma(\frac{1}{2}s+q)}, \text{ that is}$$

$$=-\frac{1}{\mathfrak{s}^{2q+1}}\varrho\frac{2(\Gamma\frac{1}{2})^s\Gamma(q+1)}{\Gamma(\frac{1}{2}s+q)};$$

and we thus have

$$\frac{dW}{d\mathfrak{s}}=-\frac{1}{\mathfrak{s}^{2q+1}}\varrho\frac{2(\Gamma\frac{1}{2})^s\Gamma(q+1)}{\Gamma(\frac{1}{2}s+q)}, \quad (\mathfrak{s}=0),$$

or, what is the same thing,

$$\varrho=-\frac{\Gamma(\frac{1}{2}s+q)}{2(\Gamma\frac{1}{2})^s\,\Gamma(q+1)}\left(\mathfrak{s}^{2q+1}\frac{dW}{d\mathfrak{s}}\right)_{\mathfrak{s}=0},$$

where, however, W being given as a function of $(x \ldots z, w)$, the notation $\frac{dW}{d\mathfrak{s}}$ requires explanation. Taking $\cos\alpha \ldots \cos\gamma$ to be the inclinations of the normal at N, in the direction NP in which the distance $\mathfrak{s}$ is measured, to the positive parts of the axes of $(x \ldots z)$, viz. these cosines denote the values of

$$\frac{dS}{dx} \cdots \frac{dS}{dz},$$

each taken with the same sign + or −, and divided by the square root of the sum of the squares of the last-mentioned quantities, then the meaning is

$$\frac{dW}{d\mathfrak{s}}=\frac{dW}{dx}\cos\alpha \ldots +\frac{dW}{dz}\cos\gamma.$$

7. The surface S may be the plane $w=0$, viz. we have then the prepotential-plane integral

$$V=\int\frac{\varrho\,dx \ldots dz}{\{(a-x)^2 \ldots +(c-z)^2+e^2\}^{\frac{1}{2}s-q}}, \quad \ldots \ldots \ldots \quad \text{(A)}$$

where e (like $\mathfrak{s}$) is positive. In afterwards writing $e=0$, we mean by 0 the limit of an indefinitely small positive quantity.

The foregoing distribution-formulæ then become

$$\varrho=\frac{\Gamma(\frac{1}{2}s+q)}{(\Gamma\frac{1}{2})^s\Gamma q}(e^{2q}W)_{e=0}, \quad \ldots \ldots \ldots \quad \text{(A)}$$

and

$$\rho=-\frac{\Gamma(\frac{1}{2}s+q)}{2(\Gamma\frac{1}{2})^s\Gamma(q+1)}\left(e^{2q+1}\frac{dW}{de}\right)_{e=0}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (A^*)$$

which will be used in the sequel.

It will be remembered that in the preceding investigation it has been assumed that q is positive, the limiting case $q=0$ being excluded†.

$q=-\frac{1}{2}$.—Nos. 8 to 13.

8. I pass to the case $q=-\frac{1}{2}$, viz. we here have the potential-surface integral

$$V=\int\frac{\rho dS}{\{(a-x)^2\ldots+(c-z)^2+(e-w)^2\}^{\frac{1}{2}s-\frac{1}{2}}}: \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (C)$$

it will be seen that the results present themselves under a remarkably different form.

The potential of the disk is, as before,

$$\rho\,\frac{2(\Gamma\frac{1}{2})^s}{\Gamma\frac{1}{2}s}\int\frac{r^{s-1}\,dr}{(r^2+\varepsilon^2)^{\frac{1}{2}s-\frac{1}{2}}},$$

where ρ here denotes the density at the point N; and the value of the r-integral

$$=R\left(1+\text{terms in }\frac{\varepsilon^2}{R^2},\frac{\varepsilon^4}{R^4},\ldots\right)-\varepsilon\,.\,\frac{\Gamma\frac{1}{2}s\,\Gamma\frac{1}{2}}{\Gamma(\frac{1}{2}s-\frac{1}{2})}.$$

Observe that this is indefinitely small, and remains so for a point P on the surface; the potential of the remaining portion of the surface (for a point P near to or on the surface) is finite, that is, neither indefinitely large nor indefinitely small, and it varies continuously as the attracted point passes through the disk (or aperture in the material surface now under consideration); hence the potential of the whole surface is finite for an attracted point P on the surface, and it varies continuously as P passes through the surface.

It will be noticed that there is in this case a term in V independent of ε; and it is on this account necessary, instead of the potential, to consider its derived function in regard to ε; viz. neglecting the indefinitely small terms which contain powers of $\frac{\varepsilon}{R}$, I write

$$\frac{dV}{d\varepsilon}=-\frac{2(\Gamma\frac{1}{2})^{s+1}}{\Gamma(\frac{1}{2}s-\frac{1}{2})}\rho.$$

The corresponding term arising from the potential of the other portion of the surface, viz. the derived function of the potential in regard to ε, is not indefinitely small; and calling it Q, the formula for the whole surface becomes

$$\frac{dV}{d\varepsilon}=Q-\frac{2(\Gamma\frac{1}{2})^{s+1}}{\Gamma(\frac{1}{2}s-\frac{1}{2})}\rho.$$

† This is, as regards q, the case throughout; a limiting value, if not expressly stated to be included, is always excluded.

9. I consider positions of the point P on the two opposite sides of the point N, say at the normal distances $\mathfrak{s}'$, $\mathfrak{s}''$, these being positive distances measured in opposite directions from the point N. The function V, which represents the potential of the surface in regard to the point P, is or may be a different function of the coordinates $(a \ldots c, e)$ of the point P, according as the point is situate on the one side or the other of the surface (as to this more presently). I represent it in the one case by V′, and in the other case by V″; and in further explanation state that $\mathfrak{s}'$ is measured *into* the space to which V′ refers, $\mathfrak{s}''$ *into* that to which V″ refers; and I say that the formulæ belonging to the two positions of the point P are

$$\frac{dW'}{d\mathfrak{s}'}=Q'-\frac{2(\Gamma\tfrac{1}{2})^{s+1}}{\Gamma(\tfrac{1}{2}s-\tfrac{1}{2})}\rho,$$

$$\frac{dW''}{d\mathfrak{s}''}=Q''-\frac{2(\Gamma\tfrac{1}{2})^{s+1}}{\Gamma(\tfrac{1}{2}s-\tfrac{1}{2})}\rho,$$

where, instead of V′, V″, I have written W′, W″ to denote that the coordinates, as well of P′ as of P″, are taken to be the values $(x \ldots z, w)$ which belong to the point N. The symbols denote

$$\frac{dW'}{d\mathfrak{s}'}=\frac{dW'}{dx}\cos\alpha'\ldots+\frac{dW''}{dz}\cos\gamma',$$

$$\frac{dW''}{d\mathfrak{s}''}=\frac{dW''}{dx}\cos\alpha''\ldots+\frac{dW''}{dz}\cos\gamma'',$$

where $(\cos\alpha'\ldots\cos\gamma')$ and $(\cos\alpha''\ldots\cos\gamma'')$ are the cosine inclinations of the normal distances $\mathfrak{s}'$, $\mathfrak{s}''$ to the positive parts of the axes of $(x\ldots z)$; since these distances are measured in opposite directions, we have $\cos\alpha''=-\cos\alpha'\ldots\cos\gamma''=-\cos\gamma'$. If we imagine a curve through N cutting the surface at right angles, or, what is the same thing, an element of the curve coinciding in direction with the normal element P′NP″, and if s denote the distance of N from a fixed point of the curve, and for the point P′ s becomes $s+\delta' s$, while for the point P″ it becomes $s-\delta'' s$, or, what is the same thing, if s increase in the direction of NP′ and decrease in that of NP″, then if any function Θ of the coordinates $(x\ldots z, w)$ of N be regarded as a function of s, we have

$$\frac{d\Theta}{ds}=\frac{d\Theta}{d\mathfrak{s}'},\quad \frac{d\Theta}{ds}=-\frac{d\Theta}{d\mathfrak{s}''}.$$

10. In particular, let Θ denote the potential of the remaining portion of the surface, that is, of the whole surface exclusive of the disk; the curve last spoken of is a curve which does not pass through the material surface, viz. the portion to which Θ has reference, and there is no discontinuity in the value of Θ as we pass along this curve through the point N. We have Q′=value of $\frac{d\Theta}{d\mathfrak{s}'}$ at the point P′, and Q″=value of $\frac{d\Theta}{d\mathfrak{s}''}$ at the point P″; and the two points P′, P″ coming to coincide together at the point

N, we have then

$$Q' = \frac{d\Theta}{d\mathfrak{s}'}, \quad = \frac{d\Theta}{ds},$$

$$Q'' = \frac{d\Theta}{d\mathfrak{s}''}, \quad = -\frac{d\Theta}{ds}.$$

We have in like manner $\frac{dW'}{d\mathfrak{s}'} = \frac{dW'}{ds}$, $\frac{dW''}{d\mathfrak{s}''} = -\frac{dW''}{ds}$; and the equations obtained above may be written

$$\frac{dW'}{ds} = \frac{d\Theta}{ds} - \frac{2(\Gamma\frac{1}{2})^{s+1}}{\Gamma(\frac{1}{2}s-\frac{1}{2})}\varrho,$$

$$\frac{dW''}{ds} = \frac{d\Theta}{ds} + \frac{2(\Gamma\frac{1}{2})^{s+1}}{\Gamma(\frac{1}{2}s-\frac{1}{2})}\varrho,$$

in which form they show that as the attracted point passes through the surface from the position P′ on the one side to P″ on the other, there is an abrupt change in the value of $\frac{dW}{ds}$, or say of $\frac{dV}{ds}$, the first derived function of the potential in regard to the orthotomic arc s, that is in the rate of increase of V in the passage of the attracted point normally to the surface. It is obvious that if the attracted point traverses the surface obliquely instead of normally, viz. if the arc s cuts the surface obliquely, there is the like abrupt change in the value of $\frac{dV}{ds}$.

Reverting to the original form of the two equations, and attending to the relation $Q'+Q''=0$, we obtain

$$\frac{dW'}{d\mathfrak{s}'} + \frac{dW''}{d\mathfrak{s}''} = \frac{-4(\Gamma\frac{1}{2})^{s+1}}{\Gamma(\frac{1}{2}s-\frac{1}{2})}\varrho,$$

or, what is the same thing,

$$\varrho = -\frac{\Gamma(\frac{1}{2}s-\frac{1}{2})}{4(\Gamma\frac{1}{2})^{s+1}}\left(\frac{dW'}{d\mathfrak{s}'} + \frac{dW''}{d\mathfrak{s}''}\right) \quad \ldots \ldots \ldots \quad \text{(C)}$$

11. I recall the signification of the symbols:—V′, V″ are the potentials, it may be different functions of the coordinates $(a \ldots c, e)$ of the attracted point, for positions of this point on the two sides of the surface (as to this more presently), and W′, W″ are what V′, V″ respectively become when the coordinates $(a \ldots c, e)$ are replaced by $(x \ldots z, w)$, the coordinates of a point N on the surface. The explanation of the symbols $\frac{dW'}{d\mathfrak{s}'}$, $\frac{dW''}{d\mathfrak{s}''}$ is given a little above; ϱ denotes the density at the point $(x \ldots z, w)$.

12. The like remarks arise as with regard to the former distribution theorem (A); the functions V′, V″ cannot be assumed at pleasure; *non constat* that there is any distribution in space, and still less any distribution on the surface, which would give such values to the potential of a point $(a \ldots c, e)$ on the two sides of the surface respectively; but assuming that the functions V′, V″ are such that they do arise from a distribution on the surface, or say that they satisfy all the conditions, whatever they are, required in

order that this may be so, then the formula determines the distribution, viz. it gives the value of ϱ, the density at a point $(x, \ldots z, w)$ of the surface.

13. In the case where the surface is the plane $w=0$, viz. in the case of the potential-plane integral,

$$V=\int\frac{\varrho dx \ldots dz}{\{(a-x)^2 \ldots +(c-z)^2+e^2\}^{\frac{1}{2}s-\frac{1}{2}}} \quad \ldots \ldots \ldots \quad \text{(B)}$$

(e assumed to be positive); then, since every thing is symmetrical on the two sides of the plane, V' and V'' are the same functions of $(a \ldots c, e)$, say they are each$=V$; W', W'' are each of them the same function, say they are each $=W$, of $(x \ldots z, e)$ that V is of $(a \ldots c, e)$, and the distribution-formula becomes

$$\varrho=-\frac{\Gamma(\frac{1}{2}s-\frac{1}{2})}{2(\Gamma\frac{1}{2})^{s+1}}\left(\frac{dW}{de}\right)_{e=0}, \quad \ldots \ldots \ldots \quad \text{(B)}$$

viz. this is also what one of the prepotential-plane formulæ becomes on writing therein $q=-\frac{1}{2}$.

$q=0$, *or Negative.*—Nos. 14 to 18.

14. Consider the case $q=0$. The prepotential of the disk is

$$\varrho \cdot \frac{2(\Gamma\frac{1}{2})^s}{\Gamma\frac{1}{2}s}(\log R+N-\log \varepsilon \ldots);$$

and to get rid of the constant term we must consider the derived function in regard to ε, viz. this is

$$=-\varrho\frac{2(\Gamma\frac{1}{2})^s}{\Gamma\frac{1}{2}s}\cdot\frac{1}{\varepsilon},$$

and we have thus for the whole surface

$$\frac{dV}{d\varepsilon}=Q-\varrho\frac{2(\Gamma\frac{1}{2})^s}{\Gamma\frac{1}{2}s}\frac{1}{\varepsilon},$$

where Q, which relates to the remaining portion of the surface, is finite; we have thence, writing, as before, W in place of V,

$$\varepsilon\frac{dW}{d\varepsilon}=-\varrho\frac{2(\Gamma\frac{1}{2})^s}{\Gamma\frac{1}{2}s},$$

or say

$$\varrho=-\frac{\Gamma\frac{1}{2}s}{2(\Gamma\frac{1}{2})^s}\left(\varepsilon\frac{dW}{d\varepsilon}\right)_{\varepsilon=0}.$$

15. Consider the case q negative, but $-q<\frac{1}{2}$. The prepotential of the disk is here

$$=\varrho\frac{2(\Gamma\frac{1}{2})^s}{\Gamma\frac{1}{2}s}\varrho\left\{\frac{R^{-2q}}{-2q}+\frac{1}{2}\varepsilon^{-2q}\frac{\Gamma\frac{1}{2}s\,\Gamma q}{\Gamma(\frac{1}{2}s+q)}+\ldots\right\};$$

and to get rid of the first term we must consider the derived function in regard to ε, viz. this is

$$-\varepsilon^{-2q-1}\varrho\frac{2(\Gamma\frac{1}{2})^s\Gamma(q+1)}{\Gamma(\frac{1}{2}s+q)};$$

whence for the potential of the whole surface

$$\frac{dV}{d\varepsilon}=Q-\varepsilon^{-2q-1}\,\varrho\,\frac{2(\Gamma\tfrac{1}{2})^s\,\Gamma(q+1)}{\Gamma(\tfrac{1}{2}s+q)},$$

where Q, the part relating to the remaining portion of the surface, is finite. Multiplying by ε^{2q+1} (where the index $2q+1$ is positive), the term in Q disappears; and writing, as before, W in place of V, this is

$$\varepsilon^{2q+1}\frac{dW}{d\varepsilon}=-\varrho\,\frac{2(\Gamma\tfrac{1}{2})^s\,\Gamma(q+1)}{\Gamma\tfrac{1}{2}s+q},$$

or say

$$\varrho=-\frac{\Gamma(\tfrac{1}{2}s+q)}{2(\Gamma\tfrac{1}{2})^s\,\Gamma(q+1)}\left(\varepsilon^{2q+1}\frac{dW}{d\varepsilon}\right)_{\varepsilon=0};$$

viz. we thus see that the formula (A*) originally obtained for the case q positive extends to the case $q=0$, and $q=-$, but $-q<\tfrac{1}{2}$; but, as already seen, it does not extend to the limiting case $q=\ \tfrac{1}{2}$.

16. If q be negative and between $-\tfrac{1}{2}$ and -1, we have in like manner a formula

$$\frac{dV}{d\varepsilon}=Q-\varrho\,\frac{2(\Gamma\tfrac{1}{2})^s\,\Gamma(q+1)}{\Gamma(\tfrac{1}{2}s+q)}\,\varepsilon^{-2q-1};$$

but here $2q+1$ being negative, the term $\varepsilon^{2q-1}Q$ does not disappear: the formula has to be treated in the same way as for $q=-\tfrac{1}{2}$, and we arrive at

$$\left\{\varepsilon'^{2q+1}\frac{dW'}{d\varepsilon'}+\varepsilon''^{2q+1}\frac{dW''}{d\varepsilon''}\right\}=-\frac{4(\Gamma\tfrac{1}{2})^s\,\Gamma(q+1)}{\Gamma(\tfrac{1}{2}s+q)},\ \varrho;$$

viz. the formula is of the same form as for the potential case $q=-\tfrac{1}{2}$. Observe that the formula does not hold good in the limiting case $q=-1$.

17. We have, in fact, here the potential of the disk

$$=\frac{2(\Gamma\tfrac{1}{2})^s}{\Gamma(\tfrac{1}{2}s)}\,\varrho\left\{\frac{R^2}{2}-\varepsilon^2\log\varepsilon\,\frac{\Gamma\tfrac{1}{2}s}{\Gamma(\tfrac{1}{2}s-1)}\right\};$$

whence

$$\frac{dV}{d\varepsilon}=Q-\frac{2(\Gamma\tfrac{1}{2})^s}{\Gamma(\tfrac{1}{2}s-1)}\,\varrho(2\varepsilon\log\varepsilon),$$

since in the complete differential coefficient $\varepsilon+2\varepsilon\log\varepsilon$ the term ε vanishes in comparison with $2\varepsilon\log\varepsilon$; and then, proceeding as before, we find

$$\frac{1}{\varepsilon'\log\varepsilon'}\,\frac{dW'}{d\varepsilon'}+\frac{1}{\varepsilon''\log\varepsilon''}\,\frac{dW''}{d\varepsilon''}=\frac{-8(\Gamma\tfrac{1}{2})^s}{\Gamma(\tfrac{1}{2}s-1)}\,\varrho;$$

but I have not particularly examined this formula.

18. If q be negative and >-1 (that is, $-q>1$), then the prepotential for the disk is

$$=\varrho\,\frac{(\Gamma\tfrac{1}{2})^s}{\Gamma\tfrac{1}{2}s}\left\{\frac{R^{-2q}}{-2q}+\frac{\tfrac{1}{2}s+q}{1}\,\frac{R^{-2q-2}}{-2q-2}\cdot\varepsilon^2\ldots+K\varepsilon^{-2q}\right\};$$

and it would seem that in order to obtain a result it would be necessary to proceed to a derived function higher than the first; but I have not examined the case.

Continuity of the Prepotential-surface Integral.—Art. Nos. 19 to 25.

19. I again consider the prepotential-surface integral

$$\int \frac{\varrho dS}{\{(a-x)^2 \ldots + (c-z)^2 + (e-w)^2\}^{\frac{1}{2}s-q}}$$

in regard to a point $(a \ldots c, e)$ not on the surface; q is either positive or negative, as afterwards mentioned.

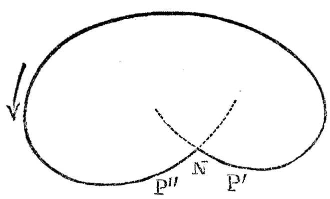
The integral or prepotential and all its derived functions, first, second, &c. *ad infinitum*, in regard to each or all or any of the coordinates $(a \ldots c, e)$ are all finite. This is certainly the case when the mass $\int \varrho dS$ is finite, and possibly in other cases also; but to fix the ideas we may assume that the mass is finite. And the prepotential and its derived functions vary continuously with the position of the attracted point $(a \ldots c, e)$, so long as this point in its course does not traverse the material surface. For greater clearness we may consider the point as moving along a continuous curve (one-dimensional locus), which curve, or the part of it under consideration, does not meet the surface; and the meaning is that the prepotential and each of its derived functions varies continuously as the point $(a \ldots c, e)$ passes continuously along the curve.

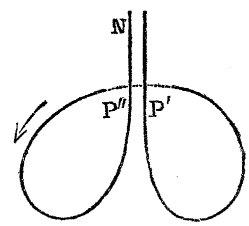
20. Consider a "region," that is, a portion of space any point of which can be by a continuous curve not meeting the material surface connected with any other point of the region. It is a legitimate inference, from what just precedes, that the prepotential is, for any point $(a \ldots c, e)$ whatever within the region, one and the same function of the coordinates $(a \ldots c, e)$, viz. the theorem, rightly understood, is true; but the theorem gives rise to a difficulty, and needs explanation.

Consider, for instance, a closed surface made up of two segments, the attracting matter being distributed in any manner over the whole surface (as a particular case $s+1=3$, a uniform spherical shell made up of two hemispheres); then, as regards the first segment (now taken as the material surface), there is no division into regions, but the whole of the $(s+1)$dimensional space is one region; wherefore the prepotential of the first segment is one and the same function of the coordinates $(a \ldots c, e)$ of the attracted point for any position whatever of this point. But in like manner the prepotential of the second segment is one and the same function of the coordinates $(a \ldots c, e)$ for any position whatever of the attracted point. And the prepotential of the whole surface, being the sum of the prepotentials of the two segments, is consequently one and the same function of the coordinates $(a \ldots c, e)$ of the attracted point for any position whatever of this point; viz. it is the same function for a point in the region inside the closed surface and for a point in the outside region. That this is not in general the case we know from the particular case, $s+1=3$, of a uniform spherical shell referred to above.

21. Consider in general an unclosed surface or segment, with matter distributed over it in any manner; and imagine a closed curve or circuit cutting the segment once; and let the attracted point $(a \ldots c, e)$ move continuously along the circuit. We may consider the circuit as corresponding to (in ordinary tridimensional space) a plane curve of

equal periphery, the corresponding points on the circuit and the plane curve being points at equal distances s along the curves from fixed points on the two curves respectively; and then treating the plane curve as the base of a cylinder, we may represent the potential as a length or ordinate, $V=y$, measured upwards from the point on the plane curve along the generating line of the cylinder, in such wise that the upper extremity of the length or ordinate y traces out on the cylinder a curve, say the prepotential curve, which represents the march of the prepotential. The attracted point may, for greater convenience, be represented as a point *on* the prepotential curve, viz. by the upper instead of the lower extremity of the length or ordinate y; and the ordinate, or height of this point above the base of the cylinder, then represents the value of the prepotential. The before-mentioned continuity-theorem is that the prepotential curve corresponding to any portion (of the circuit) which does not meet the material surface is a continuous curve, viz. that there is no abrupt change of value either in the ordinate $y(=V)$ of the prepotential curve, or in the first or any other of the derived functions $\frac{dy}{ds}, \frac{d^2y}{ds^2}$, &c. We have thus (in each of the two figures) a continuous curve as we pass



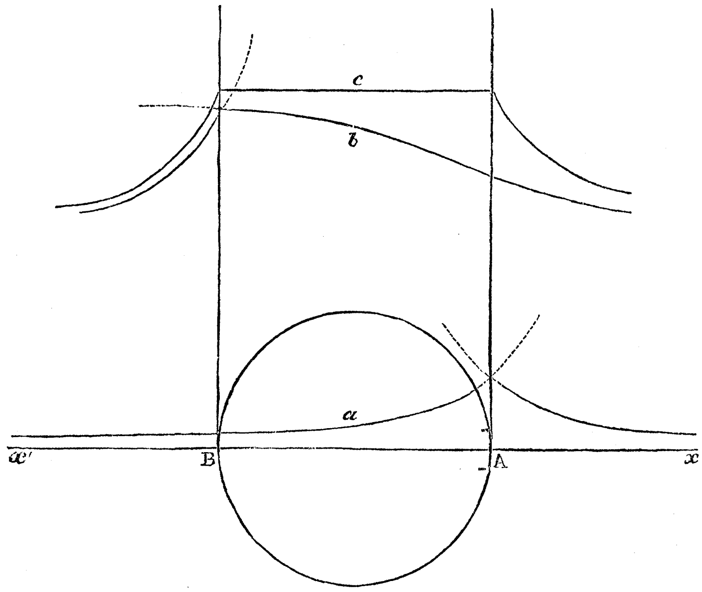


(in the direction of the arrow) from a point P$'$ on one side of the segment to a point P$''$ on the other side of the segment; but this continuity does not exist in regard to the remaining part, from P$''$ to P$'$, of the prepotential curve corresponding to the portion (of the circuit) which traverses the material surface.

22. I consider first the case $q=-\frac{1}{2}$ (see the left-hand figure): the prepotential is here a potential. At the point N, which corresponds to the passage through the material surface, then, as was seen, the ordinate y (=the Potential V) remains finite and continuous; but there is an abrupt change in the value of $\frac{dy}{ds}$, that is, in the direction of the curve: the point N is really a node with two branches crossing at this point, as shown in the figure; but the dotted continuations have only an analytical existence, and do not represent values of the potential. And by means of this branch-to-branch discontinuity at the point N, we escape from the foregoing conclusion as to the continuity of the potential on the passage of the attracted point through a closed surface.

23. To show how this is I will for greater clearness examine the case $(s+1)=3$, in ordinary tridimensional space, of the uniform spherical shell attracting according to the inverse square of the distance; instead of dividing the shell into hemispheres, I divide it by a plane into any two segments (see the figure, wherein A, B represent the

centres of the two segments respectively, and where for graphical convenience the segment A is taken to be small.



We may consider the attracted point as moving along the axis xx', viz. the two extremities may be regarded as meeting at infinity, or we may outside the sphere bend the line round, so as to produce a closed circuit. We are only concerned with what happens at the intersections with the spherical surface. The ordinates represent the potentials, viz. the curves are a, b, c for the segments A, B, and the whole spherical surface respectively. Practically, we construct the curves c, a, and deduce the curve b by taking for its ordinate the difference of the other two ordinates. The curve c is, as we know, a discontinuous curve, composed of a horizontal line and two hyperbolic branches; the curve a can be laid down approximately by treating the segment A as a plane circular disk; it is of the form shown in the figure, having a node at the point corresponding to A. [In the case where the segment A is actually a plane disk, the curve is made up of portions of branches of two hyperbolas; but taking the segment A as being what it is, the segment of a spherical surface, the curve is a single curve, having a node as mentioned above.] And from the curves c and a, deducing the curve b, we see that this is a curve without any discontinuity corresponding to the passage of the attracted point through A (but with an abrupt change of direction or node corresponding to the passage through B). And conversely, using the curves a, b to determine the curve c, we see how, on the passage of the attracted point at A into the interior of the sphere, in consequence of the branch-to-branch discontinuity of the curve a, the curve c, obtained by combination of the two curves, undergoes a change of law, passing abruptly from a hyperbolic to a rectilinear form, and how similarly on the passage of the attracted point at B from the interior to the exterior of the sphere, in consequence of the branch-to-branch discontinuity of the curve b, the curve c again undergoes a change of law, abruptly reverting to the hyperbolic form.

24. In the case q positive the prepotential curve is as shown by the right-hand figure in p. 688, viz. the ordinate is here infinite at the point N corresponding to the passage through the surface; the value of the derived function changes between + infinity and − infinity; and there is thus a discontinuity of value in the derived function. It would seem that when q is fractional this occasions a change of law on passage through the surface, but that there is no change of law when q is integral.

In illustration, consider the closed surface as made up of an infinitesimal circular disk, as before, and of a residual portion; the potential of the disk on an indefinitely near point is found as before, and the prepotential of the whole surface is

$$=\frac{1}{\mathfrak{s}^{2q}}\,\varrho\,\frac{(\Gamma\tfrac{1}{2})^s\Gamma q}{\Gamma(\tfrac{1}{2}s+q)}+\mathrm{V}_1,$$

where V_1, the prepotential of the remaining portion of the surface, is a function which varies (and its derived functions vary) continuously as the attracted point traverses the disk. To fix the ideas we may take the origin at the centre of the disk, and the axis of e as coinciding with the normal, so that $\mathfrak{s}$, which is always positive, is $=\pm e$; and the expression for the prepotential at a point $(a\ldots c,\,e)$ on the normal through the centre of the disk is

$$=\frac{1}{(\pm e)^{2q}}\cdot\varrho\,\frac{(\Gamma\tfrac{1}{2})^s.\,\Gamma q}{\Gamma(\tfrac{1}{2}s+q)}+\mathrm{V}_1,$$

viz. when q is fractional there is the discontinuity of law, inasmuch as the term changes from $\frac{1}{(+e)^{2q}}$ to $\frac{1}{(-e)^{2q}}$; but when q is integral this discontinuity disappears. The like considerations, using of course the proper formula for the attraction of the disk, would apply to the case $q=0$ or negative.

25. Or again, we might use the formulæ which belong to the case of a uniform $(s+1)$-coordinal spherical shell (see Annex No. III.), viz. we decompose the surface as follows,

$$\text{surface}=\text{disk}+\text{residue of surface};$$

and then, considering a spherical shell touching the surface at the point in question (so that the disk is in fact an element common to the surface and the spherical shell), and being of a uniform density equal to that of the disk, we have

$$\text{disk}=\text{spherical shell}-\text{residue of spherical shell};$$

and consequently

$$\text{surface}=\text{spherical shell}-\text{residue of spherical shell}+\text{residue of surface};$$

and then, considering the attracted point as passing through the disk, it does not pass through either of the two residues, and there is not any discontinuity, as regards the prepotentials of these residues respectively; there is consequently, as regards the prepotential of the surface, the same discontinuity that there is as regards the prepotential of the spherical shell. But I do not further consider the question from this point of view.

The Potential Solid Integral.—Art. No. 26.

26. We have further to consider the prepotential (and in particular the potential) of a material space; to fix the ideas, consider for the moment the case of a distribution over the space included within a closed surface, the exterior density being zero, and the interior density being, suppose for the moment, constant; we consider the discontinuity which takes place as the attracting point passes from the exterior space through the bounding surface into the interior material space. We may imagine the interior space divided into indefinitely thin shells by a series of closed surfaces similar, if we please, to the bounding surface; and we may conceive the matter included between any two consecutive surfaces as concentrated on the exterior of the two surfaces, so as to give rise to a series of consecutive material surfaces; the quantity of such matter is infinitesimal, and the density of each of the material surfaces is therefore also infinitesimal. As the attracted point comes from the external space to pass through the first of the material surfaces—suppose, to fix the ideas, it moves continuously along a curve the arc of which measured from a fixed point is $=s$—there is in the value of V (or, as the case may be, in the values of its derived functions $\frac{dV}{ds}$, &c.) the discontinuity due to the passage through the material surface; and the like as the attracted point passes through the different material surfaces respectively. Take the case of a potential, $q=-\frac{1}{2}$; then, if the surface-density were finite, there would be no finite change in the value of V, but there would be a finite change in the value of $\frac{dV}{ds}$; as it is, the changes are to be multiplied by the infinitesimal density, say ϱ, of the material surface; there is consequently no finite change in the value of the first derived function; but there is, or may be, a finite change in the value of $\frac{d^2V}{ds^2}$ and the higher derived functions. But there is in V an infinitesimal change corresponding to the passage through the successive material surfaces respectively; that is, as the attracted point enters into the material space there is a change in the law of V considered as a function of the coordinates $(a \ldots c, e)$ of the attracted point; but by what precedes this change of law takes place without any abrupt change of value either of V or of its first derived function; which derived function may be considered as representing the derived function in regard to any one of the coordinates $a \ldots c, e$. The suppositions that the density outside the bounding surface was zero and inside it constant, were made for simplicity only, and were not essential; it is enough if the density, changing abruptly at the bounding surface, varies continuously in the material space within the bounding surface*. The

* It is, indeed, enough if the density varies continuously within the bounding surface in the neighbourhood of the point of passage through the surface; but the condition may without loss of generality be stated as in the text, it being understood that for each abrupt change of density within the bounding surface we must consider the attracted point as passing through a new bounding surface, and have regard to the resulting discontinuity.

conclusion is that V′, V″ being the values at points within and without the bounding surface, V′ and V″ are in general different functions of the coordinates $(a \ldots c, e)$ of the attracting point; but that *at* the surface we have not only $V'=V''$, but that the first derived functions are also equal, viz. that we have

$$\frac{dV'}{da}=\frac{dV''}{da}, \ldots \frac{dV'}{dc}=\frac{dV''}{dc}, \quad \frac{dV'}{de}=\frac{dV''}{de}.$$

27. In the general case of a Potential,

$$V=\int\frac{\rho\, dx \ldots dz\, dw}{\{(a-x)^2 \ldots +(c-z)^2+(e-w)^2\}^{\frac{1}{2}s-\frac{1}{2}}};$$

if ϱ does not vanish at the attracted point $(a \ldots c, e)$, but has there a value ϱ' different from zero, we may consider the attracting $(s+1)$dimensional mass as made up of an indefinitely small sphere, radius ε and density ϱ', which includes within it the attracted point, and of a remaining portion external to the attracted point. Writing ∇ to denote $\frac{d^2}{da^2} \ldots +\frac{d^2}{dc^2}+\frac{d^2}{de^2}$, then, as regards the potential of the sphere, we have $\nabla V=-\frac{4(\Gamma\frac{1}{2})^{s+1}}{\Gamma(\frac{1}{2}s-\frac{1}{2})}\varrho'$ (see Annex III. No. 67), and as regards the remaining portion $\nabla V=0$; hence, as regards the whole attracting mass, ∇V has the first-mentioned value, that is we have

$$\left(\frac{d^2}{da^2} \ldots +\frac{d^2}{dc^2}+\frac{d^2}{de^2}\right)V=-\frac{4(\Gamma\frac{1}{2})^{s+1}}{\Gamma(\frac{1}{2}s-\frac{1}{2})}\varrho',$$

where ϱ' is the same function of the coordinates $(a \ldots c, e)$ that ϱ is of $(x \ldots z, w)$; viz. the potential of an attracting mass distributed not on a surface, but over a portion of space, *does not satisfy the potential equation*

$$\left(\frac{d^2}{da^2} \ldots +\frac{d^2}{dc^2}+\frac{d^2}{de^2}\right)V=0,$$

but it satisfies the foregoing equation, which only agrees with the potential equation in regard to a point $(a \ldots c, e)$ outside the material space, and for which, therefore, ϱ' is $=0$.

The equation may be written

$$\varrho'=-\frac{\Gamma(\frac{1}{2}s-\frac{1}{2})}{4(\Gamma\frac{1}{2})^{s+1}}\left(\frac{d^2}{da^2} \ldots +\frac{d^2}{dc^2}+\frac{d^2}{de^2}\right)V;$$

or, considering V as a given function of $(a \ldots c, e)$, in general a discontinuous function but subject to certain conditions as afterwards mentioned, and taking W the same function of $(x \ldots z, w)$ that V is of $(a \ldots c, e)$, then we have

$$\varrho=-\frac{\Gamma(\frac{1}{2}s-\frac{1}{2})}{4(\Gamma\frac{1}{2})^{s+1}}\left(\frac{d^2}{dx^2} \ldots +\frac{d^2}{dz^2}+\frac{d^2}{dw^2}\right)W, \qquad \text{(D)}$$

viz. this equation determines ρ as a function, in general a discontinuous function, of $(x \ldots z, w)$ such that the corresponding integral

$$V=\int \frac{\rho\, dx \ldots dz\, dw}{\{(a-x)^2 \ldots +(c-z)^2+(e-w)^2\}^{\frac{1}{2}s+q}}$$

may be the given function of the coordinates $(a \ldots c, e)$. The equation is, in fact, the distribution-theorem D.

28. It is to be observed that the given function of $(a \ldots c, e)$ must satisfy certain conditions as to value at infinity and continuity, but it is not (as in the distribution-theorems A, B, and C it is) required to satisfy a partial differential equation; the function, except as regards the conditions as to value at infinity and continuity, is absolutely arbitrary.

The potential (assuming that the matter which gives rise to it lies wholly within a finite closed surface) must vanish for points at an infinite distance, or more accurately it must for indefinitely large values of $a^2 \ldots +c^2+e^2$ be of the form, Constant $\div$ by $(a^2 \ldots +c^2+e^2)^{\frac{1}{2}s-\frac{1}{2}}$. It may be a discontinuous function; for instance outside a given closed surface it may be one function, and inside the same surface a different function of the coordinates $(a \ldots c, e)$; viz. this may happen in consequence of an abrupt change of the density of the attracting matter on the one and the other side of the given closed surface, but not in any other manner; and, happening in this manner, then V', V'' being the values for points within and without the surface respectively, it has been seen to be necessary that, at the surface, not only $V'=V''$, but also $\frac{dV'}{da}=\frac{dV''}{da} \ldots \frac{dV'}{dc}=\frac{dV''}{dc}, \frac{dV'}{de}=\frac{dV''}{de}$. Subject to these conditions as to value at infinity and continuity, V may be any function whatever of the coordinates $(a \ldots c, e)$; and then taking W, the same function of $(x \ldots z, w)$, the foregoing equation determines ρ, viz. determines it to be $=0$ for those parts of space which do not belong to the material space, and to have its proper value as a function of $(x \ldots z, w)$ for the remaining or material space.

The Prepotential Plane Theorem A.—Art. Nos. 29 to 36.

29. We have seen that if there exists on the plane $w=0$ a distribution of matter producing at the point $(a \ldots c, e)$ a given prepotential V (viz. V is to be regarded as a given function of $(a \ldots c, e)$), then that the distribution or density ρ is given by a determinate formula; but it was remarked that the prepotential V cannot be a function assumed at pleasure; it must be a function satisfying certain conditions. One of these is the condition of continuity; the function V and all its derived functions must vary continuously as we pass, without traversing the material plane, from any given point to any other given point. But it is sufficient to attend to points on one side of the plane, say the upperside, or that for which e is positive; and since any such point is accessible from any other such point by a path which does not meet the plane, it is sufficient to say that the function V must vary continuously for a passage by such path from any such point to any such point; the function V must therefore be one and the same

function (and that a continuous one in value) for all values of the coordinates $(a \ldots c)$ and positive values of the coordinate e.

If, moreover, we assume that the distribution which corresponds to the given potential V is a distribution of a finite mass $\int \rho\, dx \ldots dz$ over a finite portion of the plane $w=0$, viz. over a portion or area such that the distance of a point within the area from a fixed point, or say from the origin $(a \ldots c) = (0 \ldots 0)$, is always finite; this being so, we have the further condition that the prepotential V must for indefinitely large values of all or any of the coordinates $(a \ldots c, e)$ reduce itself to the form

$$(\int \rho\, dx \ldots dz) \div (a^2 \ldots + c^2 + e^2)^{\frac{1}{2}s+q}.$$

The assumptions upon which this last condition is obtained are perhaps unnecessary; instead of the condition in the foregoing form we, in fact, use only the condition that the prepotential vanishes for a point at infinity, that is when all or any one or more of the coordinates $(a \ldots c, e)$ are or is infinite.

Again, as we have seen, the prepotential V must satisfy the prepotential equation

$$\left(\frac{d^2}{da^2} \ldots + \frac{d^2}{dc^2} + \frac{d^2}{de^2} + \frac{e}{2q+1}\,\frac{d}{de}\right)\mathrm{V}=0.$$

These conditions satisfied, to the given prepotential V, there corresponds on the plane $w=0$, a distribution given by the foregoing formula, and which will be a distribution over a finite portion of the plane, as already mentioned.

30. The proof depends upon properties of the prepotential equation,

$$\left(\frac{d^2}{dx^2} \ldots + \frac{d^2}{dz^2} + \frac{d^2}{de^2} + \frac{2q+1}{e}\,\frac{d}{de}\right)\mathrm{W}=0,$$

or, what is the same thing,

$$\frac{d}{dx}\left(e^{2q+1}\frac{d\mathrm{W}}{dx}\right) \ldots + \frac{d}{dz}\left(e^{2q+1}\frac{d\mathrm{W}}{dz}\right) + \frac{d}{de}\left(e^{2q+1}\frac{d\mathrm{W}}{de}\right) = 0,$$

say, for shortness, $\square \mathrm{W}=0$.

Consider, in general, the integral

$$\int dx \,..\, dz\, de\, e^{2q+1}\left\{\left(\frac{d\mathrm{W}}{dx}\right)^2 \ldots + \left(\frac{d\mathrm{W}}{dz}\right)^2 + \left(\frac{d\mathrm{W}}{de}\right)^2\right\}$$

taken over a closed surface S lying altogether on the positive side of the plane $e=0$, the function W being in the first instance arbitrary.

Writing the integral under the form

$$\int dx \ldots dz\, de \left(e^{2q+1}\frac{d\mathrm{W}}{dx}\cdot\frac{d\mathrm{W}}{dx} \ldots + e^{2q+1}\frac{d\mathrm{W}}{dz}\cdot\frac{d\mathrm{W}}{dz} + e^{2q+1}\frac{d\mathrm{W}}{de}\cdot\frac{d\mathrm{W}}{de}\right),$$

we reduce the several terms by an integration by parts as follows:—

$$\text{term in } \frac{dW}{dx} \text{ is } = \int dy \ldots dz\, de\, We^{2q+1}\frac{dW}{dx} - \int dx \ldots dz\, de\, W\frac{d}{dx}\left(e^{2q+1}\frac{dW}{dx}\right),$$

$$\vdots$$

$$\frac{dW}{dz} \text{ is } = \int dx \ldots\ldots de\, We^{2q+1}\frac{dW}{dz} - \int dx \ldots dz\, de\, W\frac{d}{dz}\left(e^{2q+1}\frac{dW}{dz}\right),$$

$$\frac{dW}{de} \text{ is } = \int dx \ldots\ldots dz\, We^{2q+1}\frac{dW}{de} - \int dx \ldots dz \quad W\frac{d}{de}\left(e^{2q+1}\frac{dW}{de}\right).$$

Write dS to denote an element of surface at the point $(x \ldots z, e)$; and taking $\alpha \ldots \gamma, \delta$ to denote the inclinations of the interior normal at that point to the positive axes of coordinates, we have

$$\begin{aligned} dy \ldots dz\, de &= -dS \cos\alpha, \\ &\vdots \\ dx \ldots\ldots de &= -dS \cos\gamma, \\ dx \ldots\ldots dz &= -dS \cos\delta\,; \end{aligned}$$

and the first terms are together

$$= -\int e^{2q+1}\,W\left(\frac{dW}{dx}\cos\alpha \ldots + \frac{dW}{dz}\cos\gamma + \frac{dW}{de}\cos\delta\right) dS,$$

W here denoting the value at the surface, and the integration being extended over the whole of the closed surface: this may also be written

$$= -\int e^{2q+1}\,W\frac{dW}{d\mathfrak{s}}\,dS,$$

where $\mathfrak{s}$ denotes an element of the internal normal.

The second terms are together

$$= -\int dx \ldots dz\, de\, W\left\{\frac{d}{dx}\left(e^{2q+1}\frac{dW}{dx}\right) \ldots + \frac{d}{dz}\left(e^{2q+1}\frac{dW}{dz}\right) + \frac{d}{de}\left(e^{2q+1}\frac{dW}{de}\right)\right\} = -\int dx\,.\,dz\,de W \square W.$$

We have consequently

$$\int dx \ldots dz\, de\, e^{2q+1}\left\{\left(\frac{dW}{dx}\right)^2 \ldots + \left(\frac{dW}{dz}\right)^2 + \left(\frac{dW}{de}\right)^2\right\}$$

$$= -\int e^{2q+1}W\frac{dW}{d\mathfrak{s}}\,dS - \int dx \ldots dz\, de\, e^{2q+1}W \square W.$$

31. The second term vanishes if W satisfies the prepotential equation $\square W=0$; and this being so, if also $W=0$ for all points of the closed surface S, then the first term also vanishes, and we therefore have

$$\int dx \ldots dz\, de\, e^{2q+1}\left\{\left(\frac{dW}{dx}\right)^2 \ldots + \left(\frac{dW}{dz}\right)^2 + \left(\frac{dW}{de}\right)^2\right\} = 0,$$

where the integration extends over the whole space included within the closed surface; whence, W being a real function,

$$\frac{dW}{dx}=0, \ \ldots \ \frac{dW}{dz}=0, \ \frac{dW}{de}=0,$$

for all points within the closed surface; consequently, since W vanishes at the surface, $W=0$ for all points within the closed surface.

32. Considering W as satisfying the equation $\square W=0$, we may imagine the closed surface to become larger and larger, and ultimately infinite, at the same time flattening itself out into coincidence with the plane $e=0$, so that it comes to include the whole space above the plane $e=0$, say the surface breaks up into the surface positive infinity and the infinite plane $e=0$.

The integral $\int e^{2q+1} W \frac{dW}{d\mathfrak{z}} dS$ separates itself into two parts, the first relating to the surface positive infinity, and which vanishes if $W=0$ at infinity (that is, if all or any of the coordinates $x \ldots z, e$ are infinite); the second relating to the plane $e=0$ is $\int W\left(e^{2q+1}\frac{dW}{de}\right) dx \ldots dz$, W here denoting its value at the plane, that is when $e=0$, and the integral being extended over the whole plane. The theorem thus becomes

$$\int dx \ldots dz\, de \,.\, e^{2q+1}\left\{\left(\frac{dW}{dx}\right)^2 \ldots + \left(\frac{dW}{dz}\right)^2 + \left(\frac{dW}{de}\right)^2\right\}$$
$$= -\int W\left(e^{2q+1}\frac{dW}{de}\right) dx \ldots dz.$$

Hence also if $W=0$ at all points of the plane $e=0$, the right-hand side vanishes, and we have

$$\int dx \ldots dz\, de\, e^{2q+1}\left\{\left(\frac{dW}{dx}\right)^2 \ldots + \left(\frac{dW}{dz}\right)^2 + \left(\frac{dW}{de}\right)^2\right\}=0.$$

Consequently $\frac{dW}{dx}=0 \ldots \frac{dW}{dz}=0$, $\frac{dW}{de}=0$, for all points whatever of positive space; and therefore also $W=0$ for all points whatever of positive space.

33. Take next U, W, each of them a function of $(x \ldots z, e)$, and consider the integral

$$\int dx \ldots dz\, de\, e^{2q+1}\left(\frac{dU}{dx}\frac{dW}{dx} \ldots + \frac{dU}{dz}\frac{dW}{dz} + \frac{dU}{de}\frac{dW}{de}\right),$$

taken over the space within a closed surface S; treating this in a similar manner, we find it to be

$$= -\int e^{2q+1} W \frac{dU}{d\mathfrak{z}} dS - \int dx \ldots dz\, de\, e^{2q+1} W \square U,$$

where the integration extends over the whole of the closed surface S; and by parity of reasoning it is also

$$= -\int e^{2q+1} U \frac{dW}{d\mathfrak{z}} dS - \int dx \ldots dz\, de\, e^{2q+1} U \square W,$$

with the same limits of integration; that is, we have

$$\int e^{2q+1} W \frac{dU}{d\mathfrak{z}} dS + \int dx \ldots dz\, de \,.\, e^{2q+1} W \square U = \int e^{2q+1} U \frac{dW}{d\mathfrak{z}} dS + \int dx \ldots dz\, de \,.\, e^{2q+1} U \square W,$$

which, if U, W each satisfy the prepotential equation, becomes

$$\int e^{2q+1}\mathrm{W}\frac{d\mathrm{U}}{d\mathfrak{z}}\,d\mathrm{S}=\int e^{2q+1}\mathrm{U}\frac{d\mathrm{W}}{d\mathfrak{z}}\,d\mathrm{S}.$$

And if we now take the closed surface S to be the surface positive infinity, together with the plane $e=0$, then, provided only U and V vanish at infinity, for each integral the portion belonging to the surface positive infinity vanishes, and there remains only the portion belonging to the plane $e=0$; we have therefore

$$\int e^{2q+1}\mathrm{W}\frac{d\mathrm{U}}{de}\,dx\ldots dz=\int e^{2q+1}\mathrm{U}\frac{d\mathrm{W}}{de}\,dx\ldots dz,$$

where the functions U, W have each of them the value belonging to the plane $e=0$, viz. in U, W considered as given functions of $(x\ldots z,\ e)$ we regard e as a positive quantity ultimately put $=0$, and where the integrations extend each of them over the whole infinite plane.

34. Assume

$$\mathrm{U}=\frac{1}{\{(a-x)^2\ldots+(c-z)^2+e^2\}^{\frac{1}{2}s+q}},$$

an expression which, regarded as a function of $(x\ldots z,\ e)$, satisfies the prepotential equation in regard to these variables, and which vanishes at infinity when all or any of these coordinates $(x\ldots z,\ e)$ are infinite.

We have

$$\frac{d\mathrm{U}}{de}=\frac{-2(\frac{1}{2}s+q)e}{\{(a-x)^2\ldots+(c-z)^2+e^2\}^{\frac{1}{2}s+q+1}};$$

and we have consequently

$$\int\mathrm{W}\frac{-2(\frac{1}{2}s+q)e^{2q+2}}{\{(a-x)^2\ldots+(c-z)^2+e^2\}^{\frac{1}{2}s+q+1}}\,dx\ldots dz=\int\left(e^{2q+1}\frac{d\mathrm{W}}{de}\right)\frac{dx\ldots dz}{\{(a-x)^2\ldots+(c-z)^2+e^2\}^{\frac{1}{2}s+q}},$$

where it will be recollected that e is ultimately $=0$; to mark this we may for W write W_0.

Attend to the left-hand side; take V_0 the same function of $a\ldots c,\ e=0$, that W_0 is of $x\ldots z,\ e=0$; then first writing the expression in the form

$$\mathrm{V}_0\int\frac{-2(\frac{1}{2}s+q)e^{2q+2}dx\ldots dz}{\{(a-x)^2\ldots+(c-z)^2+e^2\}^{\frac{1}{2}s+q+1}},$$

write $x=a+e\xi\ldots z=c+e\zeta$, the expression becomes

$$=\mathrm{V}_0\int\frac{-2(\frac{1}{2}s+q)e^{2q+2}.\,e^s d\xi\ldots d\zeta}{\{e^2(1+\xi^2\ldots+\zeta^2)\}^{\frac{1}{2}s+q+1}},\quad=-2(\tfrac{1}{2}s+q)\mathrm{V}_0\int\frac{d\xi\ldots d\zeta}{\{1+\xi^2\ldots+\zeta^2\}^{\frac{1}{2}s+q+1}},$$

where the integral is to be taken from $-\infty$ to $+\infty$ for each of the new variables $\xi\ldots\zeta$.

Writing $\xi=r\alpha\ldots\zeta=r\gamma$, where $\alpha^2\ldots+\gamma^2=1$, we have $d\xi\ldots d\zeta=r^{s-1}dr\,d\mathrm{S}$ also $\xi^2\ldots+\zeta^2=r^2$, and the integral is

$$=\int\frac{r^{s-1}dr\,d\mathrm{S}}{(1+r^2)^{\frac{1}{2}s+q+1}},\quad=\int d\mathrm{S}\int_0^\infty\frac{r^{s-1}dr}{(1+r^2)^{\frac{1}{2}s+q+1}},$$

where $\int dS$ denotes the surface of the s-coordinal unit sphere $\alpha^2\ldots+\gamma^2=1$, and the r-integral is to be taken from $r=0$ to $r=\infty$; the values of the two factors thus are

$$\int dS=\frac{2(\Gamma\frac{1}{2})^s}{\Gamma(\frac{1}{2}s)}, \text{ and } \int\frac{r^{s-1}dr}{(1+r^2)^{\frac{1}{2}s+q+1}}=\frac{\frac{1}{2}\Gamma\frac{1}{2}s\Gamma(q+1)}{\Gamma(\frac{1}{2}s+q+1)}.$$

Hence the expression in question is

$$-2(\tfrac{1}{2}s+q)V_0\frac{2(\Gamma\frac{1}{2})^s}{\Gamma\frac{1}{2}s}\frac{\frac{1}{2}\Gamma\frac{1}{2}s\Gamma(q+1)}{\Gamma(\frac{1}{2}s+q+1)}, \quad =\frac{-2(\Gamma\frac{1}{2})^s\Gamma(q+1)}{\Gamma(\frac{1}{2}s+q)}V_0,$$

and we have

$$\int\left(e^{2q+1}\frac{dW}{de}\right)_0\frac{dx\ldots dz}{\{(a-x)^2\ldots+(c-z)^2+e^2\}^{\frac{1}{2}s+q}}=\frac{-2(\Gamma\frac{1}{2})^s\Gamma(q+1)}{\Gamma(\frac{1}{2}s+q)}V_0;$$

or, what is the same thing,

$$V_0=\int\frac{\frac{-\Gamma(\frac{1}{2}s+q)}{2(\Gamma\frac{1}{2})^s\Gamma(q+1)}\left(e^{2q+1}\frac{dW}{de}\right)_0}{\{(a-x)^2\ldots+(c-z)^2+e^2\}^{\frac{1}{2}s+q}}.$$

35. Take now V a function of $(a\ldots c, e)$ satisfying the prepotential equation in regard to these variables, always finite, and vanishing at infinity, and let W be the same function of $(x\ldots z, e)$, W therefore satisfying the prepotential equation in regard to the last-mentioned variables, and consider the function

$$V-\int\frac{-\frac{\Gamma(\frac{1}{2}s+q)}{2(\Gamma\frac{1}{2})^s\Gamma(q+1)}\left(e^{2q+1}\frac{dW}{de}\right)_0dx\ldots dz}{\{(a-x)^2\ldots+(c-z)^2+e^2\}^{\frac{1}{2}s+q}},$$

where the integral is taken over the infinite plane $e=0$; then this function (V — the integral) satisfies the prepotential equation (for each term separately satisfies it), is always finite, and it vanishes at infinity. It also, as has just been seen, vanishes for any point whatever of the plane $e=0$. Consequently it vanishes for all points whatever of positive space. Or, what is the same thing, if we write

$$V=\int\frac{\varrho\, dx\ldots dz}{\{(a-x)^2\ldots+(c-z)^2+e^2\}^{\frac{1}{2}s+q}}, \quad \ldots\ldots\ldots \quad \text{(A)}$$

where ϱ is a function of $(x\ldots z)$, and the integral is taken over the whole infinite plane, then if V is a function of $(a\ldots c, e)$ satisfying the above conditions, there exists a corresponding value of ϱ; viz. taking W the same function of $(x\ldots z, e)$ which V is of $(a\ldots c, e)$, the value of ϱ is

$$\varrho=-\frac{\Gamma(\frac{1}{2}s+q)}{2(\Gamma\frac{1}{2})^s\Gamma(q+1)}\left(e^{2q+1}\frac{dW}{de}\right)_0, \quad \ldots\ldots\ldots \quad \text{(A)}$$

where e is to be put $=0$ in the function $e^{2q+1}\frac{dW}{de}$. This is the prepotential-plane theorem; viz. taking for the prepotential in regard to a given point $(a, \ldots c, e)$ a function of $(a\ldots c, e)$ satisfying the prescribed conditions, but otherwise arbitrary, there exists on the plane $e=0$ a distribution ϱ given by the last-mentioned formula.

36. It is assumed in the proof that $2q+1$ is positive or zero; viz. q is positive, or if negative then $-q \not> \frac{1}{2}$; the limiting case $q=-\frac{1}{2}$ is included.

It is to be remarked that by what precedes, if q be positive (but excluding the case $q=0$) the density ϱ is given by the equivalent more simple formula

$$\varrho=\frac{\Gamma(\frac{1}{2}s+q)}{(\Gamma\frac{1}{2})^s\Gamma q}(e^{2q}\mathrm{W})_0.$$

The foregoing proof is substantially that given in GREEN'S memoir on the Attraction of Ellipsoids; it will be observed that the proof only imposes upon V the condition of vanishing at infinity, without obliging it to assume for large values of $(a\ldots c, e)$ the form $\frac{\mathrm{M}}{\{a^2\ldots+c^2+e^2\}^{\frac{1}{2}s+q}}$.

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37. In the case $q=-\frac{1}{2}$, writing here $\nabla=\frac{d^2}{dx^2}\ldots+\frac{d^2}{dz^2}+\frac{d^2}{de^2}$, we have precisely, as in the general case,

$$\int \mathrm{W}\frac{d\mathrm{U}}{d\mathfrak{s}}d\mathrm{S}+\int dx\ldots dz\, de\, \mathrm{W}\nabla\mathrm{U}=\int \mathrm{U}\frac{d\mathrm{W}}{d\mathfrak{s}}d\mathrm{S}+\int dx\ldots dz\, de\, \mathrm{U}\nabla\mathrm{W};$$

and if the functions U, W satisfy the equations $\nabla\mathrm{U}=0$, $\nabla\mathrm{W}=0$, then (subject to the exception presently referred to) the second terms on the two sides respectively each of them vanish.

But, instead of taking the surface to be the surface positive infinity together with the plane $e=0$, we now leave it an arbitrary closed surface, and for greater symmetry of notation write w in place of e; and we suppose that the functions U and W, or one of them, may become infinite at points within the closed surface; on this last account the second terms do not in every case vanish.

38. Suppose, for instance, that U at a point indefinitely near the point $(a\ldots c, e)$ within the surface becomes

$$=\frac{1}{\{(x-a)^2\ldots+(z-c)^2+(w-e)^2\}^{\frac{1}{2}s-\frac{1}{2}}};$$

then if V be the value of W at the point $(a\ldots c, e)$, we have

$$\int dx\ldots dz\, dw\, \mathrm{W}\nabla\mathrm{U}=\mathrm{V}\int dx\ldots dz\, dw\, \nabla\mathrm{U};$$

and since $\nabla\mathrm{U}=0$, except at the point in question, the integral may be taken over any portion of space surrounding this point, for instance, over the space included within the sphere, radius R, having the point $(a\ldots c, e)$ for its centre; or taking the origin at this point, we have to find $\int dx\ldots dz\, dw\, \nabla\mathrm{U}$, where

$$\mathrm{U}=\frac{1}{\{x^2\ldots+z^2+w^2\}^{\frac{1}{2}s-\frac{1}{2}}},$$

and the integration extends over the space within the sphere $x^2\ldots+z^2+w^2=\mathrm{R}^2$.

39. This may be accomplished most easily by means of a particular case of the last-mentioned theorem; viz. writing $\mathrm{W}=1$, we have

$$\int \frac{d\mathrm{U}}{d\mathfrak{s}}\, d\mathrm{S} + \int dx \ldots dz\, dw\, \nabla \mathrm{U} = 0,$$

or the required value is $= -\int \frac{d\mathrm{U}}{d\mathfrak{s}}\, d\mathrm{S}$ over the surface of the last-mentioned sphere. We have, if for a moment $r^2 = x^2 \ldots + z^2 + w^2$,

$$\frac{d\mathrm{U}}{d\mathfrak{s}} = -\left(\frac{x}{r}\frac{d}{dx} \ldots + \frac{z}{r}\frac{d}{dz} + \frac{w}{r}\frac{d}{dw}\right)\mathrm{U}, \quad = -\left(\frac{x}{r}\frac{d}{dx} \ldots + \frac{z}{r}\frac{d}{dz} + \frac{w}{r}\frac{d}{dw}\right) r \,.\, \frac{d\mathrm{U}}{dr}, \quad = -\frac{d\mathrm{U}}{dr},$$

that is, $\frac{d\mathrm{U}}{d\mathfrak{s}} = \frac{s-1}{r^s}, = \frac{s-1}{\mathrm{R}^s}$; and hence

$$\int \frac{d\mathrm{U}}{d\mathfrak{s}}\, d\mathrm{S} = \frac{s-1}{\mathrm{R}^s} \int d\mathrm{S},$$

where $\int d\mathrm{S}$ is the whole surface of the sphere $x^2 \ldots + z^2 + w^2 = \mathrm{R}^2$, viz. it is $= \mathrm{R}^s$ into the surface of the unit-sphere $x^2 \ldots + z^2 + w^2 = 1$. This spherical surface, say

$$\int d\Sigma \text{ is } = \frac{2(\Gamma\tfrac{1}{2})^{s+1}}{\Gamma\tfrac{1}{2}(s+1)}, = \frac{4(\Gamma\tfrac{1}{2})^{s+1}}{(s-1)\Gamma\tfrac{1}{2}(s-1)},$$

and we have thus $\int \frac{d\mathrm{U}}{d\mathfrak{s}}\, d\mathrm{S} = \frac{4(\Gamma\frac{1}{2})^{s+1}}{\Gamma\frac{1}{2}(s-1)}$, and consequently

$$\int dx \ldots dz\, dw\, \nabla \mathrm{U} = -\frac{4(\Gamma\tfrac{1}{2})^{s+1}}{\Gamma(\tfrac{1}{2}s - \tfrac{1}{2})}.$$

40. Treating in like manner the case where W at a point indefinitely near the point $(\mathrm{a}, \ldots \mathrm{c}, \mathrm{e})$ within the surface becomes

$$= \frac{1}{\{(x-\mathrm{a})^2 \ldots + (z-\mathrm{c})^2 + (w-\mathrm{e})^2\}^{\frac{1}{2}s - \frac{1}{2}}},$$

and writing T to denote the same function of $(\mathrm{a}, \ldots \mathrm{c}, \mathrm{e})$ that U is of $(x \ldots z, w)$, we have, instead of the foregoing, the more general theorem

$$\int \mathrm{W} \frac{d\mathrm{U}}{d\mathfrak{s}}\, d\mathrm{S} + \int dx \ldots dz\, dw\, \mathrm{W}\nabla\mathrm{U} - \frac{4(\Gamma\tfrac{1}{2})^{s+1}}{\Gamma(\tfrac{1}{2}s - \tfrac{1}{2})}\mathrm{V}$$
$$= \int \mathrm{U} \frac{d\mathrm{W}}{d\mathfrak{s}}\, d\mathrm{S} + \int dx \ldots dz\, dw\, \mathrm{U}\nabla\mathrm{W} - \frac{4(\Gamma\tfrac{1}{2})^{s+1}}{\Gamma(\tfrac{1}{2}s - \tfrac{1}{2})}\mathrm{T},$$

where in the two solid integrals respectively we exclude from consideration the space in the immediate neighbourhood of the two critical points $(a \ldots c, e)$ and $(\mathrm{a} \ldots \mathrm{c}, \mathrm{e})$ respectively.

Suppose that W is always finite within the surface, and that U is finite except at the point $(a \ldots c, e)$, and moreover that U, W are such that $\nabla\mathrm{U}=0$, $\nabla\mathrm{W}=0$, then the equation becomes

$$\int \mathrm{W} \frac{d\mathrm{U}}{d\mathfrak{s}}\, d\mathrm{S} - \frac{4(\Gamma\tfrac{1}{2})^{s+1}}{\Gamma\tfrac{1}{2}s - \tfrac{1}{2}}\mathrm{V} = \int \mathrm{U} \frac{d\mathrm{W}}{d\mathfrak{s}}\, d\mathrm{S}.$$

In particular this equation holds good if U is $=\dfrac{1}{\{(a-x)^2\ldots+(e-w)^2\}^{\frac{1}{2}s-\frac{1}{2}}}$.

41. Imagine now on the surface S a distribution ρdS producing at a point $(a'\ldots c', e')$ within the surface a potential V$'$, and at a point $(a''\ldots c'', e'')$ without the surface a potential V$''$; where, by what precedes, V$''$ is in general not the same function of $(a''\ldots c'', e'')$ that V$'$ is of $(a'\ldots c', e')$.

It is further assumed that at a point $(a\ldots c, e)$ on the surface we have V$'$=V$''$:

that V$'$, or any of its derived functions, are not infinite for any point $(a'\ldots c', e')$ within the surface:

that V$''$, or any of its derived functions, are not infinite for any point $(a''\ldots c'', e'')$ without the surface:

and that V$''$=0 for any point at infinity.

Consider V$'$ as a given function of $(a\ldots c, e)$; and take W$'$ the same function of $(x\ldots z, w)$. Then if, as before,

$$\mathrm{U}=\frac{1}{\{(a-x)^2\ldots+(c-z)^2+(e-w)^2\}^{\frac{1}{2}s-\frac{1}{2}}},$$

$$\left(\frac{d^2}{dx^2}\cdots+\frac{d^2}{de^2}+\frac{d^2}{dw^2}\right)\mathrm{U}=0,$$

then

$$\int \mathrm{U}\frac{d\mathrm{W}'}{ds'}\,d\mathrm{S}=\int \mathrm{W}'\frac{d\mathrm{U}}{ds'}\,d\mathrm{S}-\frac{4(\Gamma\frac{1}{2})^{s+1}}{\Gamma(\frac{1}{2}s-\frac{1}{2})}\mathrm{V}'.$$

Similarly, considering V$''$ as a given function of $(a\ldots c, e)$ and take W$''$ the same function of $(x\ldots z, e)$. Then, by considering the space outside the surface S, or say between this surface and infinity, and observing that U does not become infinite for any point in this space, we have

$$\int \mathrm{U}\frac{d\mathrm{W}''}{ds''}\,d\mathrm{S}=\int \mathrm{W}''\frac{d\mathrm{U}}{ds''}\,d\mathrm{S};$$

and adding these two equations, we have

$$\int \mathrm{U}\left(\frac{d\mathrm{W}'}{ds'}+\frac{d\mathrm{W}''}{ds''}\right)d\mathrm{S}=\int\left(\mathrm{W}'\frac{d\mathrm{U}}{ds'}+\mathrm{W}''\frac{d\mathrm{U}}{ds''}\right)d\mathrm{S}-\frac{4(\Gamma\frac{1}{2})^{s+1}}{\Gamma(\frac{1}{2}s-\frac{1}{2})}\mathrm{V}'.$$

But in this equation the functions W$'$ and W$''$ each of them belong to a point $(x\ldots z, w)$ on the surface, and we have at the surface W$'$=W$''$, =W suppose; the term on the right-hand side thus is $\int \mathrm{W}\left(\frac{d\mathrm{U}}{ds'}+\frac{d\mathrm{U}}{ds''}\right)d\mathrm{S}$, which vanishes in virtue of $\frac{d\mathrm{U}}{ds'}+\frac{d\mathrm{U}}{ds''}=0$; and the equation thus becomes

$$\int \mathrm{U}\left(\frac{d\mathrm{W}'}{ds'}+\frac{d\mathrm{W}''}{ds''}\right)d\mathrm{S}=-\frac{4(\Gamma\frac{1}{2})^{s+1}}{\Gamma(\frac{1}{2}s-\frac{1}{2})}\mathrm{V}':$$

that is, the point $(a\ldots c, e)$ being interior, we have

$$\mathrm{V}'=\int\frac{-\Gamma(\frac{1}{2}s-\frac{1}{2})}{4(\Gamma\frac{1}{2})^{s+1}}\left(\frac{d\mathrm{W}'}{ds'}+\frac{d\mathrm{W}''}{ds''}\right)\frac{d\mathrm{S}}{\{(a-x)^2\ldots+(c-z)^2+(e-w)^2\}^{\frac{1}{2}s-\frac{1}{2}}}.$$

In exactly the same way if $(a \ldots c, e)$ be an exterior point, then we have

$$\int \mathrm{U}\frac{d\mathrm{W}'}{d\mathfrak{s}'}\,d\mathrm{S}=\int \mathrm{W}'\frac{d\mathrm{U}}{d\mathfrak{s}'}\,d\mathrm{S},$$

$$\int \mathrm{U}\frac{d\mathrm{W}''}{d\mathfrak{s}''}\,d\mathrm{S}=\int \mathrm{W}''\frac{d\mathrm{U}}{d\mathfrak{s}''}\,d\mathrm{S}-\frac{4(\Gamma\frac{1}{2})^{s+1}}{\Gamma(\frac{1}{2}s-\frac{1}{2})}\mathrm{V}'';$$

and adding, and omitting the terms which vanish,

$$\int \mathrm{U}\left(\frac{d\mathrm{W}'}{d\mathfrak{s}'}+\frac{d\mathrm{W}''}{d\mathfrak{s}''}\right)d\mathrm{S}=-\frac{4(\Gamma\frac{1}{2})^{s+1}}{\Gamma(\frac{1}{2}s-\frac{1}{2})}\mathrm{V}'',$$

that is,

$$\mathrm{V}''=\int \frac{-\Gamma(\frac{1}{2}s-\frac{1}{2})}{4(\Gamma\frac{1}{2})^{s+1}}\left(\frac{d\mathrm{W}'}{d\mathfrak{s}'}+\frac{d\mathrm{W}''}{d\mathfrak{s}''}\right)\frac{d\mathrm{S}}{\{(a-x)^2\ldots(c-z)^2+(e-w)^2\}^{\frac{1}{2}s-\frac{1}{2}}}.$$

42. Comparing the two results with

$$\mathrm{V}=\int \frac{\rho\,d\mathrm{S}}{\{(a-x)^2\ldots+(c-z)^2+(e-w)^2\}^{\frac{1}{2}s-\frac{1}{2}}},$$

we see that V′, V″ satisfying the foregoing conditions, there exists a distribution ρ on the surface, producing the potentials V′ and V″ at an interior point and an exterior point respectively; the value of ρ in fact being

$$\rho=-\frac{\Gamma(\frac{1}{2}s-\frac{1}{2})}{4(\Gamma\frac{1}{2})^{s+1}}\left(\frac{d\mathrm{W}'}{d\mathfrak{s}'}+\frac{d\mathrm{W}}{d\mathfrak{s}''}\right), \quad \ldots \ldots \ldots \quad \text{(C)}$$

where W′, W″ are respectively the same functions of $(x \ldots z, w)$ that V′, V″ are of $(a \ldots c, e)$.

The Potential-solid Theorem D.—Art. No. 43.

43. We have as before (No. 40),

$$\int \mathrm{W}\frac{d\mathrm{U}}{d\mathfrak{s}}\,d\mathrm{S}+\int dx \ldots dz\,dw\,\mathrm{W}\nabla\mathrm{U}-\frac{4(\Gamma\frac{1}{2})^{s+1}}{\Gamma(\frac{1}{2}s-\frac{1}{2})}\mathrm{V}$$

$$=\int \mathrm{U}\frac{d\mathrm{W}}{d\mathfrak{s}}\,d\mathrm{S}+\int dx \ldots dz\,dw\,\mathrm{U}\nabla\mathrm{W}-\frac{4(\Gamma\frac{1}{2})^{s+1}}{\Gamma(\frac{1}{2}s-\frac{1}{2})}\mathrm{T},$$

where, assuming first that W is not infinite for any point $(x \ldots z, w)$ whatever, we have no term in T; and taking next $\mathrm{U}=\frac{1}{\{(a-x)^2\ldots+(c-z)^2+(e-w)^2\}^{\frac{1}{2}s-\frac{1}{2}}}$ as before, we have $\nabla\mathrm{U}=0$; the equation thus becomes

$$\int \mathrm{W}\frac{d\mathrm{U}}{d\mathfrak{s}}\,d\mathrm{S}-\int \mathrm{U}\frac{d\mathrm{W}}{d\mathfrak{s}}\,d\mathrm{S}-\frac{4(\Gamma\frac{1}{2})^{s+1}}{\Gamma(\frac{1}{2}s-\frac{1}{2})}\mathrm{V}=\int dx \ldots dz\,dw\,\mathrm{U}\nabla\mathrm{W},$$

where W may be a discontinuous function of the coordinates $(x \ldots z, w)$, provided only there is no abrupt change in the value either of W or of any of its first derived functions $\frac{d\mathrm{W}}{dx} \ldots \frac{d\mathrm{W}}{dz}, \frac{d\mathrm{W}}{dw}$, viz. it may be any function which can represent the potential of a solid mass on an attracted point $(x \ldots z, w)$; the resulting value of $\nabla\mathrm{W}$ is of course discon-

tinuous. Taking, then, for the closed surface S the boundary of infinite space, U and W each vanish at this boundary, and the equation becomes

$$-\frac{(\Gamma\frac{1}{2})^{s+1}}{\Gamma(\frac{1}{2}s-\frac{1}{2})}\text{V}=\int dx\ldots dz\,dw\,\text{U}\nabla\text{W};$$

viz. substituting for U its value, and comparing with

$$\text{V}=\int\frac{\varrho\,dx\ldots dz\,dw}{\{(a-x)^2\ldots+(c-z)^2+(e-w)^2\}^{\frac{1}{2}s-\frac{1}{2}}},$$

where the integral in the first instance extends to the whole of infinite space, but the limits may be ultimately restricted by ϱ being $=0$, we see that the value of ϱ is

$$\varrho=-\frac{\Gamma(\frac{1}{2}s-\frac{1}{2})}{(\Gamma\frac{1}{2})^{s+1}}\left(\frac{d^2}{dx^2}\ldots+\frac{d^2}{dz^2}+\frac{d^2}{dw^2}\right)\text{W},$$

W being the same function of $(x\ldots z, w)$ that V is of $(a\ldots c, e)$, which is the theorem D.

Examples of the foregoing Theorems.—Art. Nos. 44 to 49.

44. It will be remarked, as regards all the theorems, that we do not start with known limits; we start with V a function of $(a\ldots c, e)$, the coordinates of the attracted point, satisfying certain prescribed conditions, and we thence find ϱ, a function of the coordinates $(x\ldots z)$ or $(x\ldots z, w)$, as the case may be, which function is found to be $=0$ for values of $(x\ldots z)$ or $(x\ldots z, w)$ lying beyond certain limits, and to have a determinate non-evanescent value for values of $(x\ldots z)$ or $(x\ldots z, w)$ lying within these limits; and we thus, as a result, obtain these limits for the limits of the multiple integral V.

45. Thus in theorem A, in the example where the limiting equation is ultimately found to be $x^2\ldots+z^2=f^2$, we start with V a certain function of $a^2\ldots+c^2$ $(=\varkappa^2$ suppose) and e^2, viz. V is a function of these quantities through θ, which denotes the positive root of the equation

$$\frac{\varkappa^2}{f^2+\theta}+\frac{e^2}{\theta}=1,$$

the value in fact being $\text{V}=\int_\theta^\infty t^{-q-1}(t+f^2)^{-\frac{1}{2}s}dt$, and the resulting value of ϱ is found to be $=0$ for values of $(x\ldots z)$ for which $x^2\ldots+z^2>f^2$. Hence V denotes an integral

$$\int\frac{\varrho\,dx\ldots dz}{\{(a-x)^2\ldots+(c-z)^2+e^2\}^{\frac{1}{2}s+q}},$$

the limiting equation being $x^2\ldots+z^2=f^2$, say this is the s-coordinal sphere.

And similarly, in the examples where the limiting equation is ultimately found to be $\frac{x^2}{f^2}\ldots+\frac{z^2}{h^2}=1$, we start with V a certain function of $a,\ldots c, e$ through θ (or directly and through θ), where θ denotes the positive root of the equation

$$\frac{a^2}{f^2+\theta}\ldots+\frac{c^2}{h^2+\theta}+\frac{e^2}{\theta}=1,$$

and the resulting value of ϱ is found to be $=0$ for values of $(x \ldots z)$ for which

$$\frac{x^2}{f^2} \ldots + \frac{z^2}{h^2} > 1.$$

Hence V denotes an integral,

$$\int \frac{\varrho \, dx \ldots dz}{\{(a-x)^2 \ldots + (c-z)^2 + e^2\}^{\frac{1}{2}s+q}},$$

the limiting equation being $\frac{x^2}{f^2} \ldots + \frac{z^2}{h^2} = 1$, say this is the s-coordinal ellipsoid. It is clear that this includes the before-mentioned case of the s-coordinal sphere; but it is, on account of the more simple form of the θ-equation, worth while to work out directly an example for the sphere.

46. Three examples are worked out in Annex IV.; the results are as follows:—

First, θ defined for the sphere as above; $q+1$ positive;

$$\mathrm{V} = \int \frac{\left(1 - \frac{x^2 \ldots + z^2}{f^2}\right)^q dx \ldots dz}{\{(a-x)^2 \ldots + (c-z)^2 + e^2\}^{\frac{1}{2}s+q}}$$

over the sphere $x^2 \ldots + y^2 = f^2$,

$$= \frac{(\Gamma\frac{1}{2})^s \Gamma(q+1)}{\Gamma(\frac{1}{2}s+q)} f^s \int_\theta^\infty t^{-q-1} (t+f^2)^{-\frac{1}{2}s} \, dt.$$

This is included in the next-mentioned example for the ellipsoid.

Secondly, θ defined for the ellipsoid as above; $q+1$ positive;

$$\mathrm{V} = \int \frac{\left(1 - \frac{x^2}{f^2} \ldots - \frac{z^2}{h^2}\right)^q dx \ldots dz}{\{(a-x)^2 \ldots + (c-z)^2 + e^2\}^{\frac{1}{2}s+q}}$$

over the ellipsoid $\frac{x^2}{f^2} \ldots + \frac{z^2}{h^2} = 1$,

$$= \frac{(\Gamma\frac{1}{2})^s \Gamma(q+1)}{\Gamma(\frac{1}{2}s+q)} (f \ldots h) \int_\theta^\infty t^{-q-1} (t+f^2 \ldots t+h^2)^{-\frac{1}{2}} \, dt.$$

This result is included in the next-mentioned example; but the proof for the general value of m is not directly applicable to the value $m=0$ for the case in question.

Thirdly, θ and the ellipsoid as above; $q+1$ positive; $m=0$ or positive, and apparently in other cases,

$$\mathrm{V} = \int \frac{\left(1 + \frac{x^2}{f^2} \ldots - \frac{z^2}{h^2}\right)^{q+m} dx \ldots dz}{\{(a-x)^2 \ldots + (c-z)^2 + e^2\}^{\frac{1}{2}s+q}}$$

over the ellipsoid as above,

$$= \frac{(\Gamma\frac{1}{2})^s \Gamma(1+q+m)}{\Gamma(\frac{1}{2}s+q)\Gamma(1+m)} (f \ldots h) \int_0^\infty \left(1 - \frac{a^2}{f^2+\theta} \ldots - \frac{c^2}{h^2+\theta} - \frac{e^2}{\theta}\right)^m t^{-q-1} (t+f^2 \ldots t+h^2)^{-\frac{1}{2}} dt.$$

And we have in Annex V. a fourth example; here θ and the ellipsoid are as above: the result involves the Greenian functions.

47. We may in the foregoing results write $e=0$; the results, writing therein $s+1$ for s, and in the new forms taking $(a \ldots c, e)$ and $(x \ldots z, w)$ for the two sets of coordinates respectively, also writing $q-\frac{1}{2}$ for q, would give integrals of the form

$$\int \frac{\rho dx \ldots dz\, dw}{\{(a-x)^2 \ldots + (c-z)^2 + (e-w)^2\}^{\frac{1}{2}s+q}}$$

for the $(s+1)$coordinal sphere and ellipsoid $x^2 \ldots + z^2 + w^2 = f^2$ and $\frac{x^2}{f^2} \ldots + \frac{z^2}{h^2} + \frac{w^2}{k^2} = 1$; say these are prepotential solid integrals; and then, writing $q=-\frac{1}{2}$, we should obtain potential solid integrals, such as are also given by the theorem D. The change can be made if necessary; but it is more convenient to retain the results in their original forms, as relating to the s-coordinal sphere and ellipsoid.

There are two cases, according as the attracted point $(a \ldots c)$ is external or internal.

For the sphere:—For an external point $\varkappa^2 > f^2$; writing $e=0$, the equation $\frac{\varkappa^2}{f^2+\theta}=1$ has a positive root, viz. this is $\theta=\varkappa^2-f^2$; and θ will have, or it may be replaced by, this value $\varkappa^2-f^2$: for an internal point $\varkappa^2 < f^2$; as e approaches zero, the positive root of the original equation gradually diminishes and becomes ultimately $=0$, viz. in the formulæ θ is to be replaced by this value 0.

For the ellipsoid:—For an external point $\frac{a^2}{f^2} \ldots + \frac{c^2}{h^2} > 1$; writing $e=0$, the equation $\frac{a^2}{\theta+f^2} \ldots + \frac{c^2}{\theta+h^2} = 1$ has a positive root, and θ will denote this positive root: for an internal point $\frac{a^2}{f^2} \ldots + \frac{c^2}{h^2} < 1$; as e approaches zero the positive root of the original equation gradually diminishes and becomes ultimately $=0$, viz. in the formulæ θ is to be replaced by this value 0.

The resulting formulæ for the sphere $x^2 \ldots + z^2 = f^2$ may be compared with formulæ for the spherical shell, Annex VI., and each set with formulæ obtained by direct integration in Annex III.

We may in any of the formulæ write $q=-\frac{1}{2}$, and so obtain examples of theorem B.

48. As regards theorem C, we might in like manner obtain examples of potentials relating to the surfaces of the $(s+1)$coordinal sphere $x^2 \ldots + z^2 + w^2 = f^2$, and ellipsoid $\frac{x^2}{f^2} \ldots + \frac{z^2}{h^2} + \frac{w^2}{k^2} = 1$, or say to spherical and ellipsoidal shells; but I have confined myself to the sphere. We have to assume values V′ and V″ belonging to the cases of an internal and an external point respectively, and thence to obtain a value ϱ, or distribution over the spherical surface, which shall produce these potentials respectively. The result (see Annex VI.) is

$$\int \frac{dS}{\{(a-x)^2 + \ldots + (c-z)^2 + (e-w)^2\}^{\frac{1}{2}s-\frac{1}{2}}}$$

over the surface of the $(s+1)$coordinal sphere $x^2\ldots+z^2+w^2=f^2$,

$$=\frac{2(\Gamma\frac{1}{2})^{s+1}f^s}{\Gamma(\frac{1}{2}s+\frac{1}{2})}\frac{1}{\varkappa^{s-1}} \text{ for exterior point } \varkappa>f$$

and

$$=\frac{2(\Gamma\frac{1}{2})^{s+1}f^s}{\Gamma(\frac{1}{2}s+\frac{1}{2})}\frac{1}{f^{s-1}} \text{ for interior point } \varkappa<f,$$

where $\varkappa^2=a^2\ldots+c^2+e^2$. Observe that for the interior point the potential is a mere constant multiple of f.

The same Annex VI. contains the case of the s-coordinal cylinder $x^2\ldots+z^2=f^2$, which is peculiar in that the cylinder is not a finite closed surface, but the theorem C is found to extend to it.

49. As regards theorem D, we might in like manner obtain potentials relating to the $(s+1)$coordinal sphere $x^2\ldots+z^2+w^2=f^2$ and ellipsoid $\frac{x^2}{f^2}\ldots+\frac{z^2}{h^2}+\frac{w^2}{k^2}=1$; but I confine myself to the case of the sphere (see Annex VII.). We here assume values V′ and V″ belonging to an internal and an external point respectively, and thence obtain a value ρ, or distribution over the whole $(s+1)$dimensional space, which density is found to be $=0$ for points outside the sphere. The result obtained is

$$\mathrm{V}=\int\frac{dx\ldots dz\,dw}{\{(a-x)^2\ldots+(c-z)^2+(e-w)^2\}^{\frac{1}{2}s-\frac{1}{2}}}$$

over $(s+1)$coordinal sphere $x^2\ldots+z^2+w^2=f^2$,

$$=\frac{(\Gamma\frac{1}{2}s)^{s+1}}{\Gamma(\frac{1}{2}s+\frac{3}{2})}\cdot\frac{f^{s+1}}{\varkappa^{s-1}} \text{ for exterior point } \varkappa>f$$

$$=\frac{(\Gamma\frac{1}{2}s)^{s+1}}{\Gamma(\frac{1}{2}s+\frac{3}{2})}\{(\tfrac{1}{2}s+\tfrac{1}{2})f^2-(\tfrac{1}{2}s-\tfrac{1}{2})\varkappa^2\} \text{ for interior point } \varkappa<f,$$

where $\varkappa^2=a^2\ldots+c^2+e^2$.

The remaining Annexes VIII. and IX. have no immediate reference to the theorems A, B, C, D, which are the principal objects of the memoir. The subjects to which they relate will be seen from the headings and introductory paragraphs.

ANNEX I. *Surface and Volume of Sphere* $x^2\ldots+z^2+w^2=f^2$.—Nos. 51 & 52.

51. We require in $(s+1)$dimensional space, $\int dx\ldots dz\,dw$, the volume of the sphere $x^2\ldots+z^2+w^2=f^2$, and $\int d\mathrm{S}$, the surface of the same sphere.

Writing $x=f\sqrt{\xi}\ldots z=f\sqrt{\zeta}$, $w=f\sqrt{\omega}$, we have

$$dx\ldots dz\,dw=\frac{1}{2^{s+1}}f^{s+1}\xi^{-\frac{1}{2}}\ldots\zeta^{-\frac{1}{2}}\omega^{-\frac{1}{2}}d\xi\ldots d\zeta\,d\omega,$$

with the limiting condition $\xi\ldots+\zeta+\omega=1$; but in order to take account as well of the negative as the positive values of $x\ldots z, w$, we must multiply by 2^{s+1}. The value is therefore

$$=f^{s+1}\int\xi^{-\frac{1}{2}}\ldots\zeta^{-\frac{1}{2}}\omega^{-\frac{1}{2}}d\xi\ldots d\zeta\,d\omega,$$

extended to all positive values of $\xi\ldots\zeta, \omega$, such that $\xi\ldots+\zeta+\omega<1$; and we obtain this by a known theorem, viz.

$$\text{Volume of } (s+1)\text{dimensional sphere} = f^{s+1}\frac{(\Gamma\tfrac{1}{2})^{s+1}}{\Gamma(\tfrac{1}{2}s+\tfrac{3}{2})}.$$

Writing $x=f\xi\ldots z=f\zeta$, $w=f\omega$, we obtain $dS=f^s\,d\Sigma$, where $d\Sigma$ is the element of surface of the unit-sphere $\xi^2\ldots+\zeta^2+\omega^2=1$; we have element of volume $d\xi\ldots d\zeta\,d\omega=r^s\,dr\,d\Sigma$, where r is to be taken from 0 to 1, and thence

$$\int d\xi\ldots d\zeta\,d\omega=\int_0^1 r^s\,dr\,.\int d\Sigma=\frac{1}{s+1}\int d\Sigma,$$

that is,

$$\int d\Sigma=(s+1)\int d\xi\ldots d\zeta\,d\omega,\ =2(\tfrac{1}{2}s+\tfrac{1}{2})\frac{(\Gamma\tfrac{1}{2})^{s+1}}{\Gamma(\tfrac{1}{2}s+\tfrac{3}{2})}=\frac{2(\Gamma\tfrac{1}{2})^{s+1}}{\Gamma(\tfrac{1}{2}s+\tfrac{1}{2})};$$

consequently $\int dS=$ surface of $(s+1)$dimensional sphere$=f^s\dfrac{2(\Gamma\tfrac{1}{2})^{s+1}}{\Gamma(\tfrac{1}{2}s+\tfrac{1}{2})}$.

52. Writing $s-1$ for s, we have

$$\text{Volume of } (s-1)\text{dimensional sphere}=f^s\frac{(\Gamma\tfrac{1}{2})^s}{\Gamma(\tfrac{1}{2}s+1)},$$

$$\text{Surface of}\qquad \text{do.}\qquad =f^{s-1}\frac{2(\Gamma\tfrac{1}{2})^s}{\Gamma(\tfrac{1}{2}s)},$$

which forms are sometimes convenient.

Writing in the first forms $s+1=3$, or in the second forms $s=3$, we find in ordinary space

$$\text{Volume of sphere}=f^3\frac{(\Gamma\tfrac{1}{2})^3}{\Gamma(\tfrac{5}{2})}=f^3\frac{\pi^{\frac{3}{2}}}{\tfrac{3}{2}\cdot\tfrac{1}{2}\cdot\sqrt{\pi}},\ =\frac{4\pi f^3}{3},$$

and

$$\text{Surface of sphere}=f^2\frac{2(\Gamma\tfrac{1}{2})^3}{\Gamma\tfrac{3}{2}}=f^2\frac{2\pi^{\frac{3}{2}}}{\tfrac{1}{2}\sqrt{\pi}},\qquad =4\pi f^2,$$

as they should be.

ANNEX II. *The Integral* $\int_0^R \frac{r^{s-1}dr}{(r^2+e^2)^{\frac{1}{2}s+q}}$.—Nos. 53 to 63.

53. The integral in question (which occurs *antè*, No. 2) may also be considered as arising from a prepotential integral in tridimensional space; the prepotential of an element of mass dm is taken to be$=\dfrac{dm}{d^{s+2q}}$, where d is the distance of the element from the attracted point P. Hence if the element of mass be an element of the plane $z=0$, coordinates (x, y), ρ being the density, and if the attracted point be situate in the axis of z at a distance e from the origin, the prepotential is

$$V=\int\frac{\rho\,dx\,dy}{(x^2+y^2+e^2)^{\frac{1}{2}s+q}}.$$

For convenience it is assumed throughout that e is positive.

Suppose that the attracting body is a circular disk radius R, having the origin for its centre (viz. that bounded by the curve $x^2+y^2=\mathrm{R}^2$); then writing $x=r\cos\theta, y=r\sin\theta$, we have

$$\mathrm{V}=\int\frac{\varrho\, r\, dr\, d\theta}{(r^2+e^2)^{\frac{1}{2}s+q}},$$

which, if ϱ is a function of r only, is

$$=2\pi\int\frac{\varrho r\, dr}{(r^2+e^2)^{\frac{1}{2}s+q}};$$

and in particular if $\varrho=r^{s-2}$, then the value is

$$=2\pi\int\frac{r^{s-1}\, dr}{(r^2+e^2)^{\frac{1}{2}s+q}},$$

the integral in regard to r being taken from $r=0$ to $r=\mathrm{R}$. It is assumed that $s-1$ is not negative, viz. it is positive or (it may be) zero. I consider the integral

$$\int_0^{\mathrm{R}}\frac{r^{s-1}\, dr}{(r^2+e^2)^{\frac{1}{2}s+q}},$$

which I call the r-integral, more particularly in the case where e is small in comparison with R. It is to be observed that e not being $=0$, and R being finite, the integral contains no infinite element, and is therefore finite, whether q is positive, negative, or zero.

54. Writing $r=e\sqrt{v}$, the integral is

$$=\tfrac{1}{2}e^{-2q}\int\frac{v^{\frac{1}{2}s-1}\, dv}{(1+v)^{\frac{1}{2}s+q}},$$

the limits being $\frac{\mathrm{R}^2}{e^2}$ and 0.

In the case where q is positive this is

$$=\tfrac{1}{2}e^{-2q}\left(\int_0^\infty-\int_{\frac{\mathrm{R}^2}{e^2}}^\infty\right)\frac{v^{\frac{1}{2}s-1}\, dv}{(1+v)^{\frac{1}{2}s+q}};$$

viz. the first term of this is

$$=\tfrac{1}{2}e^{-2q}\frac{\Gamma\frac{1}{2}s\,.\,\Gamma q}{\Gamma(\frac{1}{2}s+q)},$$

and the second term is a term expansible in a series containing the powers $2q$, $2q+2$, &c. of the small quantity $\frac{e^2}{\mathrm{R}^2}$, as appears by effecting therein the substitution $v=\frac{1}{x}$; viz. the value of the entire integral is by this means found to be

$$=\tfrac{1}{2}e^{-2q}\left\{\frac{\Gamma\frac{1}{2}s\,.\,\Gamma q}{\Gamma(\frac{1}{2}s+q)}-\int_0^{\frac{e^2}{\mathrm{R}^2}}\frac{x^{q-1}\, dx}{(1+x)^{\frac{1}{2}s+q}}\right\}.$$

55. In the case where q is $=0$, or negative, the formula fails by reason that the element $\frac{v^{\frac{1}{2}s-1}\, dv}{(1+v)^{\frac{1}{2}s+q}}$ of the integrals $\int_0^\infty$, $\int_{\frac{\mathrm{R}^2}{e^2}}^\infty$ becomes infinite for indefinitely large values of v. Recurring to the original form $\int_0^{\mathrm{R}}\frac{r^{s-1}\, dr}{(r^2+e^2)^{\frac{1}{2}s+q}}$, it is to be observed that the integral has a

finite value when $e=0$; and it might therefore at first sight be imagined that the factor $(r^2+e^2)^{-\frac{1}{2}s-q}$ might be expanded in ascending powers of e^2, and the value of the integral consequently obtained as a series of positive powers of e^2. But the series thus obtained is of the form $e^{2k}\int_0^{\mathrm{R}} r^{-2q-2k-1}\,dr$, where $2q$ being positive, the exponent $-2q-2k-1$ is for a sufficiently small value of k at first positive, or if negative less than -1, and the value of the integral is finite; but as k increases the exponent becomes negative, and equal or greater than -1, and the value of the integral is then infinite. The inference is that the series *commences* in the form $\mathrm{A}+\mathrm{B}e^2+\mathrm{C}e^4\ldots$, but that we come at last when q is fractional to a term of the form $\mathrm{K}e^{-2q}$, and when q is $=0$, or integral, to a term of the form $\mathrm{K}e^{-2q}\log e$, the process giving the coefficients A, B, C..., so long as the exponent of the corresponding term $e^0, e^2, e^4\ldots$ is less than $-2q$ (in particular $q=0$, there is a term $k\log e$, *and the expansion-process does not give any term of the result*), and the failure of the series after this point being indicated by the values of the subsequent coefficients coming out $=\infty$.

56. In illustration, we may consider any of the cases in which the integral can be obtained in finite terms. For instance,

$s=2,\ q=-\frac{3}{2}$,

$$\text{Integral is } \int r(r^2+e^2)^{\frac{1}{2}}\,dr,\ =\tfrac{1}{3}(r^2+e^2)^{\frac{3}{2}},\ \text{from 0 to R,}$$

$$=\tfrac{1}{3}(\mathrm{R}^2+e^2)^{\frac{3}{2}}-\tfrac{1}{3}e^3;$$

viz. expanding in ascending powers of e this is

$$=\tfrac{1}{3}\mathrm{R}^3+\tfrac{1}{2}\mathrm{R}e^2\ldots-\tfrac{1}{3}e^3,$$

or we have here a term in e^3. And so,

$s=1,\ q=-2$,

$$\text{Integral is } \int(r^2+e^2)^{\frac{3}{2}}dr,\ =(\tfrac{1}{4}r^2+\tfrac{5}{8}e^2)r\sqrt{r^2+e^2}+\tfrac{3}{8}e^4\log(r+\sqrt{r^2+e^2}),\ \text{from 0 to R,}$$

$$=(\tfrac{1}{4}\mathrm{R}^2+\tfrac{5}{8}e^2)\mathrm{R}\sqrt{\mathrm{R}^2+e^2}+\tfrac{3}{8}e^4\log\frac{\mathrm{R}+\sqrt{\mathrm{R}^2+e^2}}{e};$$

viz. expanding in ascending powers of e this is

$$=\tfrac{1}{4}\mathrm{R}^4+\tfrac{3}{4}\mathrm{R}^2e^2\ldots+\tfrac{3}{8}e^4\log\frac{\mathrm{R}}{e}\,*,$$

or we have here a term in $e^4\log e$.

57. Returning to the form

$$\tfrac{1}{2}e^{-2q}\int_0^{\frac{\mathrm{R}^2}{e^2}}\frac{v^{\frac{1}{2}s-1}dv}{(1+v)^{\frac{1}{2}s+q}},$$

and writing herein $v=\dfrac{1-x}{x}$, or, what is the same thing, $x=\dfrac{1}{1+v}$, and for shortness

* Term is $\frac{3}{8}e^4\log\frac{2\mathrm{R}}{e},\ =\frac{3}{8}e^4\left(\log\frac{\mathrm{R}}{e}+\log 2\right)$, which, $\frac{\mathrm{R}}{e}$ being large, is reduced to $\frac{3}{8}e^4\log\frac{\mathrm{R}}{e}$.

$X=\frac{e^2}{e^2+R^2}, =\frac{1}{1+\frac{R^2}{e^2}}$, the value is

$$=\tfrac{1}{2}e^{-2q}\int_X^1 x^{q-1}(1-x)^{\frac{1}{2}s-1}dx,$$

where observe that $q-1$ is 0 or negative, but X being a positive quantity less than 1, the function $x^{q-1}(1-x)^{\frac{1}{2}s-1}$ is finite for the whole extent of the integration.

58. If $q=0$, this is

$$=\tfrac{1}{2}\int_X^1 \frac{1-\{1-(1-x)^{\frac{1}{2}s-1}\}dx}{x}$$

$$=\tfrac{1}{2}\log X-\tfrac{1}{2}\int_X^1 \frac{\{1-(1-x)^{\frac{1}{2}s-1}\}dx}{x}$$

$$=\tfrac{1}{2}\log\sqrt{1+\frac{R^2}{e^2}}-\tfrac{1}{2}\int_0^1 \frac{\{1-(1-x)^{\frac{1}{2}s-1}\}dx}{x}+\tfrac{1}{2}\int_0^X \frac{\{1-(1-x)^{\frac{1}{2}s-1}\}dx}{x},$$

where observe that in virtue of the change made from $\frac{1}{x}(1-x)^{\frac{1}{2}s-1}$ to $\frac{1}{x}\{1-(1-x)^{\frac{1}{2}s-1}\}$ (a function which becomes infinite, to one which does not become infinite, for $x=0$), it has become allowable in place of $\int_X^1$ to write $\int_0^1-\int_0^X$.

When e is small, the integral which is the third term of the foregoing expression is obviously a quantity of the order e^2; the first term is $\tfrac{1}{2}\left(\log\frac{R}{e}+\log\sqrt{1+\frac{e^2}{R^2}}\right)$, which, neglecting terms in e^2, is $=\tfrac{1}{2}\log\frac{R}{e}$, and hence the approximate value of the r-integral $\int_0^R \frac{r^{s-1}dr}{(r^2+e^2)^{\frac{1}{2}s}}$ is

$$=\log\frac{R}{e}-\tfrac{1}{2}\int_0^1 dx\,\frac{1-(1-x)^{\frac{1}{2}s-1}}{x},$$

or, what is the same thing, it is

$$=\log\frac{R}{e}-\tfrac{1}{2}\int_0^1 dy\,\frac{1-y^{\frac{1}{2}s-1}}{1-y},$$

where the integral in this expression is a mere numerical constant, which, when $\tfrac{1}{2}s-1$ is a positive integer, has the value

$$\tfrac{1}{1}+\tfrac{1}{2}\ldots+\frac{1}{\tfrac{1}{2}s-1};$$

and neglecting this in comparison with the logarithmic term, the approximate value is

$$=\log\frac{R}{e}.$$

59. I consider also the case $q=-\frac{1}{2}$; the integral is here

$$\tfrac{1}{2}e\int_{\mathrm{X}}^{1} x^{-\frac{3}{2}}(1-x)^{\frac{1}{2}s-1}dx$$

$$=\tfrac{1}{2}e\int_{\mathrm{X}}^{1} x^{-\frac{3}{2}}(1-\{1-(1-x)^{\frac{1}{2}s-1}\})dx$$

$$=e(\mathrm{X}^{-\frac{1}{2}}-1)+\tfrac{1}{2}e\int_{\mathrm{X}}^{1} x^{-\frac{3}{2}}\{1-(1-x)^{\frac{1}{2}s-1}\}\,dx;$$

and the first term of this being $=\sqrt{e^2+\mathrm{R}^2}-e$, this is consequently

$$=\sqrt{\mathrm{R}^2+e^2}+\tfrac{1}{2}e\int_0^{\mathrm{X}} x^{-\frac{3}{2}}\{1-(1-x)^{\frac{1}{2}s-1}\}\,dx-e\left(1+\tfrac{1}{2}\int_0^1 x^{-\frac{3}{2}}\{1-(1-x)^{\frac{1}{2}s-1}\}dx\right).$$

As regards the second term of this we have

$$-2x^{-\frac{1}{2}}\{1-(1-x)^{\frac{1}{2}s-1}\}+2(\tfrac{1}{2}s-1)\int x^{-\frac{1}{2}}(1-x)^{\frac{1}{2}s-2}dx=\int x^{-\frac{3}{2}}\{1-(1-x)^{\frac{1}{2}s-1}\}dx;$$

or taking each term between the limits 1, 0,

$$-2+2(\tfrac{1}{2}s-1)\frac{\Gamma\frac{1}{2}\Gamma(\frac{1}{2}s-1)}{\Gamma(\frac{1}{2}s-\frac{1}{2})}=\int_0^1 x^{-\frac{3}{2}}\{1-(1-x)^{\frac{1}{2}s-1}\}dx;$$

viz. this integral has the value

$$-2+\frac{2\Gamma\frac{1}{2}s\Gamma\frac{1}{2}}{\Gamma(\frac{1}{2}s-\frac{1}{2})};$$

and the value of the r-integral $\displaystyle\int_0^{\mathrm{R}}\frac{r^{s-1}dr}{(r^2+e^2)^{\frac{1}{2}s-\frac{1}{2}}}$ is consequently

$$=\sqrt{\mathrm{R}^2+e^2}+\tfrac{1}{2}e\int_0^{\mathrm{X}} x^{-\frac{3}{2}}\{1-(1-x)^{\frac{1}{2}s-1}\}dx-e\frac{\Gamma\frac{1}{2}s\Gamma\frac{1}{2}}{\Gamma(\frac{1}{2}s-\frac{1}{2})},$$

which is of the form

$$\mathrm{R}\left\{1+\text{terms in }\frac{e^2}{\mathrm{R}^2},\frac{e^4}{\mathrm{R}^4},\ldots\right\}-e\frac{\Gamma\frac{1}{2}s\Gamma\frac{1}{2}}{\Gamma(\frac{1}{2}s-\frac{1}{2})};$$

say the approximate value is

$$\mathrm{R}-e\frac{\Gamma\frac{1}{2}s\Gamma\frac{1}{2}}{\Gamma(\frac{1}{2}s-\frac{1}{2})},$$

where the first term R is the term $\displaystyle\int_0^{\mathrm{R}}dr$, given by the expansion in ascending powers of e^2; the second term is the term in e^{-2q}. And observe that term is the value of

$$\tfrac{1}{2}e\int_0^1 x^{-\frac{3}{2}}(1-x)^{\frac{1}{2}s-1}dx,$$

calculated by means of the ordinary formula for a Eulerian integral (which formula, on account of the negative exponent $-\frac{3}{2}$, is not really applicable, the value of the integral being $=\infty$) on the assumption that the Γ of a negative q is interpreted in accordance with the equation $\Gamma(q+1)=q\Gamma q$; viz. the value thus calculated is

$$=\tfrac{1}{2}e\frac{\Gamma(-\frac{1}{2})\Gamma(\frac{1}{2}s)}{\Gamma(\frac{1}{2}s-\frac{1}{2})},\quad=-e\frac{\Gamma\frac{1}{2}\Gamma\frac{1}{2}s}{\Gamma(\frac{1}{2}s-\frac{1}{2})}$$

on the assumption $\Gamma\frac{1}{2}=-\frac{1}{2}\Gamma(-\frac{1}{2})$; and this agrees with the foregoing value.

60. It is now easy to see in general how the foregoing transformed value $\frac{1}{2}e^{-2q}\int_{\mathrm{X}}^{1} x^{q-1}(1-x)^{\frac{1}{2}s-1}dx$, where q is negative and fractional, gives at once the value of the term in e^{-2q}. Observe that in the integral x is always between 1 and X $(=\frac{e^2}{e^2+\mathrm{R}^2}$, a positive quantity less than 1); the function to be integrated never becomes infinite. Imagine for a moment an integral $\int_{\mathrm{X}}^{1} x^{\alpha}dx$, where α is positive or negative. We may conventionally write this $=\int_0^1 x^{\alpha}dx-\int_0^{\mathrm{X}} x^{\alpha}dx$, understanding the first symbol to mean $\frac{1^{1+\alpha}}{1+\alpha}$, and the second to mean $\frac{\mathrm{X}^{1+\alpha}}{1+\alpha}$; they of course properly mean $\frac{1^{1+\alpha}-0^{1+\alpha}}{1+\alpha}$ and $\frac{\mathrm{X}^{1+\alpha}-0^{1+\alpha}}{1+\alpha}$; but the terms in $0^{1+\alpha}$, whether zero or infinite, destroy each other, the original form $\int_{\mathrm{X}}^{1} x^{\alpha}dx$, in fact, showing that no such terms can appear in the result.

In accordance with the convention we write

$$\int_{\mathrm{X}}^{1} x^{q-1}(1-x)^{\frac{1}{2}s-1}dx=\int_0^1 x^{q-1}(1-x)^{\frac{1}{2}s-1}dx-\int_0^{\mathrm{X}} x^{q-1}(1-x)^{\frac{1}{2}s-1}dx\,;$$

and it follows that the term in e^{-2q} is

$$\tfrac{1}{2}e^{-2q}\int_0^1 x^{q-1}(1-x)^{\frac{1}{2}s-1}dx,$$

this last expression (wherein q, it will be remembered, is a negative fraction) being understood according to the convention; and so understanding it the value of the term is

$$=\tfrac{1}{2}e^{-2q}\frac{\Gamma\frac{1}{2}s\Gamma q}{\Gamma(\frac{1}{2}s+q)},$$

where the Γ of the negative q is to be interpreted in accordance with the equation $\Gamma(q+1)=q\Gamma q$; viz. we have $\Gamma q=\frac{1}{q}\Gamma(q+1)$, $=\frac{1}{q(q+1)}\Gamma(q+2)$, &c., so as to make the argument of the Γ positive. Observe that under this convention we have

$$\Gamma q\Gamma(1-q)=\frac{\Gamma^2(\frac{1}{2})}{\sin q\pi},\text{ or the term is }\tfrac{1}{2}e^{-2q}\,.\,\frac{\Gamma^2(\frac{1}{2})}{\sin q\pi}\,\frac{\Gamma\frac{1}{2}s}{\Gamma(\frac{1}{2}s+q)\Gamma(1-q)}.$$

61. An example in which $\frac{1}{2}s-1$ is integral will make the process clearer, and will serve instead of a general proof. Suppose $q=-\frac{1}{7}$, $\frac{1}{2}s-1=4$, the expression

$$\int_0^1 x^{-\frac{8}{7}}(1-x)^4\,dx=\int_0^1(x^{-\frac{8}{7}}-4x^{-\frac{1}{7}}+6x^{\frac{6}{7}}-4x^{\frac{13}{7}}+x^{\frac{20}{7}})\,dx$$

is used to denote the value

$$-7-\tfrac{14}{3}+\tfrac{42}{13}-\tfrac{7}{5}+\tfrac{7}{27}$$

$$=7(-1-\tfrac{2}{3}+\tfrac{6}{13}-\tfrac{1}{5}+\tfrac{1}{27}),\ =7(-\tfrac{44}{27}-\tfrac{1}{5}+\tfrac{6}{13}),\ =\frac{-7\,.\,2401}{5\,.\,13\,.\,27},\ =\frac{-7^5}{5\,.\,13\,.\,27}.$$

But we have

$$\frac{\Gamma\tfrac{1}{2}s\,\Gamma q}{\Gamma(\tfrac{1}{2}s+q)}=\frac{\Gamma 5\,\Gamma(-\tfrac{1}{7})}{\Gamma(5-\tfrac{1}{7})}=\frac{24\,\Gamma(-\tfrac{1}{7})}{\tfrac{27}{7}\,.\,\tfrac{20}{7}\,.\,\tfrac{13}{7}\,.\,\tfrac{6}{7}\,.\,\tfrac{-1}{7}\,\Gamma(-\tfrac{1}{7})}=\frac{-7^5}{5\,.\,13\,.\,27},$$

agreeing with the former value.

62. The case of a negative integer is more simple; to find the logarithmic term of

$$\tfrac{1}{2}e^{-2q}\int_{\mathrm{X}}^{1}x^{q-1}(1-x)^{\frac{1}{2}s-1}\,dx,$$

we have only to expand the factor $(1-x)^{\frac{1}{2}s-1}$ so as to obtain the term involving x^{-q}; we have thus the term

$$\tfrac{1}{2}e^{-2q}\int_{\mathrm{X}}^{1}x^{q-1}(-)^q\,\frac{\Gamma\tfrac{1}{2}s}{\Gamma(1-q)\,\Gamma(\tfrac{1}{2}s+q)}\,x^{-q}\,dx$$

$$=\tfrac{1}{2}(-)^q\,e^{-2q}\frac{\Gamma\tfrac{1}{2}s}{\Gamma(1-q)\,\Gamma(\tfrac{1}{2}s+q)}\log\frac{1}{\mathrm{X}},$$

where $\log\frac{1}{\mathrm{X}}=\log\left(1+\frac{\mathrm{R}^2}{e^2}\right)$, $=2\log\frac{\mathrm{R}}{e}+2\log\sqrt{1+\frac{e^2}{\mathrm{R}^2}}$, so that neglecting the terms in $\frac{e^2}{\mathrm{R}^2}$ &c. this is $=2\log\frac{\mathrm{R}}{e}$, and the term in question is

$$=(-)^q\,e^{-2q}\frac{\Gamma\tfrac{1}{2}s}{\Gamma(1-q)\,\Gamma(\tfrac{1}{2}s+q)}\log\frac{\mathrm{R}}{e}.$$

The general conclusion is that q being negative, the r-integral

$$\int_0^{\mathrm{R}}\frac{r^{s-1}\,dr}{(r^2+e^2)^{\frac{1}{2}s+q}}$$

has for its value a series proceeding in powers of e^2, and which up to a certain point is equal to the series obtained by expanding in ascending powers of e^2 and integrating each term separately; viz. the series to the point in question is

$$\frac{\mathrm{R}^{-2q}}{-2q}-\frac{\tfrac{1}{2}s+q}{1}\,\frac{\mathrm{R}^{-2q-2}}{-2q-2}\,e^2+\frac{\tfrac{1}{2}s+q\,.\,\tfrac{1}{2}s+q+1}{1\,.\,2}\,\frac{\mathrm{R}^{-2q-4}}{-2q-4}\,e^4\ldots,$$

continued so long as the exponent of e is less than $-2q$; together with a term $\mathrm{K}e^{-2q}$ when q is fractional, and $\mathrm{K}e^{-2q}\log\frac{\mathrm{R}}{e}$ when q is integral; viz. q fractional this term is

$$=\tfrac{1}{2}e^{-2q}\frac{\Gamma\tfrac{1}{2}s\,\Gamma q}{\Gamma(\tfrac{1}{2}s+q)},\ =\tfrac{1}{2}e^{-2q}\frac{\Gamma^2\tfrac{1}{2}}{\sin q\pi}\,\frac{\Gamma\tfrac{1}{2}s}{\Gamma(\tfrac{1}{2}s+q)\,\Gamma(1-q)},$$

and q integral, it is

$$=(-)^q\,e^{-2q}\frac{\Gamma\tfrac{1}{2}s}{\Gamma(1-q)\,\Gamma(\tfrac{1}{2}s+q)}\log\frac{\mathrm{R}}{e}.$$

63. It has been tacitly assumed that $\tfrac{1}{2}s+q$ is positive; but the formulæ hold good if $\tfrac{1}{2}s+q$ is$=0$ or negative. Suppose $\tfrac{1}{2}s+q$ is 0 or a negative integer, then $\Gamma(\tfrac{1}{2}s+q)=\infty$, and the special term involving e^{-2q} or $e^{-2q}\log e$ vanishes; in fact in this case the r-integral is

$$=\int_0^{\mathrm{R}}r^{\frac{1}{2}s-1}(r^2+e^2)^{-(\frac{1}{2}s+q)}\,dr,$$

where $(r^2+e^2)^{-(\frac{1}{2}s+q)}$ has for its value a finite series, and the integral is therefore equal to a finite series $A+Be^2+Ce^4+\&c.$ If $\frac{1}{2}s+q$ be fractional, then the Γ of the negative quantity $\frac{1}{2}s+q$ must be understood as above, or, what is the same thing, we may, instead of $\Gamma(\frac{1}{2}s+q)$, write $\frac{(\Gamma\frac{1}{2})^2}{\sin(\frac{1}{2}s+q)\pi\,\Gamma(1-q-\frac{1}{2}s)}$; thus, q being integral, the exceptional term is

$$=(-)^q\,e^{-2q}\,\frac{\Gamma\frac{1}{2}s\,\sin(\frac{1}{2}s+q)\pi\,.\,\Gamma(1-q-\frac{1}{2}s)}{(\Gamma\frac{1}{2})^2\,\Gamma(1-q)}\log\frac{R}{e};$$

for instance, $s=1$, $q=-2$, the term is

$$\tfrac{1}{2}e^4\,\frac{\Gamma\frac{1}{2}\,\sin(-\frac{3}{2}\pi)\,\Gamma\frac{5}{2}}{(\Gamma\frac{1}{2})^2\,.\,\Gamma3}\log\frac{R}{e};$$

or, since $\Gamma\frac{5}{2}=\frac{3}{2}\,.\,\frac{1}{2}\,\Gamma\frac{1}{2}$, and $\Gamma3=2$, the term is $+\frac{3}{8}e^4\log\frac{R}{e}$, agreeing with a preceding result.

Annex III. *Prepotentials of Uniform Spherical Shell and Solid Sphere.*—Nos. 64 to 92.

64. The prepotentials in question depend ultimately upon two integrals, which also arise, as will presently appear, from prepotential problems in two-dimensional space, and which are for convenience termed the ring-integral and the disk-integral respectively. The analytical investigation in regard to these, depending as it does on a transformation of a function allied with the hypergeometric series, is I think interesting.

65. Consider first the prepotential of a uniform $(s+1)$dimensional spherical shell. This is

$$V=\int\frac{dS}{\{(a-x)^2\ldots+(c-z)^2+(e-w)^2\}^{\frac{1}{2}s+q}},$$

the equation of the surface being $x^2\ldots+z^2+w^2=f^2$; and there are the two cases of an internal point, $a^2\ldots+c^2+e^2<f^2$, and an external point, $a^2\ldots+c^2+e^2>f^2$.

The value is a function of $a^2\ldots+c^2+e^2$, say this is $=\varkappa^2$; and taking the axes so that the coordinates of the attracted point are $(0\ldots0,\varkappa)$, the integral is

$$=\int\frac{dS}{\{x^2\ldots+z^2+(\varkappa-w)^2\}^{\frac{1}{2}s+q}},$$

where the equation of the surface is still $x^2\ldots+z^2+w^2=f^2$. Writing $x=f\xi\ldots z=f\zeta$, $w=f\omega$, where $\xi^2\ldots+\zeta^2+\omega^2=1$, we have $dS=\frac{f^s\,d\xi\ldots d\zeta}{\omega}$, or the integral is

$$=f^s\int\frac{d\xi\ldots d\zeta}{\omega(f^2-2\varkappa f\omega+\varkappa^2)^{\frac{1}{2}s+q}}.$$

Assume $\xi=px,\ldots\zeta=pz$, where $x^2\ldots+z^2=1$; then $p^2+\omega^2=1$. Moreover, $d\xi\ldots d\zeta$, $=p^{s-1}\,dp\,d\Sigma$, where $d\Sigma$ is the element of surface of the s-dimensional unit-sphere $x^2\ldots+z^2=1$; or for p, substituting its value $\sqrt{1-\omega^2}$, we have $dp=\frac{-\omega d\omega}{\sqrt{1-\omega^2}}$; and thence

$d\xi \ldots d\zeta = -(1-\omega^2)^{\frac{1}{2}s-1}\,\omega\, d\omega\, d\Sigma$. The integral as regards p is from $p=-1$ to $+1$, or as regards ω from 1 to -1; whence reversing the sign the integral will be from $\omega=-1$ to $+1$; and the required integral is thus

$$=f^s \int_{-1}^{1} \frac{(1-\omega^2)^{\frac{1}{2}s-1}\, d\omega\, d\Sigma}{(f^2-2\kappa f\omega+\kappa^2)^{\frac{1}{2}s+q}} = f^s \int d\Sigma \int_{-1}^{1} \frac{(1-\omega^2)^{\frac{1}{2}s-1}\, d\omega}{(f^2-2\kappa f\omega+\kappa^2)^{\frac{1}{2}s+q}},$$

where $\int d\Sigma$ is the surface of the s-dimensional unit-sphere (see Annex I.), $=\frac{2(\Gamma\frac{1}{2})^s}{\Gamma\frac{1}{2}s}$; and for greater convenience transforming the second factor by writing therein $\omega=\cos\theta$, the required integral is $=\frac{(\Gamma\frac{1}{2})^s}{\Gamma(\frac{1}{2}s)}$ into

$$2f^s \int_0^\pi \frac{\sin^{s-1}\theta\, d\theta}{(f^2-2\kappa f\cos\theta+\kappa^2)^{\frac{1}{2}s+q}},$$

which last expression $\left(\text{including the factor } 2f^s, \text{but without the factor } \frac{(\Gamma\frac{1}{2})^s}{(\Gamma\frac{1}{2}s)}\right)$ is the ring-integral discussed in the present Annex. It may be remarked that the value can be at once obtained in the particular case $s=2$, which belongs to tridimensional space, viz. we then have

$$\begin{aligned} V &= 2\pi f^2 \int_0^\pi \frac{\sin\theta\, d\theta}{(f^2-2\kappa f\cos\theta+\kappa^2)^{q+1}} \\ &= \frac{2\pi f^2}{2\kappa f q}\,(f^2-2\kappa f\cos\theta+\kappa^2)^{-q} \\ &= \frac{\pi f}{\kappa q}\,\{(f-\kappa)^{-2q}-(f+\kappa)^{-2q}\}, \end{aligned}$$

which agrees with a result given, 'Mécanique Céleste,' Book XII. Chap. II.

66. Consider next the prepotential of the uniform solid $(s+1)$dimensional sphere,

$$V = \int \frac{dx \ldots dz dw}{\{(a-x)^2 \ldots + (c-z)^2 + (e-w)^2\}^{\frac{1}{2}s+q}},$$

equation of surface $x^2 \ldots + z^2 + w^2 = f^2$, and the two cases of an internal point $\kappa < f$, and an external point $\kappa > f$ ($a^2 \ldots + c^2 + e^2 = \kappa^2$ as before).

Transforming so that the coordinates of the attracted point are $0 \ldots 0, \kappa$, the integral is

$$= \int \frac{dx \ldots dz dw}{\{x^2 \ldots + z^2 + (\kappa - w)^2\}^{\frac{1}{2}s+q}},$$

where the equation is still $x^2 \ldots + z^2 + w^2 = f^2$. Writing here $x = r\xi \ldots z = r\zeta$, where $\xi^2 \ldots + \zeta^2 = 1$, we have $dx \ldots dz = r^{s-1} dr d\Sigma$, where $d\Sigma$ is an element of surface of the s-dimensional unit-sphere $\xi^2 \ldots + \zeta^2 = 1$; the integral is therefore

$$\begin{aligned} &= \int \frac{r^{s-1}\, dr d\Sigma dw}{\{r^2 + (\kappa - w)^2\}^{\frac{1}{2}s+q}} \\ &= \int d\Sigma . \int \frac{r^{s-1}\, dr dw}{\{r^2 + (\kappa - w)^2\}^{\frac{1}{2}s+q}}, \end{aligned}$$

where, as regards r and w, the integration extends over the circle $r^2 + w^2 = f^2$. The value

of the first factor (see Annex I.) is $=\dfrac{2(\Gamma\frac{1}{2})^s}{\Gamma\frac{1}{2}s}$; and writing y, x in place of r, w respectively, the integral is $=\dfrac{2(\Gamma\frac{1}{2})^s}{\Gamma\frac{1}{2}s}$ into

$$\int\frac{y^{s-1}\,dxdy}{\{(x-\kappa)^2+y^2\}^{\frac{1}{2}s+q}}$$

over the circle $x^2+y^2=f^2$; viz. this last expression $\left(\text{without the factor } \dfrac{2(\Gamma\frac{1}{2})^s}{\Gamma(\frac{1}{2}s)}\right)$ is the disk-integral discussed in the present Annex.

67. We find for the value in regard to an internal point $\kappa<f$,

$$\mathrm{V}=\frac{(\Gamma\frac{1}{2})^{s+1}}{\Gamma(\frac{1}{2}s+q)\Gamma(\frac{1}{2}-q)}f^{s+1}\int_0^\infty(t+f^2-\kappa^2)^{\frac{1}{2}-q}t^{-\frac{1}{2}-q}(t+f^2)^{-\frac{1}{2}s+q-1}dt,$$

which in the particular case $q=-\frac{1}{2}$ is

$$=\frac{(\Gamma\frac{1}{2})^{s+1}}{\Gamma(\frac{1}{2}s-\frac{1}{2})}f^{s+1}\int_0^\infty(t+f^2-\kappa^2)(t+f^2)^{-\frac{1}{2}s-\frac{3}{2}}dt;$$

viz. the integral in t is here

$$=\int_0^\infty\{(t+f^2)^{-\frac{1}{2}s-\frac{1}{2}}-\kappa^2(t+f^2)^{-\frac{1}{2}s-\frac{3}{2}}\},\ =\frac{1}{f^{s+1}}\left(\frac{f^2}{\frac{1}{2}s-\frac{1}{2}}-\frac{\kappa^2}{\frac{1}{2}s+\frac{1}{2}}\right),$$

or we have

$$\mathrm{V}=\frac{(\Gamma\frac{1}{2})^{s+1}}{\Gamma(\frac{1}{2}s-\frac{1}{2})}\left(\frac{f^2}{\frac{1}{2}s-\frac{1}{2}}-\frac{\kappa^2}{\frac{1}{2}s+\frac{1}{2}}\right).$$

It may be added that in regard to an external point $\kappa>f$, the value is

$$\mathrm{V}=\frac{(\Gamma\frac{1}{2})^{s+1}}{\Gamma(\frac{1}{2}s+q)\Gamma(\frac{1}{2}-q)}f^{s+1}\cdot\int_{\kappa^2-f^2}^\infty(t+f^2-\kappa^2)^{\frac{1}{2}-q}t^{-\frac{1}{2}-q}(t+f^2)^{-\frac{1}{2}s+q-1}dt,$$

which in the same case $q=-\frac{1}{2}$ is

$$=\frac{(\Gamma\frac{1}{2})^{s+1}}{\Gamma(\frac{1}{2}s-\frac{1}{2})}f^{s+1}\cdot\int_{\kappa^2-f^2}^\infty(t+f^2-\kappa^2)(t+f^2)^{-\frac{1}{2}s-\frac{3}{2}}dt,$$

where the t-integral is

$$=\int_{\kappa^2-f^2}^\infty\{(t+f^2)^{-\frac{1}{2}s-\frac{1}{2}}-\kappa^2(t+f^2)^{-\frac{1}{2}s-\frac{3}{2}}\}dt,\ =\frac{\kappa^{-s+1}}{\frac{1}{2}s-\frac{1}{2}}-\frac{\kappa^2\,.\,\kappa^{-s-1}}{\frac{1}{2}s+\frac{1}{2}},\ =\frac{\kappa^{-s+1}}{\frac{1}{2}s-\frac{1}{2}\,.\,\frac{1}{2}s+\frac{1}{2}};$$

and the value of V is therefore

$$=\frac{(\Gamma\frac{1}{2})^{s+1}}{\Gamma(\frac{1}{2}s+\frac{1}{2})}\frac{f^{s+1}}{\kappa^{s-1}}.$$

Recurring to the case of the internal point; then writing $\nabla=\dfrac{d^2}{da^2}\ldots+\dfrac{d^2}{dc^2}+\dfrac{d^2}{de^2}$, and observing that $\nabla(\kappa^2)=4(\frac{1}{2}s+\frac{1}{2})$, we have

$$\nabla\mathrm{V}=-\frac{4(\Gamma\frac{1}{2})^{s+1}}{\Gamma(\frac{1}{2}s-\frac{1}{2})}$$

(in particular for ordinary space $s+1=3$, or the value is $\dfrac{-4\pi^{\frac{3}{2}}}{\sqrt{\pi}}$, $=-4\pi$, which is right).

68. The integrals referred to as the ring-integral and the disk-integral arise also from the following integrals in two-dimensional space, viz. these are

$$\int \frac{y^{s-1}dS}{\{(x-\varkappa)^2+y^2\}^{\frac{1}{2}s+q}},\quad \int \frac{y^{s-1}dxdy}{\{(x-\varkappa)^2+y^2\}^{\frac{1}{2}s+q}},$$

in the first of which dS denotes an element of arc of the circle $x^2+y^2=f^2$, the integration being extended over the whole circumference, and in the second the integration extends over the circle $x^2+y^2=f^2$; y^{s-1} is written for shortness instead of $(y^2)^{\frac{1}{2}(s-1)}$, viz. this is considered as always positive, whether y is positive or negative; it is moreover assumed that $s-1$ is zero or positive.

Writing in the first integral $x=f\cos\theta$, $y=f\sin\theta$, the value is

$$=f^s\int\frac{(\sin\theta)^{s-1}\,d\theta}{(f^2-2\varkappa f\cos\theta+\varkappa^2)^{\frac{1}{2}s+q}};$$

viz. this represents the prepotential of the circumference of the circle, density varying as $(\sin\theta)^{s-1}$, in regard to a point $x=\varkappa$, $y=0$ in the plane of the circle; and similarly the second integral represents the prepotential of the circular disk, density of the element at the point $(x, y)=y^{s-1}$, in regard to the same point $x=\varkappa$, $y=0$, it being in each case assumed that the prepotential of an element of mass $\rho d\varpi$ upon a point at distance d is $=\frac{\rho d\varpi}{d^{s+2q}}$.

69. In the case of the circumference, it is assumed that the attracted point is not on the circumference, $\varkappa$ not $=f$; and the function under the integral sign, and therefore the integral itself, is in every case finite. In the case of the circle, if $\varkappa$ be an interior point, then if $2q-1$ be $=0$ or positive, the element at the attracted point becomes infinite; but to avoid this we consider not the potential of the whole circle, but the potential of the circle *less* an indefinitely small circle radius ϵ having the attracted point for its centre; which being so, the element under the integral sign, and consequently the integral itself, remains finite.

It is to be remarked that the two integrals are connected with each other; viz. the circle of the second integral being divided in rings by means of a system of circles concentric with the bounding circle $x^2+y^2=f^2$, then the prepotential of each ring or annulus is determined by an integral such as the first integral; or, analytically, writing in the second integral $x=r\cos\theta$, $y=r\sin\theta$, and therefore $dxdy=rdrd\theta$, the second integral is

$$=\int r^s\,dr\int\frac{(\sin\theta)^{s-1}\,d\theta}{(r^2+\varkappa^2-2\varkappa r\cos\theta)^{\frac{1}{2}s+q}},$$

viz. the integral in regard to θ is here the same function of r, $\varkappa$ that the first integral is of f, $\varkappa$; and the integration in regard to r is of course to be taken from $r=0$ to $r=f$. But the θ-integral is not in its original form such a function of r as to render possible the integration in regard to r; and I, in fact, obtain the second integral by a different and in some respects a better process.

70. Consider first the ring-integral, which writing therein as above $x=f\cos\theta$,

$y=f\sin\theta$, and multiplying by 2 in order that the integral, instead of being taken from 0 to 2π, may be taken from 0 to π, becomes

$$=2f^s\int\frac{(\sin\theta)^{s-1}\,d\theta}{(f^2-2\varkappa f\cos\theta+\varkappa^2)^{\frac{1}{2}s+q}}.$$

Write $\cos\frac{1}{2}\theta=\sqrt{x}$; then $\sin\frac{1}{2}\theta=\sqrt{1-x}$, $\sin\theta=2x^{\frac{1}{2}}(1-x)^{\frac{1}{2}}$; $d\theta=-x^{-\frac{1}{2}}(1-x)^{-\frac{1}{2}}dx$; $\cos\theta=-1+2x$; $\theta=0$ gives $x=1$, $\theta=\pi$ gives $x=0$, and the integral is

$$=2^{s-1}f^s\int_0^1\frac{x^{\frac{1}{2}s-1}(1-x)^{\frac{1}{2}s-1}\,dx}{\{(f+\varkappa)^2-4\varkappa fx\}^{\frac{1}{2}s+q}},$$

$$=\frac{2^{s-1}f^s}{(f+\varkappa)^{s+2q}}\int_0^1\frac{x^{\frac{1}{2}s-1}(1-x)^{\frac{1}{2}s-1}\,dx}{(1-ux)^{\frac{1}{2}s+q}},$$

if for shortness $u=\frac{4\varkappa f}{(\varkappa+f)^2}$ (obviously $u<1$).

The integral in x is here an integral belonging to the general form

$$\Pi(\alpha,\beta,\gamma,u)=\int_0^1 x^{\alpha-1}(1-x)^{\beta-1}(1-ux)^{-\gamma}dx,$$

viz. we have

$$\text{Ring-integral }=\frac{2^{s-1}f^s}{(f+\varkappa)^{s+2q}}\,\Pi(\tfrac{1}{2}s,\tfrac{1}{2}s,\tfrac{1}{2}s+q,u).$$

71. The general function $\Pi(\alpha,\beta,\gamma,u)$ is

$$\Pi(\alpha,\beta,\gamma,u)=\frac{\Gamma\alpha\Gamma\beta}{\Gamma(\alpha+\beta)}\,F(\alpha,\gamma,\alpha+\beta,u),$$

or, what is the same thing,

$$F(\alpha,\beta,\gamma,u)=\frac{\Gamma\gamma}{\Gamma\alpha\Gamma(\gamma-\alpha)}\,\Pi(\alpha,\gamma-\alpha,\beta,u),$$

and consequently transformable by means of various theorems for the transformation of the hypergeometric series; in particular the theorems

$$F(\alpha,\beta,\gamma,u)=F(\beta,\alpha,\gamma,u),$$
$$F(\alpha,\beta,\gamma,u)=(1-u)^{\gamma-\alpha-\beta}F(\gamma-\alpha,\gamma-\beta,\gamma,u);$$

and if $v=\left(\frac{1-\sqrt{1-u}}{1+\sqrt{1-u}}\right)^2$, or, what is the same thing, $u=\frac{4\sqrt{v}}{(1+\sqrt{v})^2}$, then

$$F(\alpha,\beta,2\beta,u)=(1+\sqrt{v})^{2\alpha}\,F(\alpha,\alpha-\beta+\tfrac{1}{2},v);$$

in verification observe that if $u=1$ then also $v=1$, and that with these values, calculating each side by means of the formula

$$F(\alpha,\beta,\gamma,1)=\frac{\Gamma\gamma\Gamma(\gamma-\alpha-\beta)}{\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta)}\quad\Big|\quad\Pi(\alpha,\beta,\gamma,1)=\frac{\Gamma\alpha\Gamma(\beta-\gamma)}{\Gamma(\alpha+\beta-\gamma)},$$

the resulting equation, $F(\alpha,\beta,2\beta,1)=2^{2\alpha}F(\alpha,\alpha-\beta+\frac{1}{2},\beta+\frac{1}{2},1)$, becomes

$$\frac{\Gamma2\beta\Gamma(\beta-\alpha)}{\Gamma(2\beta-\alpha)\Gamma\beta}=2^{2\alpha}\,\frac{\Gamma(\beta+\frac{1}{2})\Gamma(2\beta-2\alpha)}{\Gamma(2\beta-\alpha)\Gamma(\beta-\alpha+\frac{1}{2})},$$

that is

$$\frac{\Gamma 2\beta}{\Gamma\beta\Gamma(\beta+\frac{1}{2})}=2^{2\alpha}\frac{\Gamma(2\beta-2\alpha)}{\Gamma(\beta-\alpha)\Gamma(\beta-\alpha+\frac{1}{2})},$$

which is true, in virtue of the relation $\frac{\Gamma 2x\Gamma\frac{1}{2}}{\Gamma x\Gamma(x+\frac{1}{2})}=2^{2x-1}$.

72. The foregoing formulæ, and in particular the formula which I have written $F(\alpha, \beta, 2\beta, u)=(1+\sqrt{v})^{2\alpha}F(\alpha, \alpha-\beta+\frac{1}{2}, \beta+\frac{1}{2}, v)$, are taken from KUMMER's Memoir, "Ueber die hypergeometrische Reihe," *Crelle*, t. xv. (1836), viz. the formula in question is under a slightly different form, his formula (41) p. 76; the formula (43), p. 77, is intended to be equivalent thereto; but there is an error of transcription, $2\alpha-2\beta+1$, in place of $\beta+\frac{1}{2}$, which makes the formula (43) erroneous.

It may be remarked as to the formulæ generally, that although very probably $\Pi(\alpha, \beta, \gamma, u)$ may denote a proper function of u, whatever be the values of the indices (α, β, γ), and the various transformation-theorems hold good accordingly (the Γ-function of a negative argument being interpreted in the usual manner by means of the equation $\Gamma x=\frac{1}{x}\Gamma(1+x), =\frac{1}{x(x+1)}\Gamma(2+x)$ &c.), yet that the function $\Pi(\alpha, \beta, \gamma, u)$, used as denoting the definite integral $\int_0^1 x^{\alpha-1}(1-x)^{\beta-1}(1-ux)^{-\gamma}dx$, has no meaning except in the case where α and β are each of them positive.

In what follows we obtain for the ring-integral and the disk-integral various expressions in terms of Π-functions, which are afterwards transformed into t-integrals with a superior limit ∞ and inferior limit 0, or z^2-f^2; but for values of the variable index, q lying beyond certain limits, the indices α and β, or one of them, of the Π-function will become negative, viz. the integral represented by the Π-function, or, what is the same thing, the t-integral, will cease to have a determinate value, and at the same time, or usually so, the argument or arguments of one or more of the Γ-functions will become negative. It is quite possible that in such cases the results are not without meaning, and that an interpretation for them might be found; but they have not any obvious interpretation, and we must in the first instance consider them as inapplicable.

73. We require further properties of the Π-functions. Starting with the foregoing equation,

$$F(\alpha, \beta, 2\beta, u)=(1+\sqrt{v})^{2\alpha}F(\alpha, \alpha-\beta+\tfrac{1}{2}, \beta+\tfrac{1}{2}, v),$$

each side may be expressed in a fourfold form :—

$$\begin{array}{l|l} & = \\ F(\alpha, \beta, 2\beta, u) & (1+\sqrt{v})^{2\alpha}F(\alpha, \alpha-\beta+\frac{1}{2}, \beta+\frac{1}{2}, v) \\ =F(\beta, \alpha, 2\beta, u) & =(1+\sqrt{v})^{2\alpha}F(\alpha-\beta+\frac{1}{2}, \alpha, \beta+\frac{1}{2}, v) \\ =(1-u)^{\beta-\alpha}F(2\beta-\alpha, \beta, 2\beta, u) & =(1+\sqrt{v})^{2\alpha}(1-v)^{2\beta-2\alpha}F(\beta-\alpha+\frac{1}{2}, 2\beta-\alpha, \beta+\frac{1}{2}, v) \\ =(1-u)^{\beta-\alpha}F(\alpha, 2\beta-\alpha, 2\beta, u) & =(1+\sqrt{v})^{2\alpha}(1-v)^{2\beta-2\alpha}F(2\beta-\alpha, \beta-\alpha+\frac{1}{2}, \beta+\frac{1}{2}, v), \end{array}$$

where, instead of $(1+\sqrt{v})^{2\alpha}(1-v)^{2\beta-2\alpha}$, it is proper to write $(1+\sqrt{v})^{2\beta}(1 \quad \sqrt{v})^{2\beta-2\alpha}$; and then to each form applying the transformation

$$F(\alpha,\beta,\gamma,u)=\frac{\Gamma\gamma}{\Gamma\alpha\Gamma(\gamma-\alpha)}\,\Pi(\alpha,\gamma-\alpha,\beta,u),$$

we have

$$\frac{\Gamma 2\beta}{\Gamma\alpha\Gamma(2\beta-\alpha)}\,\Pi(\alpha,2\beta-\alpha,\beta,u)$$

$$=\frac{\Gamma 2\beta}{\Gamma\beta\,\Gamma\beta}\,\Pi(\beta,\beta,a,u)$$

$$=(1-u)^{\beta-\alpha}\frac{\Gamma 2\beta}{\Gamma(2\beta-\alpha)\Gamma\alpha}\,\Pi(2\beta-\alpha,\alpha,\beta,u)$$

$$=(1-u)^{\beta-\alpha}\frac{\Gamma 2\beta}{\Gamma\alpha\Gamma(2\beta-\alpha)}\,\Pi(\alpha,2\beta-\alpha,2\beta-\alpha,u)$$

$$=(1+\sqrt{v})^{2\alpha}\frac{\Gamma(\beta+\frac{1}{2})}{\Gamma\alpha\Gamma(\beta-\alpha+\frac{1}{2})}\,\Pi(\alpha,\beta-\alpha+\tfrac{1}{2},\alpha-\beta+\tfrac{1}{2},v)$$

$$=(1+\sqrt{v})^{2\alpha}\frac{\Gamma(\beta+\frac{1}{2})1}{\Gamma(\alpha-\beta+\frac{1}{2})\,\Gamma(2\beta-\alpha)}\,\Pi(\alpha-\beta+\tfrac{1}{2},2\beta-\alpha,\alpha,v)$$

$$=(1+\sqrt{v}^{2\beta}(1-\sqrt{v})^{2\beta-2\alpha}\frac{\Gamma(\beta+\frac{1}{2})}{\Gamma(\beta-\alpha+\frac{1}{2})\Gamma\alpha}\,\Pi(\beta-\alpha+\tfrac{1}{2},\alpha,2\beta-\alpha,v)$$

$$=(1+\sqrt{v})^{2\beta}(1-\sqrt{v})^{2\beta-2\alpha}\frac{\Gamma(\beta+\frac{1}{2})}{\Gamma(2\beta-\alpha)\Gamma(\alpha-\beta+\frac{1}{2})}\,\Pi(2\beta-\alpha,\alpha-\beta+\tfrac{1}{2},\beta-\alpha+\tfrac{1}{2},v).$$

I select on the left-hand the second form, and equating it successively to the four right-hand forms, attending to the relation $\frac{\Gamma\beta\Gamma(\beta+\frac{1}{2})}{\Gamma 2\beta}=2^{1-2\beta}\,\Gamma\frac{1}{2}$, we find

$$\Pi(\beta,\beta,\alpha,u)=(1+\sqrt{v})^{2\alpha}\,2^{1-2\beta}\frac{\Gamma\beta\Gamma\frac{1}{2}}{\Gamma\alpha\Gamma(\beta-\alpha+\frac{1}{2})}\,\Pi(\alpha,\beta-\alpha+\tfrac{1}{2},\alpha-\beta+\tfrac{1}{2},v)$$

$$=(1+\sqrt{v})^{2\alpha}\,2^{1-2\beta}\frac{\Gamma\beta\Gamma\frac{1}{2}}{\Gamma(\alpha-\beta+\frac{1}{2})\,\Gamma(2\beta-\alpha)}\,\Pi(\alpha-\beta+\tfrac{1}{2},2\beta-\alpha,\alpha,v)$$

$$=(1+\sqrt{v})^{2\beta}(1-\sqrt{v})^{2\beta-2\alpha}\,2^{1-2\beta}\frac{\Gamma\beta\Gamma\frac{1}{2}}{\Gamma(\beta-\alpha+\frac{1}{2})\Gamma\alpha}\,\Pi(\beta-\alpha+\tfrac{1}{2},\alpha,2\beta-\alpha,v)$$

$$=(1+\sqrt{v})^{2\beta}(1-\sqrt{v})^{2\beta-2\alpha}\,2^{1-2\beta}\frac{\Gamma\beta\Gamma\frac{1}{2}}{\Gamma(2\beta-\alpha)\,\Gamma(\alpha-\beta+\frac{1}{2})}\,\Pi(2\beta-\alpha,\alpha-\beta+\tfrac{1}{2},\beta-\alpha+\tfrac{1}{2},v).$$

Putting herein $\beta=\frac{1}{2}s$, $\alpha=\frac{1}{2}s+q$, the formulæ become

$$\Pi(\tfrac{1}{2}s,\tfrac{1}{2}s,\tfrac{1}{2}s+q,u)=(1+\sqrt{v})^{s+2q}\,2^{1-s}\frac{\Gamma\frac{1}{2}s\Gamma\frac{1}{2}}{\Gamma(\frac{1}{2}s+q)\Gamma(\frac{1}{2}-q)}\,\Pi(\tfrac{1}{2}s+q,\tfrac{1}{2}-q,\tfrac{1}{2}+q,v)\;.\;\;.\;\;.\quad(\text{I.})$$

$$=(1+\sqrt{v})^{s+2q}\,2^{1-s}\frac{\Gamma\frac{1}{2}s\Gamma\frac{1}{2}}{\Gamma(\frac{1}{2}+q)\Gamma(\frac{1}{2}s-q)}\,\Pi(\tfrac{1}{2}+q,\tfrac{1}{2}s-q,\tfrac{1}{2}s+q,v)\quad.\;\;.\;(\text{II.})$$

$$=(1+\sqrt{v})^{s}(1-\sqrt{v})^{-2q}\,2^{1-s}\frac{\Gamma\frac{1}{2}s\Gamma\frac{1}{2}}{\Gamma(\frac{1}{2}-q)\Gamma(\frac{1}{2}s+q)}\,\Pi(\tfrac{1}{2}-q,\tfrac{1}{2}s+q,\tfrac{1}{2}s-q,v).\quad.\;\;.(\text{III.})$$

$$=(1+\sqrt{v})^{s}(1-\sqrt{v})^{-2q}\,2^{1-s}\frac{\Gamma\frac{1}{2}s\Gamma\frac{1}{2}}{\Gamma(\frac{1}{2}s-q)\Gamma(\frac{1}{2}+q)}\,\Pi(\tfrac{1}{2}s-q,\tfrac{1}{2}+q,\tfrac{1}{2}-q,v),\quad.\;\;.(\text{IV.})$$

where observe that on the right-hand side the Π-functions of I. and IV. only differ by the sign of q, and so also the Π-functions of II. and III. only differ by the sign of q. We hence have

$$\Pi(\tfrac{1}{2}s, \tfrac{1}{2}s, \tfrac{1}{2}s-q, u)=(1+\sqrt{v})^{s-2q}\, 2^{1-s}.\frac{\Gamma\tfrac{1}{2}s\Gamma\tfrac{1}{2}}{\Gamma(\tfrac{1}{2}s-q)\Gamma(\tfrac{1}{2}+q)}\,\Pi(\tfrac{1}{2}s-q, \tfrac{1}{2}+q, \tfrac{1}{2}-q, v);$$

and comparing with (IV.),

$$\Pi(\tfrac{1}{2}s, \tfrac{1}{2}s, \tfrac{1}{2}s+q, u)=\left(\frac{1+\sqrt{v}}{1-\sqrt{v}}\right)^{2q}\Pi(\tfrac{1}{2}s, \tfrac{1}{2}s, \tfrac{1}{2}s-q, u).$$

74. The foregoing formula,

$$\text{Ring-integral}=\frac{2^{s-1}f^s}{(f+\varkappa)^{s+2q}}\,\Pi(\tfrac{1}{2}s, \tfrac{1}{2}s, \tfrac{1}{2}s+q, u),$$

where $u=\dfrac{4\varkappa f}{(f+\varkappa)^2}$, gives, as well in the case of an exterior as an interior point, a convergent series for the integral; but this series proceeds according to the powers of $\dfrac{4\varkappa f}{(f+\varkappa)^2}$. We may obtain more convenient formulæ applying to the cases of an internal and an external point respectively.

75. Internal point $\varkappa<f$, $\sqrt{1-u}=\dfrac{f-\varkappa}{f+\varkappa}$, and therefore $v=\dfrac{\kappa^2}{f^2}$.

$$\begin{aligned}
\Pi(\tfrac{1}{2}s, \tfrac{1}{2}s, \tfrac{1}{2}s+q, u)= & \left(\frac{f+\kappa}{f}\right)^{s+2q} 2^{1-s}\frac{\Gamma\tfrac{1}{2}s\Gamma\tfrac{1}{2}}{\Gamma(\tfrac{1}{2}s+q)\Gamma(\tfrac{1}{2}-q)}\;\Pi\left(\tfrac{1}{2}s+q, \tfrac{1}{2}-q, \tfrac{1}{2}+q, \frac{\kappa^2}{f^2}\right)\\
= & \left(\frac{f+\varkappa}{f}\right)^{s+2q} 2^{1-s}\frac{\Gamma\tfrac{1}{2}s\Gamma\tfrac{1}{2}}{\Gamma(\tfrac{1}{2}+q)\Gamma(\tfrac{1}{2}s-q)}\;\Pi\left(\tfrac{1}{2}+q, \tfrac{1}{2}s-q, \tfrac{1}{2}s+q, \frac{\varkappa^2}{f^2}\right)\\
= & \left(\frac{f+\varkappa}{f}\right)^{s}\left(\frac{f-\kappa}{f}\right)^{-2q} 2^{1-s}\frac{\Gamma\tfrac{1}{2}s\Gamma\tfrac{1}{2}}{\Gamma(\tfrac{1}{2}-q)\Gamma(\tfrac{1}{2}s+q)}\;\Pi\left(\tfrac{1}{2}-q, \tfrac{1}{2}s+q, \tfrac{1}{2}s-q, \frac{\varkappa^2}{f^2}\right)\\
= & \left(\frac{f+\kappa}{f}\right)^{s}\left(\frac{f-\kappa}{f}\right)^{-2q} 2^{1-s}\frac{\Gamma\tfrac{1}{2}s\Gamma\tfrac{1}{2}}{\Gamma(\tfrac{1}{2}s-q)\Gamma(\tfrac{1}{2}+q)}\;\Pi\left(\tfrac{1}{2}s-q, \tfrac{1}{2}+q, \tfrac{1}{2}-q, \frac{\varkappa^2}{f^2}\right),
\end{aligned}$$

where the Π-functions on the right-hand side are respectively

$$\begin{aligned}
&=f^{2q+1}\int_0^1 \frac{x^{\frac{1}{2}s+q-1}(1-x)^{-q-\frac{1}{2}}dx}{(f^2-\varkappa^2x)^{q+\frac{1}{2}}} &&=\frac{f^{2q+1}}{(f^2-\varkappa^2)^{2q}}\int_0^\infty t^{\frac{1}{2}s+q-1}(t+f^2-\varkappa^2)^{-\frac{1}{2}s+q}(t+f^2)^{-q-\frac{1}{2}}dt\\
&=f^{s+2q}\int_0^1 \frac{x^{q-\frac{1}{2}}(1-x)^{\frac{1}{2}s-q-1}dx}{(f^2-\varkappa^2x)^{\frac{1}{2}s+q}} &&=\frac{f^{s+2q}}{(f^2-\varkappa^2)^{2q}}\int_0^\infty t^{q-\frac{1}{2}}(t+f^2-\varkappa^2)^{q-\frac{1}{2}}(t+f^2)^{-\frac{1}{2}s-q}dt\\
&=f^{s-2q}\int_0^1 \frac{x^{-q-\frac{1}{2}}(1-x)^{\frac{1}{2}s+q-1}dx}{(f^2-\kappa^2x)^{\frac{1}{2}s-q}} &&=\frac{f^{s-2q}}{(f^2-\kappa^2)^{-2q}}\int_0^\infty t^{-q-\frac{1}{2}}(t+f^2-\varkappa^2)^{-q-\frac{1}{2}}(t+f^2)^{-\frac{1}{2}s+q}dt\\
&=f^{-2q+1}\int_0^1 \frac{x^{\frac{1}{2}s-q+1}(1-x)^{q-\frac{1}{2}}dx}{(f^2-\varkappa^2x)^{-q+\frac{1}{2}}} &&=\frac{f^{-2q+1}}{(f^2-\varkappa^2)^{-2q}}\int_0^\infty t^{\frac{1}{2}s-q-1}(t+f^2-\varkappa^2)^{-\frac{1}{2}s-q}(t+f^2)^{q-\frac{1}{2}}dt,
\end{aligned}$$

the t-forms being obtained by means of the transformation $x=\dfrac{t}{t+f^2-\varkappa^2}$; viz. this gives

$$1-x=\frac{f^2-\varkappa^2}{t+f^2-\kappa^2},\; f^2-\varkappa^2x=\frac{(f^2-\kappa^2)(t+f^2)}{t+f^2-\kappa^2},\; dx=\frac{(f^2-\kappa^2)\,dt}{(t+f^2-\varkappa^2)^2},$$

whence the results just written down.

We hence have

$$\text{Ring-integral} = \frac{f}{(f^2-\varkappa^2)^{2q}} \frac{\Gamma\tfrac{1}{2}s\Gamma\tfrac{1}{2}}{\Gamma(\tfrac{1}{2}s+q)\Gamma(\tfrac{1}{2}-q)} \int_0^\infty t^{\frac{1}{2}s+q-1}(t+f^2-\varkappa^2)^{-\frac{1}{2}s+q}(t+f^2)^{-q-\frac{1}{2}}dt$$

$$= \frac{f^s}{(f^2-\varkappa^2)^{2q}} \frac{\Gamma\tfrac{1}{2}s\Gamma\tfrac{1}{2}}{\Gamma(\tfrac{1}{2}+q)\Gamma(\tfrac{1}{2}s-q)} \int_0^\infty t^{q-\frac{1}{2}} \ (t+f^2-\varkappa^2)^{q-\frac{1}{2}} \ (t+f^2)^{-\frac{1}{2}s-q}dt$$

$$= \quad f^s \quad \frac{\Gamma\tfrac{1}{2}s\Gamma\tfrac{1}{2}}{\Gamma(\tfrac{1}{2}-q)\Gamma(\tfrac{1}{2}s+q)} \int_0^\infty t^{-q-\frac{1}{2}} \ (t+f^2-\varkappa^2)^{-q-\frac{1}{2}} \ (t+f^2)^{-\frac{1}{2}s+q}dt$$

$$= \quad f \quad \frac{\Gamma\tfrac{1}{2}s\Gamma\tfrac{1}{2}}{\Gamma(\tfrac{1}{2}s-q)\Gamma(\tfrac{1}{2}+q)} \int_0^\infty t^{\frac{1}{2}s-q-1}(t+f^2-\varkappa^2)^{-\frac{1}{2}s-q}(t+f^2)^{q-\frac{1}{2}} \ dt.$$

As a verification write $\varkappa=0$, the four integrals are

$$\int_0^\infty \frac{t^{\frac{1}{2}s+q-1}\,dt}{(t+f^2)^{\frac{1}{2}s+\frac{1}{2}}}, \quad = f^{2q-1} \quad \frac{\Gamma(\tfrac{1}{2}s+q)\Gamma(\tfrac{1}{2}-q)}{\Gamma(\tfrac{1}{2}s+\tfrac{1}{2})},$$

$$\int_0^\infty \frac{t^{\frac{1}{2}+q-1}\,dt}{(t+f^2)^{\frac{1}{2}s+\frac{1}{2}}}, \quad = f^{2q-s} \quad \frac{\Gamma(\tfrac{1}{2}+q)\Gamma(\tfrac{1}{2}s-q)}{\Gamma(\tfrac{1}{2}s+\tfrac{1}{2})},$$

$$\int_0^\infty \frac{t^{\frac{1}{2}-q-1}\,dt}{(t+f^2)^{\frac{1}{2}s+\frac{1}{2}}}, \quad = f^{-2q-s} \quad \frac{\Gamma(\tfrac{1}{2}-q)\Gamma(\tfrac{1}{2}s+q)}{\Gamma(\tfrac{1}{2}s+\tfrac{1}{2})},$$

$$\int_0^\infty \frac{t^{\frac{1}{2}s-q-1}\,dt}{(t+f^2)^{\frac{1}{2}s+\frac{1}{2}}}, \quad = f^{-2q-1} \quad \frac{\Gamma(\tfrac{1}{2}s-q)\Gamma(\tfrac{1}{2}+q)}{\Gamma(\tfrac{1}{2}s+\tfrac{1}{2})},$$

and hence from each of them

$$\text{Ring-integral} = \frac{1}{f^{2q}} \frac{\Gamma\tfrac{1}{2}s\Gamma\tfrac{1}{2}}{\Gamma(\tfrac{1}{2}s+\tfrac{1}{2})},$$

which is in fact the value obtained from

$$\text{Ring-integral} = \frac{2^{s-1}f^s}{(f+\kappa)^{s+2q}} \Pi\left(\tfrac{1}{2}s, \tfrac{1}{2}s, \tfrac{1}{2}s+q, \frac{4\varkappa f}{(\varkappa+f)^2}\right)$$

on putting therein $\varkappa=0$; viz. the value is

$$= \frac{2^{s-1}}{f^{2q}} \int_0^1 x^{\frac{1}{2}s-1}(1-x)^{\frac{1}{2}s-1}dx, \quad = \frac{1}{f^{2q}} \frac{2^{s-1}\Gamma\tfrac{1}{2}s\,.\,\Gamma\tfrac{1}{2}s}{\Gamma s}.$$

76. External point $\varkappa>f$, $\sqrt{1-u}=\dfrac{\varkappa-f}{\varkappa+f}$, and therefore $v=\dfrac{f^2}{\varkappa^2}$.

$$\Pi(\tfrac{1}{2}s, \tfrac{1}{2}s, \tfrac{1}{2}s+q, u) = \left(\frac{\varkappa+f}{\varkappa}\right)^{s+2q} \qquad 2^{1-s} \frac{\Gamma\tfrac{1}{2}s\Gamma\tfrac{1}{2}}{\Gamma(\tfrac{1}{2}s+q)\Gamma(\tfrac{1}{2}-q)} \Pi\left(\tfrac{1}{2}s+q, \tfrac{1}{2}-q, \tfrac{1}{2}+q, \frac{f^2}{\varkappa^2}\right)$$

$$= \left(\frac{\varkappa+f}{\varkappa}\right)^{s+2q} \qquad 2^{1-s} \frac{\Gamma\tfrac{1}{2}s\Gamma\tfrac{1}{2}}{\Gamma(\tfrac{1}{2}+q)\Gamma(\tfrac{1}{2}s-q)} \Pi\left(\tfrac{1}{2}+q, \tfrac{1}{2}s-q, \tfrac{1}{2}s+q, \frac{f^2}{\varkappa^2}\right)$$

$$= \left(\frac{\varkappa+f}{\varkappa}\right)^{s}\left(\frac{\varkappa-f}{\varkappa}\right)^{-2q} 2^{1-s} \frac{\Gamma\tfrac{1}{2}s\Gamma\tfrac{1}{2}}{\Gamma(\tfrac{1}{2}-q)\Gamma(\tfrac{1}{2}s+q)} \Pi\left(\tfrac{1}{2}-q, \tfrac{1}{2}s+q, \tfrac{1}{2}s-q, \frac{f^2}{\varkappa^2}\right)$$

$$= \left(\frac{\varkappa+f}{\varkappa}\right)^{s}\left(\frac{\varkappa-f}{\varkappa}\right)^{-2q} 2^{1-s} \frac{\Gamma\tfrac{1}{2}s\Gamma\tfrac{1}{2}}{\Gamma(\tfrac{1}{2}s-q)\Gamma(\tfrac{1}{2}+q)} \Pi\left(\tfrac{1}{2}s-q, \tfrac{1}{2}+q, \tfrac{1}{2}-q, \frac{f^2}{\varkappa^2}\right),$$

where the Π-functions on the right hand are respectively

$$=\kappa^{2q+1}\int_0^1 \frac{x^{\frac12 s+q-1}(1-x)^{-q-\frac12}dx}{(\kappa^2-f^2x)^{q+\frac12}} \quad\Bigg|\quad =\frac{\kappa^{2q+1}}{(\kappa^2-f^2)^{2q}}\int_{\kappa^2-f^2}^{\infty} t^{-\frac12 s+q}(t+f^2-\kappa^2)^{\frac12 s+q-1}(t+f^2)^{-q-\frac12}\,dt,$$

$$=\kappa^{s+2q}\int_0^1 \frac{x^{q-\frac12}(1-x)^{\frac12 s-q-1}dx}{(\kappa^2-f^2x)^{\frac12 s+q}} \quad\Bigg|\quad =\frac{\kappa^{s+2q}}{(\kappa^2-f^2)^{2q}}\int_{\kappa^2-f^2}^{\infty} t^{q-\frac12}\,(t+f^2-\kappa^2)^{q-\frac12}\,(t+f^2)^{-\frac12 s-q}dt,$$

$$=\kappa^{s-2q}\int_0^1 \frac{x^{-q-\frac12}(1-x)^{\frac12 s+q-1}dx}{(\kappa^2-f^2x)^{\frac12 s-q}} \quad\Bigg|\quad =\frac{\kappa^{s-2q}}{(\kappa^2-f^2)^{-2q}}\int_{\kappa^2-f^2}^{\infty} t^{-q-\frac12}\,(t+f^2-\kappa^2)^{-q-\frac12}\,(t+f^2)^{-\frac12 s+q}dt,$$

$$=\kappa^{-2q+1}\int_0^1 \frac{x^{\frac12 s-q+1}(1-x)^{q-\frac12}dx}{(\kappa^2-f^2x)^{-q+\frac12}} \quad\Bigg|\quad =\frac{\kappa^{-2q+1}}{(\kappa^2-f^2)^{-2q}}\int_{\kappa^2-f^2}^{\infty} t^{-\frac12 s-q}(t+f^2-\kappa^2)^{\frac12 s-q-1}(t+f^2)^{q-\frac12}\,dt\,;$$

we have then

$$\begin{aligned}
\text{Ring-integral}&=\frac{f^s\kappa^{1-s}}{(\kappa^2-f^2)^{2q}}\,\frac{\Gamma\frac12 s\Gamma\frac12}{\Gamma(\frac12 s+q)\Gamma(\frac12-q)}\int_{\kappa^2-f^2}^{\infty} t^{-\frac12 s+q}(t+f^2-\kappa^2)^{\frac12 s+q-1}(t+f^2)^{-q-\frac12}\,dt\\
&=\frac{f^s}{(\kappa^2-f^2)^{2q}}\,\frac{\Gamma\frac12 s\Gamma\frac12}{\Gamma(\frac12+q)\Gamma(\frac12 s-q)}\int_{\kappa^2-f^2}^{\infty} t^{q-\frac12}\,(t+f^2-\kappa^2)^{q-\frac12}\,(t+f^2)^{-\frac12 s-q}dt\\
&=\quad f^s\quad\frac{\Gamma\frac12 s\Gamma\frac12}{\Gamma(\frac12-q)\Gamma(\frac12 s+q)}\int_{\kappa^2-f^2}^{\infty} t^{-q-\frac12}\,(t+f^2-\kappa^2)^{-q-\frac12}\,(t+f^2)^{-\frac12 s+q}dt\\
&=\frac{f^s\kappa^{1-s}}{(\kappa^2-f^2)^{2q}}\,\frac{\Gamma\frac12 s\Gamma\frac12}{\Gamma(\frac12 s-q)\Gamma(\frac12+q)}\int_{\kappa^2-f^2}^{\infty} t^{-\frac12 s-q}(t+f^2-\kappa^2)^{\frac12 s-q-1}(t+f^2)^{q-\frac12}\,dt.
\end{aligned}$$

Observe that in II. and III. the integrals, except as to the limits, are the same as in the corresponding formulæ for the interior point.

If in the t-integrals we put $t+\kappa^2-f^2$ in place of t, and ultimately suppose κ indefinitely large in comparison with f, they severally become

$$\int_0^\infty (t+\kappa^2-f^2)^{-\frac12 s+q}t^{\frac12 s+q-1}(t+\kappa^2)^{-q-\frac12}\,dt=\int_0^\infty \frac{t^{\frac12 s+q-1}dt}{(t+\kappa^2)^{\frac12 s+\frac12}}=\kappa^{2q-1}\,\frac{\Gamma(\frac12 s+q)\Gamma(\frac12-q)}{\Gamma(\frac12 s+\frac12)},$$

$$\int_0^\infty (t+\kappa^2-f^2)^{q-\frac12}\,t^{q+\frac12}\,(t+\kappa^2)^{-\frac12 s-q}dt=\int_0^\infty \frac{t^{\frac12+q-1}dt}{(t+\kappa^2)^{\frac12 s+\frac12}}=\kappa^{2q-s}\,\frac{\Gamma(\frac12+q)\Gamma(\frac12 s-q)}{\Gamma(\frac12 s+\frac12)},$$

$$\int_0^\infty (t+\kappa^2-f^2)^{-q-\frac12}\,t^{-q-\frac12}\,(t+\kappa^2)^{-\frac12 s+q}dt=\int_0^\infty \frac{t^{\frac12-q-1}dt}{(t+\kappa^2)^{\frac12 s+\frac12}}=\kappa^{-2q-s}\frac{\Gamma(\frac12-q)\Gamma(\frac12 s+q)}{\Gamma(\frac12 s+\frac12)},$$

$$\int_0^\infty (t+\kappa^2-f^2)^{-\frac12 s-q}t^{\frac12 s-q-1}(t+\kappa^2)^{q-\frac12}\,dt=\int_0^\infty \frac{t^{\frac12 s-q-1}dt}{(t+\kappa^2)^{\frac12 s+\frac12}}=\kappa^{-2q-1}\frac{\Gamma(\frac12 s-q)\Gamma(\frac12+q)}{\Gamma(\frac12 s+\frac12)}\,;$$

and they all four give

$$\text{Ring-integral}=\frac{f^s}{\kappa^{s+2q}}\cdot\frac{\Gamma\frac12 s\Gamma\frac12}{\Gamma(\frac12 s+\frac12)},$$

which agrees with the value

$$\frac{2^{s-1}f^s}{(\kappa+f)^{s+2q}}\,\Pi\left(\tfrac12 s,\ \tfrac12 s,\ \tfrac12 s+q,\ \frac{4\kappa f}{(\kappa+f)^2}\right),\ =\frac{2^{s-1}f^s}{\kappa^{s+2q}}\,\Pi(\tfrac12 s,\ \tfrac12 s,\ \tfrac12 s+q,\ 0)$$

when $\frac{\kappa}{f}$ is indefinitely large.

77. We come now to the disk-integral,

$$\int \frac{y^{s-1}\,dx\,dy}{\{(x-\varkappa)^2+y^2\}^{\frac{1}{2}s+q}},$$

over the circle $x^2+y^2=f^2$. Writing $x=\varkappa+\varrho\cos\phi, y=\varrho\sin\phi$, we have $dx\,dy=\varrho\,d\varrho\,d\phi$, and the integral therefore is

$$\int \frac{\sin^{s-1}\phi\,d\varrho\,d\phi}{\varrho^{2q}},$$

where the integration in regard to ϱ is performed at once, viz. the integral is

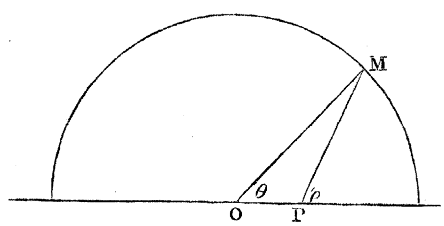
$$=\frac{1}{1-2q}\int(\varrho^{1-2q})\sin^{s-1}\phi\,d\phi\,;$$

or multiplying by 2, in order that the integration may be taken only over the semicircle, y=positive, this is

$$=\frac{1}{\frac{1}{2}-q}\int(\varrho^{1-2q})\sin^{s-1}\phi\,d\phi,$$

the term (ϱ^{1-2q}) being taken between the proper limits.

78. Consider first an interior point $\varkappa<f$. As already mentioned, we exclude an indefinitely small circle radius ε, and the limits for ϱ are from $\varrho=\varepsilon$ to ϱ=its value at the



circumference; viz. if here $x=f\cos\theta$, $y=f\sin\theta$, then we have $f\cos\theta=\varkappa+\varrho\cos\phi$, $f\sin\theta=\varrho\sin\phi$, and consequently

$$\varrho^2=\varkappa^2+f^2-2\varkappa f\cos\theta,$$

$$\sin\phi=\frac{f}{\varrho}\sin\theta,\ =\frac{f\sin\theta}{\sqrt{\varkappa^2+f^2-2\varkappa f\cos\theta}},$$

and the integral therefore is

$$=\frac{1}{\frac{1}{2}-q}\left(\frac{f^{s-1}\sin^{s-1}\theta}{\{\varkappa^2+f^2-2\varkappa f\cos\theta\}^{\frac{1}{2}s+q-1}}-\varepsilon^{1-2q}\sin^{s-1}\phi\right)d\phi.$$

As regards the second term, this is $=-\frac{\varepsilon^{1-2q}}{\frac{1}{2}-q}\int\sin^{s-1}\phi\,d\phi$, $\phi=0$ to $\phi=\pi$, or, what is the same thing, we may multiply by 2 and take the integral from $\phi=0$ to $\phi=\frac{\pi}{2}$. Writing then $\sin\phi=\sqrt{x}$, and consequently $\sin^{s-1}\phi\,d\phi=\frac{1}{2}x^{\frac{1}{2}s-1}(1-x)^{-\frac{1}{2}}\,dx$, the term is $=-\frac{\varepsilon^{1-2q}}{\frac{1}{2}-q}\frac{\Gamma\frac{1}{2}s\Gamma\frac{1}{2}}{\Gamma(\frac{1}{2}s+\frac{1}{2})}$, and the value of the disk-integral is

$$=\frac{f^{s-1}}{\frac{1}{2}-q}\int\frac{\sin^{s-1}\theta\,d\phi}{(\varkappa^2+f^2-2\varkappa f\cos\theta)^{\frac{1}{2}s+q-1}}-\frac{\varepsilon^{1-2q}}{\frac{1}{2}-q}\frac{\Gamma\frac{1}{2}s\Gamma\frac{1}{2}}{\Gamma(\frac{1}{2}s+\frac{1}{2})}.$$

But we have

$$\sin\phi=\frac{f\sin\theta}{\varrho},\quad \cos\phi=\frac{f\cos\theta-\varkappa}{\varrho},$$

and thence

$$\tan\phi=\frac{f\sin\theta}{f\cos\theta-\varkappa},\quad \sec^2\phi\,d\phi=\frac{f(f-\varkappa\cos\theta)d\theta}{(f\cos\theta-\varkappa)^2};$$

that is

$$d\phi=\frac{f(f-\varkappa\cos\theta)d\theta}{\varrho^2},=\frac{f(f-\varkappa\cos\theta)d\theta}{f^2+\varkappa^2-2\varkappa f\cos\theta},$$

or, what is the same thing,

$$=\frac{\tfrac{1}{2}\{(f^2-\varkappa^2)+(f^2+\varkappa^2-2\varkappa f\cos\theta)\}}{f^2+\varkappa^2-2\varkappa f\cos\theta};$$

and the expression for the disk-integral is therefore

$$=\frac{\tfrac{1}{2}f^{s-1}}{\tfrac{1}{2}-q}\int_0^\pi\frac{\sin^{s-1}\theta\{(f^2-\varkappa^2)+(f^2+\varkappa^2-2\varkappa f\cos\theta)\}d\theta}{\{f^2+\varkappa^2-2\varkappa f\cos\theta\}^{\frac{1}{2}s+q}}-\frac{\varepsilon^{1-2q}}{\tfrac{1}{2}-q}\frac{\Gamma\tfrac{1}{2}s\Gamma\tfrac{1}{2}}{\Gamma(\tfrac{1}{2}s+\tfrac{1}{2})}.$$

79. Writing as before $\cos\tfrac{1}{2}\theta=\sqrt{x}$, $\sin\tfrac{1}{2}\theta=\sqrt{1+x}$, &c., and $u=\frac{4\varkappa f}{(\kappa+f)^2}$, this is

$$=\frac{2^{s-2}f^{s-1}}{(\tfrac{1}{2}-q)(\varkappa+f)^{s+2q-2}}\left\{\frac{(f^2-\varkappa^2)}{(\varkappa+f)^2}\,\Pi(\tfrac{1}{2}s,\tfrac{1}{2}s,\tfrac{1}{2}s+q,u)+\Pi(\tfrac{1}{2}s,\tfrac{1}{2}s,\tfrac{1}{2}s+q-1,u)\right\}-\frac{\varepsilon^{1-2q}}{\tfrac{1}{2}-q}\frac{\Gamma\tfrac{1}{2}s\Gamma\tfrac{1}{2}}{\Gamma(\tfrac{1}{2}s+\tfrac{1}{2})}.$$

As a verification observe that if $\varkappa=0$, each of the Π-functions becomes

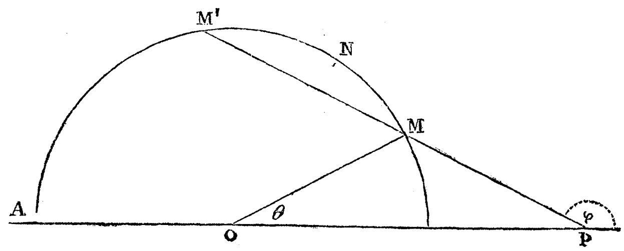
$$=\int_0^1 x^{\frac{1}{2}s-1}(1-x)^{\frac{1}{2}s-1}\,dx,=\frac{\Gamma\tfrac{1}{2}s\,\Gamma\tfrac{1}{2}s}{\Gamma s};$$

hence the whole first term is $=\frac{2\,.\,2^{s-2}\,.\,f^{1-2q}}{\tfrac{1}{2}-q}\,.\,\frac{\Gamma\tfrac{1}{2}s\,\Gamma\tfrac{1}{2}s}{\Gamma s}$, viz. this is $=\frac{f^{1-2q}}{\tfrac{1}{2}-q}\frac{\Gamma\tfrac{1}{2}s\Gamma\tfrac{1}{2}}{\Gamma(\tfrac{1}{2}s+\tfrac{1}{2})}$, and the complete value is

$$=\frac{1}{\tfrac{1}{2}-q}\frac{\Gamma\tfrac{1}{2}s\,\Gamma\tfrac{1}{2}}{\Gamma(\tfrac{1}{2}s+\tfrac{1}{2})}\{f^{1-2q}-\varepsilon^{1-2q}\},$$

vanishing, as it should do, if $f=\varepsilon$.

80. In the case of an exterior point $\varkappa>f$ the process is somewhat different, but the



result is of a like form. We have

$$\text{Disk-integral}=\frac{1}{\tfrac{1}{2}-q}\int(\varrho_1{}^{1-2q}-\varrho^{1-2q})\sin^{s-1}\phi\,d\phi,$$

ϱ_1 referring to the point M′ and ϱ to the point M. Attending first to the integral $\int\varrho^{1-2q}\sin^{s-1}\phi\,d\phi$, and writing as before $f\cos\theta=\varkappa+\varrho\cos\phi$, $f\sin\theta=\varrho\sin\phi$, this is

$$=f^{s-1}\int\frac{\sin^{s-1}\theta\,d\phi}{\{\varkappa^2+f^2-2\varkappa f\cos\theta\}^{\frac{1}{2}s+q}}$$

$$=\tfrac{1}{2}f^{s-1}\int\frac{\sin^{s-1}\theta\{(f^2-\varkappa^2)+(f^2+\varkappa^2-2\varkappa f\cos\theta)\}d\theta}{(f^2+\varkappa^2-2f\varkappa\cos\theta)^{\frac{1}{2}s+q}},$$

the inferior and superior limits being here the values of θ which correspond to the points N, A respectively, say $\theta+\alpha$, and $\theta=0$; hence, reversing the sign and interchanging the two limits, the value of $-\int\varrho^{1-2q}\sin^{s-1}\theta\,d\phi$ is the above integral taken from 0 to α. But similarly the value of $+\int\varrho_1^{1-2q}\sin^{s-1}\theta\,d\phi$ is the same integral taken from α to π; and for the two terms together the value is the same integral from 0 to π; viz. we thus find

$$\text{Disk-integral}=\frac{\frac{1}{2}f^{s-1}}{\frac{1}{2}-q}\int_0^\pi\frac{\sin^{s-1}\theta\{-(\varkappa^2-f^2)+(f^2+\varkappa^2-2\varkappa f\cos\theta)\}d\theta}{(f^2+\varkappa^2-2f\varkappa\cos\theta)^{\frac{1}{2}s+q}};$$

viz. writing as before $\cos\frac{1}{2}\theta=\sqrt{x}$ &c., and $u=\frac{4\varkappa f}{(\varkappa+f)^2}$, this is

$$=\frac{2^{s-2}f^{s-1}}{(\frac{1}{2}-q)(\varkappa+f)^{s+2q-2}}\left\{-\frac{\varkappa^2-f^2}{(\varkappa+f)^2}\cdot\Pi(\tfrac{1}{2}s,\tfrac{1}{2}s,\tfrac{1}{2}s+q,u)+\Pi(\tfrac{1}{2}s,\tfrac{1}{2}s,\tfrac{1}{2}s+q-1)\right\}.$$

81. As a verification, suppose that $\varkappa$ is indefinitely large: we must recur to the last preceding formula; the value is thus

$$=\frac{f^s}{(\frac{1}{2}-q)\varkappa^{s+2q-1}}\int_0^\pi\frac{\sin^{s-1}\theta\left(-\cos\theta+\frac{f}{\varkappa}\right)}{\left(1-\frac{2f}{\varkappa}\cos\theta\right)^{\frac{1}{2}s+q}};$$

viz. this is

$$=\frac{f^s}{(\frac{1}{2}-q)\varkappa^{s+2q-1}}\int_0^\pi\sin^{s-1}\theta\left\{-\cos\theta+[1-(s+2q)\cos^2\theta]\frac{f}{\varkappa}\right\}d\theta,$$

where the integral of the first term vanishes; the value is thus

$$=\frac{f^{s+1}}{(\frac{1}{2}-q)\varkappa^{s+2q}}\int_0^\pi\sin^{s-1}\theta[1-(s+2q)\cos^2\theta]\,d\theta,$$

where we may multiply by 2 and take the integral from 0 to $\frac{\pi}{2}$. Writing then $\sin\theta=\sqrt{x}$, the value is

$$=\frac{f^{s+1}}{(\frac{1}{2}-q)\varkappa^{s+2q}}\int_0^1 x^{\frac{1}{2}s-\frac{1}{2}}\{1-(s+2q)(1-x)\}(1-x)^{-\frac{1}{2}}\,dx,$$

where the integral is $=\frac{\Gamma\frac{1}{2}s\,\Gamma\frac{1}{2}}{\Gamma(\frac{1}{2}s+\frac{1}{2})}\left(1-\frac{\frac{1}{2}(s+2q)}{\frac{1}{2}s+\frac{1}{2}}\right)$, $=\frac{\Gamma\frac{1}{2}s\Gamma\frac{1}{2}}{\Gamma(\frac{1}{2}s+\frac{1}{2})}\cdot\frac{\frac{1}{2}-q}{\frac{1}{2}s+\frac{1}{2}}$, and hence the value is

$$=\frac{f^{s+1}}{\varkappa^{s+2q}}\cdot\frac{\Gamma\frac{1}{2}s\,\Gamma\frac{1}{2}}{\Gamma(\frac{1}{2}s+\frac{3}{2})};$$

viz. this is $=\frac{1}{\varkappa^{s+2q}}\int y^{s-1}\,dx\,dy$, over the circle $x^2+y^2=f^2$, as is easily verified.

82. Reverting to the interior point $\varkappa<f$,

Disk-integral

$$=\frac{2^{s-2}f^{s-1}}{(\frac{1}{2}-q)(\varkappa+f)^{s+2q-2}}\left\{\frac{f-\varkappa}{f+\varkappa}\Pi(\tfrac{1}{2}s,\tfrac{1}{2}s,\tfrac{1}{2}s+q,u)+\Pi(\tfrac{1}{2}s,\tfrac{1}{2}s,\tfrac{1}{2}s+q-1,u)\right\}-\frac{\varepsilon^{1-2q}}{\frac{1}{2}-q}\frac{\Gamma\frac{1}{2}s\Gamma\frac{1}{2}}{\Gamma(\frac{1}{2}s+\frac{1}{2})};$$

then reducing the expression in $\{\ \}$ by the transformations for $\Pi(\frac{1}{2}s,\frac{1}{2}s,\frac{1}{2}s+q,u)$ and the like transformations for $\Pi(\frac{1}{2}s,\frac{1}{2}s,\frac{1}{2}s+q-1,u)$, the term in $\{\ \}$ may be expressed in the four forms:—

$$2^{1-s}\frac{\Gamma\frac{1}{2}s\,\Gamma\frac{1}{2}}{\Gamma(\frac{1}{2}s+q)\Gamma(\frac{1}{2}-q)}\frac{(f+\varkappa)^{s+2q-2}}{f^{s+2q-2}}\text{ into}$$

$$\left[\left(1-\frac{\varkappa^2}{f^2}\right)\Pi\left(\tfrac{1}{2}s+q,\tfrac{1}{2}-q,\tfrac{1}{2}+q,\frac{\varkappa^2}{f^2}\right)+\frac{\frac{1}{2}s+q-1}{\frac{1}{2}-q}\Pi\left(\tfrac{1}{2}s+q-1,\tfrac{3}{2}-q,-\tfrac{1}{2}+q,\frac{\varkappa^2}{f^2}\right)\right],$$

$$2^{1-s}\frac{\Gamma\frac{1}{2}s\,\Gamma\frac{1}{2}}{\Gamma(\frac{1}{2}+q)\Gamma(\frac{1}{2}s-q)}\frac{(f+\varkappa)^{s+2q-2}}{f^{s+2q-2}}\text{ into}$$

$$\left[\left(1-\frac{\varkappa^2}{f^2}\right)\Pi\left(\tfrac{1}{2}+q,\tfrac{1}{2}s-q,\tfrac{1}{2}s+q,\frac{\varkappa^2}{f^2}\right)+\frac{-\frac{1}{2}+q}{\frac{1}{2}s+q}\Pi\left(-\tfrac{1}{2}+q,\tfrac{1}{2}s-q+1,\tfrac{1}{2}s+q-1,\frac{\varkappa^2}{f^2}\right)\right],$$

$$2^{1-s}\frac{\Gamma\frac{1}{2}s\,\Gamma\frac{1}{2}}{\Gamma(\frac{1}{2}-q)\Gamma(\frac{1}{2}s+q)}\frac{(f+\varkappa)^{s-1}(f-\varkappa)^{1-2q}}{f^{s-2q}}\text{ into}$$

$$\left[\Pi\left(\tfrac{1}{2}-q,\tfrac{1}{2}s+q,\tfrac{1}{2}s-q,\frac{\varkappa^2}{f^2}\right)+\left(1-\frac{\varkappa^2}{f^2}\right)\frac{\frac{1}{2}s+q-1}{\frac{1}{2}-q}\Pi\left(\tfrac{3}{2}-q,\tfrac{1}{2}s+q-1,\tfrac{1}{2}s-q+1,\frac{\varkappa^2}{f^2}\right)\right],$$

$$2^{1-s}\frac{\Gamma\frac{1}{2}s\,\Gamma\frac{1}{2}}{\Gamma(\frac{1}{2}s-q)\Gamma(\frac{1}{2}+q)}\frac{(f+\varkappa)^{s-1}(f-\varkappa)^{1-2q}}{f^{s-2q}}\text{ into}$$

$$\left[\Pi\left(\tfrac{1}{2}s-q,\tfrac{1}{2}+q,\tfrac{1}{2}-q,\frac{\varkappa^2}{f^2}\right)+\left(1-\frac{\varkappa^2}{f^2}\right)\frac{-\frac{1}{2}+q}{\frac{1}{2}s-q}\Pi\left(\tfrac{1}{2}s-q+1,-\tfrac{1}{2}+q,\tfrac{3}{2}-q,\frac{\varkappa^2}{f^2}\right)\right].$$

83. The first and fourth of these are susceptible of a reduction which does not appear to be applicable to the second and third. Consider in general the function

$$(1-v)\Pi(\alpha,\beta,1-\beta,v)+\frac{\alpha-1}{\beta}\Pi(\alpha-1,\beta+1,-\beta,v);$$

the second Π-function is here

$$\int_0^1 x^{\alpha-2}(1-x\,.\,1-vx)^{\beta}\,dx;$$

viz. this is

$$=\frac{x^{\alpha-1}}{\alpha-1}(1-x\,.\,1-vx)^{\beta}-\frac{1}{\alpha-1}\int_0^1 x^{\alpha-1}\frac{d}{dx}(1-x\,.\,1-vx)^{\beta}dx,$$

or, since the first term vanishes between the limits, this is

$$=\frac{\beta}{\alpha-1}\int_0^1 x^{\alpha-1}.(1-x\,.\,1-vx)^{\beta-1}(1+v-2vx)dx,$$

$$=\frac{\beta}{\alpha-1}\left\{(1+v)\Pi(\alpha,\beta,1-\beta,v)-2v\,.\int_0^1 x^{\alpha}(1-x\,.\,1-vx)^{\beta-1}dx\right\}.$$

Hence the two Π-functions together are

$$=(1-v+1+v)\int_0^1 x^{\alpha-1}(1-x.1-vx)^{\beta-1}dx-2\int_0^1 vx.x^{\alpha-1}(1-x.1-vx)^{\beta-1}dx,$$
$$=2\int_0^1 x^{\alpha-1}(1-x)^{\beta-1}(1-vx)^{\beta}dx,$$

that is

$$(1-v)\Pi(\alpha,\beta,1-\beta,v)+\frac{\alpha-1}{\beta}\Pi(\alpha-1,\beta+1,-\beta,v)=2\Pi(\alpha,\beta,-\beta,v).$$

We have therefore

$$\left(1-\frac{\varkappa^2}{f^2}\right)\Pi\left(\tfrac{1}{2}s+q,\tfrac{1}{2}-q,\tfrac{1}{2}+q,\frac{\varkappa^2}{f^2}\right)+\frac{\frac{1}{2}s+q-1}{\frac{1}{2}-q}\Pi\left(\tfrac{1}{2}s+q-1,\tfrac{3}{2}-q,-\tfrac{1}{2}+q,\frac{\varkappa^2}{f^2}\right)$$
$$=2\Pi\left(\tfrac{1}{2}s+q,\tfrac{1}{2}-q,-\tfrac{1}{2}+q,\frac{\varkappa^2}{f^2}\right);$$

and from the same equation written in the form

$$\Pi(\alpha-1,\beta+1,-\beta,v)+\frac{\beta}{\alpha-1}(1-v)\Pi(\alpha,\beta,1-\beta,v)=2\frac{\beta}{\alpha-1}\Pi(\alpha,\beta,-\beta,v),$$

we obtain

$$\Pi\left(\tfrac{1}{2}s-q,\tfrac{1}{2}+q,\tfrac{1}{2}-q,\frac{\varkappa^2}{f^2}\right)+\frac{-\frac{1}{2}+q}{\frac{1}{2}s-q}\left(1-\frac{\varkappa^2}{f^2}\right)\Pi\left(\tfrac{1}{2}s-q+1,-\tfrac{1}{2}+q,\tfrac{3}{2}-q,\frac{\varkappa^2}{f^2}\right)$$
$$=\frac{2(-\frac{1}{2}+q)}{\frac{1}{2}s-q}\Pi\left(\tfrac{1}{2}s-q+1,-\tfrac{1}{2}+q,\tfrac{1}{2}-q,\frac{\varkappa^2}{f^2}\right).$$

84. Hence the terms in [ ] are

$$=\frac{2^{2-s}\Gamma\frac{1}{2}s\Gamma\frac{1}{2}}{\Gamma(\frac{1}{2}s+q)\Gamma(\frac{1}{2}-q)}\cdot\frac{(f+\varkappa)^{s+2q-2}}{f^{s+2q-2}}\cdot\Pi\left(\tfrac{1}{2}s+q,\tfrac{1}{2}-q,-\tfrac{1}{2}+q,\frac{\varkappa^2}{f^2}\right),$$
$$=\frac{2^{2-s}(-\frac{1}{2}+q).\Gamma\frac{1}{2}s\Gamma\frac{1}{2}}{\Gamma(\frac{1}{2}s-q+1)\Gamma(\frac{1}{2}+q)}\frac{(f+\varkappa)^{s-1}(f-\varkappa)^{1-2q}}{f^{s-2q}}\Pi\left(\tfrac{1}{2}s-q+1,-\tfrac{1}{2}+q,\tfrac{1}{2}-q,\frac{\varkappa^2}{f^2}\right),$$

respectively, and the corresponding values of the disk-integral are

$$\frac{\Gamma\frac{1}{2}s\Gamma\frac{1}{2}}{\Gamma(\frac{3}{2}-q)\Gamma(\frac{1}{2}s+q)}f^{1-2q}.\Pi\left(\tfrac{1}{2}s+q,\tfrac{1}{2}-q,-\tfrac{1}{2}+q,\frac{\varkappa^2}{f^2}\right)\qquad-\frac{\epsilon^{1-2q}}{\frac{1}{2}-q}\frac{\Gamma\frac{1}{2}s\Gamma\frac{1}{2}}{\Gamma(\frac{1}{2}s+\frac{1}{2})},$$
$$\frac{-\Gamma\frac{1}{2}s\Gamma\frac{1}{2}}{\Gamma(\frac{1}{2}s-q+1)\Gamma(\frac{1}{2}+q)}\left(\frac{f^2-\varkappa^2}{f}\right)^{1-2q}.\Pi\left(\tfrac{1}{2}s-q+1,-\tfrac{1}{2}+q,\tfrac{1}{2}-q,\frac{\varkappa^2}{f^2}\right)-\frac{\epsilon^{1-2q}}{\frac{1}{2}-q}\frac{\Gamma\frac{1}{2}s\Gamma\frac{1}{2}}{\Gamma(\frac{1}{2}s+\frac{1}{2})},$$

which we may again verify by writing therein $\varkappa=0$, viz. the Π-functions thus become

$$\frac{\Gamma(\frac{1}{2}s+q).\Gamma(\frac{1}{2}-q)}{\Gamma(\frac{1}{2}s+\frac{1}{2})}\text{ and }\frac{\Gamma(\frac{1}{2}s-q+1)\Gamma(-\frac{1}{2}+q)}{\Gamma(\frac{1}{2}s+\frac{1}{2})},$$

and consequently the integral is

$$=\frac{1}{\frac{1}{2}-q}\cdot\frac{\Gamma\frac{1}{2}s\Gamma\frac{1}{2}}{\Gamma(\frac{1}{2}s+\frac{1}{2})}(f^{1-2q}-\epsilon^{1-2q}).$$

85. But the forms nevertheless belong to a system of four; from the formulæ

$$
\begin{aligned}
&\quad\Pi(\alpha,\beta,\gamma,v)\\
&=\frac{\Gamma\alpha\Gamma\beta}{\Gamma\gamma\Gamma(\alpha+\beta-\gamma)}\,\Pi(\gamma,\alpha+\beta-\gamma,\alpha,v)\\
&=(1-v)^{\beta-\gamma}\,\Pi(\beta,\alpha,\alpha+\beta-\gamma,v)\\
&=(1-v)^{\beta-\gamma}\frac{\Gamma\alpha\Gamma\beta}{\Gamma(\alpha+\beta-\gamma)\Gamma\gamma}\,\Pi(\alpha+\beta-\gamma,\gamma,\beta,v),
\end{aligned}
$$

writing therein $\alpha=\frac{1}{2}s+q$, $\beta=\frac{1}{2}-q$, $\gamma=-\frac{1}{2}+q$, we deduce

$$
\begin{aligned}
&\quad\Pi(\tfrac{1}{2}s+q,\tfrac{1}{2}-q,-\tfrac{1}{2}+q,v)\\
&=\frac{\Gamma(\frac{1}{2}s+q)\Gamma(\frac{1}{2}-q)}{\Gamma(-\frac{1}{2}+q)\Gamma(\frac{1}{2}s-q+1)}\,\Pi(-\tfrac{1}{2}+q,\tfrac{1}{2}s-q+1,\tfrac{1}{2}s+q,v)\\
&=(1-v)^{1-2q}\,\Pi(\tfrac{1}{2}-q,\tfrac{1}{2}s+q,\tfrac{1}{2}s-q+1,v)\\
&=(1-v)^{1-2q}\frac{\Gamma(\frac{1}{2}s+q)\Gamma(\frac{1}{2}-q)}{\Gamma(\frac{1}{2}s-q+1)\Gamma(-\frac{1}{2}+q)}\,\Pi(\tfrac{1}{2}s-q+1,-\tfrac{1}{2}+q,\tfrac{1}{2}-q,v);
\end{aligned}
$$

and the last-mentioned values of the disk-integral may thus be written in the four forms:

$$
\begin{array}{llll}
\dfrac{\Gamma\frac{1}{2}s\Gamma\frac{1}{2}}{\Gamma(\frac{3}{2}-q)\Gamma(\frac{1}{2}s+q)} & f^{1-2q} & \Pi\left(\frac{1}{2}s+q,\frac{1}{2}-q,-\frac{1}{2}+q,\dfrac{\varkappa^2}{f^2}\right) & -\text{term in }\varepsilon,\\
\dfrac{-\Gamma\frac{1}{2}s\Gamma\frac{1}{2}}{\Gamma(\frac{1}{2}+q)\Gamma(\frac{1}{2}s-q+1)} & f^{1-2q} & \Pi\left(-\frac{1}{2}+q,\frac{1}{2}s-q+1,\frac{1}{2}s+q,\dfrac{\varkappa^2}{f^2}\right) & -\quad ,, \quad ,\\
\dfrac{\Gamma\frac{1}{2}s\Gamma\frac{1}{2}}{\Gamma(\frac{3}{2}-q)\Gamma(\frac{1}{2}s+q)} & \left(f-\dfrac{\varkappa^2}{f}\right)^{1-2q} & \Pi\left(\frac{1}{2}-q,\frac{1}{2}s+q,\frac{1}{2}s-q+1,\dfrac{\varkappa^2}{f^2}\right) & -\quad ,, \quad ,\\
\dfrac{-\Gamma\frac{1}{2}s\Gamma\frac{1}{2}}{\Gamma(\frac{1}{2}+q)\Gamma(\frac{1}{2}s-q+1)} & \left(f-\dfrac{\varkappa^2}{f}\right)^{1-2q} & \Pi\left(\frac{1}{2}s-q+1,-\frac{1}{2}+q,\frac{1}{2}-q,\dfrac{\varkappa^2}{f^2}\right) & -\quad ,, \quad ;
\end{array}
$$

and since the last of these is in fact the second of the original forms, it is clear that if instead of the first we had taken the second of the original forms, we should have obtained again the same system of four forms.

86. Writing as before $x=\dfrac{t}{t+f^2-\varkappa^2}$ &c., the forms are

$$
\begin{array}{lll}
\dfrac{\Gamma\frac{1}{2}s\Gamma\frac{1}{2}}{\Gamma(\frac{3}{2}-q)\Gamma(\frac{1}{2}s+q)}\,(f^2-\varkappa^2)^{1-2q} & \displaystyle\int_0^\infty t^{\frac{1}{2}s+q-1}(t+f^2-\varkappa^2)^{-\frac{1}{2}s+q-1}(t+f^2)^{-\frac{1}{2}s-q}\,dt & -\text{term in }\varepsilon,\\
\dfrac{-\Gamma\frac{1}{2}s\Gamma\frac{1}{2}}{\Gamma(\frac{1}{2}+q)\Gamma(\frac{1}{2}s-q+1)}\,f^{s+1}(f^2-\varkappa^2)^{1-2q} & \displaystyle\int_0^\infty t^{-\frac{3}{2}+q}(t+f^2-\varkappa^2)^{q-\frac{1}{2}}(t+f^2)^{-\frac{1}{2}s-q}\,dt & -\quad ,, \quad ,\\
\dfrac{\Gamma\frac{1}{2}s\Gamma\frac{1}{2}}{\Gamma(\frac{3}{2}-q)\Gamma(\frac{1}{2}s+q)}\,f^{s+1} & \displaystyle\int_0^\infty t^{-q-\frac{1}{2}}(t+f^2-\varkappa^2)^{-q+\frac{1}{2}}(t+f^2)^{-\frac{1}{2}s+q-1}\,dt & -\quad ,, \quad ,\\
\dfrac{-\Gamma\frac{1}{2}s\Gamma\frac{1}{2}}{\Gamma(\frac{1}{2}s-q+1)\Gamma(\frac{1}{2}+q)} & \displaystyle\int_0^\infty t^{\frac{1}{2}s-q}(t+f^2-\varkappa^2)^{-\frac{1}{2}s-q}(t+f^2)^{-\frac{1}{2}+q}\,dt & -\quad ,, \quad ;
\end{array}
$$

87. The third of these possesses a remarkable property: write mf instead of f, and at the same time change t into m^2t, the integral becomes

$$\frac{\Gamma\frac{1}{2}s\Gamma\frac{1}{2}}{\Gamma(\frac{3}{2}-q)\Gamma(\frac{1}{2}s+q)}f^{s+1}\int_0^\infty t^{-q-\frac{1}{2}}\{m^2(t+f^2)-\varkappa^2\}^{-q+\frac{1}{2}}(t+f^2)^{-\frac{1}{2}s+q-1}dt-\text{term in } \varepsilon\,;$$

and hence writing $mf=f+\delta f$ or $m=1+\frac{\delta f}{f}$, and therefore $m^2=1+2\frac{\delta f}{f}$, the value is

$$\frac{\Gamma\frac{1}{2}s\Gamma\frac{1}{2}}{\Gamma(\frac{3}{2}-q)\Gamma(\frac{1}{2}s+q)}f^{s+1}\int_0^\infty t^{-q-\frac{1}{2}}\left\{t+f^2-\varkappa^2+\frac{2\delta f}{f}(t+f^2)\right\}^{-q+\frac{1}{2}}.(t+f^2)^{-\frac{1}{2}s+q-1}dt-\text{term in } \varepsilon.$$

Hence the term in δf is

$$=2(-q+\tfrac{1}{2})\frac{\delta f}{f}\cdot\frac{\Gamma\frac{1}{2}s\Gamma\frac{1}{2}}{\Gamma(\frac{3}{2}-q)\Gamma(\frac{1}{2}s+q)}f^{s+1}\int_0^\infty t^{-q-\frac{1}{2}}(t+f^2-\varkappa^2)^{-q-\frac{1}{2}}(t+f^2)^{-\frac{1}{2}s+q}dt,$$

$$=\delta f \text{ into expression } \frac{2\Gamma\frac{1}{2}s\,.\,\Gamma\frac{1}{2}}{\Gamma(\frac{1}{2}-q)\Gamma(\frac{1}{2}s+q)}f^s\int_0^\infty t^{-q-\frac{1}{2}}(t+f^2-\varkappa^2)^{-q-\frac{1}{2}}(t+f^2)^{-\frac{1}{2}s+q}dt,$$

where the factor which multiplies δf is, as it should be, the ring-integral; it in fact agrees with one of the expressions previously obtained for this integral.

88. Similarly for an exterior point $\varkappa>f$; starting in like manner from, Disk-integral

$$=\frac{2^{s-2}f^{s-1}}{(\frac{1}{2}-q)(\varkappa+f)^{s+2q-2}}\left\{-\frac{\varkappa-f}{\varkappa+f}\Pi(\tfrac{1}{2}s,\,\tfrac{1}{2}s,\,\tfrac{1}{2}s+q,\,u)+\Pi(\tfrac{1}{2}s,\,\tfrac{1}{2}s,\,\tfrac{1}{2}s+q-1,\,u)\right\},$$

and reducing in like manner, the term in $\{\ \}$ may be expressed in the four forms

$$2^{1-s}\frac{\Gamma\frac{1}{2}s\Gamma\frac{1}{2}}{\Gamma(\frac{1}{2}s+q)\Gamma(\frac{1}{2}-q)}\frac{(\varkappa+f)^{s+2q-2}}{\varkappa^{s+2q-2}}\text{ into}$$

$$\left[-\left(1-\frac{f^2}{\varkappa^2}\right)\Pi\left(\tfrac{1}{2}s+q,\,\tfrac{1}{2}-q,\,\tfrac{1}{2}+q,\,\frac{f^2}{\varkappa^2}\right)+\frac{\frac{1}{2}s+q-1}{\frac{1}{2}-q}\;\Pi\left(\tfrac{1}{2}s+q-1,\,\tfrac{3}{2}-q,\,-\tfrac{1}{2}+q,\,\frac{f^2}{\varkappa^2}\right)\right],$$

$$2^{1-s}\frac{\Gamma\frac{1}{2}s\Gamma\frac{1}{2}}{\Gamma(\frac{1}{2}+q)\Gamma(\frac{1}{2}s-q)}\frac{(\varkappa+f)^{s+2q-2}}{\varkappa^{s+2q-2}}\text{ into}$$

$$\left[-\left(1-\frac{f^2}{\varkappa^2}\right)\Pi\left(\tfrac{1}{2}+q,\,\tfrac{1}{2}s-q,\,\tfrac{1}{2}s+q,\,\frac{f^2}{\varkappa^2}\right)+\frac{-\frac{1}{2}+q}{\frac{1}{2}s-q}\;\Pi\left(-\tfrac{1}{2}+q,\tfrac{1}{2}s-q+1,\tfrac{1}{2}s+q-1,\frac{f^2}{\varkappa^2}\right)\right],$$

$$2^{1-s}\frac{\Gamma\frac{1}{2}s\Gamma\frac{1}{2}}{\Gamma(\frac{1}{2}-q)\Gamma(\frac{1}{2}s+q)}\left(\frac{\varkappa+f}{\varkappa}\right)^{s-1}\left(\frac{\varkappa-f}{\varkappa}\right)^{-2q+1}\text{ into}$$

$$\left[-\Pi\left(\tfrac{1}{2}-q,\,\tfrac{1}{2}s+q,\,\tfrac{1}{2}s-q,\,\frac{f^2}{\varkappa^2}\right)+\left(1-\frac{f^2}{\varkappa^2}\right)\frac{\frac{1}{2}s+q-1}{\frac{1}{2}-q}\,\Pi\left(\tfrac{1}{2}-q,\,\tfrac{1}{2}s+q,\,\tfrac{1}{2}s-q,\,\frac{f^2}{\varkappa^2}\right)\right],$$

$$2^{1-s}\frac{\Gamma\frac{1}{2}s\Gamma\frac{1}{2}}{\Gamma(\frac{1}{2}s-q)\Gamma(\frac{1}{2}+q)}\left(\frac{\varkappa+f}{\varkappa}\right)^{s-1}\left(\frac{\varkappa-f}{\varkappa}\right)^{-2q+1}\text{ into}$$

$$\left[-\Pi\left(\tfrac{1}{2}s-q,\,\tfrac{1}{2}+q,\,\tfrac{1}{2}-q,\,\frac{f^2}{\varkappa^2}\right)+\left(1-\frac{f^2}{\varkappa^2}\right)\frac{-\frac{1}{2}+q}{\frac{1}{2}s-q}\;\Pi\left(\tfrac{1}{2}s-q+1,\,-\tfrac{1}{2}+q,\,\tfrac{3}{2}-q,\,\frac{f^2}{\varkappa^2}\right)\right].$$

89. For the reduction of the first and fourth of these we have to consider

$$-(1-v)\Pi(\alpha,\ \beta,\ 1-\beta,\ v)+\frac{\alpha-1}{\beta}\ \Pi(\alpha-1,\ \beta+1,\ -\beta,\ v);$$

viz. this is

$$(-1+v+1+v)\int_0^1 x^{\alpha-1}(1-x\,.\,1-vx)^{\beta-1}dx-2\int_0^1 vx\,.\,x^{\alpha-1}(1-x\,.\,1-vx)^{\beta-1}dx,$$

$$=2v\,.\int_0^1 x^{\alpha-1}(1-x)(1-x\,.\,1-vx)^{\beta-1}dx,$$

$$=2v\,.\,\Pi(\alpha,\beta+1,\ -\beta+1,v);$$

that is,

$$-(1-v)\,\Pi(\alpha,\beta,1-\beta,\ v)+\frac{\alpha-1}{\beta}\ \Pi(\alpha-1,\beta+1,\ -\beta,\ v)=2v\Pi(\alpha,\beta+1,\ -\beta+1,\ v).$$

[I repeat for comparison the foregoing equation,

$$+(1-v)\,\Pi(\alpha,\beta,1-\beta,\ v)+\frac{\alpha-1}{\beta}\ \Pi(\alpha-1,\beta+1,\ -\beta,\ v)=2\Pi(\alpha,\ \beta,\ -\beta,\ v);$$

by adding and subtracting these we obtain two new formulæ]; for reduction of the fourth formula the equation may be written

$$-\Pi(\alpha-1,\beta+1,\ -\beta,\ v)+(1-v)\frac{\beta}{\alpha-1}\,.\,\Pi(\alpha,\ \beta,\ 1-\beta,\ v)=-2\,\frac{\beta}{\alpha-1}\ v\Pi(\alpha,\beta+1-\beta+1,v).$$

90. But it is sufficient to consider the first formula; the term in [ ] is

$$=\frac{2^{2-s}\Gamma\frac{1}{2}s\Gamma\frac{1}{2}}{\Gamma(\frac{1}{2}s+q)\Gamma(\frac{1}{2}-q)}\quad\left(\frac{\varkappa+f}{\varkappa}\right)^{s+2q-2}\frac{f^2}{\varkappa^2}\quad\Pi\left(\tfrac{1}{2}s+q,\ \tfrac{3}{2}-q,\ \tfrac{1}{2}+q,\ \frac{f^2}{\varkappa^2}\right),$$

and the corresponding value of the disk-integral is

$$=\frac{\Gamma\frac{1}{2}s\Gamma\frac{1}{2}}{\Gamma(\frac{1}{2}s+q)\Gamma(\frac{3}{2}-q)}\quad\frac{f^{s+1}}{\varkappa^{s+2q}}\quad\Pi\left(\tfrac{1}{2}s+q,\ \tfrac{3}{2}-q,\ \tfrac{1}{2}+q,\ \frac{f^2}{\kappa^2}\right),$$

which we may again verify by taking therein $\varkappa$ indefinitely large; viz. the value is then $=\frac{\Gamma\frac{1}{2}s\Gamma\frac{1}{2}}{\Gamma(\frac{1}{2}s+\frac{3}{2})}\ \frac{f^{s+1}}{\kappa^{s+2q}}$, as above. It is the first of a system of four forms, the others of which are

$$=\frac{\Gamma\frac{1}{2}s\Gamma\frac{1}{2}}{\Gamma(\frac{1}{2}+q)\Gamma(\frac{1}{2}s-q+1)}\ \frac{f^{s+1}}{\varkappa^{s+2q}}\quad\Pi\left(\tfrac{1}{2}+q,\ \tfrac{1}{2}s-q+1,\ \tfrac{1}{2}s+q,\frac{f^2}{\kappa^2}\right),$$

$$-\frac{\Gamma\frac{1}{2}s\Gamma\frac{1}{2}}{\Gamma(\frac{1}{2}s+q)\Gamma(\frac{3}{2}-q)}\quad\frac{f^{s+1}}{\varkappa^{s+2q}}\left(1-\frac{f^2}{\kappa^2}\right)^{1-2q}\Pi\left(\tfrac{3}{2}-q,\ \tfrac{1}{2}s+q,\ \tfrac{1}{2}s-q+1,\frac{f^2}{\varkappa^2}\right),$$

$$=\frac{\Gamma\frac{1}{2}s\Gamma\frac{1}{2}}{\Gamma(\frac{1}{2}s-q+1)\Gamma(\frac{1}{2}+q)}\ \frac{f^{s+1}}{\varkappa^{s+2q}}\left(1-\frac{f^2}{\kappa^2}\right)^{1-2q}\Pi\left(\tfrac{1}{2}s-q+1,\ \tfrac{1}{2}+q,\ \tfrac{3}{2}-q,\frac{f^2}{\kappa^2}\right).$$

And hence, writing as before $x=\frac{t+f^2-\kappa^2}{t}$ &c., the four values are

$$=\frac{\Gamma\frac{1}{2}s\Gamma\frac{1}{2}}{\Gamma(\frac{1}{2}s+q)\Gamma(\frac{3}{2}-q)}\quad\frac{f^{s+1}}{\varkappa^{s-1}}(\varkappa^2-f^2)^{1-2q}\int_{\kappa^2-f^2}^{\infty}t^{-\frac{1}{2}s+q-1}(t+f^2-\varkappa^2)^{\frac{1}{2}s+q-1}(t+f^2)^{-\frac{1}{2}-q}\,dt,$$

$$=\frac{\Gamma\frac{1}{2}s\Gamma\frac{1}{2}}{\Gamma(\frac{1}{2}+q)\Gamma(\frac{1}{2}s-q+1)}f^{s+1}(\varkappa^2-f^2)^{1-2q}\int_{\kappa^2-f^2}^{\infty}t^{q-\frac{3}{2}}(t+f^2-\varkappa^2)^{q-\frac{1}{2}}(t+f^2)^{-\frac{1}{2}s-q}\,dt,$$

$$=\frac{\Gamma\frac{1}{2}s\Gamma\frac{1}{2}}{\Gamma(\frac{1}{2}s+q)\Gamma(\frac{3}{2}-q)}\quad f^{s+1}\quad\int_{\kappa^2-f^2}^{\infty}t^{-q-\frac{1}{2}}(t+f^2-\varkappa^2)^{-q+\frac{1}{2}}(t+f^2)^{-\frac{1}{2}s+q-1}\,dt,$$

$$=\frac{\Gamma\frac{1}{2}s\Gamma\frac{1}{2}}{\Gamma(\frac{1}{2}s-q+1)\Gamma(\frac{1}{2}+q)}\quad\frac{f^{s+1}}{\varkappa^{s-1}}\quad\int_{\kappa^2-f^2}^{\infty}t^{-\frac{1}{2}s-q}(t+f^2-\varkappa^2)^{\frac{1}{2}s-q}(t+f^2)^{q-\frac{3}{2}}\,dt,$$

where we may in the integrals write $t+\varkappa^2-f^2$ in place of t, making the limits ∞, 0; but the actual form is preferable.

91. In the third form for f write mf, at the same time changing t into mt; the new value of the disk-integral is

$$=\frac{\Gamma\frac{1}{2}s\Gamma\frac{1}{2}}{\Gamma(\frac{1}{2}s+q)\Gamma(\frac{3}{2}-q)}\,f^{s+1}\int_{\frac{\kappa^2}{m^2}-f^2}^{\infty}t^{-q-\frac{1}{2}}(m^2(t+f^2)-\varkappa^2)^{-q+\frac{1}{2}}(t+f^2)^{-\frac{1}{2}s+q-1}\,dt.$$

Writing here $mf=f+\delta f$, that is $m=1+\frac{\delta f}{f}$, $m^2=1+\frac{2\delta f}{f}$, and observing that if $-q+\frac{1}{2}$ be positive, the factor $(m^2(t+f^2)-\varkappa^2)^{-q+\frac{1}{2}}$ vanishes for the value $t=\frac{\kappa^2}{m^2}-f^2$ at the lower limit, we see that on this supposition, $-q+\frac{1}{2}$ positive, the value is

$$=\frac{\Gamma\frac{1}{2}s\Gamma\frac{1}{2}}{\Gamma(\frac{1}{2}s+q)\Gamma\frac{3}{2}-q)}\,f^{s+1}\quad\int_{\kappa^2-f^2}^{\infty}t^{-q-\frac{1}{2}}(t+f^2-\varkappa^2+\frac{2\delta f}{f}(t+f^2))^{-q+\frac{1}{2}}(t+f^2)^{-\frac{1}{2}s+q-1}\,dt\,;$$

viz. the term in δf is $=\delta f$ into the expression

$$2(\tfrac{1}{2}-q)\cdot\frac{\Gamma\frac{1}{2}s\Gamma\frac{1}{2}}{\Gamma(\frac{1}{2}s+q)\Gamma(\frac{3}{2}-q)}\,f^{s}\int_{\kappa^2-f^2}^{\infty}t^{-q-\frac{1}{2}}(t+f^2-\varkappa^2)^{-q-\frac{1}{2}}(t+f^2)(t+f^2)^{-\frac{1}{2}s+q-1}\,dt,$$

that is into

$$2\,\frac{\Gamma\frac{1}{2}s\Gamma\frac{1}{2}}{\Gamma(\frac{1}{2}s+q)\Gamma(\frac{1}{2}-q)}\,f^{s}\int_{\kappa^2-f^2}^{\infty}t^{-q-\frac{1}{2}}(t+f^2-\varkappa^2)^{-q-\frac{1}{2}}(t+f^2)^{-\frac{1}{2}s+q}\,dt,$$

which is in fact $=\delta f$ into the value of the ring-integral.

92. Comparing for the cases of an interior point $\varkappa<f$ and an exterior point $\varkappa>f$, the four expressions for the disk-integral, it will be noticed that only the third expressions correspond precisely to each other; viz. these are: interior point, $\varkappa<f$; the value is

$$\frac{\Gamma\frac{1}{2}s\Gamma\frac{1}{2}}{\Gamma(\frac{1}{2}s+q)\Gamma(\frac{3}{2}-q)}\,f^{s+1}\cdot\int_{0}^{\infty}t^{-q-\frac{1}{2}}(t+f^2-\varkappa^2)^{-q+\frac{1}{2}}(t+f^2)^{-\frac{1}{2}s+q-1}\,dt-\frac{\epsilon^{1-2q}}{\frac{1}{2}-q}\,\frac{\Gamma\frac{1}{2}s\Gamma\frac{1}{2}s}{\Gamma(\frac{1}{2}s+q)},$$

where, if $\frac{1}{2}-q$ be positive (which is in fact a necessary condition in order to the applicability of the formula), the term in ϵ vanishes, and may therefore be omitted: and

exterior point, $\varkappa>f$; the value is

$$=\frac{\Gamma\frac{1}{2}s\Gamma\frac{1}{2}}{\Gamma(\frac{1}{2}s+q)\Gamma(\frac{3}{2}-q)}f^{s+1}\int_{\kappa^2-f^2}^{\infty}t^{-q-\frac{1}{2}}(t+f^2-\varkappa^2)^{-q+\frac{1}{2}}(t+f^2)^{-\frac{1}{2}s+q-1}\,dt,$$

differing only from the preceding one in the inferior limit $\varkappa^2-f^2$ in place of 0 of the integral. We have $\frac{1}{2}-q$ positive, and also $\frac{1}{2}s+q$ positive; viz. q may have any value diminishing from $\frac{1}{2}$ to $-\frac{1}{2}s$, the extreme values *not* admissible.

ANNEX IV. *Examples of Theorem* A.—Nos. 93 to 112.

93. It is remarked in the text that in the examples which relate to the s-coordinal sphere and ellipsoid respectively, we have a quantity θ, a function of the coordinates $(a\ldots c, e)$ of the attracted point; viz. in the case of the sphere, writing $a^2\ldots+c^2=\varkappa^2$, we have

$$\frac{\varkappa^2}{f^2+\theta}+\frac{e^2}{\theta}=1,$$

and in the case of the ellipsoid

$$\frac{a^2}{f^2+\theta}\cdots+\frac{c^2}{h^2+\theta}+\frac{e^2}{\theta}=1,$$

the equation having in each case a positive root which is called θ. The properties of the equation are the same in each case; but for the sphere, the equation being a quadric one, can be solved. The equation in fact is

$$\theta^2-\theta(e^2+\varkappa^2-f^2)-e^2f^2=0,$$

and the positive root is therefore

$$\theta=\tfrac{1}{2}\{e^2+\varkappa^2-f^2+\sqrt{(e^2+\varkappa^2-f^2)^2+4e^2f^2}\}.$$

Suppose e to gradually diminish and become $=0$; for an exterior point, $\varkappa>f$, the value of the radical is $=\varkappa^2-f^2$, and we have $\theta=\varkappa^2-f^2$; for an interior point, $\varkappa<f$, the value of the radical, supposing e only indefinitely small, is $=f^2-\varkappa^2+\frac{f^2+\kappa^2}{f^2-\varkappa^2}e^2$, and we have $\theta=\frac{1}{2}e^2\left(1+\frac{f^2+\varkappa^2}{f^2-\kappa^2}\right)$, $=\frac{e^2f^2}{f^2-\varkappa^2}$, or, what is the same thing, $\frac{e^2}{\theta}=\left(1-\frac{\kappa^2}{f^2}\right)$; viz. the positive root of the equation continually diminishes with e, and becomes ultimately $=0$.

If $\varkappa$ or e be indefinitely large, then the radical may be taken $=e^2+\varkappa^2$, and we have θ indefinitely large, $=e^2+\varkappa^2$.

94. Every thing is the same with the general equation

$$\frac{a^2}{f^2+\theta}\cdots+\frac{c^2}{h^2+\theta}+\frac{e^2}{\theta}=1;$$

the left-hand side is $=0$ for $\theta=\infty$, and (as θ decreases) continually increases, becoming infinite for $\theta=0$; there is consequently a single positive value of θ for which the value is $=1$; viz. the equation has a single positive root, and θ is taken to denote this root.

In the last-mentioned equation, let e gradually diminish and become $=0$; then for an exterior point, viz. if

$$\frac{a^2}{f^2}\dots+\frac{c^2}{h^2}>1, \text{ the equation } \frac{a^2}{f^2+\theta}\dots+\frac{c^2}{h^2+\theta}=1$$

has (as is at once seen) a single positive root, and θ becomes equal to the positive root of this equation; but for an interior point, or $\frac{a^2}{f^2}\dots+\frac{c^2}{h^2}<1$, the equation just written down has no positive root, and θ becomes $=0$, that is the positive root of the original equation continually diminishes with e, and for $e=0$ becomes ultimately $=0$; its value for e small is in fact given by $\frac{e^2}{\theta}=\left(1-\frac{a^2}{f^2}\dots-\frac{c^2}{h^2}\right)$. Also $a\dots c, e$ or any of them indefinitely large, θ is indefinitely large, $=a^2\dots+c^2+e^2$.

95. We have an interesting geometrical illustration in the case $s+1=2$; θ is here determined by the equation

$$\frac{a^2}{f^2+\theta}+\frac{b^2}{g^2+\theta}+\frac{e^2}{\theta}=1\,;$$

viz. θ is the squared z-semiaxis of the ellipsoid, confocal with the conic $\frac{x^2}{f^2}+\frac{y^2}{g^2}=1$, which passes through the point (a, b, e). Taking $e=0$, the point in question, if $\frac{a^2}{f^2}+\frac{b^2}{g^2}>1$, is a point in the plane of xy, outside the ellipse, and we have through the point a proper confocal ellipsoid, whose squared z-semiaxis does not vanish; but if $\frac{a^2}{f^2}+\frac{b^2}{g^2}<1$, then the point is within the ellipse, and the only confocal ellipsoid through the point is the indefinitely thin ellipsoid, squared semiaxes $(f^2, g^2, 0)$, which in fact coincides with the ellipse.

96. The positive root θ of the equation

$$\mathrm{J}, = 1-\frac{a^2}{f^2+\theta}\dots-\frac{c^2}{h^2+\theta}-\frac{e^2}{\theta}, =0$$

has certain properties which connect themselves with the function

$$\Theta, =\theta^{-q-1}(\theta+f^2\dots\theta+h^2)^{-\frac{1}{2}}.$$

We have (the accents denoting differentiations in regard to θ)

$$\mathrm{J}'\frac{d\theta}{da}-\frac{2a}{\theta+f^2}=0, \text{ or } \frac{d\theta}{da}=\frac{1}{\mathrm{J}'}\frac{2a}{\theta+f^2},$$

where

$$\mathrm{J}'=\frac{a^2}{(f^2+\theta)^2}\dots+\frac{c^2}{(h^2+\theta)^2}+\frac{e^2}{\theta^2},$$

and we have the like formulæ for $\dots\frac{d\theta}{dc}, \frac{d\theta}{de}$.

We deduce

$$\frac{a}{\theta+f^2}\frac{d\theta}{da}\dots+\frac{c}{\theta+h^2}\frac{d\theta}{dc}+\frac{e}{\theta}\frac{d\theta}{de}=\frac{2}{\mathrm{J}'}\left\{\frac{a^2}{(\theta+f^2)^2}\dots+\frac{c^2}{(\theta+h^2)^2}+\frac{e^2}{\theta}\right\}, =2\,;$$

and to this we may join, η being arbitrary,

$$\frac{a}{\theta+\eta+f^2}\frac{d\theta}{da}\ldots+\frac{c}{\theta+\eta+h^2}\frac{d\theta}{dc}+\frac{e}{\theta+\eta}\frac{d\theta}{de}=\frac{2}{\mathrm{J}'}\left\{\frac{a^2}{\theta+f^2.\,\theta+\eta+f^2}\ldots+\frac{c^2}{\theta+h^2.\theta+\eta+h^2}+\frac{e^2}{\theta.\theta+\eta}\right\}.$$

Again, defining $\nabla_1\theta$, $\square\theta$ as immediately appears, we have

$$\nabla_1\theta,\ =\left(\frac{d\theta}{da}\right)^2\ldots+\left(\frac{d\theta}{dc}\right)^2,\ =\frac{1}{\mathrm{J}'^2}.4\mathrm{J}',\ =\frac{4}{\mathrm{J}'};$$

and passing to the second differential coefficients, we have

$$\frac{d^2\theta}{da^2}=\frac{2}{\mathrm{J}'(\theta+f^2)}-\frac{8a^2}{\mathrm{J}'^2(\theta+f^2)^3}-\frac{4a^2\mathrm{J}''}{\mathrm{J}'^3(\theta+f^2)^2},$$

where

$$\mathrm{J}''=-2\left\{\frac{a^2}{(\theta+f^2)^3}\ldots+\frac{c^2}{(\theta+h^2)^3}+\frac{e^2}{\theta^3}\right\},$$

with the like formulæ for $\ldots\frac{d^2\theta}{de^2},\frac{d^2\theta}{de^2}$. Joining to these $\frac{2q+1}{e}\frac{d\theta}{de}=\frac{4q+2}{\mathrm{J}'\theta}$, we obtain

$$\begin{aligned}\square\theta,\ &=\left(\frac{d^2\theta}{da^2}\ldots+\frac{d^2\theta}{dc^2}+\frac{d^2\theta}{de^2}+\frac{2q+1}{e}\frac{d\theta}{de}\right),\\ &=\frac{2}{\mathrm{J}'}\left\{\frac{1}{\theta+f^2}\ldots+\frac{1}{\theta+h^2}+\frac{1+(2q+1)}{\theta}\right\}\\ &-\frac{8}{\mathrm{J}'^2}(-\tfrac{1}{2}\mathrm{J}'')-\frac{4\mathrm{J}''}{\mathrm{J}'^3}(\mathrm{J}'),\end{aligned}$$

where the last two terms destroy each other; and observing that we have

$$\frac{\Theta'}{\Theta}=-\tfrac{1}{2}\left(\frac{1}{\theta+f^2}\ldots+\frac{1}{\theta+h^2}+\frac{2q+1}{\theta}\right),$$

the result is

$$\square\theta=\frac{2}{\mathrm{J}'}\left(-\frac{2\Theta'}{\Theta}\right),\ =-\frac{4\Theta'}{\mathrm{J}'\Theta}.$$

97. First example. $\varkappa^2=a^2\ldots+c^2$, and θ is the positive root of $\frac{\varkappa^2}{f^2+\theta}+\frac{e^2}{\theta}=1$.

V is assumed $=\int_\theta^\infty t^{-q-1}(t+f^2)^{-\frac{1}{2}s}dt$, where $q+1$ is positive.

I do not work the example out; it corresponds step by step with, and is hardly more simple than, the next example, which relates to the ellipsoid. The result is

$$\begin{aligned}&\rho=0,\ \text{if } x^2\ldots+z^2>f^2,\\ &\rho=\frac{\Gamma(\frac{1}{2}s+q)}{(\Gamma\frac{1}{2})^s\Gamma(q+1)}f^{-s}\left(1-\frac{x^2\ldots+z^2}{f^2}\right)^q,\ \text{if } x^2\ldots+z^2<f^2;\end{aligned}$$

hence the integral

$$\int\frac{\left(1-\frac{x^2\ldots+z^2}{f^2}\right)^q dx\ldots dz}{\{(a-x)^2\ldots+(c-z)^2+e^2\}^{\frac{1}{2}s+q}},$$

taken over the sphere $x^2 \ldots + z^2 = f^2$,

$$= \frac{(\Gamma\frac{1}{2})^s \Gamma(q+1)}{\Gamma(\frac{1}{2}s+q)} \int_0^\infty t^{-q-1}(t+f^2)^{-\frac{1}{2}s} dt.$$

98. Second example. θ the positive root of $\frac{a^2}{f^2+\theta} \ldots + \frac{c^2}{h^2+\theta} + \frac{e^2}{\theta} = 1$; $q+1$ positive.

Consider here the function

$$\mathrm{V} = \int_\theta^\infty t^{-q-1}(t+f^2 \ldots t+h^2)^{-\frac{1}{2}} dt;$$

this satisfies the prepotential equation. We have in fact

$$\frac{d\mathrm{V}}{da} = -\Theta \frac{d\theta}{da};\ \frac{d^2\mathrm{V}}{da^2} = -\Theta \frac{d^2\theta}{da^2} - \Theta' \left(\frac{d\theta}{da}\right)^2,$$

with the like expressions for $\frac{d^2\mathrm{V}}{dc^2}, \frac{d^2\mathrm{V}}{de^2}$; also

$$\frac{2q+1}{e} \frac{d\mathrm{V}}{de} = -\Theta \frac{2q+1}{e} \frac{d\theta}{de}.$$

Hence

$$\square \mathrm{V} = -\Theta \square \theta - \Theta' \nabla_1 \theta,$$

or, substituting for $\square \theta$ and $\nabla_1 \theta$ their values, this is

$$= -\Theta \left(-\frac{4\Theta'}{\mathrm{J}'\Theta}\right) - \Theta' . 4\mathrm{J}, = 0.$$

Moreover V does not become infinite for any values of $(a \ldots c, e)$, e not $=0$; and it vanishes for points at ∞; and not only so, but for indefinitely large values of any of the coordinates $(a \ldots e, e)$ it reduces itself to a numerical multiple of $(a^2 \ldots + c^2 + e^2)^{-\frac{1}{2}s+q}$; in fact in this case θ is indefinitely large, $= a^2 \ldots + c^2 + e^2$: consequently throughout the integral t is indefinitely large, and we may therefore write

$$\mathrm{V} = \int_\theta^\infty t^{-q-1} . t^{-\frac{1}{2}s} dt, \quad = -\frac{1}{\frac{1}{2}s+q} \left(t^{-\frac{1}{2}s-q}\right)_\theta^\infty, \quad = \frac{1}{\frac{1}{2}s+q} \theta^{-\frac{1}{2}s-q},$$

that is

$$\mathrm{V} = \frac{1}{\frac{1}{2}s+q} (a^2 \ldots + c^2 + e^2)^{-\frac{1}{2}s-q}.$$

The conditions of the theorem are thus satisfied, and we have for ϱ either of the formulæ,

$$\varrho = \frac{\Gamma(\frac{1}{2}s+q)}{(\Gamma\frac{1}{2})^s \Gamma q} (e^{2q}\mathrm{W})_0, \quad \varrho = \frac{-\Gamma(\frac{1}{2}s+q)}{2(\Gamma\frac{1}{2})^s \Gamma(q+1)} \left(e^{2q+1} \frac{d\mathrm{W}}{de}\right)_0$$

(in the former of them q must be positive; in the latter it is sufficient if $q+1$ be positive).

99. We have W the same function of $(x \ldots z, e)$ that V is of $(a \ldots c, e)$; viz. writing λ for the positive root of

$$\frac{x^2}{f^2+\lambda} \cdots + \frac{z^2}{h^2+\lambda} + \frac{e^2}{\lambda} = 1,$$

the value of W is

$$= \int_\lambda^\infty t^{-q-1}(t+f^2 \ldots t+h^2)^{-\frac{1}{2}} dt.$$

Considering the formula which involves e^{2q}W,—first, if $\frac{x^2}{f^2} \ldots + \frac{z^2}{h^2} > 1$, then when e is $=0$ the value of λ is not $=0$; the integral W is therefore finite (not indefinitely large), and we have $e^{2q}\mathrm{W}=0$, consequently $\varrho=0$.

But if $\frac{x^2}{f^2} \ldots + \frac{z^2}{h^2} < 1$, then when e is indefinitely small, λ is also indefinitely small; viz. we then have $\frac{e^2}{\lambda} = 1 - \frac{x^2}{f^2} \ldots - \frac{z^2}{h^2}$; the value of W is

$$\mathrm{W} = (f \ldots h)^{-1} \int_\lambda^\infty t^{-q-1} dt, \quad = (f \ldots h)^{-1} . \frac{1}{q} \lambda^{-q},$$

and hence

$$\varrho = \frac{\Gamma(\frac{1}{2}s+q)}{(\Gamma\frac{1}{2})^s \Gamma q} \cdot \frac{1}{q} \left(\frac{e^2}{\lambda}\right)^q . (f \ldots h)^{-1}, \quad = \frac{\Gamma(\frac{1}{2}s+q)}{(\Gamma\frac{1}{2})^s \Gamma(q+1)} (f \ldots h)^{-1} \left(1 - \frac{x^2}{f^2} \ldots - \frac{z^2}{h^2}\right)^q.$$

100. Again, using the formula which involves $\left(e^{2q+1} \frac{d\mathrm{W}}{de}\right)$; we have here $\frac{d\mathrm{V}}{de} = -\Theta \frac{d\theta}{de}$, or substituting for Θ and $\frac{d\theta}{de}$ their values and multiplying by e^{2q+1}, we find

$$e^{2q+1} \frac{d\mathrm{V}}{de} = 2e^{2q+2} \theta^{-1} \mathrm{J}'^{-1} \Theta,$$

$$= 2e^{2q+2} \theta^{-q-2} \left[\frac{a^2}{(f^2+\theta)^2} \cdots + \frac{c^2}{(h^2+\theta)^2} + \frac{e^2}{\theta^2}\right]^{-1} (\theta+f^2 \ldots \theta+h^2)^{-\frac{1}{2}},$$

and therefore

$$e^{2q+1} \frac{d\mathrm{W}}{de} = 2e^{2q+2} \lambda^{-q-2} \left[\frac{x^2}{(f^2+\lambda)^2} \cdots + \frac{c^2}{(h^2+\lambda)^2} + \frac{e^2}{\lambda^2}\right]^{-1} (\lambda+f^2 \ldots \lambda+h^2)^{-\frac{1}{2}}.$$

Hence, writing $e=0$, first for an exterior point or $\frac{x^2}{f^2} \ldots + \frac{z^2}{h^2} > 1$, λ is not $=0$, and the expression vanishes in virtue of the factor e^{2q+2}; whence also $\varrho=0$; next for an interior point or $\frac{x^2}{f^2} \ldots + \frac{z^2}{h^2} < 1$, λ is $=0$, hence also $\frac{e^2}{\lambda^2} = \frac{1}{\lambda}\left(1 - \frac{x^2}{f^2} \ldots - \frac{z^2}{h^2}\right)$ is infinite; and neglecting in comparison with it the terms $\frac{x^2}{(f^2+\lambda)^2}$ &c., the value is

$$2\left(\frac{e^2}{\lambda}\right)^q (f \ldots h)^{-1}, \quad = 2\left(1 - \frac{x^2}{f^2} \ldots - \frac{z^2}{h^2}\right)^q (f \ldots h)^{-1},$$

and we have as before,

$$\varrho=\frac{\Gamma(\frac{1}{2}s+q)}{(\Gamma\frac{1}{2})^s\Gamma(q+1)}(f\ldots h)^{-1}\left(1-\frac{x^2}{f^2}\cdots-\frac{x^2}{h^2}\right)^q.$$

101. Hence in the formula

$$\mathrm{V}=\int\frac{\varrho\, dx\ldots dz}{\{(a-x)^2\ldots+(c-z)^2+e^2\}^{\frac{1}{2}s+q}}$$

$$=\int_\theta^\infty t^{-q-1}(t+f^2\ldots t+h^2)^{-\frac{1}{2}}dt,$$

ϱ has the value just found, or, what is the same thing, we have

$$\int\frac{\left(1-\frac{x^2}{f^2}\cdots-\frac{z^2}{h^2}\right)^q dx\ldots dz}{\{(a-x)^2\ldots+(c-z)^2+e^2\}^{\frac{1}{2}s+q}}$$

over ellipsoid $\frac{x^2}{f^2}\cdots+\frac{z^2}{h^2}=1$,

$$=\frac{(\Gamma\frac{1}{2})^s\,.\,\Gamma(q+1)}{\Gamma(\frac{1}{2}s+q)}(f\ldots h)\int_\theta^\infty t^{-q-1}(t+f^2\ldots t+h^2)^{-\frac{1}{2}}dt.$$

102. We may in this result write $e=0$. There are two cases, according as the attracted point is exterior or interior: if it is exterior, $\frac{a^2}{f^2}\cdots+\frac{c^2}{h^2}>1$, θ will denote the positive root of the equation $\frac{a^2}{f^2+\theta}\cdots+\frac{c^2}{h^2+\theta}=1$; if it be interior, $\frac{a^2}{f^2}\cdots+\frac{c^2}{h^2}<1$, θ will be$=0$; and we thus have

$$\int\frac{\left(1-\frac{x^2}{f^2}\cdots-\frac{z^2}{h^2}\right)^q dx\ldots dz}{\{(a-x)^2\ldots+(c-z)^2\}^{\frac{1}{2}s+q}}$$

$$=\frac{(\Gamma\frac{1}{2})^s\Gamma(q+1)}{\Gamma(\frac{1}{2}s+q)}(f\ldots h)\int_\theta^\infty t^{-q-1}(t+f^2\ldots t+h^2)^{-\frac{1}{2}}dt,\ \text{for exterior point}\ \frac{a^2}{f^2}\cdots+\frac{c^2}{h^2}>1,$$

$$=\frac{(\Gamma\frac{1}{2})^s\Gamma(q+1)}{\Gamma(\frac{1}{2}s+q)}(f\ldots h)\int_0^\infty t^{-q-1}(t+f^2\ldots t+h^2)^{-\frac{1}{2}}dt,\ \text{for interior point}\ \frac{a^2}{f^2}\cdots+\frac{c^2}{h^2}<1;$$

but as regards the value for an interior point it is to be observed that unless q be negative (between 0 and -1, since $1+q$ is positive by hypothesis) the two sides of the equation will be each of them infinite.

103. Third example. We assume here

$$\mathrm{V}=\int_\theta^\infty dt\,\mathrm{I}^m\,\mathrm{T},$$

where

$$\mathrm{I}=1-\frac{a^2}{f^2+t}\cdots-\frac{c^2}{h^2+t}-\frac{e^2}{t},$$

$$\mathrm{T}=t^{-q-1}(t+f^2\ldots t+h^2)^{-\frac{1}{2}},$$

and, as before, θ is the positive root of the equation

$$J=1-\frac{a^2}{f^2+\theta}\cdots-\frac{c^2}{h^2+\theta}-\frac{e^2}{\theta},\ =0.$$

$\frac{1}{2}s+q$ is positive in order that the integral may be finite; also m is positive.

104. In order to show that V satisfies the prepotential equation $\square V=0$, I shall, in the first place, consider the more general expression,

$$V=\int_{\theta+\eta}^{\infty}dt\,I^m\,T,$$

where η is a constant positive quantity which will be ultimately put $=0$. The functions previously called J and Θ will be written J_0 and Θ_0, and J, Θ will now denote

$$J,\ =1-\frac{a^2}{\theta+\eta+f^2}\cdots-\frac{c^2}{\theta+\eta+h^2}-\frac{e^2}{\theta+\eta},$$

$$\Theta,\ =(\theta+\eta)^{-q-1}(\theta+\eta+f^2\ldots\theta+\eta+h^2)^{-\frac{1}{2}};$$

whence also, subtracting from J the evanescent function J_0, we have

$$J=\eta\left(\frac{a^2}{\theta+f^2\,.\,\theta+\eta+f^2}\cdots+\frac{c^2}{\theta+h^2\,.\,\theta+\eta+h^2}+\frac{e^2}{\theta\,.\,\theta+\eta}\right),$$

say this is

$$J=\eta P;$$

and we have thence, by former equations and in the present notation,

$$\frac{a}{\theta+\eta+f^2}\frac{d\theta}{da}\cdots+\frac{c}{\theta+\eta+h^2}\frac{d\theta}{dc}+\frac{e}{\theta+\eta}\frac{d\theta}{de}=\frac{2}{J_0}.\,P,$$

$$\nabla_1\theta=\frac{4}{J_0},$$

$$\square\theta=\frac{-4\Theta_0'}{J_0'\Theta_0}.$$

In virtue of the equation which determines θ, we have

$$\frac{dV}{da}=\int_{\theta+\eta}^{\infty}dt\,mI^{m-1}\frac{-2a}{t+f^2}T-J^m\Theta\frac{d\theta}{da};$$

and thence

$$\begin{aligned}\frac{d^2V}{da^2}=\int_{\theta+\eta}^{\infty}dt\left\{mI^{m-1}\frac{-2}{t+f^2}+m(m-1)\,I^{m-2}\frac{4a^2}{(t+f^2)^2}\right\}T\\ \left.\begin{aligned}&-mJ^{m-1}\left(-\frac{2a}{\theta+\eta+f^2}\right)\Theta\frac{d\theta}{da}\\ &-mJ^{m-1}\left(\frac{-2a}{\theta+\eta+f^2}\right)\Theta\frac{d\theta}{da}\end{aligned}\right\}\\ -\frac{d}{d\theta}(J^m\Theta)\left(\frac{d\theta}{da}\right)^2\\ -J^m\Theta\frac{d^2\theta}{da^2},\end{aligned}$$

with like expressions for $\ldots\frac{d^2V}{dc^2},\frac{d^2V}{de^2}$.

Also

$$\frac{2q+1}{e}\frac{d\mathrm{V}}{de}=\int_{\theta+\eta}^{\infty} dt\, m\mathrm{I}^{m-1}\frac{-4q-2}{t}\mathrm{T}$$
$$-\frac{2q+1}{e}\,\mathrm{J}^m\Theta\frac{d\theta}{de};$$

and hence

$$\square\mathrm{V}=\int_{\theta+\eta}^{\infty} dt\Big[-2m\,\mathrm{I}^{m-1}\Big\{\frac{1}{t+f^2}\cdots-\frac{1}{t+h^2}+\frac{1+(2q+1)}{t}\Big\}\mathrm{T}$$
$$+m(m-1)\,\mathrm{I}^{m-2}.\,4\Big\{\frac{a^2}{(t+f^2)^2}\cdots+\frac{c^2}{(t+h^2)^2}+\frac{e^2}{t^2}\Big\}\mathrm{T}\Big]$$
$$+4m\mathrm{J}^{m-1}\Theta\left(\frac{a}{\theta+\eta+f^2}\frac{d\theta}{da}\cdots+\frac{c}{\theta+\eta+h^2}\frac{d\theta}{dc}+\frac{e}{\theta+\eta}\frac{d\theta}{de}\right)$$
$$-\frac{d}{d\theta}(\mathrm{J}^m\Theta)\left(\left(\frac{d\theta}{da}\right)^2\cdots+\left(\frac{d\theta}{dc}\right)^2+\left(\frac{d\theta}{de}\right)^2\right)$$
$$-\mathrm{J}^m\Theta\left(\frac{d^2\theta}{da^2}\cdots+\frac{d^2\theta}{dc^2}+\frac{d^2\theta}{de^2}+\frac{2q+1}{e}\frac{d\theta}{de}\right).$$

105. Writing I′, T′ for the first derived coefficients of I, T in regard to t, we have

$$\mathrm{I}'=\frac{a^2}{(t+f^2)^2}\cdots+\frac{c^2}{(t+h^2)^2}+\frac{e^2}{t^2},\qquad \frac{\mathrm{T}'}{\mathrm{T}}=-\tfrac{1}{2}\left(\frac{1}{t+f^2}\cdots+\frac{1}{t+h^2}+\frac{2q+2}{t}\right),$$

and the integral is therefore

$$\int_{\theta+\eta}^{\infty} dt\left(2m\mathrm{I}^{m-1}\frac{2\mathrm{T}'}{\mathrm{T}}\mathrm{T}+m(m-1)\mathrm{I}^{m-2}.\,4\mathrm{I}'\mathrm{T}\right),$$
$$=\int_{\theta+\eta}^{\infty} dt(4m\mathrm{I}^{m-1}\mathrm{T}'+4m(m-1)\mathrm{I}^{m-2}\mathrm{I}'\mathrm{T}),$$
$$=\int_{\theta+\eta}^{\infty} dt\,4m\frac{d}{dt}(\mathrm{I}^{m-1}\mathrm{T});$$

viz., $\mathrm{I}^{m-1}\mathrm{T}$ vanishing for $t=\infty$, this is

$$=-4m\,\mathrm{J}^{m-1}\Theta.$$

Hence, writing $(\mathrm{J}^m\Theta)'$ instead of $\frac{d}{d\theta}(\mathrm{J}^m\Theta)$, we have

$$\square\mathrm{V}=-4m\,\mathrm{J}^{m-1}\Theta$$
$$+4m\,\mathrm{J}^{m-1}\Theta\left(\frac{a}{\theta+\eta+f^2}\frac{d\theta}{da}\cdots+\frac{c}{\theta+\eta+f^2}\frac{d\theta}{dc}+\frac{e}{\theta+\eta}\frac{d\theta}{de}\right)$$
$$-(\mathrm{J}^m\Theta)'\,\nabla_1\theta$$
$$-\mathrm{J}^m\Theta\;\square\theta;$$

viz. this is

$$\square\mathrm{V}=-4m\mathrm{J}^{m-1}\Theta$$
$$+8m\mathrm{J}^{m-1}\Theta.\frac{\mathrm{P}}{\mathrm{J}_0'}$$
$$-4(\mathrm{J}^m\Theta)'\frac{1}{\mathrm{J}_0'}$$
$$+4\mathrm{J}^m\Theta\frac{\Theta_0'}{\mathrm{J}_0'\Theta_0};$$

or, instead of $(J^m\Theta)'$, writing $mJ^{m-1}J'\Theta+J^m\Theta'$, this is

$$\square V=-\frac{4mJ^{m-1}\Theta}{J_0'}(J'-2P+J)-\frac{4J^m}{J_0'\Theta_0}(\Theta'\Theta_0-\Theta\Theta'_0).$$

We have here

$$J'-2P+J=a^2\left\{\frac{1}{(\theta+\eta+f^2)^2}-\frac{2}{(\theta+\eta+f^2)(\theta+f^2)}+\frac{1}{(\theta+f^2)^2}\right\}\ldots+e^2\left\{\frac{1}{(\theta+\eta)^2}-\frac{2}{(\theta+\eta)\theta}+\frac{1}{\theta^2}\right\}$$

$$=\eta^2\left\{\frac{a^2}{(\theta+f^2)^2(\theta+\eta+f^2)^2}\cdots+\frac{c^2}{(\theta+h^2)^2(\theta+\eta+h^2)^2}+\frac{e^2}{\theta^2(\theta+\eta)^2}\right\}$$

$$=\eta^2\,.\,Q,\text{ suppose.}$$

Also $\Theta'\Theta_0-\Theta\Theta_0'$ contains the factor η, is $=\eta M$ suppose.

106. Substituting for J, $J'-2P+J$, and $\Theta'\Theta_0-\Theta\Theta_0'$ their values ηP, ηQ, and ηM, the whole result contains the factor η^{m+1}, viz. we have

$$\square V=-\frac{4\eta^{m+1}P^{m-1}}{J_0'}\left(Q\Theta+\frac{PM}{\Theta_0}\right);$$

and if here, except in the term η^{m+1}, we write $\eta=0$, we have

$$P=\frac{a^2}{(\theta+f^2)^2}\cdots+\frac{c^2}{(\theta+h^2)^2}+\frac{e^2}{\theta^2},=J_0,$$

$$Q=\frac{a^2}{(\theta+f^2)^4}\cdots+\frac{c^2}{(\theta+h^2)^4}+\frac{e^2}{\theta^4},=\tfrac{1}{6}J_0''',$$

$$M=\Theta_0\Theta_0''-\Theta_0'^2,$$

and the formula becomes

$$\square V=-4\eta^{m+1}J_0'^{m-2}\left\{\tfrac{1}{6}J_0'''\Theta_0+J_0'\left(\Theta_0''-\frac{\Theta_0'^2}{\Theta_0}\right)\right\};$$

or (instead of J_0, Θ_0) using now J, Θ in their original significations,

$$J=1-\frac{a^2}{\theta+f^2}\cdots-\frac{c^2}{\theta+h^2}-\frac{e^2}{\theta},\text{ and }\Theta=\theta^{-q-1}(\theta+f^2\ldots\theta+h^2)^{-\frac{1}{2}},$$

this is

$$\square V=-4\eta^{m+1}J'^{m-2}\left\{\tfrac{1}{6}J'''\Theta+J'\left(\Theta''-\frac{\Theta'^2}{\Theta}\right)\right\},$$

or, what is the same thing,

$$=-4\eta^{m+1}J'^{m-2}\Theta\left\{\tfrac{1}{6}J'''+J'\left(\frac{\Theta'}{\Theta}\right)'\right\};$$

viz. the expression in { } is

$$=\left[\frac{a^2}{(\theta+f^2)^4}\cdots+\frac{c^2}{(\theta+h^2)^4}+\frac{e^2}{\theta^4}\right]+\tfrac{1}{2}\left[\frac{a^2}{(\theta+f^2)^2}\cdots+\frac{c^2}{(\theta+h^2)^2}+\frac{e^2}{\theta^2}\right]\left[\frac{1}{(\theta+f^2)^2}\cdots+\frac{1}{(\theta+h^2)^2}+\frac{e^2}{\theta^2}\right].$$

We thus see that η being infinitesimal $\square V$ is infinitesimal of the order η^{m+1}; and hence η being $=0$, we have

$$\square V=0;$$

viz. the prepotential equation is satisfied by the value

$$\mathrm{V}=\int_\theta^\infty dt\, \mathrm{I}^m \mathrm{T},$$

where $m+1$ is positive.

107. We have consequently a value of ϱ corresponding to the foregoing value of V; and this value is

$$\varrho=-\frac{\Gamma(\tfrac{1}{2}s+q)}{2\pi^{\frac{1}{2}s}\Gamma(q+1)}\left(e^{2q+1}\frac{d\mathrm{W}}{de}\right)_{e=0},$$

where, writing λ for the positive root of

$$1-\frac{x^2}{\lambda+f^2}\cdots-\frac{z^2}{\lambda+h^2}-\frac{e^2}{\lambda}=0,$$

we have

$$\mathrm{W}=\int_\lambda^\infty dt\left(1-\frac{x^2}{t+f^2}\cdots-\frac{z^2}{t+h^2}-\frac{e^2}{t}\right)^m t^{-q-1}(t+f^2\ldots+t+h^2)^{-\frac{1}{2}};$$

we thence obtain

$$\frac{d\mathrm{W}}{de}=\int_\lambda^\infty dt\,.-\frac{2me}{t}\left(1-\frac{x^2}{t+f^2}\cdots-\frac{z^2}{t+h^2}-\frac{e^2}{t}\right)^{m-1} t^{-q-1}(t+f^2\ldots t+h^2)^{-\frac{1}{2}}$$
$$-\left(1-\frac{x^2}{\lambda+f^2}\cdots-\frac{z^2}{\lambda+h^2}-\frac{e^2}{\lambda}\right)^m \lambda^{-q-1}(\lambda+f^2\ldots\lambda+h^2)^{-\frac{1}{2}}.\frac{d\lambda}{de};$$

or multiplying by e^{2q+1}, and substituting for $\frac{d\lambda}{de}$ its value

$$=\frac{\frac{2e}{\lambda}}{\left\{\frac{x^2}{(\lambda+f^2)^2}\cdots+\frac{z^2}{(\lambda+h^2)^2}+\frac{e^2}{\lambda^2}\right\}},$$

we have

$$e^{2q+1}\frac{d\mathrm{W}}{de}=\int_\lambda^\infty dt\,.-\frac{2me^{2q+2}}{t^{q+2}}\left(1-\frac{x^2}{t+f^2}\cdots-\frac{z^2}{t+h^2}-\frac{e^2}{t}\right)^{m+1}(t+f^2\ldots t+h^2)^{-\frac{1}{2}}$$
$$-\frac{\frac{2e^{2q+2}}{\lambda^{q+2}}}{\left\{\frac{x^2}{(\lambda+f^2)^2}\cdots+\frac{z^2}{(\lambda+h^2)^2}+\frac{e^2}{\lambda^2}\right\}}\left(1-\frac{x^2}{\lambda+f^2}\cdots-\frac{z^2}{\lambda+h^2}-\frac{e^2}{\lambda}\right)^m.(\lambda+f^2\ldots\lambda+h^2)^{-\frac{1}{2}},$$

where the second term, although containing the evanescent factor

$$\left(1-\frac{x^2}{\lambda+f^2}\cdots-\frac{z^2}{\lambda+h^2}-\frac{e^2}{\lambda}\right)^m,$$

is for the present retained.

108. I attend to the second term.

1°. Suppose $\frac{x^2}{f^2}\cdots+\frac{z^2}{h^2}>1$, then as e diminishes and becomes $=0$, λ does not become zero, but it becomes the positive root of the equation

$$1-\frac{x^2}{\lambda+f^2}\cdots-\frac{z^2}{\lambda+h^2}=0;$$

hence the term, containing as well the evanescent factor e^{2q+2} as the other evanescent factor $\left(1-\frac{x^2}{\lambda+f^2}\cdots-\frac{z^2}{\lambda+h^2}-\frac{e^2}{\lambda}\right)^m$, is $=0$.

2°. Suppose $\frac{x^2}{f^2}\cdots+\frac{z^2}{h^2}<1$, then as e diminishes to zero, λ tends to become $=0$, but $\frac{e^2}{\lambda}$ is finite and $=1-\frac{x^2}{f^2}\cdots-\frac{z^2}{h^2}$, whence $\frac{e^2}{\lambda^2}$ is indefinitely large; and since $\frac{x^2}{(\lambda+f^2)^2}\cdots+\frac{z^2}{(\lambda+h^2)^2}$ becomes $=\frac{x^2}{f^4}\cdots+\frac{z^2}{h^4}$, which is finite, the denominator may be reduced to $\frac{e^2}{\lambda^2}$, and the term therefore is

$$=-2\left(\frac{e^2}{\lambda}\right)^q\left(1-\frac{x^2}{\lambda+f^2}\cdots-\frac{z^2}{\lambda+h^2}-\frac{e^2}{\lambda}\right)^m.(\lambda+f^2\ldots\lambda+h^2)^{-\frac{1}{2}},$$

$$=-2\left(1-\frac{x^2}{f^2}\cdots-\frac{z^2}{h^2}\right)^q\left(1-\frac{x^2}{\lambda+f^2}\cdots-\frac{z^2}{\lambda+h^2}-\frac{e^2}{\lambda}\right)^m(f\ldots h)^{-1},$$

which, the other factor being finite, vanishes in virtue of the evanescent factor

$$\left(1-\frac{x^2}{\lambda+f^2}\cdots-\frac{z^2}{\lambda+h^2}-\frac{e^2}{\lambda}\right)^m.$$

Hence the second term always vanishes, and we have (e being $=0$)

$$e^{2q+1}\frac{dW}{de}=\int_\lambda^\infty dt.-\frac{2me^{2q+2}}{t^{q+2}}\left(1-\frac{x^2}{t+f^2}\cdots-\frac{z^2}{t+h^2}-\frac{e^2}{t}\right)^m(t+f^2\ldots t+h^2)^{-\frac{1}{2}}.$$

109. Considering first the case $\frac{x^2}{f^2}\cdots+\frac{z^2}{h^2}>1$, then as e diminishes to zero, λ does not become $=0$; the integral contains no infinite element, and it consequently vanishes in virtue of the factor e^{2q+2}.

But if $\frac{x^2}{f^2}\cdots+\frac{z^2}{h^2}<1$, then introducing instead of t the new variable ξ, $=\frac{e^2}{t}$, that is $t=\frac{e^2}{\xi}$, $dt=\frac{-e^2}{\xi^2}$, and writing for shortness,

$$R=1-\frac{x^2}{f^2+\frac{e^2}{\xi}}\cdots-\frac{z^2}{h^2+\frac{e^2}{\xi}},$$

the term becomes

$$=\int d\xi.\,2m(R-\xi)^{m-1}\xi^q\left(f^2+\frac{e^2}{\xi}\ldots h^2+\frac{e^2}{\xi}\right)^{-\frac{1}{2}},$$

where, as regards the limits corresponding to $t=\infty$, we have $\xi=0$, and corresponding to $t=\lambda$ we have ξ the positive root of $R-\xi=0$. But e is indefinitely small; except for indefinitely small values of ξ, we have

$$R=1-\frac{x^2}{f^2}\cdots-\frac{z^2}{h^2}, \text{ and } \left(f^2+\frac{e^2}{\xi}\ldots h^2+\frac{e^2}{\xi}\right)^{-\frac{1}{2}}=(f\ldots h)^{-1};$$

and if ξ be indefinitely small, then whether we take the accurate or the reduced

expressions, the elements are finite, and the corresponding portion of the integral is indefinitely small. We may consequently reduce as above; viz. writing now

$$R=1-\frac{x^2}{f^2}\ldots-\frac{z^2}{h^2},$$

the formula is

$$e^{2q+1}\frac{dW}{de}=\int_R^0 d\xi\,.\,2m(R-\xi)^{m-1}\xi^q(f\ldots h)^{-1},$$

$$=-2m(f\ldots h)^{-1}.\int_0^R d\xi\,.\,\xi^q(R-\xi)^{m-1};$$

or writing $\xi=Ru$, the integral becomes $=R^{q+m}\int_0^1 du\,u^q(1-u)^{m-1}$, which is

$$=\frac{\Gamma(1+q)\Gamma(m)}{\Gamma(1+q+m)}R^{q+m};$$

that is, we have

$$e^{2q+1}\frac{dW}{de}=-2(f\ldots h)^{-1}.\frac{\Gamma(1+q)\Gamma(1+m)}{\Gamma(1+q+m)},$$

and consequently

$$\rho=\frac{\Gamma(\tfrac{1}{2}s+q)}{2(\Gamma\tfrac{1}{2})^s\Gamma(1+q)}\,.\,2(f\ldots h)^{-1}.\frac{\Gamma(1+q)\Gamma(1+m)}{\Gamma(1+q+m)}R^{q+m},$$

that is

$$\rho=(f\ldots h)^{-1}\frac{\Gamma(\tfrac{1}{2}s+q)\Gamma(1+m)}{(\Gamma\tfrac{1}{2})^s\Gamma(1+q+m)}R^{q+m},$$

viz. ρ has this value for values of $(x\ldots z)$ such that $\frac{x^2}{f^2}\ldots+\frac{z^2}{h^2}<1$, but is $=0$ if $\frac{x^2}{f^2}\ldots+\frac{z^2}{h^2}>1$.

110. Multiplying by a constant factor so as to reduce ρ to the value R^{q+m}, the final result is

$$V=\int\frac{\left(1-\frac{x^2}{f^2}\ldots-\frac{z^2}{h^2}\right)^{q+m}dx\ldots dz}{[(a-x)^2\ldots+(c-z)^2\ldots+e^2]^{\frac{1}{2}s+q}},$$

the limits being given by the equation

$$\frac{x^2}{f^2}\ldots+\frac{z^2}{h^2}=1$$

is

$$=\frac{\Gamma(\tfrac{1}{2})^s\Gamma(1+q+m)}{\Gamma(\tfrac{1}{2}s+q)\Gamma(1+m)}(f\ldots h)\int_\theta^\infty dt\,t^{-q-1}\left(1-\frac{a^2}{t+f^2}\ldots-\frac{c^2}{t+h^2}\right)^m(t+f^2\ldots t+h^2)^{-\frac{1}{2}},$$

where θ is the positive root of

$$1-\frac{a^2}{\theta+f^2}\ldots-\frac{c^2}{\theta+h^2}-\frac{e^2}{\theta}=0.$$

In particular if $e=0$, or

$$V=\int\frac{\left(1-\frac{x^2}{f^2}\ldots-\frac{z^2}{h^2}\right)^{q+m}dx\ldots dz}{\{(a-x)^2\ldots+(c-z)^2\}^{\frac{1}{2}s+q}},$$

there are two cases,

exterior, $\frac{a^2}{f^2}\ldots+\frac{c^2}{h^2}>1$, θ is positive root of $1-\frac{a^2}{f^2}\ldots-\frac{c^2}{2}=0$,

interior, $\frac{a^2}{f^2}\ldots+\frac{c^2}{h^2}<1$, θ vanishes, viz. the limits in the integral are ∞, 0;

q must be *negative*, $1+q$ positive as before, in order that the t-integral may not be infinite in regard to the element $t=0$.

It is assumed in the proof that m and $1+q$ are each of them positive; but, as appears by the second example, the theorem is true for the extreme value $m=0$; it does not, however, appear that the proof can be extended to include the extreme value $q=-1$. The formula seems, however, to hold good for values of m, q beyond the foregoing limits; and it would seem that the only necessary conditions are $\frac{1}{2}s+q$, $1+m$, and $1+q+m$, each of them positive. The theorem is in fact a particular case of the following one, proved, Annex X. No. 162, viz.

$$V=\int\frac{\phi\left(1-\frac{x^2}{f^2}\ldots-\frac{z^2}{h^2}\right)dx\ldots dz}{\{(a-x)^2\ldots+(c-z)^2+e^2\}^{\frac{1}{2}s+q}}$$

over the ellipsoid $\frac{x^2}{f^2}\ldots+\frac{z^2}{h^2}=1$,

$$=\frac{(\Gamma\frac{1}{2})^s(f..h)}{\Gamma(-q)\Gamma(\frac{1}{2}s+q)}\int_\theta^\infty dt\,t^{-q-1}(t+f^2\ldots t+h^2)^{-\frac{1}{2}}(1-\sigma)^{-q}\int_0^1 x^{-q-1}\phi(\sigma+(1-\sigma)x)dx,$$

where σ denotes $\frac{a^2}{f^2+t}\ldots+\frac{c^2}{h^2+t}+\frac{e^2}{t}$: assuming $\phi u=(1-u)^{q+m}$, we have

$$\phi(\sigma+(1-\sigma)x)=(1-\sigma)^{q+m}(1-x)^{q+m},$$

and the theorem is thus proved.

111. Particular cases:

$$m=0;\quad\int\frac{\left(1-\frac{x^2}{f^2}\ldots-\frac{z^2}{h^2}\right)^q dx\ldots dz}{[(a-x)^2\ldots+(c-z)^2+e^2]^{\frac{1}{2}s+q}}=\frac{(\Gamma\frac{1}{2})^s\Gamma(1+q)}{\Gamma(\frac{1}{2}s+q)}(f\ldots h)\int_\theta^\infty dt\,t^{-q-1}(t+f^2\ldots t+h^2)^{-\frac{1}{2}}.$$

Cor. In a somewhat similar manner it may be shown that

$$\int\frac{\left(1-\frac{x^2}{f^2}\ldots-\frac{z^2}{h^2}\right)^q x\,dx\ldots dz}{\{(a-x)^2\ldots+(c-z)^2+e^2\}^{\frac{1}{2}s+q}}=\frac{(\Gamma\frac{1}{2})^s\Gamma(1+q)}{\Gamma(\frac{1}{2}s+q)}.(f\ldots h)\int_\theta^\infty dt\frac{af^2}{t+f^2}t^{-q-1}(t+f^2\ldots t+h^2)^{-\frac{1}{2}}.$$

Multiply the first by a and subtract the second, we have

$$\int\frac{\left(1-\frac{x^2}{f^2}\ldots-\frac{z^2}{h^2}\right)^q(a-x)\,dx\ldots dz}{\{(a-x)^2\ldots+(c-z)^2+e^2\}^{\frac{1}{2}s+q}}=\frac{(\Gamma\frac{1}{2})^s\Gamma(1+q)}{\Gamma(\frac{1}{2}s+q)}(f\ldots h)\int_\theta^\infty dt\,.\,\frac{a}{t+f^2}t^{-q}(t+f^2\ldots t+h^2)^{-\frac{1}{2}};$$

or writing $q+1$ for q, this is

$$\int\frac{\left(1-\frac{x^2}{f^2}\ldots-\frac{z^2}{h^2}\right)^{q+1}(a-x)\,dx\,..\,dz}{\{(a-x)^2\ldots+(c-z)^2+e^2\}^{\frac{1}{2}s+q+1}}=\frac{(\Gamma\frac{1}{2})^s\Gamma(2+q)}{\Gamma(\frac{1}{2}s+q+1)}(f\ldots h)\int_\theta^\infty dt\,.\,\frac{a}{t+f^2}t^{-q-1}(t+f^2\ldots t+h^2)^{-\frac{1}{2}};$$

and we have similar formulæ with (instead of $(a-x)$) $\ldots c-z$, e in the numerator.

112. If $m=1$, we have

$$\int \frac{\left(1-\frac{x^2}{f^2}\dots-\frac{z^8}{h^2}\right)^{q+1} dx\dots dz}{\{(a-x)^2\dots+(c-z)^2+e^2\}^{\frac{1}{2}s+q}}=\frac{(\Gamma\frac{1}{2})^s\Gamma(2+q)}{\Gamma(\frac{1}{2}s+q)}(f\dots h)\int_\theta^\infty\left\{1-\frac{a^2}{t+f^2}\dots-\frac{c^2}{t+h^2}-\frac{e^2}{t}\right\}t^{-q-1}(t+f^2\dots t+h^2)^{-\frac{1}{2}},$$

which, differentiated in respect to a, gives the $(a-x)$formula; hence conversely, assuming the $a-x, \dots c-z, e$ formulæ, we obtain by integration the last preceding formula to a constant *près*, viz. we thereby obtain the multiple integral $=\mathrm{C}+$ right-hand function, where C is independent of $(a\dots c, e)$; and by taking these all infinite, observing that then $\theta=\infty$, the two integrals each vanish, and we obtain $\mathrm{C}=0$.

In particular $s=3$, $q=-1$, then

$$\int\frac{dx\,dy\,dz}{\{(a-x)^2+(b-y)^2+(c-z)^2+e^2\}^{\frac{1}{2}}}=\pi fgh\int_\theta^\infty\left\{1-\frac{a^2}{t+f^2}-\frac{b^2}{t+g^2}-\frac{c^2}{t+h^2}-\frac{e^2}{t}\right\}(t+f^2.\,t+g^2.\,t+h^2)^{-\frac{1}{2}},$$

which, putting therein $e=0$, gives the potential of an ellipsoid for the cases of an exterior point and an interior point respectively.

ANNEX V. GREEN'S *Integration of the Prepotential Equation*

$$\left(\frac{d^2}{da^2}\dots+\frac{d^2}{dc^2}+\frac{d^2}{de^2}+\frac{2q+1}{e}\frac{d}{de}\right)\mathrm{V}=0.\text{—Nos. 113 to 128.}$$

113. In the present Annex I in part reproduce GREEN'S process for the integration of this equation by means of a series of functions analogous to LAPLACE'S Functions, and which may be termed "Greenians" (see his Memoir on the Attraction of Ellipsoids, referred to above); each such function gives rise to a Prepotential Integral.

GREEN shows, by a complicated and difficult piece of general reasoning, that there exist solutions of the form $\mathrm{V}=\Theta\phi$ (see *post*, No. 116), where ϕ is a function of the s new variables $\alpha, \beta\dots\gamma$ without θ, such that $\nabla\phi=\varkappa\phi$, $\varkappa$ being a function of θ only; these functions ϕ of the variables $\alpha, \beta\dots\gamma$ are in fact the Greenian Functions in question. The function of the order 0 is $\phi=1$; those of the order 1 are $\phi=\alpha, \phi=\beta\dots\phi=\gamma$; those of the order 2 are $\phi=\alpha\beta$, &c., and s-functions each of the form

$$\tfrac{1}{2}\{\mathrm{A}\alpha^2+\mathrm{B}\beta^2\dots+\mathrm{C}\gamma^2\}+\mathrm{D}.$$

The existence of the functions just referred to other than the s-functions involving the squares of the variables is obvious enough; the difficulty first arises in regard to these s-functions; and the actual development of them appears to me important by reason of the light which is thereby thrown upon the general theory. This I accomplish in the present Annex; and I determine by GREEN'S process the corresponding prepotential integrals. I do not go into the question of the Greenian Functions of orders superior to the second.

114. I write for greater clearness $(a, b\dots c, e)$ instead of $(a\dots c, e)$ to denote the series of $(s+1)$ variables; viz. $(a, b\dots c)$ will denote a series of s variables; corresponding to these we have the semiaxes $(f, g\dots h)$, and the new variables $(\alpha, \beta\dots\gamma)$;

these last, with the before-mentioned function θ, are the $s+1$ new variables of the problem; and for convenience there is introduced also a quantity ε; viz. we have

$$\begin{aligned} a&=\sqrt{f^2+\theta}\ \alpha,\\ b&=\sqrt{g^2+\theta}\ \beta,\\ &\vdots\\ c&=\sqrt{h^2+\theta}\ \gamma,\\ e&=\sqrt{\theta}\quad\ \ \varepsilon, \end{aligned}$$

where $1=\alpha^2+\beta^2\ldots+\gamma^2+\varepsilon^2$.

That is, we have θ a function of $a, b\ldots c, e$ determined by

$$\frac{a^2}{f^2+\theta}+\frac{b^2}{g^2+\theta}\ldots+\frac{c^2}{h^2+\theta}+\frac{e^2}{\theta}=1;$$

and then $\alpha, \beta\ldots\gamma$ are given as functions of the same quantities $a, b\ldots c, e$ by the equations

$$\alpha^2=\frac{a^2}{f^2+\theta},\quad \beta^2=\frac{b^2}{g^2+\theta}\ldots\gamma^2=\frac{c^2}{h^2+\theta};$$

also ε, considered as a function of the same quantities, is

$$=\sqrt{1-\frac{a^2}{(f^2+\theta)}-\frac{b^2}{(g^2+\theta)}\ldots-\frac{c^2}{(h^2+\theta)}}.$$

115. Introducing instead of $a, b\ldots c, e$ the new variables $\alpha, \beta\ldots\gamma, \theta$, the transformed differential equation is

$$4\theta\frac{d^2\mathrm{V}}{d\theta^2}+2\frac{d\mathrm{V}}{d\theta}\left(s+2q+2-\frac{f^2}{f^2+\theta}\ldots-\frac{h^2}{h^2+\theta}\right)+\nabla\mathrm{V}=0,$$

where for shortness

$$\begin{aligned} \nabla\mathrm{V}=&\frac{1}{f^2+\theta}\left\{-\alpha^2-\frac{g^2}{g^2+\theta}\beta^2\ldots-\frac{h^2}{h^2+\theta}\gamma^2+1\right\}\frac{d^2\mathrm{V}}{d\alpha^2}\\ &+\frac{1}{g^2+\theta}\left\{-\frac{f^2}{f^2+\theta}\alpha^2-\beta^2\ldots-\frac{h^2}{h^2+\theta}\gamma^2+1\right\}\frac{d^2\mathrm{V}}{d\beta^2}\\ &\quad\vdots\\ &+\frac{1}{h^2+\theta}\left\{-\frac{f^2}{f^2+\theta}\alpha^2-\frac{g^2}{g^2+\theta}\beta^2\ldots-\gamma^2+1\right\}\frac{d^2\mathrm{V}}{d\gamma^2}\\ &-\frac{2\theta}{f^2+\theta\,.\,g^2+\theta}\frac{d^2\mathrm{V}}{d\alpha d\beta}-\&\text{c.}\\ &+\frac{1}{f^2+\theta}\left\{-2q-2-\theta\left(\frac{1}{g^2+\theta}\ldots+\frac{1}{h^2+\theta}\right)\right\}\alpha\frac{d\mathrm{V}}{d\alpha}\\ &+\frac{1}{g^2+\theta}\left\{-2q-2-\theta\left(\frac{1}{f^2+\theta}\ldots+\frac{1}{h^2+\theta}\right)\right\}\beta\frac{d\mathrm{V}}{d\beta}\\ &\quad\vdots\\ &+\frac{1}{h^2+\theta}\left\{-2q-2-\theta\left(\frac{1}{f^2+\theta}+\frac{1}{g^2+\theta}\ldots\right)\right\}\gamma\frac{d\mathrm{V}}{d\gamma}. \end{aligned}$$

Also

$$e^{2q+1}\frac{dV}{de}=-\theta^{q+1}\varepsilon^{2q+2}\left\{1-\frac{f^2\alpha^2}{f^2+\theta}\cdots-\frac{h^2\gamma^2}{\gamma^2+\theta}\right\}^{-1}\left(\frac{1}{f^2+\theta}\,\alpha\frac{dV}{d\alpha}\cdots+\frac{1}{h^2+\theta}\,\gamma\frac{dV}{d\gamma}-2\frac{dV}{d\theta}\right).$$

116. To integrate the equation for V we assume

$$V=\Theta\phi,$$

where Θ is a function of θ only, and ϕ a function of $\alpha, \beta\ldots\gamma$ (without θ), such that

$$\nabla\phi=\varkappa\phi,$$

$\varkappa$ being a function of θ only. Assuming that this is possible, the remaining equation to be satisfied is obviously

$$4\theta\frac{d^2\Theta}{d\theta^2}+2\frac{d\Theta}{d\theta}\left\{2q+2+\theta\left(\frac{1}{f^2+\theta}\cdots+\frac{1}{h^2+\theta}\right)\right\}+\varkappa\Theta=0.$$

Solutions of the form in question are

$$\begin{aligned}
&\phi=1\ , \quad \varkappa=0,\\
&\phi=\alpha\ , \quad \varkappa=\frac{1}{f^2+\theta}\left\{-2q-2-\frac{\theta}{g^2+\theta}\cdots-\frac{\theta}{h^2+\theta}\right\},\\
&\phi=\beta\ , \quad \varkappa=\quad ,, \qquad\qquad ,,\\
&\quad\vdots\\
&\phi=\alpha\beta, \quad \varkappa=\frac{-2\theta}{f^2+\theta\,.\,g^2+\theta}+\frac{1}{f^2+\theta}\cdot\left\{-2q-2-\frac{\theta}{g^2+\theta}\cdots-\frac{\theta}{h^2+\theta}\right\},\\
&\qquad\qquad\qquad\qquad\qquad +\frac{1}{g^2+\theta}\cdot\left\{-2q-2-\frac{\theta}{f^2+\theta}\cdots-\frac{\theta}{g^2+\theta}\right\};
\end{aligned}$$

and it can be shown next that there is a solution of the form

$$\phi=\tfrac{1}{2}(A\alpha^2+B\beta^2\ldots+C\gamma^2)+D.$$

117. In fact, assuming that this satisfies $\nabla\phi-\varkappa\phi=0$, we must have identically

$$\begin{aligned}
&\frac{A}{f^2+\theta}\left\{-\alpha^2-\frac{g^2}{g^2+\theta}\beta^2\ldots-\frac{h^2}{h^2+\theta}\gamma^2+1\right\}\\
&+\frac{B}{g^2+\theta}\left\{-\frac{f^2}{f^2+\theta}\alpha^2-\beta^2\ldots-\frac{h^2}{h^2+\theta}\gamma^2+1\right\}\\
&\quad\vdots\\
&+\frac{C}{h^2+\theta}\left\{-\frac{f^2}{f^2+\theta}\alpha^2-\frac{g^2}{g^2+\theta}\beta^2\ldots-\gamma^2+1\right\}\\
&+\frac{A}{f^2+\theta}\left\{-s-2q-1\ +\frac{g^2}{g^2+\theta}\cdots+\frac{h^2}{h^2+\theta}\right\}\\
&+\frac{B}{g^2+\theta}\left\{-s-2q-1\ +\frac{f^2}{f^2+\theta}\cdots+\frac{h^2}{h^2+\theta}\right\}\\
&\quad\vdots\\
&+\frac{C}{h^2+\theta}\left\{-s-2q-1\ +\frac{f^2}{f^2+\theta}+\frac{g^2}{g^2+\theta}\cdots\right\}\\
&+\varkappa\{\tfrac{1}{2}(A\alpha^2+B\beta^2\ldots+C\gamma^2)\ +\ D\ \};
\end{aligned}$$

so that from the term in α^2 we have

$$\frac{\mathrm{A}}{f^2+\theta}\left\{-s-2q-2+\frac{g^2}{g^2+\theta}\ldots+\frac{h^2}{h^2+\theta}\right\}-\tfrac{1}{2}\varkappa\mathrm{A}-\frac{\mathrm{B}f^2}{f^2+\theta\,.\,g^2+\theta}\ldots-\frac{\mathrm{C}f^2}{f^2+\theta\,.\,h^2+\theta}=0\,;$$

or, what is the same thing,

$$\mathrm{A}\left\{-2q-3-\frac{\theta}{g^2+\theta}\ldots+\frac{\theta}{h^2+\theta}-\tfrac{1}{2}\varkappa(f^2+\theta)\right\}-\mathrm{B}\frac{f^2}{g^2+\theta}\ldots-\frac{\mathrm{C}f^2}{h^2+\theta}=0,$$

with the like equations from $\beta^2\ldots\gamma^2$; and from the constant term we have

$$\mathrm{A}\frac{1}{f^2+\theta}\qquad\qquad+\mathrm{B}\frac{1}{g^2+\theta}\ldots+\frac{\mathrm{C}}{h^2+\theta}-\varkappa\mathrm{D}=0.$$

118. Multiplying this last by f^2, and adding it to the first, we obtain

$$\mathrm{A}\left\{-2q-2-\frac{\theta}{f^2+\theta}-\frac{\theta}{g^2+\theta}\ldots-\frac{\theta}{h^2+\theta}-\tfrac{1}{2}\varkappa(f^2+\theta)\right\}-\varkappa f^2\mathrm{D}=0\,;$$

viz. putting for shortness $\Omega=\theta\left(\frac{1}{f^2+\theta}+\frac{1}{g^2+\theta}\ldots+\frac{1}{h^2+\theta}\right)$, this is

$$\mathrm{A}\{2q+2+\Omega+\tfrac{1}{2}\varkappa(f^2+\theta)\}+\varkappa f^2\mathrm{D}=0,$$

and similarly

$$\mathrm{B}\{2q+2+\Omega+\tfrac{1}{2}\varkappa(g^2+\theta)\}+\varkappa g^2\mathrm{D}=0,$$

$$\vdots$$

$$\mathrm{C}\{2q+2+\Omega+\tfrac{1}{2}\varkappa(h^2+\theta)\}+\varkappa h^2\mathrm{D}=0,$$

and to these we join the foregoing equation

$$\frac{\mathrm{A}}{f^2+\theta}+\frac{\mathrm{B}}{g^2+\theta}\ldots+\frac{\mathrm{C}}{h^2+\theta}\qquad-\varkappa\mathrm{D}=0.$$

Eliminating A, B ... C, D we have an equation which determines $\varkappa$ as a function of θ; and the equations then determine the ratios of A, B ... C, D, so that these quantities will be given as determinate multiples of an arbitrary quantity M. The equation for $\varkappa$ is in fact

$$\frac{f^2}{(f^2+\theta)\{2q+2+\Omega+\frac{1}{2}\kappa(f^2+\theta)\}}+\frac{g^2}{(g^2+\theta)\{2q+2+\Omega+\frac{1}{2}\kappa(g^2+\theta)\}}\ldots$$
$$+\frac{h^2}{(h^2+\theta)\{2q+2+\Omega+\frac{1}{2}\kappa(h^2+\theta)\}}+1=0\,;$$

and the values of A, B .. C, D are then

$$\frac{\mathrm{M}f^2}{2q+2+\Omega+\frac{1}{2}\varkappa(f^2+\theta)},\ \frac{\mathrm{M}g^2}{2q+2+\Omega+\frac{1}{2}\varkappa(g^2+\theta)},\ \ldots\frac{\mathrm{M}h^2}{2q+2+\Omega+\frac{1}{2}\varkappa(h^2+\theta)},\quad-\frac{\mathrm{M}}{\varkappa},$$

values which seem to be dependent on θ: if they were so, it would be fatal to the success of the process; but they are really independent of θ.

119. That they are independent of θ depends on the theorem that we have

$$\varkappa=\frac{(2q+2+\Omega)\varkappa_0}{2q+2-\frac{1}{2}\varkappa_0\theta},$$

where $\varkappa_0$ is a quantity independent of θ determined by the equation

$$\frac{1}{2q+2+\frac{1}{2}\varkappa_0 f^2}+\frac{1}{2q+2+\frac{1}{2}\varkappa_0 g^2}\cdots+\frac{1}{2q+2+\frac{1}{2}\varkappa_0 h^2}+1=0,$$

($\varkappa_0$ is in fact the value of $\varkappa$ on writing $\theta=0$), and that, omitting the arbitrary multiplier, the values of A, B ... C, D then are

$$\frac{f^2}{2q+2+\frac{1}{2}\varkappa_0 f^2},\ \frac{g^2}{2q+2+\frac{1}{2}\varkappa_0 g^2},\ \cdots\frac{h^2}{2q+2+\frac{1}{2}\varkappa_0 h^2},\ -\frac{1}{\varkappa_0};$$

or, what is the same thing, the value of ϕ is

$$=\frac{\frac{1}{2}f^2\alpha^2}{2q+2+\frac{1}{2}\varkappa_0 f^2}+\frac{\frac{1}{2}g^2\beta^2}{2q+2+\frac{1}{2}\varkappa_0 g^2}\cdots+\frac{\frac{1}{2}h^2\gamma^2}{2q+2+\frac{1}{2}\varkappa_0 h^2}-\frac{1}{\varkappa_0}.$$

120. [To explain the ground of the assumption

$$\varkappa=\frac{(2q+2+\Omega)\varkappa_0}{2q+2-\frac{1}{2}\varkappa_0\theta},$$

observe that, assuming

$$\frac{2q+2+\Omega+\frac{1}{2}\varkappa(f^2+\theta)}{2q+2+\frac{1}{2}\varkappa_0 f^2}=\frac{2q+2+\Omega+\frac{1}{2}\varkappa(g^2+\theta)}{2q+2+\frac{1}{2}\varkappa_0 g^2},$$

then multiplying out and reducing, we obtain

$$\tfrac{1}{2}\varkappa_0(2q+2+\Omega)(g^2-f^2)+(2q+2)\,.\,\tfrac{1}{2}\varkappa(f^2-g^2)+\tfrac{1}{4}\varkappa_0\varkappa(g^2-f^2)\theta=0;$$

viz. the equation divides out by the factor g^2-f^2, thereby becoming

$$\varkappa_0(2q+2+\Omega)-(2q+2)\varkappa+\tfrac{1}{2}\varkappa\varkappa_0\theta=0,$$

that is, it gives for $\varkappa$ the foregoing value: hence clearly, $\varkappa$ having this value, we obtain by symmetry

$$2q+2+\Omega+\tfrac{1}{2}\varkappa(f^2+\theta),\ 2q+2+\Omega+\tfrac{1}{2}\varkappa(g^2+\theta),\ldots 2q+2+\Omega+\tfrac{1}{2}\varkappa(h^2+\theta),$$

proportional to

$$2q+2+\tfrac{1}{2}\varkappa_0 f^2,\qquad 2q+2+\tfrac{1}{2}\varkappa_0 g^2,\ldots\qquad 2q+2+\tfrac{1}{2}\varkappa_0 h^2;$$

viz. the ratios, not only of A : B, but of A : B ... : C will be independent of θ.]

121. To complete the transformation, starting with the foregoing value of $\varkappa$, we have

$$2q+2+\Omega+\tfrac{1}{2}\varkappa(f^2+\theta)=(2q+2+\Omega)\,.\,\frac{2q+2+\frac{1}{2}\varkappa_0 f^2}{2q+2-\frac{1}{2}\varkappa_0\theta},\ \text{\&c.};$$

so that we have

$$\begin{aligned}&\mathrm{A}\{2q+2+\tfrac{1}{2}\varkappa_0 f^2\}+\varkappa_0 f^2\mathrm{D}=0,\\ &\mathrm{B}\{2q+2+\tfrac{1}{2}\varkappa_0 g^2\}+\varkappa_0 g^2\mathrm{D}=0,\\ &\vdots\\ &\mathrm{C}\{2q+2+\tfrac{1}{2}\varkappa_0 h^2\}+\varkappa_0 h^2\mathrm{D}=0,\end{aligned}$$

and

$$\frac{\mathrm{A}}{f^2+\theta}+\frac{\mathrm{B}}{g^2+\theta}\cdots+\frac{\mathrm{C}}{h^2+\theta}-\frac{(2q+2+\Omega)\varkappa_0\mathrm{D}}{2q+2-\frac{1}{2}\varkappa_0\theta}=0.$$

Substituting for A, B, ... C their values, this last becomes

$$-\frac{\varkappa_0 \mathrm{D}}{2q+2+\frac{1}{2}k_0\theta}\left\{\frac{2q+2}{2q+2+\frac{1}{2}\varkappa_0 f^2}-\frac{\theta}{f^2+\theta}\right\}\cdots-\frac{\varkappa_0 \mathrm{D}}{2q+2-\frac{1}{2}\varkappa_0\theta}\left\{\frac{2q+2}{2q+2+\frac{1}{2}\varkappa_0 h^2}-\frac{\theta}{h^2+\theta}\right\}$$
$$-\frac{\varkappa_0 \mathrm{D}}{2q+2-\frac{1}{2}\varkappa_0\theta}\{2q+2+\Omega\}=0;$$

viz. this is

$$\left\{\frac{2q+2}{2q+2+\frac{1}{2}\varkappa_0 f^2}-\frac{\theta}{f^2+\theta}\right\}\cdots+\left\{\frac{2q+2}{2q+2+\frac{1}{2}\varkappa_0 h^2}-\frac{\theta}{h^2+\theta}\right\}+2q+2+\Omega=0;$$

or substituting for Ω its value, and dividing out by $2q+2$, we have

$$\frac{1}{2q+2+\frac{1}{2}\varkappa_0 f^2}+\frac{1}{2q+2+\frac{1}{2}\varkappa_0 g^2}\cdots+\frac{1}{2q+2+\frac{1}{2}\varkappa_0 h^2}+1=0,$$

the equation for the determination of $\varkappa_0$.

122. The equation for $\varkappa_0$ is of the order s; there are consequently s functions of the form in question, and each of the terms $\alpha^2, \beta^2, \ldots \gamma^2$ can be expressed as a linear function of these. It thus appears that any quadric function of $\alpha, \beta, \ldots \gamma$ can be expressed as a sum of Greenian functions; viz. the form is

$$\begin{aligned}
&\mathrm{A}\\
&+\mathrm{B}\alpha+\&\mathrm{c}.\\
&+\mathrm{C}\alpha\beta+\&\mathrm{c}.\\
&+\mathrm{D}'\left(\frac{\frac{1}{2}f^2\alpha^2}{2q+2+\frac{1}{2}\varkappa_0' f^2}+\frac{\frac{1}{2}g^2\beta^2}{2q+2+\frac{1}{2}\varkappa_0' g^2}+\cdots\frac{\frac{1}{2}h^2\gamma^2}{2q+2+\frac{1}{2}\varkappa_0' h^2}-\frac{1}{\varkappa_0'}\right)\\
&+\mathrm{D}''(\quad ,, \qquad\qquad ,, \qquad\qquad ,, \qquad)\\
&\quad (s \text{ lines}),
\end{aligned}$$

viz. the terms multiplied by $\mathrm{D}', \mathrm{D}''$, &c. respectively are those answering to the roots $\varkappa_0', \varkappa_0'', \ldots$ of the equation in $\varkappa_0$.

The general conclusion is that any rational and integral function of $\alpha, \beta, \ldots \gamma$ can be expressed as a sum of Greenian functions.

123. We have next to integrate the equation

$$4\theta\frac{d^2\Theta}{d\theta^2}+2\frac{d\Theta}{d\theta}\left(2q+2+\frac{\theta}{f^2+\theta}+\frac{\theta}{g^2+\theta}\cdots+\frac{\theta}{h^2+\theta}\right)-\varkappa\Theta=0.$$

Suppose $\varkappa=0$, a particular solution is $\Theta=1$;

$$\varkappa=\frac{1}{f^2+\theta}\left(-2q-2-\frac{\theta}{g^2+\theta}\cdots-\frac{\theta}{h^2+\theta}\right), \text{ a particular solution is } \frac{\sqrt{f^2+\theta}}{\sqrt{f^2+g^2\ldots+h^2}};$$

in fact, omitting the constant denominator, or writing $\Theta=\sqrt{f^2+\theta}$, and therefore

$$\frac{d\Theta}{d\theta}=\frac{1}{2\sqrt{f^2+\theta}}, \quad \frac{d^2\Theta}{d\theta^2}=-\frac{1}{4(f^2+\theta)^{\frac{3}{2}}},$$

the equation to be verified is

$$-\frac{\theta}{(f^2+\theta)^{\frac{3}{2}}}+\frac{1}{\sqrt{f^2+\theta}}\left\{\ \ 2q+2+\frac{\theta}{f^2+\theta}+\frac{\theta}{g^2+\theta}\cdots+\frac{\theta}{h^2+\theta}\right\}$$

$$+\frac{1}{\sqrt{f^2+\theta}}\left\{-2q-2\qquad\qquad-\frac{\theta}{g^2+\theta}\cdots-\frac{\theta}{h^2+\theta}\right\}=0,\text{ which is right.}$$

Again, suppose $\varkappa=\frac{-2\theta}{f^2+\theta\,.\,g^2+\theta}+\&\text{c.}$ (value belonging to $\phi=\alpha\beta$, see No. 116), a particular solution is $\frac{\sqrt{f^2+\theta}\ \sqrt{g^2+\theta}}{f^2+g^2\ldots+h^2}$; in fact omitting the constant factor, or writing

$$\Theta=\sqrt{f^2+\theta}\,\sqrt{g^2+\theta},$$

and therefore

$$\frac{d\Theta}{d\theta}=\tfrac{1}{2}\left\{\frac{\sqrt{g^2+\theta}}{\sqrt{f^2+\theta}}+\frac{\sqrt{f^2+\theta}}{\sqrt{g^2+\theta}}\right\},$$

$$\frac{d^2\Theta}{d\theta^2}=\tfrac{1}{4}\left\{-\frac{\sqrt{g^2+\theta}}{(f^2+\theta)^{\frac{3}{2}}}+\frac{2}{\sqrt{f^2+\theta}\,\sqrt{g^2+\theta}}-\frac{\sqrt{f^2+\theta}}{(g^2+\theta)^{\frac{3}{2}}}\right\},$$

the equation to be verified is

$$\theta\left\{-\frac{\sqrt{g^2+\theta}}{(f^2+\theta)^{\frac{3}{2}}}+\frac{2}{\sqrt{f^2+\theta}\,\sqrt{g^2+\theta}}-\frac{\sqrt{f^2+\theta}}{(g^2+\theta)^{\frac{3}{2}}}\right\}$$

$$+\left(\frac{\sqrt{g^2+\theta}}{\sqrt{f^2+\theta}}+\frac{\sqrt{f^2+\theta}}{\sqrt{g^2+\theta}}\right)\left\{2q+2+\frac{\theta}{f^2+\theta}+\frac{\theta}{g^2+\theta}\cdots+\frac{\theta}{h^2+\theta}\right\}$$

$$+\sqrt{f^2+\theta}\,\sqrt{g^2+\theta}\ \ \left\{\frac{-2\theta}{f^2+\theta . g^2+\theta}+\frac{1}{f^2+\theta}\left(-2q-2-\frac{\theta}{g^2+\theta}\cdots-\frac{\theta}{h^2+\theta}\right)\right.$$

$$\left.+\frac{1}{g^2+\theta}\left(-2q-2-\frac{\theta}{f^2+\theta}\cdots+\frac{\theta}{h^2+\theta}\right)\right\}=0\,;$$

or putting for shortness $\Omega=\frac{\theta}{f^2+\theta}+\frac{\theta}{g^2+\theta}\cdots+\frac{\theta}{h^2+\theta}$, this is

$$-\frac{\theta\sqrt{g^2+\theta}}{(f^2+\theta)^{\frac{3}{2}}}+\frac{2\theta}{\sqrt{f^2+\theta}\,\sqrt{g^2+\theta}}-\frac{\theta\sqrt{f^2+\theta}}{(g^2+\theta)^{\frac{3}{2}}}\qquad+\left(\frac{\sqrt{g^2+\theta}}{\sqrt{f^2+\theta}}+\frac{\sqrt{f^2+\theta}}{\sqrt{g^2+\theta}}\right)(2q+2+\Omega)$$

$$-\frac{2\theta}{\sqrt{f^2+\theta}\,\sqrt{g^2+\theta}}+\frac{\sqrt{g^2+\theta}}{\sqrt{f^2+\theta}}\left(-2q-2+\frac{\theta}{f^2+\theta}-\Omega\right)+\frac{\sqrt{f^2+\theta}}{\sqrt{g^2+\theta}}\left(-2q-2+\frac{\theta}{g^2+\theta}-\Omega\right)=0,$$

which is true.

And generally the particular solution is deduced from the value of ϕ by writing therein

$$\frac{\sqrt{f^2+\theta}}{\sqrt{f^2+g^2\ldots+h^2}},\ \frac{\sqrt{g^2+\theta}}{\sqrt{f^2+g^2\ldots+h^2}},\ldots\frac{\sqrt{h^2+\theta}}{\sqrt{f^2+g^2\ldots+h^2}}$$

in place of $\alpha,\beta,\ldots\gamma$ respectively: say the value thus obtained is $\Theta=\mathrm{H}$, where H is what ϕ becomes by the above substitution.

124. Represent for a moment the equation in Θ by

$$4\theta\frac{d^2\Theta}{d\theta^2}+2\frac{d\Theta}{d\theta}\mathrm{P}+\varkappa\Theta=0,$$

and assume that this is satisfied by $\Theta = \mathrm{H}\int z\,d\theta$, then we have

$$4\theta\left(\frac{d^2\mathrm{H}}{d\theta^2}\int z\,d\theta + 2\,\frac{d\mathrm{H}}{d\theta}\,z + \mathrm{H}\,\frac{dz}{d\theta}\right)$$

$$+2\mathrm{P}\left(\frac{d\mathrm{H}}{d\theta}\int z\,d\theta + \quad \mathrm{H}\,z\right)$$

$$+\ \varkappa\,.\ \ \mathrm{H}\int z\,d\theta \qquad\qquad = 0\,;$$

and therefore

$$\left(8\theta\frac{d\mathrm{H}}{d\theta} + 2\mathrm{PH}\right)z + 4\theta\mathrm{H}\,\frac{dz}{d\theta} = 0\,;$$

viz., multiplying by $\frac{\mathrm{H}}{4\theta}$, this is

$$\frac{d}{d\theta}(\mathrm{H}^2 z) + \frac{1}{2\theta}\,\mathrm{PH}^2 z = 0,$$

or

$$\frac{1}{\mathrm{H}^2 z}\,\frac{d}{d\theta}\,(\mathrm{H}^2 z) + \frac{1}{2\theta}\,\mathrm{P} = 0\,;$$

viz. substituting for P its value, this is

$$\frac{1}{\mathrm{H}^2 z}\,\frac{d}{d\theta}(\mathrm{H}^2 z) + \frac{1}{2\theta}\left(2q + 2 + \frac{\theta}{f^2+\theta} + \frac{\theta}{g^2+\theta}\cdots + \frac{\theta}{h^2+\theta}\right) = 0.$$

Hence, integrating,

$$\mathrm{H}^2 z = \frac{\mathrm{C}\theta^{-q-1}}{\sqrt{f^2+\theta\,.\,g^2+\theta\,\ldots\,h^2+\theta}}, \qquad \text{C an arbitrary constant,}$$

and

$$\Theta = \mathrm{CH}\int_{\chi}\frac{\theta^{-q-1}\,d\theta}{\mathrm{H}^2\,\sqrt{f^2+\theta\,.\,g^2+\theta\,\ldots\,h^2+\theta}}, \qquad \chi \text{ arbitrary,}$$

where the constants of integration are C, λ; or, what is the same thing, taking T the same function of t that H is of θ (viz. T is what ϕ becomes on writing therein

$$\frac{\sqrt{f^2+t}}{\sqrt{f^2+g^2\ldots+h^2}}\cdots\frac{\sqrt{g^2+t}}{\sqrt{f^2+g^2\ldots+h^2}}\cdots\frac{\sqrt{h^2+t}}{\sqrt{f^2+g^2\ldots+h^2}},$$

in place of $\alpha, \beta, \ldots \gamma$ respectively), then

$$\Theta = -\mathrm{CH}\int_{\theta}^{\chi}\frac{t^{-q-1}\,dt}{\mathrm{T}^2\,\sqrt{f^2+t\,.\,g^2+t\,.\,.\,h^2+t}},$$

where χ may be taken $=\infty$: we thus have

$$\mathrm{V} = \Theta\phi = -\mathrm{CH}\phi\int_{\theta}^{\infty}\frac{t^{-q-1}\,dt}{\mathrm{T}^2\,\sqrt{f^2+t\,.\,g^2+t\,\ldots\,h^2+t}}.$$

Recollecting that

$$1 = \frac{a^2}{f^2+\theta} + \frac{b^2}{g^2+\theta}\cdots + \frac{c^2}{h^2+\theta} + \frac{e^2}{\theta},$$

so that for $\theta=\infty$ we have $a^2+b^2\ldots+c^2+e^2=\theta$, the assumption $\chi=\infty$ comes to making V vanish for infinite values of $(a, b, \ldots c, e)$.

125. We have to find the value of ϱ corresponding to the foregoing value of V; viz. W being the value of V, on writing therein $(x, y, \ldots z)$ in place of $(a, b, \ldots c)$, then (theorem A)

$$\varrho = -\frac{\Gamma(\frac{1}{2}s+q)}{2(\Gamma\frac{1}{2})^s\,\Gamma(q+1)}\left(e^{2q+1}\frac{d\mathrm{W}}{de}\right)_0.$$

Take λ the same function of $(x, y, \ldots z, e)$ that θ is of $(a, b, \ldots c, e)$, viz. λ the positive root of

$$\frac{x^2}{f^2+\lambda}+\frac{y^2}{g^2+\lambda}\cdots+\frac{z^2}{h^2+\lambda}-\frac{e^2}{\lambda}=1,$$

and $(\xi, \eta, \ldots \zeta, \tau)$ corresponding to $(\alpha, \beta, \ldots \gamma, \varepsilon)$, viz.

$$\xi=\frac{x}{\sqrt{f^2+\lambda}},\ \eta=\frac{y}{\sqrt{g^2+\lambda}}\cdots\zeta=\frac{z}{\sqrt{h^2+\lambda}},\ \tau=\sqrt{1-\frac{x^2}{f^2+\lambda}-\frac{y^2}{g^2+\lambda}\cdots-\frac{z^2}{h^2+\lambda}},$$

so that W is the same function of $(\xi, \eta, \ldots \lambda)$ that V is of $(\alpha, \beta, \ldots \theta)$: say this is

$$\mathrm{W}=-\mathrm{C}\Lambda\psi\int_\lambda^\infty\frac{t^{-q-1}\,dt}{\mathrm{T}^2\sqrt{f^2+t\,.\,g^2+t\ldots h^2+t}},$$

then we have for ϱ the value

$$\varrho=\frac{\Gamma(\frac{1}{2}s+q)}{2(\Gamma\frac{1}{2})^s\Gamma(q+1)}\lambda^{q+1}\tau^{2q+2}\left(1-\frac{f^2\xi^2}{f^2+\lambda}\cdots-\frac{h^2\zeta^2}{h^2+\lambda}\right)^{-1}.\left(\frac{1}{f^2+\lambda}\,\xi\frac{d\mathrm{W}}{d\xi}\cdots+\frac{1}{h^2+\lambda}\,\zeta\frac{d\mathrm{W}}{d\zeta}-2\frac{d\mathrm{W}}{d\lambda}\right),$$

where e is to be put $=0$.

126. Suppose e is $=0$, then if $\frac{x^2}{f^2}+\frac{y^2}{g^2}\ldots+\frac{z^2}{h^2}>1$, λ is not $=0$, but is the positive root of $\frac{x^2}{f^2+\lambda}+\frac{y^2}{g^2+\lambda}\cdots+\frac{z^2}{h^2+\lambda}=1$, τ, $=\sqrt{1-\frac{x^2}{f^2+\lambda}-\frac{y^2}{g^2+\lambda}\cdots-\frac{z^2}{h^2+\lambda}}$, is $=0$, and we have $\varrho=0$, viz. ϱ is $=0$ for all points outside the ellipsoid $\frac{x^2}{f^2}+\frac{y^2}{g^2}\ldots+\frac{z^2}{h^2}=1$.

But if $\frac{x^2}{f^2}+\frac{y^2}{g^2}\ldots+\frac{z^2}{h^2}<1$, then on writing $e=0$, we have $\lambda=0$, $\tau^2=\frac{e^2}{\lambda}$,

$$\varrho=\frac{\Gamma(\frac{1}{2}s+q)}{2\pi^{\frac{1}{2}s}\Gamma(q+1)}\,.\,\lambda^{q+1}\frac{e^{2q+2}}{\lambda^{q+1}}\,.\,\frac{\lambda}{e^2}\left(\frac{1}{f^2}\,\xi\frac{d\mathrm{W}}{d\xi}+\frac{1}{g^2}\,\eta\frac{d\mathrm{W}}{d\eta}+\ldots+\frac{1}{h^2}\,\zeta\frac{d\mathrm{W}}{d\zeta}-2\frac{d\mathrm{W}}{d\lambda}\right)_{\lambda=0}$$

$$=\frac{\Gamma(\frac{1}{2}s+q)}{2\pi^{\frac{1}{2}s}\Gamma(q+1)}\,.\quad e^{2q}\lambda\ .\ \left(\frac{1}{f^2}\,\xi\frac{d\mathrm{W}}{d\xi}+\frac{1}{g^2}\,\eta\frac{d\mathrm{W}}{d\eta}\cdots+\frac{1}{h^2}\,\zeta\frac{d\mathrm{W}}{d\zeta}-2\frac{d\mathrm{W}}{d\lambda}\right)_{\lambda=0},$$

where term in ( ) is

$$=-\mathrm{C}\Lambda_0\psi_0\,.+\frac{2\lambda^{-q-1}}{\Lambda_0{}^2fg\ldots h}$$

$$=-2\mathrm{C}\frac{\psi_0}{\Lambda_0 fg\ldots h}\,.\,\frac{1}{\lambda^{q+1}}.$$

Hence

$$\varrho=\frac{\Gamma(\tfrac{1}{2}s+q)}{2\pi^{\frac{1}{2}s}\Gamma(q+1)}\cdot\frac{2\mathrm{C}\psi_0}{\Lambda_0 fg\ldots h}\left(\frac{e^2}{\lambda}\right)^q$$
$$=\frac{-\Gamma(\tfrac{1}{2}s+q)}{2\pi^{\frac{1}{2}s}\Gamma(q+1)}\cdot\frac{2\mathrm{C}\psi_0}{\Lambda_0 fg\ldots h}\left(1-\frac{x^2}{f^2}-\frac{y^2}{g^2}\ldots-\frac{z^2}{h^2}\right)^q,$$

where ψ_0, Λ_0 are what ψ, Λ become on writing therein $\lambda=0$. It will be remembered that Λ is what H becomes on changing therein θ into λ; hence Λ_0 is what H becomes on writing therein $\theta=0$.

Moreover ψ is what ϕ becomes on changing therein $\alpha, \beta\ldots\gamma$ into $\xi, \eta\ldots\zeta$: writing $\lambda=0$, we have $\xi=\frac{x}{f}$, $\eta=\frac{y}{g}\ldots\zeta=\frac{z}{h}$; hence ψ_0 is what ϕ becomes on changing therein $\alpha, \beta\ldots\gamma$ into $\frac{x}{f}, \frac{y}{g}\ldots\frac{z}{h}$. And it is proper in ϕ to restore the original variables by writing $\frac{a}{\sqrt{f^2+\theta}}, \frac{b}{\sqrt{g^2+\theta}}\ldots\frac{c}{\sqrt{h^2+\theta}}$ in place of $\alpha, \beta\ldots\gamma$.

127. Recapitulating,

$$\mathrm{V}=\int\frac{\varrho\,dx\ldots dz}{[(a-x)^2\ldots+(c-z)^2+e^2]^{\frac{1}{2}s+q}},$$

where, since for the value of V about to be mentioned ϱ vanishes for points outside the ellipsoid, the integral is to be taken over the ellipsoid

$$\frac{x^2}{f^2}\ldots+\frac{z^2}{h^2}=1,$$

and then (transferring a constant factor) if

$$\mathrm{V}=\frac{(\Gamma\tfrac{1}{2})^s\Gamma(q+1)}{\Gamma(\tfrac{1}{2}s+q)}\Lambda_0(f\ldots h)\,.\,\mathrm{H}\phi\int_\theta^\infty\frac{t^{-q-1}dt}{\mathrm{T}^2\sqrt{t+f^2\ldots t+h^2}},$$

the corresponding value of ϱ is

$$\varrho=\psi_0\left(1-\frac{x^2}{f^2}\ldots-\frac{z^2}{h^2}\right)^q,$$

where Λ_0 is what H becomes on writing therein $\theta=0$, and ψ_0 is what ψ becomes on writing

$$\frac{x}{f}\ldots\frac{z}{h}\text{ in place of }\alpha\,.\,.\,\gamma.$$

128. Thus putting for shortness $\Omega=t^{-q-1}(t+f^2\ldots t+h^2)^{-\frac{1}{2}}$, we have in the three several cases $\phi=1$, $\phi=\frac{a}{\sqrt{f^2+\theta}}$, $\phi=\frac{ab}{\sqrt{f^2+\theta}\,.\,\sqrt{g^2+\theta}}$ respectively,

$$\mathrm{H}=1,\qquad \varrho=\left(1-\frac{x^2}{f^2}\ldots-\frac{z^2}{h^2}\right)^q,\ \mathrm{V}=\frac{(\Gamma\tfrac{1}{2})^s\Gamma(1+q)}{\Gamma(\tfrac{1}{2}s+q)}(f\ldots h)\int_\theta^\infty\Omega\,dt,$$

$$\mathrm{H}=\frac{\sqrt{f^2+\theta}}{\sqrt{f^2\ldots+h^2}},\qquad \varrho=x\left(\quad,,\quad,,\quad\right)^q,\ \mathrm{V}=\quad,,\quad,,\quad a\int_\theta^\infty\frac{f^2}{f^2+t}\Omega\,dt,$$

$$\mathrm{H}=\frac{\sqrt{f^2+\theta}\,.\,\sqrt{g^2+\theta}}{f^2\ldots+h^2},\ \varrho=xy\left(\quad,,\quad,,\quad\right)^q,\ \mathrm{V}=\quad,,\quad,,\quad ab\int_\theta^\infty\frac{f^2g^2}{f^2+t\ldots g^2+t}\Omega\,dt,$$

and for the case last considered

$$\phi=\frac{\frac{1}{2}\frac{f^2a^2}{f^2+\theta}}{2q+2+\frac{1}{2}\varkappa_0 f^2}\cdots+\frac{\frac{1}{2}\frac{h^2c^2}{h^2+\theta}}{2q+2+\frac{1}{2}\varkappa_0 f^2}-\frac{1}{\varkappa_0},$$

$$\mathrm{H}=\frac{\frac{1}{2}f^2(f^2+\theta)^2}{2q+2+\frac{1}{2}\varkappa_0 f^2}\cdots+\frac{\frac{1}{2}h^2(h^2+\theta)}{2q+2+\frac{1}{2}\varkappa_0 h^2}-\frac{1}{\varkappa_0},\ \text{T same function with } t \text{ for } \theta,$$

$$\psi_0=\frac{\frac{1}{2}x^2}{2q+2+\frac{1}{2}\varkappa_0 f^2}\cdots+\frac{\frac{1}{2}z^2}{2q+2+\frac{1}{2}\varkappa_0 h^2}-\frac{1}{\varkappa_0},$$

$$\Lambda_0=\frac{\frac{1}{2}f^4}{2q+2+\frac{1}{2}\varkappa_0 f^2}\cdots+\frac{\frac{1}{2}h^4}{2q+2+\frac{1}{2}\varkappa_0 h^2}-\frac{1}{\varkappa_0},$$

where $\varkappa_0$ is the root of the equation $\frac{1}{2q+2+\frac{1}{2}\varkappa_0 f^2}\cdots+\frac{1}{2q+2+\frac{1}{2}\varkappa_0 h^2}+1=0$,

$$\varrho=\left(1-\frac{x^2}{f^2}\cdots-\frac{z^2}{h^2}\right)^q\psi_0,\quad \mathrm{V}=\frac{(\Gamma\frac{1}{2})^s\Gamma(1+q)}{\Gamma(\frac{1}{2}s+q)}(f\ldots h)\Lambda_0\mathrm{H}\phi\int_\theta^\infty \mathrm{T}^{-2}t^{-q-1}(t+f^2\ldots t+h^2)^{-\frac{1}{2}}dt.$$

ANNEX VI. *Examples of Theorem* C.—Nos. 129 to 132.

129. First example relating to the $(s+1)$ coordinal sphere $x^2\ldots+z^2+w^2=f^2$. Assume

$$\mathrm{V}'=\frac{\mathrm{M}}{(a^2\ldots+c^2+e^2)^{\frac{1}{2}(s-1)}},\quad \mathrm{V}''=\frac{\mathrm{M}}{f^{s-1}},\ \text{(a constant)},$$

these values each satisfy the potential equation.

V′ is not infinite for any point outside the surfaces, and for indefinitely large distances it is of the proper form.

V″ is not infinite for any point inside the surface; and at the surface V′=V″.

The conditions of the theorem are therefore satisfied; and writing

$$\mathrm{V}=\int\frac{\varrho\,d\mathrm{S}}{\{(a-x^2)\ldots+(c-z)^2+(e-w)^2\}^{\frac{1}{2}s-\frac{1}{2}}},$$

we have

$$\varrho=-\frac{\Gamma(\frac{1}{2}s-\frac{1}{2})}{4(\Gamma\frac{1}{2})^{s+1}}\left(\frac{d\mathrm{W}'}{d\mathfrak{s}'}+\frac{d\mathrm{W}''}{d\mathfrak{s}''}\right),$$

where

$$\mathrm{W}'=\frac{\mathrm{M}}{(x^2\ldots+z^2+w^2)^{\frac{1}{2}s-\frac{1}{2}}},\quad \mathrm{W}''=\frac{\mathrm{M}}{f^{s-1}};\ \text{hence}\ \frac{d\mathrm{W}''}{d\mathfrak{s}''}=0,$$

$$\frac{d\mathrm{W}'}{d\mathfrak{s}'}=\left(\frac{x}{f}\frac{d}{dx}\cdots+\frac{z}{f}\frac{d}{dz}+\frac{w}{f}\frac{d}{dw}\right)\frac{\mathrm{M}}{(x^2\ldots z^2+w^2)^{\frac{1}{2}s-\frac{1}{2}}}$$

$$=-\frac{(s-1)\frac{1}{f}(x^2\ldots+z^2+w^2)\,\mathrm{M}}{(x^2\ldots+z^2+w^2)^{\frac{1}{2}s+\frac{1}{2}}},$$

which at the surface is $=\dfrac{-(s-1)\text{M}}{f^s}$.

Hence

$$\rho=\frac{(s-1)\Gamma(\tfrac{1}{2}s-\tfrac{1}{2})\,.\,\text{M}}{4(\Gamma\tfrac{1}{2})^{s+1}f^s},\quad =\frac{\Gamma(\tfrac{1}{2}s+\tfrac{1}{2})\,.\,\text{M}}{2(\Gamma\tfrac{1}{2})^{s+1}f^s}\ \text{(viz. } \rho \text{ is constant).}$$

130. Writing for convenience $\text{M}=\dfrac{2(\Gamma\frac{1}{2})^{s+1}f^s}{\Gamma(\frac{1}{2}s+1)}\delta f$ (δf a constant which may be put $=1$), also $a^2\ldots+c^2+e^2=\varkappa^2$, we have $\rho=\delta f$, and consequently

$$\int\frac{\delta f\,d\text{S}}{\{(a-x)^2\ldots+(c-z)^2+(e-w)^2\}^{\frac{1}{2}s-\frac{1}{2}}}$$

$$=\frac{2(\Gamma\tfrac{1}{2})^{s+1}f^s\delta f}{\Gamma(\tfrac{1}{2}s+\tfrac{1}{2})}\frac{1}{\varkappa^{s-1}}\ \text{for exterior point } \varkappa>f,$$

$$=\frac{2(\Gamma\tfrac{1}{2})^{s+1}f^s\delta f}{\Gamma(\tfrac{1}{2}s+\tfrac{1}{2})}\frac{1}{f^{s-1}}\ \text{for interior point } \varkappa<f.$$

By making $a\ldots c$, e all indefinitely large we find

$$\int\delta f\,d\text{S}=\frac{2(\Gamma\tfrac{1}{2})^{s+1}f^s\delta f}{\Gamma(\tfrac{1}{2}s+\tfrac{1}{2})},$$

viz. the expression on the right-hand side is here the mass of the shell thickness δf.

Taking $s=3$ we have the ordinary formulæ for the Potential of a uniform spherical shell.

131. Suppose $s=3$, but let the surface be the infinite cylinder $x^2+y^2=f^2$. Take here

$$\text{V}'=\text{M}\log\sqrt{a^2+b^2},\quad \text{V}''=\text{M}\log f,$$

each satisfying the potential equation $\dfrac{d^2\text{V}}{da^2}+\dfrac{d^2\text{V}}{db^2}=0$; but V′, instead of vanishing, is infinite at infinity, and the conditions of the theorem are not satisfied; the Potential of the cylinder is in fact infinite. But the failure is a mere consequence of the special value of s, viz. this is such that $s-2$, instead of being positive, is $=0$. Reverting to the general case of $(s+1)$dimensional space, let the surface be the infinite cylinder $x^2\ldots+z^2=f^2$; and assume

$$\text{V}'=\frac{\text{M}}{(a^2\ldots+c^2)^{\frac{1}{2}(s-2)}},\quad \text{V}''=\frac{\text{M}}{f^{s-2}}\ \text{(a constant),}$$

these satisfy the potential equation; viz. as regards V′, we have

$$\left(\frac{d^2}{da^2}\ldots+\frac{d^2}{dc^2}+\frac{d^2}{de^2}\right)\text{V}'=0,\ \text{that is}\ \left(\frac{d^2}{da^2}\ldots+\frac{d^2}{dc^2}\right)\text{V}'=0.$$

V′ is not infinite at any point outside the cylinder, and it vanishes at infinity, except indeed when only the coordinate e is infinite, and its form at infinity is not

$$=\text{M}\div(a^2\ldots+c^2+e^2)^{\frac{1}{2}(s-1)}.$$

V″ is not infinite for any point within the cylinder; and at the surface we have V′=V″.

We have

$$\rho = -\frac{\Gamma(\frac{1}{2}s-\frac{1}{2})}{4(\Gamma\frac{1}{2})^{s+1}}\left(\frac{dW'}{d\mathfrak{s}'}+\frac{dW''}{d\mathfrak{s}''}\right),$$

where

$$\frac{dW'}{d\mathfrak{s}'} = \frac{-(s-2)\frac{1}{f}(x^2\ldots+z^2)\mathrm{M}}{(x^2+\ldots z^2)^{\frac{1}{2}s}}, = \frac{-(s-2)\mathrm{M}}{f^{s-1}} \text{ at the surface; } \frac{dW''}{d\mathfrak{s}''}=0,$$

and therefore

$$\rho = \frac{(s-2)\Gamma(\frac{1}{2}s-\frac{1}{2})\mathrm{M}}{4(\Gamma\frac{1}{2})^{s+1}f^{s-1}} \text{ (viz. } \rho \text{ is constant);}$$

or, what is the same thing, writing $\mathrm{M}=\frac{4(\Gamma\frac{1}{2})^{s+1}f^{s-1}\delta f}{(s-2)\Gamma\frac{1}{2}(s-1)}$, whence $\rho=\delta f$, and writing also $a^2\ldots+c^2=\kappa^2$, we have

$$\int\frac{\delta f\, d\mathrm{S}}{\{(a-x)^2\ldots+(c-z)^2+(e-w)^2\}^{\frac{1}{2}s-\frac{1}{2}}}$$

$$=\frac{4(\Gamma\frac{1}{2})^{s+1}f^{s-1}\delta f}{(s-2)\Gamma(\frac{1}{2}s-\frac{1}{2})}\frac{1}{\kappa^{s-2}} \text{ for an exterior point } \kappa>f,$$

$$=\frac{4(\Gamma\frac{1}{2})^{s+1}f^{s-1}\delta f}{(s-2)\Gamma(\frac{1}{2}s-\frac{1}{2})}\frac{1}{f^{s-2}} \text{ for interior point } \kappa<f.$$

132. This is right; but we can without difficulty bring it to coincide with the result obtained for the $(s+1)$dimensional sphere with only $s-1$ in place of s; we may in fact, by a single integration, pass from the cylinder $x^2\ldots+z^2=f^2$ to the s-dimensional sphere or circle $x^2\ldots+z^2f^2$, which is the base of this cylinder. Writing first $d\mathrm{S}=d\Sigma dw$, where $d\Sigma$ refers to the s variables $(x\ldots z)$ and the sphere $x^2\ldots+z^2=f^2$; or using now $d\mathrm{S}$ in this sense, then in place of the original $d\mathrm{S}$ we have $d\mathrm{S}dw$: and the limits of w being ∞, $-\infty$, then in place of $e-w$ we may write simply w. This being so, and putting for shortness $(a-x)^2\ldots+(c-z)^2=\mathrm{A}^2$, the integral is

$$\int_{-\infty}^{\infty}dw\int\frac{\delta f\, d\mathrm{S}\, dw}{(\mathrm{A}^2+w^2)^{\frac{1}{2}(s-1)}},$$

and we have without difficulty

$$\int_{-\infty}^{\infty}\frac{d\omega}{(\mathrm{A}^2+w^2)^{\frac{1}{2}(s-1)}}=\frac{1}{\mathrm{A}^{s-2}}\frac{\Gamma\frac{1}{2}\Gamma\frac{1}{2}(s-2)}{\mathrm{F}\frac{1}{2}(s-1)}.$$

[To prove it write $w=\mathrm{A}\tan\theta$, then the integral is in the first place converted into $\frac{2}{\mathrm{A}^{s-2}}\int_0^{\frac{\pi}{2}}\cos^{s-3}\theta d\theta$, which, putting $\cos\theta=\sqrt{x}$ and therefore $\sin\theta=\sqrt{1-x}$, becomes

$$=\frac{1}{\mathrm{A}^{s-2}}\int_0^1 x^{\frac{1}{2}-1}(1-x)^{\frac{1}{2}(s-2)-1}dx,$$

which has the value in question.]

Hence replacing A by its value we have

$$\frac{\Gamma\frac{1}{2}\Gamma\frac{1}{2}(s-2)}{\Gamma\frac{1}{2}(s-1)}\int\frac{\delta f d\mathrm{S}}{\{(a-x)^2\ldots+(c-z)^2\}^{\frac{1}{2}(s-2)}}=\frac{4\pi^{\frac{1}{2}s}\Gamma(\frac{1}{2})f^{s-1}\delta f}{(s-2)\Gamma\frac{1}{2}(s-1)}\left\{\frac{1}{(a^2\ldots+c^2)^{\frac{1}{2}(s-2)}}\text{ or }\frac{1}{f^{s-2}}\right\};$$

that is

$$\int\frac{\delta f\,d\mathrm{S}}{\{(a-x)^2\ldots+(c-z)^2\}^{\frac{1}{2}(s-2)}}=\frac{4\pi^{\frac{1}{2}s}f^{s-1}\,\delta f}{(s-2)\Gamma\frac{1}{2}(s-2)}\left\{\frac{1}{(a^2\ldots+c^2)^{\frac{1}{2}(s-2)}}\text{ or }\frac{1}{f^{s-2}}\right\}$$

$$=\frac{2\pi^{\frac{1}{2}s}f^{s-1}\,\delta f}{\Gamma\frac{1}{2}s}\left\{\frac{1}{(a^2\ldots+c^2)^{\frac{1}{2}(s-2)}}\text{ or }\frac{1}{f^{s-2}}\right\};$$

viz. this is the formula for the sphere with $s-1$ instead of s.

Annex VII. *Example of Theorem* D.—Nos. 133 & 134.

133. The example relates to the $(s+1)$dimensional sphere $x^2\ldots+z^2+w^2=f^2$. Instead of at once assuming for V a form satisfying the proper conditions as to continuity, we assume a form with indeterminate coefficients, and make it satisfy the conditions in question. Write

$$\mathrm{V}=\frac{\mathrm{M}}{(a^2\ldots+c^2+e^2)^{\frac{1}{2}s-\frac{1}{2}}}\qquad\text{for } a^2\ldots+c^2+e^2>f^2;$$

$$=\mathrm{A}(a^2\ldots+c^2+e^2)+\mathrm{B}\text{ for } a^2\ldots+c^2+e^2<f^2:$$

In order that the two values may be equal at the surface, we must have

$$\frac{\mathrm{M}}{f^{s-1}}=\mathrm{A}f^2+\mathrm{B},$$

and in order that the derived functions $\frac{d\mathrm{V}}{da}$ &c. may be equal, we must have

$$\frac{-(s-1)a\mathrm{M}}{f^{s+1}}=2\mathrm{A}a,\ \&\text{c.};$$

viz. these are all satisfied if only $\frac{-(s-1)\mathrm{M}}{f^{s+1}}=2\mathrm{A}$.

We have thus the values of A and B, or the exterior potential being as above

$$=\frac{\mathrm{M}}{(a^2\ldots+c^2+e^2)^{\frac{1}{2}s-\frac{1}{2}}},$$

the value of the interior potential must be

$$=\frac{\mathrm{M}}{f^{s-1}}\left\{(\tfrac{1}{2}s+\tfrac{1}{2})-(\tfrac{1}{2}s-\tfrac{1}{2})\cdot\frac{a^2\ldots+c^2+e^2}{f^2}\right\}.$$

The corresponding values of W are of course

$$\frac{\mathrm{M}}{(x^2\ldots+z^2+w^2)^{\frac{1}{2}s-\frac{1}{2}}}\text{ and }\frac{\mathrm{M}}{f^{s-1}}\left\{(\tfrac{1}{2}s+\tfrac{1}{2})-(\tfrac{1}{2}s-\tfrac{1}{2})\,\frac{x^2\ldots+z^2+w^2}{f^2}\right\},$$

and we thence find

$$\rho=0\qquad\qquad\text{if } x^2\ldots+z^2+w^2>f^2,$$

$$\rho=-\frac{\Gamma(\frac{1}{2}s-\frac{1}{2})}{4(\Gamma\frac{1}{2})^{s+1}}\{-4(\tfrac{1}{2}s-\tfrac{1}{2})(\tfrac{1}{2}s+\tfrac{1}{2})\}\frac{\mathrm{M}}{f^{s+1}},\ =\frac{\Gamma(\frac{1}{2}s+\frac{3}{2})}{(\Gamma\frac{1}{2})^{s+1}}\frac{\mathrm{M}}{f^{s+1}}$$

$$\text{if } x^2\ldots+z^2+w^2<f^2.$$

Assuming for M the value $\frac{(\Gamma\frac{1}{2})^{s+1}}{\Gamma(\frac{1}{2}s+\frac{3}{2})}f^{s+1}$, the last value becomes $\varrho=1$; and writing for shortness $a^2\ldots+c^2+e^2=\varkappa^2$, we have

$$V=\int\frac{dx\ldots dz\,dw}{\{(a-x)^2\ldots+(c-z)^2+(e-w)^2\}^{\frac{1}{2}s+\frac{1}{2}}}\text{ over }(s+1)\text{dimensional sphere }x^2\ldots+z^2+w^2=f^2,$$

$$=\frac{(\Gamma\frac{1}{2})^{s+1}}{\Gamma(\frac{1}{2}s+\frac{3}{2})}\frac{f^{s+1}}{\varkappa^{s-1}}\qquad\text{, for exterior point }\varkappa>f,$$

$$=\frac{(\Gamma\frac{1}{2})^{s+1}}{\Gamma(\frac{1}{2}s+\frac{3}{2})}\{(\tfrac{1}{2}s+\tfrac{1}{2})f^2-(\tfrac{1}{2}s-\tfrac{1}{2})\varkappa^2\},\text{ for interior point }\varkappa<f.$$

134. The case of the ellipsoid $\frac{x^2}{f^2}\ldots+\frac{z^2}{h^2}=1$ for $s+1$-dimensional space may be worked out by the theorem; this is in fact what is done in tridimensional space by LEJEUNE-DIRICHLET in his Memoir of 1846 above referred to.

ANNEX VIII. *Prepotentials of the Homaloids.*—Nos. 135 to 137.

135. We have in tridimensional space the series of figures—the plane, the line, the point; and there is in like manner in $(s+1)$dimensional space a corresponding series of $(s+1)$ terms; the $(s+1)$coordinal plane—the line, the point: say these are the homaloids or homaloidal figures. And (taking the density as uniform, or, what is the same thing, $=1$) we may consider the prepotentials of these several figures in regard to an attracted point, which, for greater simplicity, is taken not to be on the figure.

136. The integral may be written

$$V=\int\frac{dw\ldots dt}{\{(a-x)^2\ldots+(c-z)^2+(d-w)^2\ldots+(e-t)^2+u^2\}^{\frac{1}{2}s+q}},$$

which still relates to a $(s+1)$dimensional space: the $(s+1)$ coordinates of the attracted point instead of being $(a\ldots c,\ e)$ are $(a\ldots c,\ d\ldots e,\ u)$; viz. we have the s' coordinates $(a\ldots c)$, the $s-s'$ coordinates $(d\ldots e)$, and the $(s+1)$th coordinate u: and the integration is extended over the $(s-s')$dimensional figure $w=-\infty$ to $+\infty,\ldots t=-\infty$ to $+\infty$. And it is also assumed that q is positive.

It is at once clear that we may reduce the integral to

$$V=\int\frac{dw\ldots dt}{\{(a-x)^2\ldots+(c-z)^2+u^2+w^2\ldots+t^2\}^{\frac{1}{2}s+q}},$$

say for shortness

$$=\int\frac{dw\ldots dt}{(A^2+w^2\ldots+t^2)^{\frac{1}{2}s+q}},$$

where A^2, $=(a-x)^2\ldots+(c-z)^2+u^2$, is a constant as regards the integration, and where the limits in regard to each of the $s-s'$ variables are $-\infty$, $+\infty$.

We may for these variables write $r\xi\ldots r\zeta$, where $\xi^2\ldots+\zeta^2=1$; and we then have

$w^2\ldots+t^2=r^2$, $dw\ldots dt=r^{s-s'-1}dr\,dS$, where dS is the element of surface of the $(s-s')$-coordinal unit-sphere $\xi^2\ldots+\zeta^2=1$. We thus obtain

$$V=\int\frac{r^{s-s'-1}\,dr}{\{A^2+r^2\}^{\frac{1}{2}s+q}}\int dS,$$

where the integral in regard to r is taken from 0 to ∞, and the integral $\int dS$ over the surface of the unit-sphere; hence by Annex I. the value of this last factor is $=\frac{2(\Gamma\frac{1}{2})^{s-s'}}{\Gamma\frac{1}{2}(s-s')}$. The integral represented by the first factor will be finite, provided only $\frac{1}{2}s'+q$ be positive; which is the case for any value whatever of s' if only q be positive.

The first factor is an integral such as is considered in Annex II.; to find its value we have only to write $r=A\sqrt{x}$, and we thus find it to be

$$=\frac{1}{(A^2)^{\frac{1}{2}s'+q}}\,\frac{1}{2}\int_0^\infty\frac{x^{\frac{1}{2}s-\frac{1}{2}s'-1}\,dx}{(1+x)^{\frac{1}{2}s+q}},\ \text{viz.}=\frac{1}{A^{s'+2q}}\cdot\frac{\frac{1}{2}\Gamma\frac{1}{2}(s-s')\Gamma(\frac{1}{2}s'+q)}{\Gamma(\frac{1}{2}s+q)},$$

and we thus have

$$V=\frac{1}{A^{s'+2q}}\cdot\frac{(\Gamma\frac{1}{2})^{s-s'}\Gamma(\frac{1}{2}s'+q)}{\Gamma(\frac{1}{2}s+q)},$$

$$=\frac{(\Gamma\frac{1}{2})^{s-s'}\Gamma(\frac{1}{2}s'+q)}{\Gamma(\frac{1}{2}s+q)}\ \frac{1}{\{(a-x)^2\ldots+(c-z)^2+u^2\}^{\frac{1}{2}s'+q}}.$$

137. As a verification observe that the prepotential equation $\square V=0$, that is

$$\left(\frac{d^2}{da^2}\ldots+\frac{d^2}{dc^2}+\frac{d^2}{dd^2}\ldots+\frac{d^2}{de^2}+\frac{d^2}{du^2}+\frac{2q+1}{u}\,\frac{d}{du}\right)V=0\,;$$

for a function V which contains only the $s'+1$ variables $(a\ldots c,\ u)$ becomes

$$\left(\frac{d^2}{da^2}\ldots+\frac{d^2}{dc^2}+\frac{d^2}{du^2}+\frac{2q+1}{u}\,\frac{d}{du}\right)V=0,$$

which is satisfied by V a constant multiple of $\{(a-x)^2\ldots+(c-z)^2+u^2\}^{\frac{1}{2}-s'-q}$.

ANNEX IX. *The* GAUSS-JACOBI *Theory of Epispheric Integrals.*—No. 138.

138. The formula obtained (Annex IV. No. 110) is proved only for positive values of m; but writing therein $q=0$, $m=-\frac{1}{2}$, it becomes

$$\int\frac{dx\ldots dz}{\sqrt{1-\frac{x^2}{f^2}\ldots-\frac{z^2}{h^2}}\{(a-x)^2\ldots+(c-z)^2+e^2\}^{\frac{1}{2}s}}$$

$$=\frac{(\Gamma\frac{1}{2})^s}{\Gamma\frac{1}{2}s}f\ldots h\int^\infty dt\,.\,t^{-1}\left(1-\frac{a^2}{t+f^2}\ldots-\frac{c^2}{t+h^2}-\frac{e}{t}\right)^{-\frac{1}{2}}(t+f^2\ldots t+h^2)^{-\frac{1}{2}},$$

a formula which is obtainable as a particular case of a more general one

$$\int\frac{dS}{\{(*\between x\ldots z,\ w)^2\}^{\frac{1}{2}s}}=\frac{2(\Gamma\frac{1}{2})^s}{\Gamma(\frac{1}{2}s)}\int_{-A}^\infty dt\,\frac{1}{\sqrt{-\text{Disct.}\{(*\between X\ldots Z,W,T)^2+t(X^2\ldots+Z^2+W^2+T^2)\}}}$$

(notation to be presently explained), being a result obtained by JACOBI by a process which is in fact the extension to any number of variables of that made use of by GAUSS in his Memoir 'Determinatio attractionis quam ..... exerceret planeta, &c.' (1818). I proceed to develop this theory.

139. JACOBI's process has reference to a class of s-tuple integrals (including some of those here previously considered) which may be termed "epispheric": viz. considering the $(s+1)$ variables $(x\ldots z, w)$ connected by the equation $x^2\ldots+z^2+w^2=1$, or say they are the coordinates of a point on a $(s+1)$tuple unit-sphere, then the form is $\int \mathrm{U}d\mathrm{S}$, where $d\mathrm{S}$ is the element of the surface of the unit-sphere, and U is any function of the $s+1$ coordinates; the integral is taken to be of the form $\int \frac{d\mathrm{S}}{\{(*\between x\ldots z, w, 1)^2\}^{\frac{1}{2}s}}$, and we then obtain the general result above referred to.

Before going further it is convenient to remark that taking as independent variables the s coordinates $x\ldots z$, we have $d\mathrm{S}=\frac{dx\ldots dz}{dw}$ where w stands for $\pm\sqrt{1-x^2\ldots-z^2}$; we must in obtaining the integral take account of the two values of w, and finally extend the integral to the values of $x\ldots z$ which satisfy $x^2\ldots+z^2<1$.

If, as is ultimately done, in place of $x\ldots z$ we write $\frac{x}{f}\ldots\frac{z}{h}$ respectively, then the value of $d\mathrm{S}$ is $=\frac{1}{f\ldots h}\,\frac{dx\ldots dz}{w}$, where w now stands for $\pm\sqrt{1-\frac{x^2}{f^2}\ldots-\frac{z^2}{h^2}}$; we must in finding the value of the integral take account of the two values of w, and finally extend the integral to the values of $x\ldots z$ which satisfy $\frac{x^2}{f^2}\ldots+\frac{z^2}{h^2}<1$.

140. The determination of the integral depends upon formulæ for the transformation of the spherical element $d\mathrm{S}$, and of the quadric function $(x, y\ldots z, w, 1)^2$.

First, as regards the spherical element $d\mathrm{S}$; let the $s+1$ variables $x, y\ldots z, w$ which satisfy $x^2+y^2\ldots+z^2+w^2=1$ be regarded as functions of the s independent variables $\theta, \phi, \ldots \psi$; then we have

$$d\mathrm{S}=\begin{vmatrix} x, & y & \ldots & z, & w \\ \frac{dx}{d\theta}, & \frac{dy}{d\theta} & \ldots & \frac{dz}{d\theta}, & \frac{dw}{d\theta} \\ \frac{dx}{d\phi}, & \frac{dy}{d\phi} & \ldots & \frac{dz}{d\phi}, & \frac{d}{d\phi} \\ \vdots & & & & \\ \frac{dx}{d\psi}, & \frac{dy}{d\psi} & \ldots & \frac{dz}{d\psi}, & \frac{dw}{d\psi} \end{vmatrix} d\theta d\phi\ldots d\psi, =\frac{\partial(x, y\ldots z, w)}{\partial(\theta, \phi\ldots\psi, *)}\,d\theta\, d\phi\ldots d\psi, \text{ for shortness.}$$

Suppose we effect on the $s+1$ variables $(x, y\ldots z, w)$ a transformation

$$x, y\ldots z, w=\frac{\mathrm{X}}{\mathrm{T}}, \frac{\mathrm{Y}}{\mathrm{T}}, \ldots \frac{\mathrm{Z}}{\mathrm{T}}, \frac{\mathrm{W}}{\mathrm{T}},$$

thus introducing for the moment $s+2$ variables X, Y, ... Z, W, T, which satisfy identically $X^2+Y^2\ldots+Z^2+W^2-T^2=0$, then considering these as functions of the foregoing s independent variables $\theta, \phi, \ldots \psi$, we have

$$dS=\frac{1}{T^{s+1}}\cdot\begin{vmatrix} X, & Y & \ldots & Z, & W \\ \frac{dX}{d\theta}, & \frac{dY}{d\theta} & \ldots & \frac{dZ}{d\theta}, & \frac{dW}{d\theta} \\ \frac{dX}{d\phi}, & \frac{dY}{d\phi} & \ldots & \frac{dZ}{d\phi}, & \frac{dW}{d\phi} \\ \vdots & & & & \\ \frac{dX}{d\psi}, & \frac{dY}{d\psi} & \ldots & \frac{dZ}{d\psi}, & \frac{dW}{d\psi} \end{vmatrix} d\theta\, d\phi \ldots d\psi=\frac{1}{T^{s+1}}\frac{\partial(X, Y\ldots Z, W)}{\partial(\theta, \phi\ldots\psi, *)}\, d\theta\, d\phi\ldots d\psi.$$

141. Considering next the $s+2$ variables X, Y, ... Z, W, T as linear functions (with constant terms) of the $s+1$ new variables $\xi, \eta \ldots \zeta, \omega$, or say as linear functions of the $s+2$ quantities $\xi, \eta \ldots \zeta, \omega, 1$, which implies between them a linear relation

$$aX+bY\ldots+cZ+dW+eT=1;$$

and assuming that we have *identically*

$$X^2+Y^2\ldots+Z^2+W^2-T^2=\xi^2+\eta^2\ldots+\zeta^2+\omega^2-1,$$

so that in consequence of the left-hand side being $=0$, the right-hand side is also $=0$; viz. $\xi, \eta \ldots \zeta, \omega$ are connected by

$$\xi^2+\eta^2\ldots+\zeta^2+\omega^2=1:$$

let $d\Sigma$ represent the spherical element belonging to the coordinates $\xi, \eta, \ldots \zeta, \omega$. Considering these as functions of the foregoing s independent variables $\theta, \phi, \ldots \psi$, we have

$$d\Sigma=\begin{vmatrix} \xi, & \eta & \ldots & \zeta, & \omega \\ \frac{d\xi}{d\theta}, & \frac{d\eta}{d\theta} & \ldots & \frac{d\zeta}{d\omega}, & \frac{d\omega}{d\theta} \\ \frac{d\xi}{d\phi}, & \frac{d\eta}{d\phi} & \ldots & \frac{d\zeta}{d\phi}, & \frac{d\omega}{d\phi} \\ \vdots & & & & \\ \frac{d\xi}{d\psi}, & \frac{d\eta}{d\psi} & \ldots & \frac{d\zeta}{d\psi}, & \frac{d\omega}{d\psi} \end{vmatrix} d\theta\, d\phi\ldots d\psi=\frac{\partial(\xi, \eta\ldots\zeta, \omega)}{\partial(\theta, \phi\ldots\psi, *)}d\theta\, d\phi\ldots d\psi.$$

142. We have in this expression $\xi, \eta \ldots \zeta, \omega$, each of them a linear function of the $s+2$ quantities X, Y, ... Z, W, T; the determinant is consequently a linear function of $s+2$ like determinants obtained by substituting for the variables any $s+1$ out of the $s+2$ variables X, Y ... Z, W, T; but in virtue of the equation $X^2+Y^2\ldots+Z^2+W^2-T^2=0$,

these $s+2$ determinants are proportional to the quantities X, Y ... Z, W, T respectively, and the determinant thus assumes the form

$$\frac{a\mathrm{X}+b\mathrm{Y}\ldots+c\mathrm{Z}+d\mathrm{W}+e\mathrm{T}}{\mathrm{T}}\Delta,$$

where Δ is the like determinant with $(\mathrm{X}, \mathrm{Y}, \ldots \mathrm{Z}, \mathrm{W})$, and where the coefficients $a, b, \ldots c, d, e$ are precisely those of the linear relation $a\mathrm{X}+b\mathrm{Y}\ldots+c\mathrm{Z}+d\mathrm{W}+e\mathrm{T}=1$; the last-mentioned expression is thus $=\frac{1}{\mathrm{T}}\Delta$, or, substituting for Δ its value, we have

$$d\Sigma=\frac{1}{\mathrm{T}}\frac{\partial(\mathrm{X}, \mathrm{Y}\ldots \mathrm{Z}, \mathrm{W})}{\partial(\theta, \phi\ldots\psi, *)}d\theta\, d\phi\ldots d\psi;$$

viz. comparing with the foregoing expression for $d\mathrm{S}$ we have

$$d\mathrm{S}=\frac{1}{\mathrm{T}^s}d\Sigma,$$

which is the requisite formula for the transformation of $d\mathrm{S}$.

143. Consider the integral

$$\int\frac{d\mathrm{S}}{\{(*\between x, y\ldots z, w, 1)^2\}^{\frac{1}{2}s}},$$

which, from its containing a single quadric function, may be called "one-quadric." Then effecting the foregoing transformation,

$$x, y\ldots z, w=\frac{\mathrm{X}}{\mathrm{T}}, \frac{\mathrm{Y}}{\mathrm{T}}, \ldots \frac{\mathrm{Z}}{\mathrm{T}}, \frac{\mathrm{W}}{\mathrm{T}},$$

and observing that

$$(*\between x, y\ldots z, w, 1)^2=\frac{1}{\mathrm{T}^2}(*\between \mathrm{X}, \mathrm{Y}\ldots \mathrm{Z}, \mathrm{W}, \mathrm{T})^2,$$

the integral becomes

$$=\int\frac{d\Sigma}{\{(*\between \mathrm{X}, \mathrm{Y}\ldots \mathrm{Z}, \mathrm{W}, \mathrm{T})^2\}^{\frac{1}{2}s}},$$

where X, Y ... Z, W, T denote given linear functions (with constant terms) of the $s+1$ variables $\xi, \eta\ldots\zeta, \omega$, or, what is the same thing, given linear functions of the $s+2$ quantities $\xi, \eta\ldots\zeta, \omega, 1$, such that identically $\mathrm{X}^2+\mathrm{Y}^2\ldots+\mathrm{Z}^2+\mathrm{W}^2-\mathrm{T}^2=\xi^2+\eta^2\ldots+\zeta^2+\omega^2-1$. We have then $\xi^2+\eta^2\ldots+\zeta^2+\omega^2-1=0$, and $d\Sigma$ as the corresponding spherical element.

144. We may have X, Y ... Z, W, T such linear functions of $\xi, \eta\ldots\zeta, \omega, 1$ that not only

$$\mathrm{X}^2+\mathrm{Y}^2\ldots+\mathrm{Z}^2+\mathrm{W}^2-\mathrm{T}^2=\xi^2+\eta^2\ldots+\zeta^2+\omega^2-1$$

as above, but also

$$(*\between \mathrm{X}, \mathrm{Y}, \ldots \mathrm{Z}, \mathrm{W}, \mathrm{T})^2=\mathrm{A}\xi^2+\mathrm{B}\eta^2\ldots+\mathrm{C}\zeta^2+\mathrm{E}\mathrm{W}^2-\mathrm{L};$$

and this being so, the integral becomes

$$\int\frac{d\Sigma}{\{A\xi^2+B\eta^2\ldots+C\zeta^2+EW^2-L\}^{\frac{1}{2}s}},$$

where the $s+2$ coefficients $A, B\ldots C, E, L$ are given by means of the identity

$$\begin{aligned}&-(\theta+A)(\theta+B)\ldots(\theta+C)(\theta+E)(\theta+L)\\ &\quad=\text{Disct.}\{(*\between X, Y\ldots Z, W, T)^2+\theta(X^2+Y^2\ldots+Z^2+W^2-T^2)\};\end{aligned}$$

viz. equating the discriminant to zero, we have an equation in θ, the roots whereof are $-A, -B\ldots-C, -E, -L$.

The integral is

$$\int\frac{d\Sigma}{\{(A-L)\xi^2+(B-L)\eta^2\ldots+(C-L)\zeta^2+(E-L)\omega^2\}^{\frac{1}{2}s}},$$

which is of the form

$$\int\frac{d\Sigma}{\{a\xi^2+b\eta^2\ldots+c\zeta^2+e\omega^2\}^{\frac{1}{2}s}},$$

where I provisionally assume that $a, b\ldots c, e$ are all positive.

145. To transform this, in place of the $s+1$ variables $\xi, \eta\ldots\zeta, \omega$ connected by $\xi^2+\eta^2\ldots+\zeta^2+\omega^2=1$, we introduce the $s+1$ variables $x, y\ldots z, w$ such that

$$x=\frac{\xi\sqrt{a}}{\varrho},\quad y=\frac{\eta\sqrt{b}}{\varrho},\quad\ldots z=\frac{\zeta\sqrt{c}}{\varrho},\quad w=\frac{\omega\sqrt{d}}{\varrho},$$

where

$$\varrho^2=a\xi^2+b\eta^2\ldots+c\zeta^2+ew^2,$$

and consequently

$$x^2+y^2\ldots+z^2+w^2=1.$$

Hence writing dS to denote the spherical element corresponding to the point $(x, y\ldots z, w)$, we have by a former formula

$$\begin{aligned}dS&=\frac{1}{\varrho^{s+1}}\frac{\partial(\xi\sqrt{a}, \eta\sqrt{b}\ldots\zeta\sqrt{c}, \omega\sqrt{e})}{\partial(\theta, \phi\ldots\psi, *)}\,d\theta\,d\phi\ldots d\psi\\ &=\frac{(ab\ldots ce)^{\frac{1}{2}}}{\varrho^{s+1}}\,d\Sigma;\end{aligned}$$

or, what is the same thing,

$$\frac{d\Sigma}{\{a\xi^2+b\eta^2\ldots+c\zeta^2+e\omega^2\}^{\frac{1}{2}(s+1)}}=\frac{1}{(ab\ldots ce)^{\frac{1}{2}}}\,dS.$$

Hence integrating each side, and observing that $\int dS$, taken over the whole spherical surface $x^2+y^2\ldots+z^2+w^2=1$, is $=2(\Gamma\frac{1}{2})^{s+1}\div\Gamma(\frac{1}{2}s+\frac{1}{2})$, we have

$$\int\frac{d\Sigma}{\{a\xi^2+b\eta^2\ldots c\zeta^2+e\omega^2\}^{\frac{1}{2}(s+1)}}=\frac{2(\Gamma\frac{1}{2})^{s+1}}{\Gamma(\frac{1}{2}s+\frac{1}{2})}\cdot\frac{1}{(ab\ldots ce)^{\frac{1}{2}}}.$$

146. For $a, b \dots c, e$ write herein $a+\theta, b+\theta \dots c+\theta, e+\theta$ respectively, and multiplying each side by θ^{q-1}, where q is any positive integer or fractional number less than $\frac{1}{2}s$, integrate from $\theta=0$ to $\theta=\infty$. On the left-hand side, attending to the relation $\xi^2+\eta^2 \dots +\zeta^2+\omega^2=1$, the integral in regard to θ is

$$\int_0^\infty \frac{\theta^{q-1}\,d\theta}{\{\varrho^2+\theta\}^{\frac{1}{2}(s+1)}},$$

where $\varrho^2, =a\xi^2+b\eta^2 \dots +c\zeta^2+e\omega^2$, as before is independent of θ; the value of the definite integral is

$$=\frac{\Gamma(\frac{1}{2}(s+1)-q)\Gamma(q)}{\Gamma\frac{1}{2}(s+1)}\,\frac{1}{\varrho^{s+1-2q}},$$

which, replacing ϱ by its value and multiplying by $d\Sigma$, and prefixing the integral sign, gives the left-hand side; hence forming the equation and dividing by a numerical factor, we have

$$\int\frac{d\Sigma}{(a\xi^2 \dots +c\zeta^2+e\omega^2)^{\frac{1}{2}(s+1)-q}}=\frac{2(\Gamma\frac{1}{2})^{s+1}}{\Gamma q\Gamma\frac{1}{2}(s+1)-q}\int_0^\infty dt\,.\,t^{-q-1}(t+a \dots t+c\,.\,t+e)^{-\frac{1}{2}},$$

and in particular if $q=-\frac{1}{2}$, then

$$\int\frac{d\Sigma}{(a\xi^2 \dots +c\zeta^2+e^2\omega^2)^{\frac{1}{2}s}}=\frac{2(\Gamma\frac{1}{2})^s}{\Gamma\frac{1}{2}s}\int_0^\infty dt\,.\,t^{-\frac{1}{2}}(t+a \dots t+c\,.\,t+e)^{-\frac{1}{2}},$$

or, if for $a \dots c, e$ we restore the values $\mathrm{A-L} \dots \mathrm{C-L}, \mathrm{E-L}$, then

$$\int\frac{d\Sigma}{(\mathrm{A}\xi^2 \dots +\mathrm{C}\zeta^2+\mathrm{E}\omega^2-\mathrm{L})^{\frac{1}{2}s}}=\frac{2(\Gamma\frac{1}{2})^s}{\Gamma\frac{1}{2}s}\int_0^\infty dt\,.\,t^{-\frac{1}{2}}(t+\mathrm{A-L} \dots t+\mathrm{C-L}\,.\,t+\mathrm{E-L})^{-\frac{1}{2}},$$

$$=\frac{2(\Gamma\frac{1}{2})^s}{\Gamma\frac{1}{2}s}\int_{-\mathrm{L}}^\infty dt\,.\,(t+\mathrm{A} \dots t+\mathrm{C}\,.\,t+\mathrm{E}\,.\,t+\mathrm{L})^{-\frac{1}{2}};$$

viz. we thus have

$$\int\frac{d\mathrm{S}}{\{(*\between x \dots z, w, 1)^2\}^{\frac{1}{2}s}}=\frac{2(\Gamma\frac{1}{2})^s}{\Gamma\frac{1}{2}s}\int_{-\mathrm{L}}^\infty dt(t+\mathrm{A} \dots t+\mathrm{C}\,.\,t+\mathrm{E}\,.\,t+\mathrm{L})^{-\frac{1}{2}},$$

where $t+\mathrm{A} \dots t+\mathrm{C}\,.\,t+\mathrm{E}\,.\,t+\mathrm{L}$ is in fact a given rational and integral function of t; viz. it is

$$=-\text{Disct.}\{(*\between \mathrm{X} \dots \mathrm{Z}, \mathrm{W}, \mathrm{T})^2+t(\mathrm{X}^2 \dots +\mathrm{Z}^2+\mathrm{W}^2-\mathrm{T}^2)\}.$$

147. Consider in particular the integral

$$\int\frac{d\mathrm{S}}{\{(a-fx)^2 \dots +(c-hz)^2+(e-kw)^2+l^2\}^{\frac{1}{2}s}};$$

here

$$\begin{aligned}(*\between \mathrm{X} \dots \mathrm{Z}, \mathrm{W}, \mathrm{T})^2&+t(\mathrm{X}^2 \dots +\mathrm{Z}^2+\mathrm{W}^2-\mathrm{T}^2)\\ &=(a\mathrm{T}-f\mathrm{X})^2 \dots +(c\mathrm{T}-h\mathrm{Z})^2+(e\mathrm{T}-k\mathrm{W})^2+l^2\mathrm{T}^2\\ &\qquad+t(\mathrm{X}^2 \dots +\mathrm{Z}^2+\mathrm{W}^2-\mathrm{T}^2)\\ &=(f^2+t)\mathrm{X}^2 \dots +(h^2+t)\mathrm{Z}^2+(k^2+t)\mathrm{W}^2+(a^2 \dots +c^2+e^2+l^2-t)\mathrm{T}^2\\ &\qquad-2af\mathrm{XT} \dots 2ch\mathrm{ZT}-2ek\mathrm{WT};\end{aligned}$$

viz. the discriminant taken negatively is

$$\begin{vmatrix} t+f^2 \ldots & & , -af \\ \vdots & & \\ & t+h^2, & -ch \\ -af \ldots & -ch & -(a^2 \ldots +c^2+e^2+l^2)+t \end{vmatrix}$$

which is

$$=t+f^2 \ldots t+h^2 \,.\, t+k^2 \left(t-a^2 \ldots -c^2-e^2-l^2+\frac{a^2f^2}{t+f^2} \cdots +\frac{c^2h^2}{t+h^2}+\frac{e^2k^2}{t+k^2}\right),$$

$$=t\,.\,(t+f^2 \ldots t+h^2 \,.\, t+k^2)\left(1-\frac{a^2}{t+f^2} \cdots -\frac{c^2}{t+h^2}-\frac{e^2}{t+k^2}-\frac{l^2}{t}\right)$$

$$=t+\mathrm{A} \ldots t+\mathrm{C} \,.\, t+\mathrm{E} \,.\, t+\mathrm{L},$$

and consequently $-\mathrm{A} \ldots -\mathrm{C}, -\mathrm{E}, -\mathrm{L}$ are the roots of the equation

$$1-\frac{a^2}{t+f^2} \cdots -\frac{c^2}{t+h^2}-\frac{e^2}{t+k^2}-\frac{l^2}{t}=0.$$

148. The roots are all real; moreover there is one and only one positive root. Hence taking $-\mathrm{L}$ to be the positive root, we have $\mathrm{A} \ldots \mathrm{C}$, E, $-\mathrm{L}$ all positive; and therefore *à fortiori* $\mathrm{A}-\mathrm{L}, \ldots \mathrm{C}-\mathrm{L}, \mathrm{E}-\mathrm{L}$ all positive, which agrees with a foregoing provisional assumption. Or, writing for greater convenience θ to denote the positive quantity $-\mathrm{L}$, that is taking θ to be the positive root of the equation

$$1-\frac{a^2}{\theta+f^2} \cdots -\frac{c^2}{\theta+h^2}-\frac{e^2}{\theta+k^2}-\frac{l^2}{\theta}=0,$$

we have

$$\int \frac{d\mathrm{S}}{\{(a-fx)^2 \ldots +(c-hz)^2+(e-kw)^2+l^2\}^{\frac{1}{2}s}}$$

$$=\frac{2(\Gamma\frac{1}{2})^s}{\Gamma\frac{1}{2}s}\int_\theta^\infty dt \; \frac{1}{\sqrt{t\,.\,t+f^2 \ldots t+h^2 \,.\, t+k^2\left(1-\frac{a^2}{t+f^2} \cdots -\frac{c^2}{t+h^2}-\frac{e^2}{t+k^2}-\frac{l^2}{t}\right)}};$$

or, what is the same thing, we have

$$\frac{1}{f \ldots h}\int \frac{dx \ldots dz}{\pm w\{(a-x)^2 \ldots +(c-z)^2+(e\mp kw)^2+l^2\}^{\frac{1}{2}s}}$$

$$=\frac{\Gamma\frac{1}{2}s}{2(\Gamma\frac{1}{2})^s}\int_\theta^\infty dt\left(1-\frac{a^2}{t+f^2} \cdots -\frac{c^2}{t+h^2}-\frac{e^2}{t+k^2}-\frac{l^2}{t}\right)^{-\frac{1}{2}}(t\,.\,t+f^2 \ldots t+h^2 \,.\, t+k^2)^{-\frac{1}{2}},$$

where on the left-hand side w now denotes $\sqrt{1-\frac{x^2}{f^2} \cdots -\frac{z^2}{h^2}}$, and the limiting equation is $\frac{x^2}{f^2} \cdots +\frac{z^2}{h^2}=1$.

149. Suppose $l=0$, then if

$$\frac{a^2}{f^2} \cdots +\frac{c^2}{h^2}+\frac{e^2}{k^2}>1,$$

the equation

$$1-\frac{a^2}{\theta+f^2}\dots-\frac{c^2}{\theta+h^2}-\frac{e^2}{\theta+k^2}=0$$

has a positive root differing from zero, which may be represented by the same letter θ; but if

$$\frac{a^2}{f^2}\dots+\frac{c^2}{h^2}+\frac{e^2}{k^2}<1,$$

then the positive root of the original equation becomes $=0$; viz. as l gradually diminishes to zero the positive root θ also diminishes, and becomes ultimately zero.

Hence writing $l=0$, we have

$$\int\frac{d\mathrm{S}}{\{(a-fx)^2\dots+(c-hz)^2+(e-kw)^2\}^{\frac{1}{2}s}},$$

or, what is the same thing,

$$\frac{1}{f\dots h}\int\frac{dx\dots dz}{\pm w\{(a-x)^2\dots+(c-z)^2+(e\mp kw)^2\}^{\frac{1}{2}s}}$$

$$=\frac{2(\Gamma\tfrac{1}{2})^s}{\Gamma\tfrac{1}{2}s}\int_\theta^\infty dt\left(1-\frac{a^2}{t+f^2}\dots-\frac{c^2}{t+h^2}-\frac{e^2}{t+k^2}\right)^{-\frac{1}{2}}(t\,.\,t+f^2\dots t+h^2\,.\,t+k^2)^{-\frac{1}{2}},$$

θ now denoting the positive root of the equation

$$1-\frac{a^2}{\theta+f^2}\dots-\frac{c^2}{\theta+h^2}-\frac{e^2}{\theta+k^2}=0,$$

or else denoting 0, according as

$$\frac{a^2}{f^2}\dots+\frac{c^2}{h^2}+\frac{e^2}{k^2}>1 \text{ or } <1.$$

In the case $\frac{a^2}{f^2}\dots+\frac{e^2}{k^2}<1$, the inferior limit being then 0, this is in fact JACOBI'S theorem (Crelle, t. xii. p. 69, 1834); but JACOBI does not consider the general case where l is not $=0$, nor does he give explicitly the formula in the other case

$$l=0,\quad \frac{a^2}{f^2}\dots+\frac{c^2}{h^2}+\frac{e^2}{k^2}>1.$$

150. Suppose $k=0$, e being in the first instance not $=0$, then the former alternative holds good; and observing, in regard to the form which contains $\pm w$ in the denominator, that we can now take account of the two values by simply multiplying by 2, we have

$$\int\frac{d\mathrm{S}}{\{(a-fx)^2\dots+(c-hz)^2+e^2\}^{\frac{1}{2}s}},\quad =\frac{2}{f\dots h}\int\frac{dx\dots dz}{w\{(a-x)^2\dots+(c-z)^2+e^2\}^{\frac{1}{2}s}},$$

(w on the right-hand side denoting $\sqrt{1-\frac{x^2}{f^2}\dots-\frac{z^2}{h^2}}$, and the limiting equation being $\frac{x^2}{f^2}\dots+\frac{z^2}{h^2}=1$), each

$$=\frac{2(\Gamma\frac{1}{2})^s}{\Gamma\frac{1}{2}s}\int_\theta^\infty dt\left(1-\frac{a^2}{t+f^2}\cdots-\frac{c^2}{t+h^2}-\frac{e^2}{t}\right)^{-\frac{1}{2}}t^{-1}(t+f^2\ldots t+h^2)^{-\frac{1}{2}},$$

where θ is here the positive root of the equation $1-\frac{a^2}{\theta+f^2}\cdots-\frac{c^2}{\theta+h^2}-\frac{e^2}{\theta}=0$, which is the formula referred to at the beginning of the present Annex. We may in the formula write $e=0$, thus obtaining the theorem under two different forms for the cases $\frac{a^2}{f^2}\cdots+\frac{c^2}{h^2}>1$ and <1 respectively.

ANNEX X. *Methods of* LEJEUNE-DIRICHLET *and* BOOLE.—Nos. 151 to 162.

151. The notion that the density ϱ is a discontinuous function vanishing for points outside the attracting mass has been made use of in a different manner by LEJEUNE-DIRICHLET (1839) and BOOLE (1857): viz. supposing that ϱ has a given value $f(x\ldots z)$ within a given closed surface S and is $=0$ outside the surface, these geometers in the expression of a potential or prepotential integral replace ϱ by a definite integral which possesses the discontinuity in question, viz. it is $=f(x\ldots z)$ for points inside the surface and $=0$ for points outside the surface; and then in the potential or prepotential integral they extend the integration over the whole of infinite space, thus getting rid of the equation of the surface as a limiting equation for the multiple integral.

152. LEJEUNE-DIRICHLET'S paper "Sur une nouvelle méthode pour la détermination des intégrales multiples" is published in 'Comptes Rendus,' t. viii. pp. 155–160 (1839), and Liouv. t. iv. pp. 164–168 (same year). The process is applied to the form

$$-\frac{1}{p-1}\frac{d}{da}\int\frac{dx\,dy\,dz}{\{(a-x)^2+(b-y)^2+(c-z)^2\}^{\frac{1}{2}(p-1)}}$$

over the ellipsoid $\frac{x^2}{\alpha^2}+\frac{y^2}{\beta^2}+\frac{z^2}{\gamma^2}=1$; but it would be equally applicable to the triple integral itself, or say to the s-tuple integral

$$\int\frac{dx\ldots dz.}{\{(a-x)^2\ldots+(c-z)^2\}^{\frac{1}{2}s+q}},$$

or, indeed, to

$$\int\frac{dx\ldots dz}{\{(a-x)^2\ldots+(c-z)^2+e^2\}^{\frac{1}{2}s+q}}$$

over the ellipsoid $\frac{x^2}{f^2}\cdots+\frac{z^2}{h^2}=1$; but it may be as well to attend to the first form, as more resembling that considered by the author.

153. Since $\frac{2}{\pi}\int_0^\infty\frac{\sin\phi}{\phi}\cos\lambda\phi\,d\phi$ is $=1$ or 0, according as λ is <1 or >1, it follows that the integral is equal to the real part of the following expression,

$$\frac{2}{\pi}\int_0^\infty d\phi\frac{\sin\phi}{\phi}\int e^{i\left(\frac{x^2}{f^2}\cdots+\frac{z^2}{h^2}\right)}\frac{dx\ldots dz}{\{(a-x)^2\ldots+(c-z)^2\}^{\frac{1}{2}s+q}},$$

where the integrations in regard to $x\ldots z$ are now to be extended from $-\infty$ to $+\infty$ for each variable. A further transformation is necessary: since

$$\frac{1}{\sigma^r}=\frac{1}{\Gamma r}e^{-r\pi i}\int_0^\infty d\psi\,.\,\psi^{r-1}\,e^{i\sigma\psi},\qquad \sigma \text{ positive and } r \text{ positive and } <1,$$

writing herein $(a-x)^2\ldots+(c-z)^2$ for σ, and $\frac{1}{2}s+q$ for r, we have

$$\frac{1}{\{(a-x)^2\ldots+(c-z)^2\}^{\frac{1}{2}s+q}}=\frac{1}{\Gamma(\frac{1}{2}s+q)}\,e^{-(\frac{1}{2}s+q)\pi i}\int_0^\infty d\psi\,.\,\psi^{\frac{1}{2}s+q-1}\,e^{i\psi\{(a-x)^2\ldots+(c-z)^2\}},$$

and the value is thus

$$=\frac{2}{\pi\Gamma(\frac{1}{2}s+q)}\,e^{-(\frac{1}{2}s+q)\frac{\pi i}{2}}\int_0^\infty d\phi\,\frac{\sin\phi}{\phi}\int_0^\infty d\psi\,.\,\psi^{\frac{1}{2}s+q-1}\int e^{i\left(\frac{x^2}{f^2}\ldots+\frac{z^2}{h^2}\right)\phi}\,e^{-i\psi\{(a-x)^2\ldots+(c-z)^2\}}\,dx\ldots dz,$$

where the integral in regard to the variables $(x\ldots z)$ is

$$=e^{i\psi(a^2\ldots+c^2)}\int dx\,e^{i\left\{\left(\psi+\frac{\phi}{f^2}\right)x^2+2a\psi x\right\}}\ldots\int dz\,e^{i\left\{\left(\psi+\frac{\phi}{h^2}\right)z^2-2c\psi z\right\}};$$

and here the x-integral is

$$=e^{\frac{1}{4}i\pi}\sqrt{\frac{f^2\pi}{f^2\psi+\phi}}\,e^{-\frac{a^2f^2\psi^2 i}{f^2\psi+\phi}},$$

and the like for the other integrals up to the z-integral. The resulting value is thus

$$=\frac{2}{\pi\Gamma(\frac{1}{2}s+q)}\,e^{-\frac{1}{2}q\pi i}\int_0^\infty\frac{\sin\phi}{\phi}\,.\int_0^\infty d\psi\,.\,\psi^{\frac{1}{2}s+q-1}\,e^{\psi\phi i\left(\frac{a^2}{\phi+f^2\psi}\ldots+\frac{c^2}{\phi+h^2\psi}\right)}\frac{\pi^{\frac{1}{2}s}f\ldots h}{\sqrt{\phi+f^2\psi\ldots\phi+h^2\psi}},$$

which, putting therein $\psi=\frac{\phi}{t}$, $d\psi=-\frac{\phi}{t^2}\,dt$, is

$$=\frac{2\pi^{\frac{1}{2}s-1}}{\Gamma(\frac{1}{2}s+q)}(f\ldots h)e^{-\frac{1}{2}q\pi i}\int_0^\infty dt\,\frac{t^{-q-1}}{\sqrt{f^2+t\ldots h^2+t}}\int_0^\infty e^{i\phi\left(\frac{a^2}{f^2+t}\ldots+\frac{c^2}{h^2+t}\right)}\sin\phi\,.\,\phi^{q-1}\,d\phi.$$

154. But we have to consider only the real part of this expression; viz. writing for shortness $\sigma=\frac{a^2}{f^2+t}\ldots+\frac{c^2}{h^2+t}$, we require the real part of

$$e^{-\frac{1}{2}q\pi i}\int_0^\infty e^{i\sigma\phi}\,.\,\phi^{q-1}\sin\phi\,d\phi.$$

Writing here for $\sin\phi$ its exponential value $\frac{1}{2i}(e^{i\phi}-e^{-i\phi})$, and using the formula

$$\frac{1}{\sigma^q}=\frac{1}{\Gamma q}\,e^{-q\pi i}\int_0^\infty d\phi\,.\,\phi^{q-1}\,.\,e^{i\sigma\phi}\qquad(\sigma\text{ positive}),$$

and the like one

$$\frac{1}{(-\sigma)^q}=\frac{1}{\Gamma q}\,e^{q\pi i}\int_0^\infty d\phi\,.\,\phi^{q-1}\,e^{i\sigma\phi}\qquad(\sigma\text{ negative})$$

(in which formulæ q must be positive and less than 1), we see that the real part in question is $=0$, or is

$$-\frac{\Gamma q\sin(q+1)\pi}{2(1-\sigma)^q},\ =\frac{\pi}{2\Gamma(1-q)}\,\frac{1}{(1-\sigma)^q},$$

according as $\sigma>1$ or $\sigma<1$.

155. If the point is interior, $\frac{a^2}{f^2}\ldots+\frac{c^2}{h^2}<1$, and consequently also $\sigma<1$, and the value, writing $(\Gamma\frac{1}{2})^2$ instead of π, is

$$=\frac{(\Gamma\frac{1}{2})^s}{\Gamma(\frac{1}{2}s+q)\Gamma(1-q)}(f\ldots h)\,.\int_0^\infty dt\,.\,t^{-q-1}(t+f^2\ldots t+h^2)^{-\frac{1}{2}}\left(1-\frac{a^2}{f^2+t}\cdots-\frac{c^2}{h^2+t}\right)^{-q}.$$

But if the point be exterior, $\frac{a^2}{f^2}\ldots+\frac{c^2}{h^2}>1$, and hence, writing θ for the positive root of the equation, $\sigma=1$; viz. θ is the positive root of the equation $\frac{a^2}{f^2+\theta}\cdots+\frac{c^2}{h^2+\theta}=1$; then $t=0$, σ is greater than 1, and continues so as t increases, until, for $t=\theta$, σ becomes $=1$, and for larger values of t we have $\sigma<1$; and the expression thus is

$$=\frac{(\Gamma\frac{1}{2})^s}{\Gamma(\frac{1}{2}s+q)\Gamma(1-q)}(f\ldots h)\int_\theta^\infty dt\,.\,t^{-q-1}(t+f^2\ldots t+h^2)^{-\frac{1}{2}}\left(1-\frac{a^2}{f^2-t}\cdots-\frac{c^2}{h^2+t}\right)^{-q};$$

viz. the two expressions in the cases of an interior point and an exterior point respectively give the value of the integral

$$\int\frac{dx\ldots dz}{\{(a-x)^2\ldots+(c-z)^2\}^{\frac{1}{2}s+q}}.$$

This is in fact the formula of Annex IV. No. 110, writing therein $e=0$ and $m=-q$.

156. BOOLE's researches are contained in two memoirs dated 1846, "On the Analysis of Discontinuous Functions," Trans. Royal Irish Academy, vol. xxi. (1848), pp. 124–139, and "On a certain Multiple Definite Integral," do. pp. 140–150 (the particular theorem about to be referred to is stated in the postscript of this memoir), and in the memoir "On the Comparison of Transcendents, with certain applications to the theory of Definite Integrals," Phil. Trans. vol. 147, for 1857, pp. 745–803, the theorem being the third example, p. 794. The method is similar to that of, and was in fact suggested by, LEJEUNE-DIRICHLET; the auxiliary theorem made use of in the memoir of 1857 for the representation of the discontinuity being

$$\frac{f(x)}{t^i}=\frac{1}{\pi\Gamma i}\int_{-\infty}^\infty\int_0^\infty\int_0^\infty da\,dv\,ds\cos\{(a-x-ts)v+\tfrac{1}{2}i\pi\}v^is^{i-1}f(a),$$

which is a deduction from FOURIER's theorem.

Changing the notation (and in particular writing s and $\frac{1}{2}s+q$ for his n and i) the method is here applied to the determination of the s-tuple integral

$$\mathrm{V}=\int dx\ldots dz\,\frac{\phi\left(\frac{x^2}{f^2}\cdots+\frac{z^2}{h^2}\right)}{\{(a-x)^2\ldots+(c-z)^2+e^2\}^{\frac{1}{2}s+q}}$$

(where ϕ is an arbitrary function) over the ellipsoid $\frac{x^2}{f^2}\cdots+\frac{z^2}{h^2}=1$.

157. The process is as follows: we have

$$\frac{\phi\left(\frac{x^2}{f^2}\cdots+\frac{z^2}{h^2}\right)}{\{(a-x)^2\ldots+(c-z)^2+e^2\}^{\frac{1}{2}s+q}}=\frac{1}{\pi\Gamma(\frac{1}{2}s+q)}\int_0^1\int_0^\infty\int_0^\infty du\,dv\,d\tau\,v^{\frac{1}{2}s+q}t^{\frac{1}{2}s+q-1}$$
$$\cos\left\{\left(u-\frac{x^2}{f^2}\cdots-\frac{z^2}{h^2}-\tau\Big((a-x)^2\ldots+(c-z)^2+e^2\Big)v\right)+\tfrac{1}{2}(\tfrac{1}{2}s+q)\pi\right\}\phi u;$$

viz. the right-hand side is here equal to the left-hand side or is $=0$, according as $\frac{x^2}{f^2}\ldots+\frac{z^2}{h^2}<1$ or >1. V is consequently obtained by multiplying the right-hand side by $dx\ldots dz$ and integrating from $-\infty$ to $+\infty$ for each variable.

Hence, changing the order of the integration,

$$\mathrm{V}=\frac{1}{\pi\Gamma(\frac{1}{2}s+q)}\int_0^1\int_0^\infty\int_0^\infty du\,dv\,d\tau\, v^{\frac{1}{2}s+q}\tau^{\frac{1}{2}s+q-1}\phi u\,.\,\Omega,$$

where

$$\Omega=\int dx\ldots dz\cos\left\{\left(u-e^2\tau-\frac{x^2}{f^2}\ldots-\frac{z^2}{h^2}+\tau\{(a-x)^2\ldots+(c-z)^2\}\right)v+\tfrac{1}{2}(\tfrac{1}{2}s+q)\pi\right\}.$$

Now

$$\frac{x^2}{f^2}+\tau(a-x)^2=\frac{1+f^2\tau}{f^2}\xi^2+\frac{\tau a^2}{1+f^2\tau},\qquad\ldots\frac{z^2}{h^2}+\tau(c-z)^2=\frac{1+h^2\tau}{h^2}\zeta^2+\frac{\tau c^2}{1+h^2\tau},$$

if

$$\xi=x-\frac{f^2\tau a}{1+f^2\tau},\qquad\ldots\zeta=z-\frac{h^2\tau c}{1+h^2\tau}.$$

158. Substituting, and integrating with respect to $\xi\ldots\zeta$ between the limits $-\infty$, $+\infty$, we have

$$\Omega=\frac{(f\ldots h)\pi^{\frac{1}{2}s}}{(1+f^2\tau\ldots1+h^2\tau)^{\frac{1}{2}}v^{\frac{1}{2}s}}\cos\left\{\left(u-e^2\tau-\frac{a^2\tau}{1+f^2\tau}\ldots-\frac{c^2\tau}{1+h^2\tau}\right)v+\tfrac{1}{2}q\pi\right\};$$

or, what is the same thing, writing $\frac{1}{t}$ in place of τ, this is

$$\Omega=\frac{(f\ldots h)\pi^{\frac{1}{2}s}t^{\frac{1}{2}}}{(f^2+t\ldots h^2+t)^{\frac{1}{2}}v^{\frac{1}{2}s}}\cos\left\{\left(u-\frac{a^2}{f^2+t}\ldots-\frac{c^2}{h^2+t}-\frac{e^2}{t}\right)v+\tfrac{1}{2}q\pi\right\};$$

that is, writing

$$\sigma=\frac{a^2}{f^2+t}\ldots+\frac{c^2}{h^2+t}+\frac{e^2}{t},$$

we have

$$\mathrm{V}=\frac{\pi^{\frac{1}{2}s-1}(f\ldots h)}{\Gamma(\frac{1}{2}s+q)}\int_0^1\int_0^\infty\int_0^\infty du\,dv\,dt\,\frac{t^{-q-1}v^q\cos\{(u-\sigma)v+\frac{1}{2}q\pi\}\phi u}{(t+f^2\ldots t+h^2)^{\frac{1}{2}}};$$

or, writing $\pi^{\frac{1}{2}s-1}=\frac{1}{\pi}(\Gamma\frac{1}{2})^s$, this is

$$=\frac{(\Gamma\frac{1}{2})^s(f\ldots h)}{\Gamma(\frac{1}{2}s+q)}\int_0^\infty dt\,.\,t^{-q-1}(t+f^2\ldots t+h^2)^{-\frac{1}{2}}.\frac{1}{\pi}\int_0^1\int_0^\infty du\,dv\,.\,v^q\cos\{(u-\sigma)v+\tfrac{1}{2}q\pi\}\phi u$$

159. BOOLE writes

$$\frac{1}{\pi}\int_0^1\int_0^\infty du\,dv\,d^q\cos\{(u-\sigma)v+\tfrac{1}{2}q\pi\}\phi u=\left(-\frac{d}{d\sigma}\right)^q\phi(\sigma);$$

viz. starting from FOURIER's theorem,

$$\frac{1}{\pi}\int_0^1\int_0^\infty du\,dv\cos(u-\sigma)v\,.\,\phi u=\phi(\sigma)$$

(where $\phi(\sigma)$ is regarded as vanishing except when σ is between the limits 0, 1, and the limits of u are taken to be 1, 0 accordingly), then, according to an admissible theory of

general differentiation, we have the result in question. He has in the formula $\frac{1}{s}$ instead of my t; and he proceeds, "Here σ increases continually with s. As s varies from 0 to ∞, σ also varies from 0 to ∞. To any positive limits of σ will correspond positive limits of s; and these, as will hereafter appear [refers to his note B], will in certain cases replace the limits 0 and ∞ in the expression for V."

160. It seems better to deal with the result in the following manner, as in part shown p. 803 of BOOLE's memoir. Writing the integral in the form

$$V=\frac{(\Gamma\frac{1}{2})^s.(f\ldots h)}{\pi\Gamma(\frac{1}{2}s+q)}\int_0^1\int_0^\infty du\,dt\,.\,t^{-q-1}(t+f^2\ldots t+h^2)^{-\frac{1}{2}}\phi(u)\int_0^\infty dv\,.\,v^q\cos\{(u-\sigma)v+\tfrac{1}{2}q\pi\},$$

effect the integration in regard to v; viz. according as u is greater or less than σ, then

$$\int_0^\infty dv\,.\,v^q\cos\{(u-\sigma)v+\tfrac{1}{2}q\pi\}=\frac{\Gamma(q+1)\sin(q+1)\pi}{(u-\sigma)^{q+1}},\text{ or } 0,$$

$$=\frac{\pi}{\Gamma(-q)\,(u-\sigma)^{q+1}}\quad,\text{ or } 0\,;$$

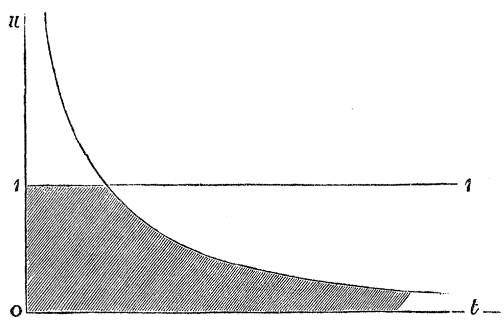
and consequently, writing for σ its value,

$$V=\frac{(\Gamma\frac{1}{2})^s(f\ldots h)}{\Gamma(-q)\Gamma(\frac{1}{2}s+q)}\int_0^1\int_0^\infty du\,dt\left\{t^{-q-1}(t+f^2\ldots t+h^2)^{-\frac{1}{2}}\left(u-\frac{a^2}{f^2+t}\cdots-\frac{c^2}{h^2+t}-\frac{e^2}{t}\right)^{-q-1}\phi u,\ \text{or 0 as above}\right\}.$$

161. To further explain this, consider t as an x-coordinate and u as a y-coordinate; then tracing the curve

$$y=\frac{a^2}{f^2+x}\cdots+\frac{c^2}{h^2+x}+\frac{e^2}{x}$$

for positive values of x, this is a mere hyperbolic branch, as shown in the figure, viz. $x=0$, $y=\infty$; and as x continually increases to ∞, y continually decreases to zero.



The limits are originally taken to be from $u=0$ to $u=1$ and $t=0$ to $t=\infty$, viz. over the infinite strip bounded by the lines tO, O1, 11; but within these limits the function under the integral sign is to be replaced by zero whenever the values u, t are such that u is less than $\frac{a^2}{f^2+t}\cdots+\frac{c^2}{h^2+t}+\frac{e^2}{t}$, viz. when the values belong to a point in the shaded

portion of the strip; the integral is therefore to be extended only over the unshaded portion of the strip; viz. the value is

$$\mathrm{V}=\frac{(\Gamma\frac{1}{2})^s(f\ldots h)}{\Gamma(-q)\Gamma(\frac{1}{2}s+q)}\iint du\,dt\,.\,t^{-q-1}(t+f^2\ldots t+h^2)^{-\frac{1}{2}}\left(u-\frac{a^2}{f^2+t}\ldots-\frac{c^2}{h^2+t}-\frac{e^2}{t}\right)^{-q-1}\phi u,$$

the double integral being taken over the unshaded portion of the strip; or, what is the same thing, the integral in regard to u is to be taken from $u=\frac{a^2}{f^2+t}\ldots+\frac{c^2}{h^2+t}+\frac{e^2}{t}$ (say from $u=\sigma$) to $u=1$, and then the integral in regard to t is to be taken from $t=\theta$ to $t=\infty$, where, as before, θ is the positive root of the equation $\sigma=1$, that is of $\frac{a^2}{f^2+\theta}\ldots+\frac{c^2}{h^2+\theta}+\frac{e^2}{\theta}=1.$

162. Write $u=\sigma+(1-\sigma)x$, and therefore $u-\sigma=(1-\sigma)x$, $1-u=(1-\sigma)(1-x)$ and $du=(1-\sigma)dx$; then the limits $(1, 0)$ of x correspond to the limits $(1, \sigma)$ of u, and the formula becomes

$$\mathrm{V}=\frac{(\Gamma\frac{1}{2})^s.(f\ldots h)}{\Gamma(-q)\Gamma(\frac{1}{2}s+q)}\int_\theta^\infty dt\,.\,t^{-q-1}(t+f^2\ldots t+h^2)^{-\frac{1}{2}}(1-\sigma)^{-q-1}\int_0^1 dx\,.\,x^{-q-1}.\,\phi\{\sigma+(1-\sigma)x\},$$

where σ is retained in place of its value $\frac{a^2}{f^2+t}\ldots+\frac{c^2}{h^2+t}+\frac{e^2}{t}$. This is in fact a form (deduced from BOOLE's result in the memoir of 1846) given by me, Cambridge and Dublin Mathematical Journal, vol. ii. (1847), p. 219.

If in particular $\phi u=(1-u)^{q+m}$, then $\phi\{\sigma+(1-\sigma)x\}=(1-\sigma)^{q+m}(1-x)^{q+m}$, and thence

$$\int_0^1 x^{-q-1}\{\phi\sigma+(1-\sigma)x\}\,dx=(1-\sigma)^m\int_0^1 x^{-q-1}(1-x)^{q+m}dx,$$
$$=\frac{\Gamma(-q)\Gamma(1+q+m)}{\Gamma(1+m)}(1-\sigma)^m;$$

and thence restoring for σ its value, we have

$$\mathrm{V}=\frac{(\Gamma\frac{1}{2})^s\Gamma(1+q+m)}{\Gamma(\frac{1}{2}s+q)\Gamma(1+m)}(f\ldots h)\int_\theta^\infty dt\,.\,t^{-q-1}(t+f^2\ldots t+h^2)^{-\frac{1}{2}}\left(1-\frac{a^2}{f^2+t}\ldots-\frac{c^2}{h^2+t}-\frac{e^2}{t}\right)^m$$

as the value of the integral

$$\int\frac{\left(1-\frac{x^2}{f^2}\ldots-\frac{z^2}{h^2}\right)^{q+m}dx\ldots dz}{\{(a-x)^2\ldots+(c-z)^2+e^2\}^{\frac{1}{2}s+q}}$$

over the ellipsoid $\frac{x^2}{f^2}\ldots+\frac{z^2}{h^2}=1$. This is in fact the theorem of Annex IV. No. 110 in its general form; but the proof assumes that q is positive.