IX. *On Multiple Contact of Surfaces.*

By WILLIAM SPOTTISWOODE, *M.A., Treas. R.S.*

Received May 24,—Read June 17, 1875.

IN a paper "On the Contact of Quadrics with other Surfaces," published in the Proceedings of the London Mathematical Society (May 14, 1874, p. 70), I have shown that it is not in general possible to draw a quadric surface V so as to touch a given surface U in more than two points, but that a condition must be fulfilled for every additional point. The equations expressing these conditions, being interpreted in one way, show that two points being taken arbitrarily the third point of contact, if such there be, must lie on a curve the equation whereof is there given. The same formulæ, interpreted in another way, serve to determine the conditions which the coefficients of the surface U must fulfil in order that the contact may be possible for three or more points taken arbitrarily upon it; and, in particular, the degrees of these conditions give the number of surfaces of different kinds which satisfy the problem.

In another paper, "Sur les Surfaces osculatrices" (Comptes Rendus, 6 Juillet, 1874, p. 24), the corresponding conditions for the osculation of a quadric with a given surface are discussed.

In the present paper I have regarded the question in a more general way; and having shown how the formulæ for higher degrees of contact are obtained, I have developed more in detail some special cases of interest.

For the convenience of the reader, I have in § 1 briefly recapitulated the principal parts of the two papers above quoted. In § 2 I have given, at all events, a first sketch of a general theory of multiple contact with quadrics; in § 3 the particular cases of three-, four-, five-, and six-pointic contact are discussed; and in § 4 some conditions for the existence of points of four-, five-, six-branched single (*i. e.* not multiple) contact are established.

Thus far the investigation concerns the contact of quadrics only with other surfaces. The concluding part of the paper is concerned with the corresponding problem for cubics, in which case conditions of possibility do not arise for either simple or two-branched contact, but are first met with for three-branched contact. The conditions in question, with some of their consequences, are here given; but it is perhaps hardly worth while to prosecute the subject much further in this direction.

It will be observed in the course of the paper that some of the numerical results must be taken as subject to limitations to be expected from further research; but the intricate nature of the investigation will, I hope, be considered as affording some justification for submitting it thus rough-hewn to the notice of the Society.

§ 1. *Recapitulation of former Methods and Results.*

Let x, y, z, t; x_1, y_1, z_1, t_1; . . be the coordinates of the points P, P_1, . . respectively; and let

$$U=(x, y, z, t)^n=0,\ V=(x, y, z, t)^m=0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

be the equations of the two surfaces whose contact is the subject of consideration. The conditions of contact may, as is well known, be written thus:

$$\left.\begin{array}{l} \partial_x U : \partial_y U : \partial_z U : \partial_t U \\ =\partial_x V : \partial_y V : \partial_z V : \partial_t V. \end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Another form of these conditions is, however, better adapted to our present purpose. In fact, writing

$$\left.\begin{array}{l} \partial_y V\partial_z U-\partial_z V\partial_y U=\delta\, U,\ \partial_x V\partial_t U-\partial_t V\partial_x U=\delta_3 U, \\ \partial_z V\partial_x U-\partial_x V\partial_z U=\delta_1 U,\ \partial_y V\partial_t U-\partial_t V\partial_y U=\delta_4 U, \\ \partial_x V\partial_y U-\partial_y V\partial_x U=\delta_2 U,\ \partial_z V\partial_t U-\partial_t V\partial_z U=\delta_5 U, \end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

or, more briefly,

$$\left\Vert\begin{array}{llll} \partial_x U, & \partial_y U, & \partial_z U, & \partial_t U \\ \partial_x V, & \partial_y V, & \partial_z V, & \partial_t V \end{array}\right\Vert=\delta U,\ \delta_1 U, \ldots \delta_5 U, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

we may take as the conditions necessary, in order that V shall touch U at the point P, any two of the six following, viz.:

$$\delta U=0,\ \delta_1 U=0, \ldots \delta_5 U=0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Similarly, as I have shown in a memoir "On the Contact of Surfaces" (Phil. Trans. 1872, p. 259), we may take as the further conditions that V (assumed to touch U at the point P) may have a three-branch contact* with V at the point P, any three independent equations of the following system:

$$\delta^2 U=0,\ \delta_1{}^2 U=0, \ldots \delta\delta_1 U=0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

and so on for higher degrees of contact.

For the purpose of the present inquiry it will be convenient to transform the equations (5) and (6) into yet another shape. Thus, if we write

$$\left.\begin{array}{l} (x, y, z, t)^n=0^n,\ (x, y, z, t)^{n-1}(x_1, y_1, z_1, t_1)=0^{n-1}1, \\ (x, y, z, t)^2=0^2,\ (x, y, z, t)\quad (x_1, y_1, z_1, t_1)=01, \end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

and multiply each number of the system (2), first by x_1, y_1, z_1, t_1 respectively, and add, then by x_2, y_2, z_2, t_2, and add, and so on, we shall obtain the systems

$$\left.\begin{array}{l} 0^{n-1}1 : 0^{n-1}2 : \ldots \\ =\ 01\ :\ 02\ : \ldots \end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

* The terms 2-branch, 3-branch, &c. contact, already used by Professors CAYLEY and CLIFFORD, have the following signification:—in ordinary contact the curve of intersection has at the point a double point, or two branches; in contact of the second order, a triple point, or three branches, and so on.

This result may be generalized by introducing operative symbols, as follows:—

Let $$P_{10}=x_1\partial_x+y_1\partial_y+z_1\partial_z+t_1\partial_t$$

be a differential operator, capable of operating on functions of x, y, z, t, and containing x_1, y_1, z_1, t_1 as mere constants; then

$$\square_{12}\ (\text{or more completely expressed } \boxed{\circ}_{12})$$
$$=P_{10}V\,.\,P_{20}-P_{20}V\,.\,P_{10}$$

is a differential operator capable of operating on any function of x_1, y_1, z_1, t_1; x_2, y_2, z_2, t_2; in which, when V is a quadric in x, y, z, t, $P_{10}V=P_{01}V_1$. Operating with $\square_{12}$ on $U=(x, y, z, t)^n$, we have $\square_{12}V$=a function of the form

$$(x_1, y_1, z_1, t_1)(x_2, y_2, z_2, t_2)(x, y, z, t)^n,$$

linear in the coefficients of V, and also in those of U.

With this explanation we may write

$$\left.\begin{array}{l} 01\,.\,0^{n-1}2-02\,.\,0^{n-1}1=\square_{12}U,\\ 01\,.\,0^{n-1}3-03\,.\,0^{n-1}1=\square_{13}U,\\ \quad ..\qquad\quad ..\qquad\quad ..\\ 02\,.\,0^{n-1}3-03\,.\,0^{n-1}2=\square_{23}U,\\ \quad ..\qquad\quad ..\qquad\quad .. \end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad (9)$$

or, more briefly,

$$\left\| \begin{array}{ll} 0^{n-1}1, & 0^{n-1}2, \ .. \\ 01\ , & 02\ , \ .. \end{array} \right\| = \square_{12}U,\ \square_{13}U, \ ..\ \square_{23}U, \ ..; \quad . \quad . \quad . \quad (10)$$

and the system (5) may then be replaced by the following, viz.

$$\square_{12}U=0,\ \square_{13}U=0,\ ..\ \square_{23}U=0, \ . \ . \ . \ . \ . \ . \ . \ (11)$$

any two of which may be taken as the conditions required. Similarly the system (6) may be replaced by the following, viz.

$$\square^2_{12}U=0,\ \square^2_{13}U=0,\ ..\ \square_{12}\square_{13}U=0, \quad . \ . \ . \ . \ . \ . \ (12)$$

any three independent members of which may be taken as the conditions required.

If the two surfaces touch at a second point P_1, we may form expressions similar to (10) but involving the coordinates of P_1 in the place of those of P, thus:—

$$\left\| \begin{array}{ll} 1^{n-1}0, & 1^{n-1}2, \ .. \\ 10\ , & 12\ , \ .. \end{array} \right\| = \square_{02}U_1,\ \square_{03}U_1, \ ..\ \square_{23}U_1, \ .. \quad . \ . \ . \ . \ (13)$$

and the conditions for contact at the point P_1 will be comprised in the system

$$\square_{02}U_1=0,\ \square_{03}U_1=0,\ ..\ \square_{23}U_1=0. \ . \ . \ . \ . \ . \ . \ . \ (14)$$

Similarly, the conditions for contact at a third point P_2 will be comprised in the system

$$\square_{01}U_2=0,\ \square_{03}U_2=0,\ ..\ \square_{13}U_2=0, \ .. \ . \ . \ . \ . \ . \ . \ (15)$$

and so on for any number of points.

In the same way the conditions for osculation at a second point P_1 will be comprised in the system

$$\square^2_{02}U_1=0, \quad \square^2_{03}U_1=0, \ldots \square_{02}\square_{03}U_1=0, \ldots \quad . \quad . \quad . \quad . \quad . \quad (16)$$

and so on for any number of points.

Returning to the equations (11), (14), (15), which express the conditions for contact at the three points P, P_1, P_2, and selecting a suitable member from each system, we may form the following group:

$$\square_{12}U=0, \quad \square_{20}U_1=0, \quad \square_{01}U_2=0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

These written in full are as follows:—

$$\begin{aligned} 01 \,.\, 0^{n-1}2&=02 \,.\, 0^{n-1}1 \\ 12 \,.\, 1^{n-1}0&=10 \,.\, 1^{n-1}2 \\ 20 \,.\, 2^{n-1}1&=21 \,.\, 2^{n-1}0, \end{aligned}$$

whence, multiplying together the dexter and sinister sides of these equations respectively, and rejecting the common factor 12 . 20 . 01, we obtain as a condition for the possibility of a quadric V touching a surface U in the three points P, P_1, P_2, the following equation:—

$$0^{n-1}1 \,.\, 1^{n-1}2 \,.\, 2^{n-1}0=0^{n-1}2 \,.\, 1^{n-1}0 \,.\, 2^{n-1}1 \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

This equation shows that the three points must be so situated that each lies on one of the intersections of the first polars of the other two with respect to the surface U; for it may be written in each of the following forms, viz.

$$\begin{aligned} &0^{n-1}1-\lambda 0^{n-1}2=0, \\ &1^{n-1}2-\mu 1^{n-1}0=0, \\ &2^{n-1}0-\nu 2^{n-1}1=0. \end{aligned}$$

In order to account geometrically for the existence of the equation of condition, we may observe, as Professor CAYLEY has remarked, that, drawing through the points P, P_1, P_2 a plane, this plane meets the three tangent-planes of the surface U in the three sides of a triangle, which sides pass respectively through the points P, P_1, P_2; and also meets the surface V in a conic touching the sides of the triangle in these points P, P_1, P_2 respectively. Considering these points as given, a relation is implied between the directions of the three sides; viz. the triangle must be such that if each summit be joined to the opposite point the three joining lines will meet in a point.

In order that contact may subsist for a fourth point P_3 we may take either the points P_1, P_2, P_3, giving as the condition

$$1^{n-1}2 \,.\, 2^{n-1}3 \,.\, 3^{n-1}1=2^{n-1}1 \,.\, 3^{n-1}2 \,.\, 1^{n-1}3 \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

or the points P_2, P, P_3, giving

$$2^{n-1}0 \,.\, 0^{n-1}3 \,.\, 3^{n-1}2=0^{n-1}2 \,.\, 3^{n-1}0 \,.\, 2^{n-1}3 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

or the points P, P_1, P_3, giving

$$0^{n-1}1 \cdot 1^{n-1}3 \cdot 3^{n-1}0 = 1^{n-1}0 \cdot 3^{n-1}1 \cdot 0^{n-1}3 \ldots \qquad (21)$$

But since in any one of these conditions (19), (20), (21) combined with (18) will ensure contact at the four points, it follows that the four equations (18)..(21) must be equivalent to only two independent relations. It is perhaps worth while to verify this analytically. In fact, if we multiply together the dexter and sinister sides of (19), (20), (21) respectively, and reject the common factor

$$k = 0^{n-1}3 \cdot 1^{n-1}3 \cdot 2^{n-1}3 \cdot 3^{n-1}0 \cdot 3^{n-1}1 \cdot 3^{n-1}2,$$

we shall reproduce the equation (18). Again, if we represent the equations (19), (20), (21), (18) respectively by the following expressions,

$$a = a', \; b = b', \; c = c', \; d = d',$$

the multiplications above indicated give as their result

$$abc = a'b'c' = kd = kd';$$

and consequently if any two of the equations $a=a'$, $b=b'$, $c=c'$ be satisfied, the third, as well as the equation $d=d'$, will be so also; which is the verification required.

This, as Professor CAYLEY has pointed out, gives rise to an interesting theorem of Solid Geometry; viz. writing for greater symmetry α, β, γ, δ, instead of P, P_1, P_2, P_3, and considering ABCD the tetrahedron formed by the tangent planes of U at the same four points respectively, we have the tetrahedron ABCD, the planes whereof contain the points α, β, γ, δ, respectively. Now the plane of $\alpha\beta\gamma$ determines with the planes BCD, CDA, DAB, a triangle, the sides whereof contain the points α, β, γ, and which is such that the lines joining the summits with the opposite points meet in a point. Similarly the planes of $\beta\gamma\delta$, $\gamma\delta\alpha$, $\delta\alpha\beta$ determine three other triangles having similar properties. And the theorem is that, if the foregoing relation be satisfied for any two of the triangles, it will be satisfied for the other two.

The equation (18) may be regarded either as a relation between the coordinates of the three points P, P_1, P_2, or as a relation between the coefficients of U. In the first case it shows that if two of the points be taken arbitrarily, the third must be on a curve defined by (18) together with the equation $U=0$. As to the fourth point of contact, if such there be, suppose that we represent the equations (19), (20), (21), (18) by the symbols $(1, 2, 3)=0$, $(0, 2, 3)=0$, $(0, 1, 3)=0$, $(0, 1, 2)=0$; then taking arbitrarily the points P, P_1, the curve on which P_2 must lie will be given by the equation $(0, 1, 2)=0$. The equation $(0, 1, 3)=0$ merely shows that P_3 must lie on the same curve; but the equation $(0, 2, 3)$ shows that, if P_2 be taken arbitrarily on the curve in question, P_3 will lie at one of a finite number of points on the curve. In fact, if we take $U_3=0$ as the condition that P_3 shall lie on the surface, $(0, 1, 3)=0$ as the equation expressing the condition that P_3 shall lie on the curve, and $(0, 2, 3)$ the additional condition for contact at P_3, we shall have three equations each of the degree n for determining the coordinates of P_3. The number of positions for P_3 will therefore

apparently be n^3; but this number doubtless admits of some reduction in consequence of the particular form of the equations. In fact, if a, b .. be the coefficients of the highest powers of $x_3, y_3, \ldots$ in the equation $3^n=0$, then the terms involving $a^3, b^3, \ldots$ in (18) all vanish. But I have not as yet fully investigated this question.

Turning to the second view of the case, which is in fact the more interesting, viz. that in which (18) is to be considered as a relation between the coefficients of U, we have for every point of contact four equations, viz.

$$U=0, \quad \partial_x U : \partial_y U : \partial_z U = \partial_x V : \partial_y V : \partial_z V, \quad V=0,$$

or their equivalents; *i. e.* for p points of contact $4p$ equations, viz.

p equations involving the coefficients of U alone,
$2p$,, ,, U & V,
p ,, ,, V alone.

But of the $2p$ equations which involve the coefficients of U & V, $p-2$ may be cleared of those belonging to V, and reduced to the form (18); and we hence have finally

$2p-2$ equations involving the coefficients of U alone,
$p+2$,, ,, U & V,
p ,, ,, V alone.

By means of the last $2p+2$ equations let us determine as many as possible of the coefficients of V. Then if $2p+2>9$ we shall be able to eliminate those coefficients in $2p-7$ different ways, and obtain $2p-7$ equations involving the coefficients of U to the degree $10-p$; p being supposed $\not> 9$. We may thence form the following table of conditions to which the surface U will be subject:—

Number of points.	Number of coefficients of V determined.	Number of conditions in U.	Numbers and degrees of conditions in coefficients of U.		
			Of degree 1.	Of degree 3.	Also
3	8	4	3	1	
4	9	7	4	2	1 of degree 6
5	9	11	5	3	3 ,, ,, 5
6	9	15	6	4	5 ,, ,, 4
7	9	19	7	5	7 ,, ,, 3
8	9	23	8	6	9 ,, ,, 2
9	9	27	9	7	11 ,, ,, 1

It must, however, be owned that these numbers, although doubtless true as superior limits, must evidently undergo some reduction after a detailed examination of the equations upon which they depend. And on this account I abstain at present from writing down the geometrical theorems which will obviously occur to the reader on perusing these results.

Thus far for simple or two pointic contact. In order to find the conditions for three-branch contact at several points, we must, as in the second paper above quoted,

employ the equations (12) and (16); and for this purpose it is necessary in the first place to develop the formula $\square^2_{12}\mathrm{U}$; viz. we have

$$\left.\begin{aligned}\square^2_{12}\mathrm{U}=&(02)^2 0^{n-2}1^2-2\,.\,01\,.\,02\,.\,0^{n-2}12+(01)^2 0^{n-2}2^2\\ &-02\,.\,0^{n-1}2\,.\,1^2+(02\,.\,0^{n-1}1+01\,.\,0^{n-1}2)12-01\,.\,0^{n-1}1\,.\,2^2\end{aligned}\right\}\quad .\quad .\quad (22)$$

But having reference to the conditions $\square_{20}\mathrm{U}=0$, $\square_{21}\mathrm{U}=0$, .. we may put (θ being an indeterminate quantity)

$$01=\theta 0^{n-1}1,\ 02=\theta 0^{n-1}2,\ .\ .\ .\ .\ .\ .\ .\ .\ .\ .\ (23)$$

whence, by substituting these values in the developed form of the equation $\square^2_{12}\mathrm{U}=0$, and dividing throughout by θ, we obtain

$$\left.\begin{aligned}&\theta\{(0^{n-1}2)^2 0^{n-2}1^2-2\,.\,0^{n-1}1\,.\,0^{n-1}2\,.\,0^{n-2}12+(0^{n-1}1)^2 0^{n-2}2^2\}\\ &=(0^{n-1}2)^2 1^2-2\,.\,0^{n-1}1\,.\,0^{n-1}2\,.\,12+(0^{n-1}1)^2 2^2\}\end{aligned}\right\}\quad .\ .\ .\ .\quad (24)$$

But since P_1, P_2 are supposed to lie on the quadric V, we have $1^2=0$, $2^2=0$; so that the equation last above written is reduced to the following, viz.

$$2\,.\,0^{n-1}1\,.\,0^{n-1}2\,.\,12=-\theta\{(0^{n-1}2)^2 0^{n-2}1^2-\,..\}=\theta\begin{vmatrix}. & 0^{n+1}1, & 0^{n-1}\,2\\ 0^{n-1}, & 0^{n-2}1^2, & 0^{n-2}12\\ 0^{n-1}2, & 0^{n-2}21, & 0^{n-2}\,2^2\end{vmatrix}$$

or, as this may be written for greater symmetry,

$$\begin{aligned}&=\theta\begin{vmatrix}0^{n-2}\,0^2, & 0^{n-2}01\,, & 0^{n-2}02\\ 0^{n-2}10\,, & 0^{n-2}\,1^2, & 0^{n-2}12\\ 0^{n-2}20\,, & 0^{n-2}21\,, & 0^{n-2}\,2^2\end{vmatrix}.\\ &=\theta\,[0',\,1,\,2]\ \text{suppose};\end{aligned}$$

and one expression for three-branched contact at the point P will be

$$2\,.\,0^{n-1}1\,.\,0^{n-1}2\,.\,12=\theta[0',\,1,\,2].$$

A similar transformation may be applied to $\square^2_{13}\mathrm{U}$, ..; and also to $\square_{12}\square_{13}\mathrm{U}$, ..; but the latter forms, which are rather more complicated, are not necessary for the present purpose. This being so, if the surfaces touch at the four points P, P_1, P_2, P_3, we may take for the conditions of osculation at the point P the three equations $\square^2_{23}\mathrm{U}=0$, $\square^2_{21}=0$, $\square^2_{12}\mathrm{U}=0$, which being transformed in the manner above explained give the following results, viz.:—

$$\left.\begin{aligned}2\,.\,0^{n-1}2\,.\,0^{n-1}3\,.\,23=\theta[0',\,2,\,3]\\ 2\,.\,0^{n-1}3\,.\,0^{n-1}1\,.\,31=\theta[0',\,3,\,1]\\ 2\,.\,0^{n-1}1\,.\,0^{n-1}2\,.\,12=\theta[0',\,1,\,2]\end{aligned}\right\};\quad .\ .\ .\ .\ .\ .\ .\quad (25)$$

whence, eliminating the indeterminate quantity θ, we obtain

$$23:31:12=0^{n-1}1[0',\,2,\,3]:0^{n-1}2[0',\,3,\,1]:0^{n-1}3[0',\,1,\,2].$$

We also obtain

$$(\theta=)\frac{0^{n-1}1}{01}=\frac{0^{n-1}2}{02}=\frac{0^{n-1}3}{03}$$

$$=\frac{2\,.\,0^{n-1}2\,.\,0^{n-1}3\,.\,23}{[0', 2, 3]}$$

$$=\frac{2\,.\,0^{n-1}3\,.\,0^{n-1}1\,.\,31}{[0', 3, 1]}$$

$$=\frac{2\,.\,0^{n-1}1\,.\,0^{n-1}2\,.\,12}{[0', 1, 2]},$$

which are the five conditions in order that U, assumed to pass through P, and also through P_1, P_2, P_3, may have a three-branch contact at P.

Similarly for three-branched contact at the point P, we should find

$$30:02:23=1^{n-1}2[1', 3, 0]:1^{n-1}3[1', 0, 2]:1^{n-1}0[1', 2, 3],$$

and for osculation at the point P_2,

$$30:01:13=2^{n-1}1[2', 30]:2^{n-1}3[2', 0, 1]:2^{n-1}0[2', 1, 3].$$

Substituting from these equations the values of $30:23$, $13:30$, $23:31$ in the identical equation

$$(30:23)(13:30)(23:31)=1,$$

we obtain the following relation,

$$1^{n-1}2[1', 3, 0]2^{n-1}0[2', 1, 3]0^{n-1}1[0', 2, 3]$$
$$=1^{n-1}0[1', 2, 3]2^{n-1}1[2', 3, 0]0^{n-1}2[0', 3, 1],$$

which, in virtue of the condition (18), may be reduced to the form

$$[0', 2, 3][1', 3, 0][2', 1, 3]=[0', 3, 1][1', 2, 3][2', 3, 0]. \quad . \quad . \quad . \quad . \quad (26)$$

This therefore is a condition which, in addition to those found before, must be fulfilled in order that it may be possible to draw through four points P, P_1, P_2, P_3 a quadric surface which will touch the surface U in those points, and have three-branched contact with it in three of them.

The total number of conditions for two-branched contact at four points, and three-branched at one of them, may be calculated as follows:—For three-branched contact at three points we shall have the following numbers of equations in the coefficients:

3	equations in U of the degree	1,
1	,, ,,	3,
1	,, ,,	9,
5	,, U and V of the degrees in U,	1; in V, 1,
8	,, ,, ,,	1; ,, 2,
3	,, V of the degree 1.	

Beside this we shall have for the two-branched contact at the fourth point, one equation in U of the degree 1, one of the degree 3, one in U and V of the degrees 1, 1, and one in V of the degree 1; so that the total number will be

4	equations in U of the degree 1,					(a)
2	,,	,,	3,			(b)
1	,,	,,	9,			(c)
6	,,	U and V of the degrees	in U, 1;	in V, 1,		(d)
8	,,	,,	,, 1;	,, 2,		(e)
4	,,	V of the degree 1.				(f)

Taking any five of the equations marked (d) and the four marked (f), we see that all the coefficients of the quadric surface are determinate and unique; also that the quantities to which they are severally proportional are apparently of the degree 6 in the coefficients of U. Subject to future reduction of this number, the equations of condition in the coefficients of U and their degrees will be,

4	of the degree		1,
2	,,	,,	3,
1	,,	,,	7,
1	,,	,,	9,
8	,,	,,	13.

But having these ulterior reductions of degree in view, I at present abstain from enunciating the geometrical theorems which these results suggest.

§ 2. *General Theory of Multiple Contact of Quadrics with other Surfaces.*

The question may be considered from rather a more general point of view. Dropping, for the moment, the suffixes, and taking the symbol $\square$ to represent any of the operations $\square_{12}$, $\square_{13}$, .. (say $\square_{12}$), let $\square'$ represent that part of $\square$ which operates on U, and $\square''$ that part which operates on V; so that we may write symbolically $\square=\square'+\square''$. It is first required to find the values of $\square\mathrm{U}$, $\square^2\mathrm{U}$, .. in terms of $\square'\mathrm{U}$, $\square''\mathrm{U}$, .. The transformation will perhaps be better understood by examining two or three special instances before entering upon the general case. Thus

$$\left.\begin{aligned}\square\mathrm{U}&=\square'\mathrm{U},\\ \square^2\mathrm{U}&=\square'^2\mathrm{U}+\square''\square'\mathrm{U},\\ \square^3\mathrm{U}&=\square'^3\mathrm{U}+\square''\square'^2\mathrm{U}+\square'\square''\square'\mathrm{U}+\square''^2\square'\mathrm{U}.\end{aligned}\right\}\quad\ldots\ldots\quad(1)$$

But it will be found on developing the expression that $\square''^2\square'\mathrm{U}=(12)^2\square'\mathrm{U}$; so that if, as is supposed in the present problem, the two surfaces touch at the point under consideration (a condition which is expressed by the equation $\square'\mathrm{U}=0$), we have always

$\square''^2\square'\mathrm{U}=0$. Moreover

$$\begin{aligned}\square'\square''\square'\mathrm{U}&=n(12)\square'(02\,.\,0^{n-1}1+01\,.\,0^{n-1}2)\\&=n(n-1)(12)\{(02)^2 0^{n-2}1^2-(01)^2 0^{n-2}2^2\},\\ \square''\square'^2\mathrm{U}&=n(n-1)\square''\{(02)^2 0^{n-2}1^2-2\,.\,01\,.\,02\,.\,0^{n-2}12+(01)^2 0^{n-2}2^2\}\\&=2n(n-1)(12)\{(02)^2 0^{n-2}1^2-(01)^2 0^{n-2}2^2\};\end{aligned}$$

so that we have generally the relation

$$\square'\square''\square'\mathrm{U}=\tfrac{1}{2}\square''\square'^2\mathrm{U},$$

and consequently

$$\square^3\mathrm{U}=\square'^3\mathrm{U}+\tfrac{3}{2}\square''\square'^2\mathrm{U}. \qquad (2)$$

Proceeding to the fourth degree, we obtain, by means of the formulæ already established, the following expression:—

$$\square^4\mathrm{U}=\square'^4\mathrm{U}+\square''\square'^3\mathrm{U}+\tfrac{3}{2}\square'\square''\square'^2\mathrm{U}+\tfrac{3}{2}\square''^2\square'^2\mathrm{U}.$$

But operating with $\square'$ on the expression given above for $\square''\square'^2\mathrm{U}$, we obtain

$$\begin{aligned}\tfrac{3}{2}\square'\square''\square'^2\mathrm{U}=3n(n-1)(n-2)(12)\{&(02)^3 0^{n-3}1^3-01(02)^2 0^{n-3}1^2 2\\&-(01)^2 02\,.\,0^{n-3}12^2+(01)^3 0^{n-3}2^3\};\end{aligned}$$

also

$$\begin{aligned}\square'^3\mathrm{U}\quad=\quad n(n-1)(n-2)\quad\{&(02)^3 0^{n-3}1^3-3(02)^2 01\,.\,0^{n-3}1^2 2\\&+3(01)^2 02\,.\,0^{n-3}12^2-(01)^3 0^{n-3}2^3\}.\end{aligned}$$

Hence, operating upon this with $\square''$, we should find

$$\tfrac{3}{2}\square'\square''\square'^2\mathrm{U}=\square''\square'^3\mathrm{U};$$

and consequently, collecting the former results,

$$\begin{aligned}\square\mathrm{U}&=\square'\mathrm{U},\\ \square^2\mathrm{U}&=\square'^2\mathrm{U}+\square''\square'\mathrm{U},\\ \square^3\mathrm{U}&=\square'^3\mathrm{U}+\tfrac{3}{2}\square''\square'^2\mathrm{U},\\ \square^4\mathrm{U}&=\square'^4\mathrm{U}+\tfrac{4}{2}\square''\square'^3\mathrm{U}+\tfrac{3}{2}\square''^2\square'^2\mathrm{U}.\end{aligned} \qquad (3)$$

Having now seen, in the special cases $m=2$, 3, 4, the nature of the transformations contemplated, we may now proceed to the more general case; viz. since the order of the operations $\square'$, $\square''$ is not indifferent, we have in general for $\square^m\mathrm{U}$ an expression of the form

$$\begin{aligned}\square^m\mathrm{U}=\;&\square'^m\mathrm{U}\\&+\square''\square'^{m-1}\mathrm{U}+\square'\square''\square^{m-2}\mathrm{U}+\dots\\&+\square''^2\square'^{m-2}\mathrm{U}+\square''\square'\square''\square^{m-3}\mathrm{U}+\dots\\&+\dots;\end{aligned} \qquad (4)$$

and our first object is to show how this may be reduced to the form

$$\mathrm{A}\square'^m\mathrm{U}+\mathrm{B}\square''\square^{m-1}\mathrm{U}+\mathrm{C}\square''^2\square'^{m-2}\mathrm{U}+\dots\mathrm{A}'\square'^{m-2}\mathrm{U}+\dots$$

Now the general term of $\square'^m\mathrm{U}$ is

$$n(n-1)\ldots(n-m+1)(-)^i\frac{m(m-1)\ldots(m-i+1)}{1.2\ldots i}(02)^{m-i}(01)^i0^{n-m}1^{m-i}2^i;$$

and it will be found on examination that the general term of $\square''\square'^m\mathrm{U}$ differs from that of $\square'^m\mathrm{U}$ only in respect of the factor $(12)(m-2i)$. Similarly the general terms of $\square''^2\square'^m\mathrm{U}, \ldots \square''^p\square'^m\mathrm{U}$ differ from that of $\square'^m\mathrm{U}$ only in respect of the factors $(12)^2(m-2i)^2, \ldots (12)^p(m-2i)^p$, respectively.

This being so, let $\delta_1=x_1\partial_x+\ldots$, $\delta_2=x_2\partial_x+\ldots$; and let it be understood that δ_1, δ_2 affect the external subject of operation alone, then we may write

$$\square'^m\mathrm{U}=(02\delta_1-01\delta_2)^m\mathrm{U};$$

and if we write the above expression in the form of a quantic, thus,

$$\square'^m\mathrm{U}=(1, 1, \ldots)(02\delta_1, -01\delta_2)^m\mathrm{U},$$

we may at once write down the expression for $\square''^p\square'^m\mathrm{U}$ thus,

$$(12)^p(m^p, (m-2)^p, (m-4)^p, \ldots)(02\delta_1, -01\delta_2)^m\mathrm{U}. \quad \ldots \ldots \quad (5)$$

Having thus exhibited the form of the function $\square''^p\square'^m\mathrm{U}$, we must next calculate the effect of the operation $\square'$ upon this function; and if for this purpose we operate with $\square'$ upon the general term of $\square''^p\square'^m\mathrm{U}$, we shall find for the general term of $\square'\square''^p\square'^m\mathrm{U}$ the following expression:—

$$n(n-i)\ldots(n-m)(-)^i(12)^p\left\{\begin{array}{l}\dfrac{m(m-1)\ldots(m-i+1)}{1.2\ldots i}(m-2i)^p\\ +\dfrac{m(m-1)\ldots(m-i+2)}{1.2\ldots(i-1)}(m-2i+2)^p\end{array}\right\}(02)^{m-i+1}(01)^i0^{n-m-1}1^{m-i+1}2^i$$

$$=n(n-1)\ldots(n-m)(-)^i(12)^p\frac{m(m-1)\ldots(m-i+1)}{1.2\ldots i}$$

$$\{(m-i+1)(m-2i)^p+i(m-2i+2)^p\}(02)^{m-i+1}(01)^i0^{n-m-1}1^{m-i+1}2^i.$$

As regards the coefficient between the brackets $\{\ \}$, let $m-2i+1=\mu$; then for $p=1, 2, \ldots$ we have successively

$$(m-i+1)(\mu-1)+i(\mu+1)=m\mu,$$

$$(m-i+1)(\mu-1)^2+i(\mu+1)^2=(m+1)\mu^2-2\mu^2+(m+1)$$
$$=(m-1)\mu^2+(m+1),$$

$$(m-i+1)(\mu-1)^3+i(\mu+1)^3=(m+1)\mu^3-3\mu^3+3(m+1)\mu-\mu$$
$$=(m-2)\mu^3+(3m+2)\mu,$$

$$(m-i+1)(\mu-1)^4+i(\mu+1)^4=(m+1)\mu^4-4\mu^4+6(m+1)\mu^2-4\mu^2+(m+1)$$
$$=(m-3)\mu^4+2(3m+1)\mu^2+(m+1);$$

and generally

$$\begin{aligned}(m-i+1)(\mu-1)^p+i(\mu+1)^p=(m-p+1)\mu^p & \\ +\frac{p(p-1)}{1.2.3}(3m-p+5)\mu^{p-2} & \\ +\frac{p(p-1)(p-2)(p-3)}{1.2.3.4.5}(5m-p+9)\mu^{p-4} & \\ +\ldots & \\ +\frac{p(p-1)\ldots(p-j+2)}{1.2\ldots j}(jm-p+2j-1)\mu^{p-j+1} & \\ +\ldots, &\end{aligned}$$

in which it is to be understood that j is always an odd number.

From these transformations we obtain the following results, viz.:—

$$\left.\begin{aligned}&\square'\square''\square'^m\mathrm{U}=\frac{m}{m+1}\square''\square'^{m+1}\mathrm{U},\\ &\square'\square''^2\square'^m\mathrm{U}=\frac{m-1}{m+1}\square''^2\square'^{m+1}\mathrm{U}+(12)^2\square'^{m+1}\mathrm{U},\\ &\square'\square''^3\square'^m\mathrm{U}=\frac{m-2}{m+1}\square''^3\square'^{m+1}\mathrm{U}+(12)^2\frac{3m+2}{m+1}\square''\square'^{m+1}\mathrm{U},\\ &\quad\ldots\qquad\ldots\\ &\square'\square''^p\square'^m\mathrm{U}=\frac{m-p+1}{m+1}\square''^p\square'^{m+1}\mathrm{U}+\frac{p(p-1)}{1.2.3}\,\frac{3m-p+5}{m+1}(12)^2\square''^{p-2}\square'^{m+1}\mathrm{U}+\ldots\\ &\qquad+\frac{p(p-1)\ldots(p-j+2)}{1.2\ldots j}\,\frac{jm-p+2j-1}{m+1}(12)^{j-1}\square''^{p-j+1}\square'^{m+1}\mathrm{U}+\ldots\end{aligned}\right\}\quad(6)$$

But since the general term of the expansion of $\square^m\mathrm{U}$ will be of the form $\square''^\beta\square'^b\square''^\alpha\square'^a\mathrm{U}$, it follows from these formulæ that every such term, and consequently the whole expansion, may by successive steps be reduced to a series of terms of the form $\square''^{p-q}\square'^{m-p}\mathrm{U}$.

Before proceeding further with the main question it will be worth while to notice a few consequences of these reductions. Thus, in the first place,

$$\square'^2\square''\square'^m\mathrm{U}=\frac{m}{m+1}\square'\square''\square'^{m+1}\mathrm{U}=\frac{m}{m+1}\cdot\frac{m+1}{m+2}\square''\square'^{m+2}\mathrm{U};$$

whence we obtain the following system:—

$$\left.\begin{aligned}&\square'\ \square''\square^m\mathrm{U}=\frac{m}{m+1}\square''\square'^{m+1}\mathrm{U},\\ &\square'^2\square''\square'^m\mathrm{U}=\frac{m}{m+2}\square''\square'^{m+2}\mathrm{U},\\ &\square'^3\square''\square'^m\mathrm{U}=\frac{m}{m+3}\square''\square'^{m+3}\mathrm{U},\\ &\quad\ldots\qquad\ldots\\ &\square'^{m_1}\square''\square'^m\mathrm{U}=\frac{m}{m+m_1}\square''\square'^{m+m_1}\mathrm{U}.\end{aligned}\right\}\quad\ldots\ldots\ldots\quad(7)$$

Again,

$$\square'^2\square''^2\square'^m\mathrm{U}=\square'\left\{\frac{m-1}{m+1}\square''^2\square'^{m+1}\mathrm{U}+(12)^2\square'^{m+1}\mathrm{U}\right\}$$

$$=\frac{m(m-1)}{(m+2)(m+1)}\square''^2\square'^{m+2}\mathrm{U}+(12)^2\left(\frac{m-1}{m+1}+1\right)\square'^{m+2}\mathrm{U}\,;$$

whence the system

$$\left.\begin{aligned}\square'^2\square''^2\square'^m\mathrm{U}&=\frac{m(m-1)}{(m+2)(m+1)}\square''^2\square'^{m+2}\mathrm{U}+\frac{2m}{m+1}(12)^2\square'^{m+2}\mathrm{U},\\ \square'^3\square''^2\square'^m\mathrm{U}&=\frac{m(m-1)}{(m+3)(m+2)}\square''^2\square'^{m+3}\mathrm{U}+\frac{3m}{m+2}(12)^2\square'^{m+3}\mathrm{U},\\ \dots\quad&\quad\dots\end{aligned}\right\}\quad .\ .\ .\quad (8)$$

Assuming this to be true for m_1-1, we should have

$$\square'^{m_1-1}\square''^2\square'^m\mathrm{U}=\frac{m(m-1)}{(m+m_1-1)(m+m_1-2)}\square''^2\square'^{m+m_1-1}\mathrm{U}+\frac{(m_1-1)m}{m+m_1-2}(12)^2\square'^{m+m_1-1}\,;$$

then operating again with $\square'$, we obtain

$$\square'^{m_1}\square''^2\square'^m\mathrm{U}=\frac{m(m-1)}{(m+m_1-1)(m+m_1-2)}\square''^2\square'^{m+m_1}\mathrm{U}$$

$$+\left\{\frac{m(m-1)}{(m+m_1-1)(m+m_1-2)}+\frac{(m_1-1)m}{m+m_1-2}\right\}(12)^2\square'^{m+m_1}\mathrm{U},$$

and the numerator of the expression within the brackets

$$=m(m-1+mm_1-m+m_1^2-2m_1+1)$$

$$=m_1m(m+m_1-2)\,;$$

so that the expression sought finally becomes

$$\frac{m(m-1)}{(m+m_1-1)(m+m_1-2)}\square''^2\square'^{m+m_1}\mathrm{U}-\frac{m_1m}{m+m_1-1}(12)^2\square'^{m+m_1}\mathrm{U},\quad .\ .\ .\quad (9)$$

which proves the law.

Once more, operating with $\square'$ upon the expression given above for $\square'\square''^3\square'^m\mathrm{U}$, we obtain

$$\square'^2\square''^3\square'^m\mathrm{U}=\frac{m-2}{m+1}\left\{\frac{m-1}{m+2}\square''^2\square'^{m+2}\mathrm{U}+\frac{3m+5}{m+2}(12)^2\square''\square'^{m+2}\mathrm{U}\right\}$$

$$+\frac{3m+2}{m+1}\cdot\frac{m+1}{m+2}(12)^2\square''\square'^{m+2}\mathrm{U};$$

whence the system

$$\left.\begin{aligned}\square'^2\square''^3\square'^m\mathrm{U}&=\frac{(m-1)(m-2)}{(m+2)(m+1)}\square''^3\square'^{m+2}\mathrm{U}+\frac{2(3m^2+2m-4)}{(m+2)(m+1)}(12)^2\square''\square'^{m+2}\mathrm{U},\\ \square'^3\square''^3\square'^m\mathrm{U}&=\frac{m(m-1)(m-2)}{(m+3)(m+2)(m+1)}\square''^3\square'^{m+3}\mathrm{U}+\frac{3m(3m^2+5m-6)}{(m+3)(m+2)(m+1)}(12)^2\square''\square'^{m+3}\mathrm{U},\\ \square'^4\square''^3\square'^m\mathrm{U}&=\frac{m(m-1)(m-2)}{(m+4)(m+3)(m+2)}\square''^3\square'^{m+4}\mathrm{U}+\frac{4m(3m^2+8m-8)}{(m+4)(m+3)(m+2)}(12)^2\square''\square'^{m+4}\mathrm{U}.\\ \dots\quad&\quad\dots\end{aligned}\right\}\ (10)$$

Assuming this to be true for m_1-1, we should have

$$\square'^{m_1-1}\square''^3\square'^m\mathrm{U}=\frac{m(m-1)(m-2)}{(m+m_1-1)(m+m_1-2)(m+m_1-3)}\square''^3\square'^{m+m_1-1}\mathrm{U}$$
$$+\frac{(m_1-1)m\{3m^2+(3m_1-7)m-2m_1+2\}}{(m+m_1-1)(m+m_1-2)(m+m_1-3)}(12)^2\square''\square'^{m+m_1-1}\mathrm{U};$$

then operating with $\square'$ we obtain

$$\square'^{m_1}\square''^3\square'^m\mathrm{U}=\frac{m(m-1)(m-2)}{(m+m_1-1)(m+m_1-2)(m+m_1-3)}\left\{\frac{m+m_1-3}{m+m_1}\square''^3\square'^{m+m_1}\mathrm{U}\right.$$
$$\left.+\frac{3m+3m_1-1}{m+m_1}(12)^2\square''\square'^{m+m_1}\mathrm{U}\right\}$$
$$+\frac{(m_1-1)m\{(3m^2+3m_1-7)m-2m_1+2\}}{(m+m_1-1)(m+m_1-2)(m+m_1-3)}\;\frac{m+m_1-1}{m+m_1}(12)^2\square''\square'^{m+m_1}\mathrm{U}$$
$$=\frac{m(m-1)(m-2)}{(m+m_1)(m+m_1-1)(m+m_1-2)}\square''^3\square'^{m+m_1}\mathrm{U}$$
$$+\frac{m}{(m+m_1)(m+m_1-1)(m-m_1-2)}\;\frac{1}{(m+m_1-3)}(12)^2\square''\square'^{m+m_1}\mathrm{U}$$
$$\{(m-1)(m-2)(3m+3m_1-1)+(m_1-1)(3m^2+3mm_1-7m-2m_1+2)(m+m_1-1)\};$$

and the quantity within the brackets $\{\;\}$

$$=3m_1(m-1)(m-2)+(3m-1)(m-1)(m-2)$$
$$+\{m_1(3m-2)+(3m-1)(m-2)\}\{m_1(m_1+m-2)-(m-1)\},$$

of which the part independent of m_1 obviously vanishes, and the remainder

$$=m_1\{3(m-1)(m-2)+m_1(3m-2)(m_1+m-2)-(3m-2)(m-1)+(3m-1)(m-2)(m_1+m-2)\};$$

or if we put $m+m_1-3=\mu$, $m+m_1-2=\mu+1$, $m_1=\mu-(m-3)$, the above expression becomes

$$=m_1\{3(m-1)(m-2)+(\mu-m+3)(3m-2)(\mu+1)-(3m-2)(m-1)+(3m-1)(m-2)(\mu+1)\}$$
$$=m_1\{3(m-1)(m-2)-(m-3)(3m-2)-(3m-2)(m-1)+(3m-1)(m-2)$$
$$+\mu[-(3m-2)(m-4)+(3m-1)(m-2)]+\mu^2(3m-2)\}$$
$$=m_1(m+m_1-3)\{(m+m_1-3)(3m-2)+7m-6\}$$
$$=m_1(m+m_1-3)\{3m^2+(3m_1-4)m+2m_1\};$$

so that the expression for the value of $\square'^{m_1}\square''^3\square'^m\mathrm{U}$ finally becomes

$$\left.\begin{aligned}&\frac{m(m-1)(m-2)}{(m+m_1)(m+m_1-1)(m+m_1-2)}\square''^3\square^{m+m_1}\mathrm{U}\\&+\frac{m_1m\{3m^2+(3m_1-4)m+2m_1\}}{(m+m_1)(m+m_1-1)(m+m_1-2)}(12)^2\square''\square'^{m+m_1}\mathrm{U},\end{aligned}\right\}\quad\ldots\ldots\quad(11)$$

which proves the law.

It seems unnecessary to pursue these developments further.

It has now been shown that the expression for $\square^m$ may be reduced to the form

$$\square'^m+\beta\square''\square'^{m-1}+(\gamma\square''^2+\gamma'\square'')\square^{m-2}+\ldots(\Sigma_i\varkappa\square''^{m-i})\square'^2.$$

Hence, on replacing each term of this series by its value given by (5), we find that the expression for $\square^m$U may ultimately be reduced to the form

$$\{A(02\delta_1,\ -01\delta_2)^m+B(12)(02\delta_1,\ -01\delta_2)^{m-1}+\ldots K(12)^{m-2}(02\delta_1,\ -01\delta_2)^2\}U,\qquad(12)$$

or, still more symbolically,

$$(A,\ B,\ldots K)\big((02\delta_1,\ -01\delta_2),\ (12)\big)^{m-2}(02\delta_1,\ -01\delta_2)^2U,$$

excepting in the case of $m=2$, when there is one extra term, as was seen at the outset and as will be noticed again below. This being so, we may eliminate the quantities 01, 02 by the formulæ $01=\theta0^{n-1}1$, $02=\theta0^{n-1}2$, and then divide out θ^2 throughout. The expression is then reduced to the following form,

$$(A,\ B,\ldots K)\big(\theta(0^{n-1}2\delta_1,\ -0^{n-1}1\delta_2),\ (12)\big)^{m-2}(0^{n-1}2\delta_1,\ -0^{n-1}1\delta_2)^2U.$$

But if the surfaces touch at either of the points P_1, P_2 (say P_1), we shall have

$$12=\theta_11^{n-1}2,\ \theta=\frac{01}{0^{n-1}1}=\frac{1^{n-1}0}{0^{n-1}1}\theta_1,$$

or

$$\theta:1^{n-1}0=12:0^{n-1}1\,.\,1^{n-1}2\,;\qquad\ldots\ldots\ldots\qquad(13)$$

so that the expression in question may be finally cleared of all quantities relating to the quadric V (01, 02, 12), and reduced to the form

$$(A,\ B,\ldots K)(1^{n-1}0(0^{n-1}2\delta_1,\ -0^{n-1}\delta_2),\ 0^{n-1}1\,.\,1^{n-1}2)^{m-2}(0^{n-1}2\delta_1,\ -0^{n-1}1\delta_2)^2U,\quad(14)$$

in which it is to be remembered that δ_1, δ_2 operate only on the external subject U, and not upon any of its derivatives occurring in the operative factors themselves.

It is in the eliminations effected by means of the formula (13) that the main difference between the methods of this and of the former papers consists. The conditions for multiple contact here established are more numerous, and at the same time of lower degrees, and therefore more stringent, than those found before; but they appear to carry the subject to its limit.

If the surfaces touch also at the point P_2, we may in like manner use as the formulæ for elimination the following, viz.

$$21=\theta_22^{n-1}1,\ \theta=\frac{02}{0^{n-1}2}=\frac{2^{n-1}0}{0^{n-1}2}\theta_2.$$

The results obtained by these two methods cannot of course be independent. In fact the equivalence of the two forms may be readily shown as follows. The first result is a rational function, homogeneous in the two quantities $1^{n-1}0$, $0^{n-1}1\,.\,1^{n-1}2$, say

$$(1^{n-1}0,\ 0^{n-1}1\,.\,1^{n-1}2)^{m-2}=0\,;$$

and in like manner the second may be represented by the equation

$$(2^{n-1}0,\ 0^{n-1}2\,.\,2^{n-1}1)^{m-2}=0.$$

On multiplying the first by $(2^{n-1}0)^{m-2}$, and the second by $(1^{n-1}0)^{m-2}$, we obtain the two expressions

$$(1^{n-1}0\,.\,2^{n-1}0,\ 0^{n-1}1\,.\,1^{n-1}2\,.\,2^{n-1}0)^{m-2}=0,$$

$$(1^{n-1}0\,.\,2^{n-1}0,\ 0^{n-1}2\,.\,2^{n-1}1\,.\,1^{n-1}0)^{m-2}=0.$$

But since the surfaces touch in the three points P, P_1, P_2, it follows that

$$0^{n-1}1\,.\,1^{n-1}2\,.\,2^{n-1}0=0^{n-1}2\,.\,2^{n-1}1\,.\,1^{n-1}0.$$

Hence the two expressions are equivalent.

It is further to be noticed that the last term of the expression is of the form

$$\{(02)^2\delta_1^2 \pm (02)^2\delta_2^2\}\,\mathrm{U},$$

according as m is even or odd. Consequently from $m=5$ and upwards the last term of $\Box^m\mathrm{U}$ may always be eliminated by means of the expression for $\Box^{m-2}\mathrm{U}$; and the equation finally depressed by one degree in θ and (12).

The expression (14) when equated to zero will form one condition, which must be satisfied, either by the coordinates of the points, or by the coefficients of U, in order that it may be possible to draw a quadric having $(m+1)$-pointic contact with U at the point P, and contact of the same or of lower degrees at other points P_1, P_2, ...

Such is the general theory; but the subject will perhaps become more easily intelligible by the aid of the next section, in which the cases of $m=2$, 3, 4, 5 are examined in some detail.

It will probably have been remarked that we have here developed only expressions of the form $\Box^m\mathrm{U}$, and have taken no account of those of the form $\Box^p_{13}\Box^m_{12}\mathrm{U}$. But the latter, which would have been more complicated, are happily unnecessary; since the eliminations above indicated will be always possible for expressions of the form $\Box^m_{ij}\mathrm{U}=0$, provided only that one of the subscript numbers shall always correspond with that of one of the points at which contact takes place. And this may always be ensured, because in all the investigations of this paper, except those contained in § 4, contact, two-pointic at least, is supposed to subsist at more than one point. Thus, if there be contact at a second point P_1, we may use the operators $\Box_{12}$, $\Box_{13}$, ..; if there be contact also at P_2, we may use also the operators $\Box_{23}$, $\Box_{24}$, ..; and so on for any number of points.

§ 3. *Conditions for the cases* $n=3$, 4, 5, 6.

With a view to examining more in detail the cases of $m=1$, 2, 3, 4, 5, we may write down the developments indicated in the preceding section thus:—

$$\left.\begin{aligned}
&\Box\,\mathrm{U}=(02\delta_1-01\delta_2)\,\mathrm{U},\\
&\Box^2\mathrm{U}=(02\delta_1-01\delta_2)^2\mathrm{U}+\ (12)(02\,\delta_1+01\,\delta_2)\mathrm{U},\\
&\Box^3\mathrm{U}=(02\delta_1-01\delta_2)^3\mathrm{U}+3(12)(\overline{02}^2\delta_1^2-\overline{01}^2\delta_2^2)\mathrm{U},\\
&\Box^4\mathrm{U}=(02\delta_1-01\delta_2)^4\mathrm{U}+2(12)(3,\,1,\,-1,\,-3)(02\delta_1,-01\delta_2)^3\mathrm{U}+6(12)^2(\overline{02}^2\delta_1^2+\overline{01}^2\delta_2^2)\mathrm{U}
\end{aligned}\right\}(1)$$

Again, operating with $\Box=\Box'+\Box''$ upon the expression for $\Box^4U$ in terms of $\Box'$, $\Box''$, we obtain

$$\begin{aligned}\Box^5U&=\Box'^5U+\Box''\Box'^4U+2\Box'\Box''\Box'^3U+2\Box''^2\Box'^3U+\tfrac{3}{2}\Box'\Box''^2\Box'^2U+\tfrac{3}{2}\Box''^3\Box'^2U\\&=\Box'^5U+\tfrac{5}{2}\Box''\Box'^4U+\tfrac{5}{2}\Box''^2\Box'^3U+\tfrac{3}{2}\Box''^3\Box'^2U+\tfrac{3}{2}(12)^2\Box'^3U.\end{aligned}$$

But

$$\begin{aligned}\Box''\Box'^4U&=(12)(4,2,0,-2,-4)(02\delta_1,-01\delta_2)^4U\\&=2(12)(2,1,0,-1,-2)(02\delta_1,-01\delta_2)^4U,\\\Box''^2\Box'^3U&=(12)^2(9,1,1,9)(02\delta_1,-01\delta_2)^3U,\\\Box''^3\Box'^2U&=(12)^3 2^3(\overline{02}^2\delta_1^2-\overline{01}^2\delta_2^2)U,\end{aligned}$$

which, in virtue of the condition $\Box^3U=0$,

$$=-(12)^2\frac{2^3}{3}(02\delta_1-01\delta_2)^3U;$$

hence

$$\begin{aligned}\Box^5U=&(02\delta_1-01\delta_2)^5U\\&+10(12)(1,2,0,-2,-1\text{Ɔ}(02\delta_1,-01\delta_2)^4U\\&+\tfrac{15}{2}(12)^2(3,1,1,3\text{Ɔ}(02\delta_1,-01\delta_2)^3U\\&-4(12)^2(02\delta_1-01\delta_2)^3U\\&+\tfrac{3}{2}(12)^2(02\delta_1-01\delta_2)^3U.\end{aligned}$$

But collecting the terms of the third degree, we have for the coefficient of $(12)^2$

$$\begin{aligned}&\{(\tfrac{15}{2}\cdot3-4+\tfrac{3}{2})(02)^3\delta_1^3+(-\tfrac{15}{2}+12-\tfrac{3}{2}\cdot3)(02)^2(01)\delta_1^2\delta^2+\,..\}U\\&=20\big((02)^3\delta_1^3-(01)^3\delta_2^3\big)U;\end{aligned}$$

so that, finally,

$$\left.\begin{aligned}\Box^5U=&(02\delta_1-01\delta_2)^5U\\&+10(12)(1,2,0,-2,-3\text{Ɔ}(02\delta_1,-01\delta_2)^4U\\&+20(12)^2\big((02)^3\delta_1^3-(01)^3\delta_2^3\big)U,\end{aligned}\right\}\quad.\;\;.\;\;.\quad(2)$$

which, for greater symmetry, may be written thus:—

$$\begin{aligned}&(1,5,10,10,5,1\text{Ɔ}(02\delta_1-01\delta_2)^5\\&+10(12)(1,2,0,-2,-1\text{Ɔ}(02\delta_1-01\delta_2)^4\\&+20(12)^2(1,0,0,1\text{Ɔ}(02\delta_1-01\delta_2)^3.\end{aligned}$$

These expressions may be rendered somewhat more compact by writing as follows:—

$$\left.\begin{aligned}A&=0^{n-1}2\delta_1-0^{n-1}\delta_2,\\B&=0^{n-1}2\delta_1+0^{n-1}\delta_2;\end{aligned}\right\}\quad.\;\;.\;\;.\;\;.\;\;.\;\;.\;\;.\;\;.\;\;.\;\;.\;\;.\;\;.\;\;.\;\;.\quad(3)$$

and referring to the developments given above it will be found that

$$\left.\begin{aligned}\square^2\mathrm{U}&=\theta\mathrm{A}^2\mathrm{U}+(12)\mathrm{BU},\\ \square^3\mathrm{U}&=\theta\mathrm{A}^3\mathrm{U}+(12)\mathrm{BAU},\\ \square^4\mathrm{U}&=\theta^2\mathrm{A}^4\mathrm{U}+6\theta(12)\mathrm{BA}^2\mathrm{U}+3(12)^2(\mathrm{A}^2+\mathrm{B}^2)\mathrm{U},\\ \square^5\mathrm{U}&=\theta^2\mathrm{A}^5\mathrm{U}+10\theta(12)\mathrm{BA}^3\mathrm{U}+5(12)^2(\mathrm{A}^2+3\mathrm{B}^2)\mathrm{U}.\end{aligned}\right\}\quad\ldots\ldots\quad(4)$$

For three-branch contact at P the number of equations, in addition to those for ordinary contact, is three; and replacing the suffixes, we may take for these equations the following,

$$\square^2_{12}\mathrm{U}=0,\quad \square^2_{13}\mathrm{U}=0,\quad \square^2_{14}=0;$$

or, transforming as above,

$$\left.\begin{aligned}\theta\mathrm{A}^2_{12}\mathrm{U}+(12)\mathrm{B}_{12}\mathrm{U}&=0,\\ \theta\mathrm{A}^2_{13}\mathrm{U}+(13)\mathrm{B}_{13}\mathrm{U}&=0,\\ \theta\mathrm{A}^2_{14}\mathrm{U}+(14)\mathrm{B}_{14}\mathrm{U}&=0.\end{aligned}\right\}\quad\ldots\ldots\quad(5)$$

Now since the surfaces are supposed to touch at the point P, we shall have

$$\theta=01:0^{n-1}1=02:0^{n-1}2=\ldots$$

If they touch also at the point P_1, we shall have also

$$10:1^{n-1}0=12:1^{n-1}2=\ldots,$$

and so on for other points; so that when the surfaces touch at the point P_1, we shall have

$$\left.\begin{aligned}\theta:1^{n-1}0&=12:0^{n-1}1\,.\,1^{n-1}2,\\ \theta:1^{n-1}0&=13:0^{n-1}1\,.\,1^{n-1}3,\\ \ldots\quad&\quad\ldots;\end{aligned}\right\}\quad\ldots\ldots\quad(6)$$

when they touch also at the point P_2 we shall have

$$\left.\begin{aligned}\theta:2^{n-1}0&=21:0^{n-1}2\,.\,2^{n-1}1,\\ \theta:2^{n-1}0&=23:0^{n-1}2\,.\,2^{n-1}3,\\ \ldots\quad&\quad\ldots,\end{aligned}\right\}\quad\ldots\ldots\quad(7)$$

and so on for other points of contact.

These equations show that if the surfaces touch only at the point P, there is no means of eliminating any of the ratios $\theta:12$, $\theta:13$, .. If, however, they touch also at a second point P_1, we can eliminate all the ratios of the form $\theta:12$; *i. e.* those containing the symbol 1 in the denominator. If the surfaces touch at a third point P_2, we can eliminate all the ratios of the form $\theta:12$, $\theta:13$, $\theta:23$; viz. all those containing either of the symbols 1 or 2 in the denominator. It may be remarked that when the surfaces

touch at the three points P, P_1, P_2, the ratio $\theta : 12$ may be eliminated by either of the two formulæ

$$\left.\begin{array}{l}\theta : 1^{n-1}0=12 : 0^{n-1}1 \,.\, 1^{n-1}2,\\ \theta : 2^{n-1}0=21 : 0^{n-1}2 \,.\, 2^{n-1}1.\end{array}\right\} \quad \ldots\ldots\ldots \quad (8)$$

That the conditions so obtained are equivalent to one another, and not independent, is both obvious *à priori* and is capable of being shown by multiplying the denominators of the first of the equations by $2^{n-1}0$ and those of the second by $1^{n-1}0$. The equations then take the form

$$\begin{aligned}\theta : 1^{n-1}0 \,.\, 2^{n-1}0 &= 12 : 0^{n-1}1 \,.\, 1^{n-1}2 \,.\, 2^{n-1}0\\ &= 21 : 0^{n-1}2 \,.\, 2^{n-1}1 \,.\, 1^{n-1}0.\end{aligned}$$

But since the surfaces touch at the three points P, P_1, P_2, the last two denominators are in virtue of (18) of § 1 equal. Hence each one of the two equations (8) involves the other as a consequence.

Returning to the equations (5) and eliminating the ratios $\theta : 12, \ldots$, we obtain the three conditions:

$$\left.\begin{array}{l}1^{n-1}0A^2_{12}U+0^{n-1}1 \,.\, 1^{n-1}2B_{12}U=0,\\ 1^{n-1}0A^2_{13}U+0^{n-1}1 \,.\, 1^{n-1}3B_{13}U=0,\\ 1^{n-1}0A^2_{14}U+0^{n-1}1 \,.\, 1^{n-1}4B_{14}U=0.\end{array}\right\} \quad \ldots\ldots\ldots \quad (9)$$

If the surfaces touch at a third point P_2, we may replace the equation $\square^2_{14}U=0$ by $\square^2_{23}U=0$. If they touch also at the point P_3, the results of elimination may be put into two forms, viz.

$$\left.\begin{array}{ll}2^{n-1}0A^2_{23}U+0^{n-1}2 \,.\, 2^{n-1}3B_{23}U=0, & 3^{n-1}0A^2_{23}U+0^{n-1}3 \,.\, 3^{n-1}2B_{23}U=0,\\ 3^{n-1}0A^2_{31}U+0^{n-1}3 \,.\, 3^{n-1}1B_{31}U=0, & 1^{n-1}0A^2_{31}U+0^{n-1}1 \,.\, 1^{n-1}3B_{31}U=0,\\ 1^{n-1}0A^2_{12}U+0^{n-1}1 \,.\, 1^{n-1}2B_{12}U=0, & 2^{n-1}0A^2_{12}U+0^{n-1}2 \,.\, 2^{n-1}1B_{12}U=0.\end{array}\right\} \;.\; (10)$$

If the surfaces touch only at the three points P, P_1, P_2, we shall have only the first form of the first equation, the second of the second, and either form of the third.

If the surfaces have three-branch contact at a second point P_1, we may derive the conditions to be fulfilled, by a similar process, from the system

$$\square^2_{02}U_1=0, \quad \square^2_{03}U_1=0, \quad \square^2_{04}U_1=0,$$

and so on for any number of points.

For four-branch contact the number of equations is four, which may be written thus:

$$\left.\begin{array}{l}\theta A^3_{12}U+3(12)B_{12}A_{12}U=0,\\ \theta A^3_{13}U+3(13)B_{13}A_{13}U=0,\\ \theta A^3_{14}U+3(14)B_{14}A_{14}U=0,\\ \theta A^3_{15}U+3(15)B_{15}A_{15}U=0;\end{array}\right\} \quad \ldots\ldots\ldots \quad (11)$$

and if the surfaces touch not only at the two points P, P_1, but also at a third point P_2, the last equation may be replaced by

$$\theta A_{23}^3 U + 3(23) B_{23} A_{23} U = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

The results of the elimination of the ratios $\theta : 12, ..$ will be of the form

$$1^{n-1}0 A_{12}^3 U + 3 \,.\, 0^{n-1}1 \,.\, 1^{n-1}2 B_{12} A_{12} U = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

Similarly the five conditions for five-branch contact will be of the form

$$(1^{n-1}0)^2 A_{12}^4 U + 6 \,.\, 1^{n-1}0 \,.\, 0^{n-1}1 \,.\, 1^{n-1}2 B_{12} A_{12}^2 U + 3(0^{n-1}1 \,.\, 1^{n-1}2)^2 (A^2 + B^2) U = 0, \quad (14)$$

and the six conditions for six-branch contact will be of the form

$$(1^{n-1}0)^2 A^5 U + 10 \,.\, 1^{n-1}0 \,.\, 0^{n-1}1 \,.\, 1^{n-1}2 B A^3 U + 5(0^{n-1}1 \,.\, 1^{n-1}2)^2 B (A^2 + 3B^2) U = 0. \quad (15)$$

Recapitulating the results now obtained, we may form the subjoined Table for the possibility of contact of a quadric V with a given surface U, viz. simple contact at a point P_1, &c.

For contact at the point P.	Number of conditions.	Degrees of conditions in		
		Coefficients of U.	Coordinates of P.	Coordinates of P_1; of P_2, ..
3 branch . .	3	4	$3n-3$	$n+1$; 2
4 do. . .	4	5	$4n-5$	$n+2$; 3
5 do. . .	5	7	$5n-6$	$2n+2$; 4
6 do. . .	6	8	$6n-8$	$2n+4$; 5

To these of course must be added the conditions that P and P_1 lie on the surface U, viz. $U=0$, $U_1=0$, and that U and V touch at P.

If the surfaces have simple contact at a third point P_2, we must add the condition $U_2=0$, and the condition (18) of § 1; and similarly for every additional point at which they have simple contact.

If the surfaces have 3, 4, .. pointic contact at a second, third, .. point we must double, triple, .. the number of conditions for each such degree of contact; the degrees of the conditions remaining unchanged.

Suppose, then, that the quadric V touches the surface U in two points, P, P_1; then in order that the contact may become three-pointic at either of these points (say P), three conditions are necessary. And if $a, b, ..$ be the coefficients of U, these conditions may be expressed thus:

$$(a, b, ..)^4 (x, y, ..)^{3n-3} (x_1, y_1, ..)^{n+1} (x_2, y_2, ..)^2 = 0,$$
$$(a, b, ..)^4 (x, y, ..)^{3n-3} (x_1, y_1, ..)^{n+1} (x_3, y_3, ..)^2 = 0,$$
$$(a, b, ..)^4 (x, y, ..)^{3n-3} (x_1, y_1, ..)^{n+1} (x_4, y_4, ..)^2 = 0;$$

and if by means of these equations, together with the equation $U_1=0$, we eliminate the

coordinates of P_1, we shall have an equation in $x, y, ..$ which will determine a curve upon U at every point of which (P_1 being taken arbitrarily) it is possible to draw a quadric having three-pointed contact. The degree of the curve would appear to be $3(n-1)\,3n(n+1)^2=9n(n+1)(n^2-1)$.

But regarding the conditions in question as relations between the coefficients of U, we have three equations of the degree 4 and one of the degree 1. Hence we may conclude that through any two points in space we may describe 64 surfaces, whose equations contain 5 independent constants (*e.g.* quartic scrolls having twisted cubics for their nodal lines), such that a quadric may be drawn touching them in two points each, and having three-pointic contact at one of the two points.

This theorem admits of obvious generalizations; but, having reference to the provisional nature of the numerical results, it seems hardly worth while to make a statement of the theorems which will probably require qualification hereafter.

There is yet another way in which the equations of condition may be regarded. The conditions for three-pointic contact involve the coordinates of four points, P, P_1, P_2, P_3; *i. e.* twelve disposable quantities. These may be determined so as to satisfy twelve equations. Hence it appears that on a given surface we may take four points (4 equations), such that a quadric may be drawn touching the surface at the four points (2 equations), and having three-pointic contact at two of them (6 equations); *i. e.* $4+2+6=12$ equations in all.

Again, the conditions for four-pointic contact involve the coordinates of five points, $P, P_1, .. P_4$; *i. e.* fifteen disposable quantities. Hence we may conclude that on a given surface we may take five points (5 equations), such that a quadric may be drawn, and having three-pointic contact at three of them ($1+3\times3=10$ equations); *i. e.* $5+10=15$ equations in all.

Or, again, on a given surface we may take five points (5 equations), such that a quadric may be drawn touching the surface in the five points (3 equations), and having four-pointic contact at one of them (3+4 equations); *i. e.* $5+3+3+4=15$ equations in all.

To these theorems others might doubtless be added.

§ 4. *On Points of Four-, Five-, Six-pointic Contact with a Quadric Surface.*

If, setting aside for the moment the question of multiple contact, we fix our attention upon a single point P, the formulæ established in the preceding section suggest certain conditions necessary for the existence on a surface of points of four-, five-, six-branch contact with a quadric. These may be described as the conditions for a quartitactic, quintactic, sextactic point on a surface. Now, referring to the equations (4), (6), and (7) of § 3, and to the process there used for the elimination of the quantities relating to the quadric V (*i. e.* θ, 12, 13, ..), it appears that, when the contact subsists for a single point only, we have available for the purposes of elimination the relations

$$\theta=01:0^{n-1}1=02:0^{n-1}2=\,..,$$

but not the relations

$$\theta_1=10:1^{n-1}0=12:1^{n-1}2=\ldots$$

Consequently we cannot eliminate, in the manner there effected, the quantities relating to the quadric V from any of the six equations derived from the forms $U=0$, $\square U=0$, $\square^2 U=0$. We may, in fact, by means of these equations, determine six out of the nine constants of V, but that is all.

When, however, P is a quartitactic point, we have the four additional equations, say

$$\square^3_{23}U=0,\ \square^3_{31}U=0,\ \square^3_{12}U=0,\ \square^3_{14}U=0,$$

ten in all. From one, or two, or three of these we may, by means of the equations

$$\square^2_{23}U=0,\ \square^2_{31}U=0,\ \square^2_{12}U=0,$$

eliminate one, or two, or three of the quantities $\theta:23$, $\theta:31$, $\theta:12$, and obtain conditions of the form

$$3A^2U\,.\,BAU-A^3U\,.\,BU=0,$$

to which the proper suffixes 23, 31, 12 are to be appended.

By this means we may either determine all nine of the coefficients of the quadric, and have one condition of the above form; or we may determine eight of the coefficients, and have two such conditions; or, lastly, we may determine seven of the coefficients, and have three such conditions.

From this it would appear that, if we regard the conditions as equations in the coordinates of the point P, there will be a curve on U every point of which will be quartactic, with a single quadric at each point. Again, there will be a definite number of points which will be also quartactic, having a singly infinite number of quadrics having four-branch contacts at the points.

When P is a quintactic point, we have the five additional equations,

$$\square^4_{23}U=0,\ \square^4_{31}U=0,\ \square^4_{12}U=0,\ \square^4_{14}U=0,\ \square^4_{24}U=0.$$

From three, or four, of these we may eliminate three, or four, of the quantities $\theta:23$, $\theta:31$, $\theta:12$, $\theta:14$, by means of the equations

$$\square^2_{23}U=0,\ \square^2_{31}U=0,\ \square^2_{12}U=0,\ \square^3_{14}U=0,$$

and obtain results of the forms

$$(BU)^2A^4U-6BU\,.\,A^2U\,.\,BAU+3(A^2U)^2(A^2+B^2)U=0,$$
$$3(BAU)^2A^4U-6BAU\,.\,A^3U\,.\,BA^2U+(A^3U)^2(A^2+B^2)U=0.$$

If we have previously determined the nine coefficients, or if we have determined eight only, and use one of the new equations for determining the ninth, or if we have determined only seven, and use two of the new equations for determining the two remaining coefficients, we shall have six conditions; or, lastly, if having determined seven only, we now determine one more, we shall have seven conditions.

When P is a sextactic point, we have the six additional equations,

$$\square^{5}_{23}U=0,\ \square^{5}_{31}U=0,\ \square^{5}_{12}U=0,\ \square^{5}_{14}U=0,\ \square^{5}_{24}U=0,\ \square^{5}_{34}U=0.$$

From four, or five, of these we may eliminate four, or five, of the quantities $\theta:23$, $\theta:31$, $\theta:12$, $\theta:14$, $\theta:24$, $\theta:34$, by means of the equations

$$\square^{2}_{23}U=0,\ \square^{2}_{31}U=0,\ \square^{2}_{12}U=0,\ \square^{3}_{14}U=0,\ \square^{4}_{24}U=0,$$

and obtain results of the forms

$$(BU)^2A^5U-10BU\,.\,A^2U\,.\,BA^3U+5(A^2U)^2(A^2+3B^2)U=0,$$

$$(BAU)^2A^5U-10BAU\,.\,A^5U\,.\,BA^3U+5(A^3U)^2(A^2+3B^2)U=0,$$

$$\begin{vmatrix} A^4U, & 6BA^2U, & 3(A^2+B^2)U\ \ , & . \\ . & A^4U, & 6BA^2U\ \ , & 3(A^2+B^2)U \\ A^5U, & 10BA^3U, & 5B(A^2+3B^2)U, & . \\ . & A^5U, & 10BA^3U, & 5B(A^2+3B^2)U \end{vmatrix}=0.$$

If either we have determined all the coefficients of the quadric, or if having determined only eight we use one of the last equations for determining the results, we shall have twelve conditions.

Recapitulating, the following is a Table of the number of conditions so found, and of their degrees in the several quantities contained in them:—

For a	No. of constants determined.	No. of conditions.	Degree of condition in the		
			Coordinates of P.	Coordinates of P_1, . .	Coefficients of U.
Quartitactic point. .	7	3	$2(3n-4)$	4	6
	8	2			
	9	1			
Quintactic point. .	7	$3+5=8$	$3(3n-4)$	6	9
	8	$2+5=3+4=7$	$11n-16$	5	11
	9	$1+5=2+4=3+3=6$			
Sextactic point. .	7	$8+6=14$	$2(5n-7)$	7	10
	8	$7+6=8+5=13$	$6(2n-3)$	9	12
	9	$6+6=7+5=8+4=12$	$2(9n-14)$	11	18

§ 5. *On Multiple Contact of Cubics with other Surfaces.*

Hitherto we have considered in detail only the case of quadrics, that is to say, the conditions which must be fulfilled by the coefficients of a surface U in order that it may be possible to draw a quadric having contact of given orders with U at more than one point. There is of course a corresponding problem with cubics, and indeed with surfaces of any degree; but the question soon becomes so complicated that it may be doubted

whether the results would be worth having even if means were found for pushing the investigation much further. There is, however, one case, namely the osculation of cubics, in which it is possible within a moderate compass to arrive at a solution.

Thus, when V is a cubic,

$$\left.\begin{aligned}&\square_{12}U=0^22\,.\,0^{n-1}1-0^21\,.\,0^{n-1}2,\\&\square^2_{12}U=(0^22)^2 0^{n-2}1^2-2\,.\,0^22\,.\,0^21\,.\,0^{n-2}12+(0^21)^2 0^{n-2}2^2\\&\qquad-0^21\,.\,02^2\,.\,0^{n-1}1+(0^22\,.\,0^{n-1}1+0^21\,.\,0^{n-1}2)012-0^22\,.\,01^2\,.\,0^{n-1}2.\end{aligned}\right\}\quad .\ .\ (1)$$

But in the same way, as in the case of quadrics (22), we may put

$$\left.\begin{aligned}&0^21=\theta\,0^{n-1}1,\quad 0^22=\theta\,0^{n-1}2,\\&1^20=\theta_1 1^{n-1}0,\quad 2^20=\theta_2 2^{n-1}0\,;\end{aligned}\right\}\ .\ .\ .\ .\ .\ .\ .\ .\ .\ .\ (2)$$

and the conditions for osculation at the point P, viz. $\square^2_{23}U=0$, $\square^2_{31}U=0$, $\square^2_{12}U=0$, will then become

$$\left.\begin{aligned}&2\,.\,0^{n-1}2\,.\,0^{n-1}3\,.\,023=-\theta[0',2,3]\quad .\quad+\theta_2 2^{n-1}0(0^{n-1}3)^2+\theta_3 3^{n-1}0(0^{n-1}2)^2,\\&2\,.\,0^{n-1}3\,.\,0^{n-1}1\,.\,031=-\theta[0',3,1]+\theta_1 1^{n-1}0(0^{n-1}3)^2+\quad .\quad+\theta_3 3^{n-1}0(0^{n-1}1)^2,\\&2\,.\,0^{n-1}1\,.\,0^{n-1}2\,.\,012=-\theta[0',1,2]+\theta_1 1^{n-1}0(0^{n-1}2)^2+\theta_2 2^{n-1}0(0^{n-1}1)^2+\quad .\end{aligned}\right\}(3)$$

Similarly for osculation at a second point P, we should have

$$\left.\begin{aligned}&2\,.\,1^{n-1}3\,.\,1^{n-1}0\,.\,130=\theta 0^{n-1}1(1^{n-1}3)^2-\theta_1[1',3,0]\quad .\quad+\theta_3 3^{n-1}1(1^{n-1}0)^2,\\&2\,.\,1^{n-1}0\,.\,1^{n-1}2\,.\,102=\theta 0^{n-1}1(1^{n-1}2)^2-\theta_1[1',0,2]+\theta_2 2^{n-1}1(1^{n-1}0)^2\quad .\\&2\,.\,1^{n-1}2\,.\,1^{n-1}3\,.\,123=\quad .\quad-\theta_1[1',2,3]+\theta_2 2_{n-1}1(1^{n-1}3)^2+\theta_3 3^{n-1}1(1^{n-1}2)^2\,;\end{aligned}\right\}\ (4)$$

and for osculation at a third point P_1, we should have

$$\left.\begin{aligned}&2\,.\,2^{n-1}0\,.\,2^{n-1}1\,.\,201=\theta 0^{n-1}2(2^{n-1}1)^2+\theta_1 1^{n-1}2(2^{n-1}0)^2-\theta_2[2',0,1]\quad .\\&2\,.\,2^{n-1}1\,.\,2^{n-1}3\,.\,213=\quad .\quad+\theta_1 1^{n-1}2(2^{n-1}3)^2-\theta_2[2',1,3]+\theta_3 3^{n-1}2(2^{n-1}1)^2,\\&2\,.\,2^{n-1}3\,.\,2^{n-1}0\,.\,230=\theta 0^{n-1}2(2^{n-1}3)^2\quad .\quad-\theta_2[2',3,0]+\theta_3 3^{n-1}2(2^{n-1}0)^2.\end{aligned}\right\}\ (5)$$

These together form a system of nine equations involving the seven quantities $123:230:301:012:\theta:\theta_1:\theta_2:\theta_3$. We can therefore eliminate these quantities in two different ways; in other words, there will be two relations between the coefficients of the equation of the surface U. In order to determine the degrees of these resultants in the coefficients in question, consider first the three equations in 012, θ, θ_1, θ_2. The degrees of the coefficients of these quantities (in the coefficients of U) are 2, 3, 3, 3 respectively; hence the quantities θ, θ_1, θ_2 will be proportionate to expressions which are of the degrees $3+3+2=8$. Next taking the two equations involving 123, and eliminating that quantity between them, we shall obtain an equation in θ_1, θ_2, θ_3, the coefficients of which are of the degree 5 in those of U. Similarly from the others we should obtain equations in θ, θ_1, θ_2; θ, θ_1, θ_3; whose coefficients are of the same degree.

From these equations we can, by means of the values of the ratios $\theta:\theta_1:\theta_2$, eliminate $\theta, \theta_1, \theta_2, \theta_3$ in two different ways; and the resulting equations will be of the degree $5+5+8=18$.

Proceeding to perform the actual eliminations, and taking the three equations involving 012, we find that the ratios

$$-2\,.\,012 : -\theta : \theta_1 : -\theta_2$$

are proportional to the determinants of the matrix,

$$\begin{array}{llll} 0^{n-1}1\,.\,0^{n-1}2, & -[0', 1, 2], & 1^{n-1}0(0^{n-1}2)^2, & 2^{n-1}0(0^{n-1}1)^2, \\ 1^{n-1}2\,.\,1^{n-1}0, & 0^{n-1}1(1^{n-1}2)^2, & -[0, 1', 2], & 2^{n-1}1(1^{n-1}0)^2, \\ 2^{n-1}0\,.\,2^{n-1}1, & 0^{n-1}2(2^{n-1}1)^2, & 1^{n-1}2(2^{n-1}0)^2, & -[0, 1, 2']\,; \end{array}$$

viz. the quantity to which θ is proportional will be

$$\begin{aligned} &0^{n-1}1[0, 1', 2]0^{n-1}2[0, 1, 2'] \\ &+0^{n-1}1[0, 1', 2](2^{n-1}0)^2 2^{n-1}1\,.\,0^{n-1}1 \\ &+0^{n-1}2[0, 1, 2'](1^{n-1}0)^2 0^{n-1}2\,.\,1^{n-1}2 \\ &-0^{n-1}1\,.\,0^{n-1}2\,.\,(1^{n-1}0)^2(2^{n-1}0)^2 1^{n-1}2\,.\,2^{n-1}1 \\ &+1^{n-1}0(0^{n-1}1)^2(1^{n-1}2)^2(2^{n-1}0)^2 2^{n-1}0 \\ &+2^{n-1}0(1^{n-1}0)^2(0^{n-1}2)^2(2^{n-1}1)^2 1^{n-1}0\,; \end{aligned}$$

and writing, for brevity,

$$\left.\begin{aligned} 2^{n-1}3\,.\,3^{n-1}1\,.\,1^{n-1}2-3^{n-1}2\,.\,1^{n-1}3\,.\,2^{n-1}1&=P_{123}, \\ 2^{n-1}3\,.\,3^{n-1}0\,.\,0^{n-1}2-3^{n-1}2\,.\,0^{n-1}3\,.\,2^{n-1}0&=P_{023}, \\ 1^{n-1}3\,.\,3^{n-1}0\,.\,0^{n-1}1-3^{n-1}1\,.\,0^{n-1}3\,.\,1^{n-1}0&=P_{013}, \\ 1^{n-1}2\,.\,2^{n-1}0\,.\,0^{n-1}1-2^{n-1}1\,.\,0^{n-1}2\,.\,1^{n-1}0&=P_{012}, \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

and forming expressions symmetrical to that written above, we shall find that they may all be put into the following shape:—

$$\left.\begin{aligned} &\theta : \theta_1 : \theta_2 \\ &=\big([0, 1', 2]0^{n-1}1+(1^{n-1}0)^2 0^{n-1}2\,.\,1^{n-1}2\big)\big([0, 1, 2']0^{n-1}2+(2^{n-1}0)^2 0^{n-1}1\,.\,2^{n-1}1\big) \\ &\qquad\qquad +1^{n-1}0\,.\,2^{n-1}0P^2_{012} \\ &:\big([0, 1, 2']1^{n-1}2+(2^{n-1}1)^2 1^{n-1}0\,.\,2^{n-1}0\big)\big([0', 1, 2]1^{n-1}0+(0^{n-1}1)^2 0^{n-1}2\,.\,1^{n-1}2\big) \\ &\qquad\qquad +2^{n-1}1\,.\,0^{n-1}1P^2_{012} \\ &:\big([0', 1, 2]2^{n-1}0+(0^{n-1}2)^2 2^{n-1}1\,.\,0^{n-1}1\big)\big([0, 1', 2]2^{n-1}1+(1^{n-1}2)^2 1^{n-1}0\,.\,2^{n-1}0\big) \\ &\qquad\qquad +0^{n-1}2\,.\,1^{n-1}2P^2_{012} \end{aligned}\right\} \quad (7)$$

Again, from the two equations involving 123, we obtain by elimination

$$\left.\begin{aligned} &-\big([1', 2, 3]2^{n-1}+(1^{n-1}2)^2 1^{n-1}3\,.\,2^{n-1}3\big)2^{n-1}3\theta_1 \\ &+\big([1, 2', 3]1^{n-1}2+(2^{n-1}1)^2 1^{n-1}3\,.\,2^{n-1}3\big)1^{n-1}3\theta_2+1^{n-1}2\,.\,2^{n-1}1P_{123}\theta_3=0. \end{aligned}\right\} \quad . \quad (8)$$

Similarly from the two equations in 023 we obtain

$$\left.\begin{array}{l} -\left([0', 2, 3]2^{n-1}0+(0^{n-1}2)^2 2^{n-1}3 \,.\, 0^{n-1}3\right)2^{n-1}3\theta \\ +\left([0, 2', 3]0^{n-1}2+(2^{n-1}0)^2 2^{n-1}3 \,.\, 0^{n-1}3\right)0^{n-1}3\theta_2+0^{n-1}2 \,.\, 2^{n-1}0\mathrm{P}_{023}\theta_3=0; \end{array}\right\} \quad . \quad . \quad (9)$$

and from the two equations in 013

$$\left.\begin{array}{l} -\left([0', 1, 3]\,1^{n-1}0+(0^{n-1}1)^2 0^{n-1}3 \,.\, 1^{n-1}3\right)1^{n-1}3\theta \\ +\left([0, 1', 3]\,0^{n-1}1+(1^{n-1}0)^2 0^{n-1}3 \,.\, 1^{n-1}3\right)0^{n-1}3\theta_1+0^{n-1}1 \,.\, 1^{n-1}0\,\mathrm{P}_{0133}\theta=0. \end{array}\right\} \quad . \quad . \quad (10)$$

If we now put

$$\left.\begin{array}{l}
[1, 2', 3]1^{n-1}2+(2^{n-1}1)^2 2^{n-1}3 \,.\, 1^{n-1}3=\mathrm{B}, \\
[1, 2, 3']1^{n-1}3+(3^{n-1}1)^2 3^{n-1}2 \,.\, 1^{n-1}2=\mathrm{B}', \\
[1, 2, 3']2^{n-1}3+(3^{n-1}2)^2 3^{n-1}1 \,.\, 2^{n-1}1=\mathrm{C}, \\
[1', 2, 3]2^{n-1}1+(1^{n-1}2)^2 1^{n-1}3 \,.\, 2^{n-1}3=\mathrm{C}', \\
[1', 2, 3]3^{n-1}1+(1^{n-1}3)^2 1^{n-1}2 \,.\, 3^{n-1}2=\mathrm{D}, \\
[1, 2', 3]3^{n-1}2+(2^{n-1}3)^2 2^{n-1}1 \,.\, 3^{n-1}1=\mathrm{D}', \\
[2', 3, 0]0^{n-1}2+(2^{n-1}0)^2 2^{n-1}3 \,.\, 0^{n-1}3=\mathrm{F}, \\
[2, 3', 0]0^{n-1}3+(3^{n-1}0)^2 3^{n-1}2 \,.\, 0^{n-1}2=\mathrm{F}', \\
[2, 3', 0]2^{n-1}3+(3^{n-1}2)^2 3^{n-1}0 \,.\, 2^{n-1}0=\mathrm{H}, \\
[2, 3, 0']2^{n-1}0+(0^{n-1}2)^2 0^{n-1}3 \,.\, 2^{n-1}3=\mathrm{H}', \\
[2, 3, 0']3^{n-1}0+(0^{n-1}3)^2 0^{n-1}2 \,.\, 3^{n-1}2=\mathrm{K}, \\
[2', 3, 0]3^{n-1}2+(2^{n-1}3)^2 2^{n-1}0 \,.\, 3^{n-1}0=\mathrm{K}', \\
[3, 0, 1']0^{n-1}1+(1^{n-1}0)^2 1^{n-1}3 \,.\, 0^{n-1}3=\mathrm{L}, \\
[3', 0, 1]0^{n-1}3+(3^{n-1}0)^2 3^{n-1}1 \,.\, 0^{n-1}1=\mathrm{L}', \\
[3', 0, 1]1^{n-1}3+(3^{n-1}1)^2 3^{n-1}0 \,.\, 1^{n-1}0=\mathrm{M}, \\
[3, 0', 1]1^{n-1}0+(0^{n-1}1)^2 0^{n-1}3 \,.\, 1^{n-1}3=\mathrm{M}', \\
[3, 0', 1]3^{n-1}0+(0^{n-1}3)^2 0^{n-1}1 \,.\, 3^{n-1}1=\mathrm{O}, \\
[3, 0, 1']3^{n-1}1+(1^{n-1}3)^2 1^{n-1}0 \,.\, 3^{n-1}0=\mathrm{O}', \\
[0, 1', 2]0^{n-1}1+(1^{n-1}0)^2 1^{n-1}2 \,.\, 0^{n-1}2=\mathrm{P}, \\
[0, 1, 2']0^{n-1}2+(2^{n-1}0)^2 2^{n-1}1 \,.\, 0^{n-1}1=\mathrm{P}', \\
[0, 1, 2']1^{n-1}2+(2^{n-1}1)^2 2^{n-1}0 \,.\, 1^{n-1}0=\mathrm{Q}, \\
[0', 1, 2]1^{n-1}0+(0^{n-1}1)^2 0^{n-1}2 \,.\, 1^{n-1}2=\mathrm{Q}', \\
[0', 1, 2]2^{n-1}0+(0^{n-1}2)^2 0^{n-1}1 \,.\, 2^{n-1}1=\mathrm{R}, \\
[0, 1', 2]2^{n-1}1+(1^{n-1}2)^2 1^{n-1}0 \,.\, 2^{n-1}0=\mathrm{R}'.
\end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

Substituting these values in the equations (7) .. (10) for determining θ, θ_1, θ_2, θ_3, we shall obtain the following expressions:—

For osculation at the points P_1, P_2, P_3,

$$\left.\begin{aligned}\theta_1:\theta_2:\theta_3= & BB'+2^{n-1}1\ .\ 3^{n-1}1P^2_{123}\\ & :CC'+3^{n-1}2\ .\ 1^{n-1}2P^2_{123}\\ & :DD'+1^{n-1}3\ .\ 2^{n-1}3P^2_{123},\\ K'3^{n-1}0\theta_2-H2^{n-1}0\theta_3-2^{n-1}3\ .\ 3^{n-1}2P_{023}\theta=0,\\ M'1^{n-1}0\theta_3-O'3^{n-1}0\theta_1-3^{n-1}1\ .\ 1^{n-1}3P_{013}\theta=0,\\ R'2^{n-1}0\theta_1-Q\,1^{n-1}0\theta_2-1^{n-1}2\ .\ 2^{n-1}1P_{012}\theta=0.\end{aligned}\right\}\quad\ldots\ldots\quad(12)$$

Similarly for osculation at the points P, P_2, P_3,

$$\left.\begin{aligned}\theta:\theta_2:\theta_3= & FF'+2^{n-1}0\ .\ 3^{n-1}0P^2_{023}\\ & :HH'+3^{n-1}2\ .\ 0^{n-1}2P^2_{023}\\ & :KK'+0^{n-1}3\ .\ 2^{n-1}3P^2_{023},\\ D'3^{n-1}1\theta_2-C\,2^{n-1}1\theta_3-2^{n-1}3\ .\ 3^{n-1}2P_{123}\theta_1=0,\\ L'0^{n-1}1\theta_3-O\,3^{n-1}1\theta-3^{n-1}0\ .\ 0^{n-1}3P_{013}\theta_1=0,\\ R\,2^{n-1}1\theta-P'\,0^{n-1}1\theta_2-0^{n-1}2\ .\ 2^{n-1}0P_{012}\theta_1=0.\end{aligned}\right\}\quad\ldots\ldots\quad(13)$$

Similarly for osculation at the points P, P_1, P_3,

$$\left.\begin{aligned}\theta:\theta_1:\theta_3= & LL'+1^{n-1}0\ .\ 3^{n-1}0P^2_{013}\\ & :MM'+3^{n-1}1\ .\ 0^{n-1}1P^2_{013}\\ & :OO'+0^{n-1}3\ .\ 1^{n-1}3P^2_{013},\\ D3^{n-1}2\theta_1-B'\,1^{n-1}2\theta_3-1^{n-1}3\ .\ 3^{n-1}1P_{123}\theta_2=0,\\ F'0^{n-1}2\theta_3-K\,3^{n-1}2\theta-3^{n-1}0\ .\ 0^{n-1}3P_{023}\theta_2=0,\\ Q'1^{n-1}2\theta-P\,0^{n-1}2\theta_1-0^{n-1}1\ .\ 1^{n-1}0P_{012}\theta_2=0.\end{aligned}\right\}\quad\ldots\ldots\quad(14)$$

And for osculation at the points P, P_1, P_2,

$$\left.\begin{aligned}\theta:\theta_1:\theta_2= & PP'+1^{n-1}0\ .\ 2^{n-1}0P^2_{012}\\ & :QQ'+2^{n-1}1\ .\ 0^{n-1}1P^2_{012}\\ & :RR'+0^{n-1}2\ .\ 1^{n-1}2P^2_{012},\\ C'2^{n-1}3\theta_1-B\,1^{n-1}3\theta_2-1^{n-1}2\ .\ 2^{n-1}1P_{123}\theta_3=0,\\ F\,0^{n-1}3\theta_2-H'2^{n-1}3\theta-2^{n-1}0\ .\ 0^{n-1}2P_{023}\theta_3=0,\\ M'1^{n-1}3\theta-L\,0^{n-1}3\theta_1-0^{n-1}1\ .\ 1^{n-1}0P_{013}\theta_3=0.\end{aligned}\right\}\quad\ldots\ldots\quad(15)$$

Whence, finally, the two conditions for osculation at the points P_1, P_2, P_3, together with simple contact at the four points P, P_1, P_2, P_3, will be

$$\left.\begin{aligned}&\{K'(CC'+3^{n-1}2\ .\ 1^{n-1}2P^2_{123})3^{n-1}0-H(DD'+1^{n-1}3\ .\ 2^{n-1}3P^2_{123})2^{n-1}0\}:2^{n-1}3\ .\ 3^{n-1}2P_{023}\\ =&\{M(DD'+1^{n-1}3\ .\ 2^{n-1}3P^2_{123})1^{n-1}0-O'(BB'+2^{n-1}1\ .\ 3^{n-1}1P^2_{123})3^{n-1}0\}:3^{n-1}1\ .\ 1^{n-1}3P_{013}\\ =&\{B'(BB'+2^{n-1}1\ .\ 3^{n-1}1P^2_{123})2^{n-1}0-Q(CC'+3^{n-1}2\ .\ 1^{n-1}2P^2_{123})1^{n-1}0\}:1^{n-1}2\ .\ 2^{n-1}1P_{012},\end{aligned}\right\}\ (16)$$

and similarly for the other groups of points. The degree of these equations, when cleared of fractions, is obviously 18, as stated above.

The requisite conditions are consequently, four of the form $1^n=0$, $2^n=0$, . . , of the degrees 1, 1, 1, 1, in the coefficients of U; and two of the degree 18 in the same quantities; six in all. And from these data theorems corresponding to those enunciated for contact by a quadric may be written down.

It is, however, to be noticed that if osculation subsist at four points P, P_1, P_2, P_3, then we have simultaneously the equations

$$\left.\begin{array}{ll} \theta_1:\theta_2:\theta_3= & \;.\; : BB'+\ldots : CC'+\ldots : DD'+\ldots, \\ \theta:\;.\;:\theta_2:\theta_3= & FF'+\ldots :\;.\;: HH'+\ldots : KK'+\ldots, \\ \theta:\theta_1:\;.\;:\theta_3= & LL'+\ldots : MM'+\ldots :\;.\;: OO'+\ldots, \\ \theta:\theta_1:\theta_2:\;.\;= & PP'+\ldots : QQ'+\ldots : RR'+\ldots :\;. \end{array}\right\} \quad . \quad . \quad . \quad . \quad (17)$$

any two rows of which being taken as independent the remaining two are consequences. Taking any two we can eliminate one of the ratios $\theta:\theta_1:\theta_2:\theta_3$, and thus obtain as one of the conditions an equation of the degree 16.

In this case, therefore, there will be four conditions of the degree 1, three of the degree 18, and one of the degree 16.

In certain special cases these expressions undergo considerable modification. Thus, if the surface U be capable of being touched by a quadric, as well as being osculated by a cubic in the four points P, P_1, P_2, P_3, we shall have, as proved above,

$$P_{123}=0,\quad P_{023}=0,\quad P_{013}=0,\quad P_{012}=0;$$

and the system of conditions for the osculation between the cubic V and the surface U will take the following form:—

$$\left.\begin{array}{l} \left.\begin{array}{l} K'CC'\,3^{n-1}0-HDD'\,2^{n-1}0=0, \\ MDD'\,1^{n-1}0-O'BB'\,3^{n-1}0=0, \\ R'BB'\,2^{n-1}0-QCC'\,1^{n-1}0=0, \end{array}\right\} \\ \left.\begin{array}{l} D'HH'\,3^{n-1}1-CKK'\,2^{n-1}1=0, \\ L'KK'\,0^{n-1}1-OFF'\,3^{n-1}1=0, \\ RFF'\,2^{n-1}1-P'HH'\,0^{n-1}1=0, \end{array}\right\} \\ \left.\begin{array}{l} DMM'\,3^{n-1}2-B'OO'\,1^{n-1}2=0, \\ FOO'\,0^{n-1}2-KLL'\,3^{n-1}2=0, \\ QLL'\,1^{n-1}2-PMM'\,0^{n-1}2=0, \end{array}\right\} \\ \left.\begin{array}{l} C'QQ'\,2^{n-1}3-BRR'\,1^{n-1}3=0, \\ FRR'\,0^{n-1}3-HPP'\,2^{n-1}3=0, \\ MPP'\,1^{n-1}3-LQQ'\,0^{n-1}3=0, \end{array}\right\} \end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

of which four groups two only will of course be independent. It appears, therefore, that for osculation at three points we have three conditions, and for osculation at four points six, when the surface can be touched by a quadric at the four points, instead of two, or four, as in the case when such contact is not presupposed. The degree of these conditions is 13; but one of each three may be depressed to the degree 12 by multiplying together the dexter and sinister sides of these expressions and dividing out the common factor $BB'CC'DD'1^{n-1}0 \,.\, 2^{n-1}0 \,.\, 3^{n-1}0$, &c. The results, by the help of the relations (17), may in the case of osculation at the four points be put into either of the following forms, viz.:—

$$\left.\begin{array}{ll} K'MR'=HO'Q, & KM'R=H'OQ', \\ D'L'R=COP', & DLR'=C'O'P, \\ DF'Q'=B'KP, & D'FQ=BK'P', \\ C'FM=BH'L, & CF'M=B'HL'. \end{array}\right\} \quad \ldots\ldots \quad (19)$$

The arrangement of the letters in these equations will be perhaps more readily apprehended by reference to the matrix written below; by which it appears that the combinations, accents apart, follows that of the principal minors of the determinant,

$$\begin{array}{cccc} ., & B, & C, & D, \\ F, & . & H, & K, \\ L, & M, & . & O, \\ P, & Q, & R, & . \end{array}$$

The total conditions will in this case be, four of the degree 1, four of the degree 13, and two of the degree 12.

The form of the above equations will perhaps be best seen by actually writing down the first, viz.

$$\left|\begin{array}{ccc} . & [01'2]2^{n-1}1+\ldots & [01'3]3^{n-1}1+\ldots \\ [02'1]1^{n-1}2+\ldots & . & [02'3]3^{n-1}2+\ldots \\ [03'1]1^{n-1}3+\ldots & [03'2]2^{n-1}3+\ldots & \end{array}\right| =0. \quad \ldots \quad (20)$$