# XIII. *Addition to Memoir on the Transformation of Elliptic Functions.*

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I HAVE recently succeeded in completing a theory considered in my 'Memoir on the Transformation of Elliptic Functions,' Phil. Trans., vol. 164 (1874), pp. 397–456—that of the septic transformation, $n=7$. We have here

$$\frac{1-y}{1+y}=\frac{1-x}{1+x}\left(\frac{\alpha-\beta x+\gamma x^2-\delta x^3}{\alpha+\beta x^2+\gamma x^2+\delta x^3}\right)^2,$$

a solution of

$$\frac{\mathrm{M}dy}{\sqrt{1-y^2\,.\,1-v^8y^2}}=\frac{dx}{\sqrt{1-x^2\,.\,1-u^8x^2}},$$

where $\frac{1}{\mathrm{M}}=1+\frac{2\beta}{\alpha}$; and the ratios $\alpha:\beta:\gamma:\delta$, and the $uv$-modular equation are determined by the equations

$$\begin{gathered}u^{14}\alpha^2=v^2\delta^2,\\ u^6(2\alpha\gamma+2\alpha\beta+\beta^2)=v^2(\gamma^2+2\gamma\delta+2\beta\delta),\\ \gamma^2+2\beta\gamma+2\alpha\delta+2\beta\delta=v^2u^2(2\alpha\gamma+2\beta\gamma+2\alpha\delta+\beta^2),\\ \delta^2+2\gamma\delta=v^2u^{10}(\alpha^2+2\alpha\beta);\end{gathered}$$

or, what is the same thing, writing $\alpha=1$, the first equation may be replaced by $\delta=\frac{u^7}{v}$, and then, $\alpha$, $\delta$ having these values, the last three equations determine $\beta$, $\gamma$ and the modular equation. If instead of $\beta$ we introduce M, by means of the relation $\frac{1}{\mathrm{M}}=1+2\beta$, that is $2\beta=\frac{1}{\mathrm{M}}-1$, then the last equation gives $2\gamma=u^3v^3\left(\frac{1}{\mathrm{M}}-\frac{u^4}{v^4}\right)$; and $\alpha$, $\beta$, $\gamma$, $\delta$ having these values, we have the residual two equations

$$\begin{gathered}u^6(2\alpha\gamma+2\alpha\beta+\beta^2)=\ v^2(\gamma^2+2\gamma\delta+2\beta\delta),\\ \gamma^2+2\beta\gamma+2\alpha\delta+\beta\delta=v^2u^2(2\alpha\gamma+2\beta\gamma+2\alpha\delta+\beta^2),\end{gathered}$$

viz., each of these is a quadric equation in $\frac{1}{\mathrm{M}}$; hence eliminating $\frac{1}{\mathrm{M}}$, we have the modular equation; and also (linearly) the value of $\frac{1}{\mathrm{M}}$, and thence the values of $\alpha$, $\beta$, $\gamma$, $\delta$ in terms of $u$, $v$.

Before going further it is proper to remark that, writing as above $\alpha=1$, then if $\delta=\beta\gamma$, we have

$$1-\beta x+\gamma x^2-\delta x^3=(1-\beta x)(1+\gamma x^2),$$
$$1+\beta x+\gamma x^2+\delta x^3=(1+\beta x)(1+\gamma x^2),$$

and the equation of transformation becomes

$$\frac{1-y}{1+y}=\frac{1-x}{1+x}\left(\frac{1-\beta x}{1+\beta x}\right)^2,$$

viz., this belongs to the cubic transformation. The value of $\beta$ in the cubic transformation was taken to be $\beta=\frac{u^3}{v}$, but for the present purpose it is necessary to pay attention to an omitted double sign, and write $\beta=\pm\frac{u^3}{v}$; this being so, $\delta=\beta\gamma$, and giving to $\gamma$ the value $\mp u^4$, $\delta$ will have its foregoing value $=\frac{u^7}{v}$. And from the theory of the cubic equation, according as $\beta=\frac{u^3}{v}$ or $=-\frac{u^3}{v}$, the modular equation must be $u^4-v^4+2uv(1-u^2v^2)=0$, or $u^4-v^4-2uv(1-u^2v^2)=0$.

We thus see *à priori*, and it is easy to verify that the equations of the septic transformation are satisfied by the values

$$\alpha=1,\ \beta=\ \ \frac{u^3}{v},\ \gamma=\ \ u^4, \delta=\frac{u^7}{v},\ \text{and } u^4-v^4+2uv(1-u^2v^2)=0\ ;$$
$$\alpha=1,\ \beta=-\frac{u^3}{v},\ \gamma=-u^4, \delta=\frac{u^7}{v},\ \text{and } u^4-v^4-2uv(1-u^2v^2)=0\ ;$$

and it hence follows that in obtaining the modular equation for the septic transformation, we shall meet with the factors $u^4-v^4\pm2uv(1-u^2v^2)$. Writing for shortness $uv=\theta$, these factors are $u^4-v^4\pm2\theta(1-\theta^2)$, the factor for the proper modular equation is $u^8+v^8-\Theta$, where

$$\Theta=8\theta-28\theta^2+56\theta^3-70\theta^4+56\theta^5-28\theta^6+8\theta^7$$

[viz., the equation $(1-u^8)(1-v^8)-(1-uv)^8=0$ is $u^8+v^8-\Theta=0$], and the modular equation as obtained by the elimination from the two quadric equations in fact presents itself in the form

$$(u^4-v^4+2\theta-2\theta^3)^2(u^4-v^4-2\theta+2\theta^3)^2(u^8+v^8-\Theta)=0.$$

Proceeding to the investigation, we substitute the values

$$\alpha=1,\ \beta=\tfrac{1}{2}\left(\frac{1}{\mathrm{M}}-1\right),\ \gamma=\tfrac{1}{2}u^3v^3\left(\frac{1}{\mathrm{M}}-\frac{u^4}{v^4}\right),\ \delta=\frac{u^7}{v}$$

in the residual two equations, which thus become

$$\frac{1}{M^2}(1-v^8) \qquad +\frac{2}{M}(1-uv)^3(1+uv) \qquad +\left\{(1-u^8)-4(1-uv)\left(1+\frac{u^7}{v}\right)\right\}=0,$$

$$\frac{1}{M^2}\left\{-u^2v^2(1-uv)^3(1+uv)\right\}+\frac{2}{M}\left\{u^2v^2(1-u^8)+\frac{u^3}{v}(1+u^2v^2)(u^4-v^4)\right\}$$
$$+\left\{\frac{u^{14}}{v^2}+6\frac{u^7}{v}(1-u^2v^2)-u^2v^2\right\}\quad=0,$$

the first of which is given p. 432 of the 'Memoir.' Calling them

$$\left(a,\ b,\ c\between\frac{1}{M},\ 1\right)^2=0,\ \left(a',\ b',\ c'\between\frac{1}{M},\ 1\right)^2=0,$$

we have

$$\frac{1}{M^2}:\frac{2}{M}:1=bc'-b'c:ca'-c'a:ab'-a'b,$$

and the result of the elimination therefore is

$$(ca'-c'a)^2-4(bc'-b'c)(ab'-a'b)=0.$$

Write as before $uv=\theta$. In forming the expressions $ca'-c'a$, &c., to avoid fractions we must in the first instance introduce the factor $v^2$, thus

$$\begin{aligned} v^2(ca'-c'a)&=v\{v(1-u^8)-4(1-\theta)(v+u^7)\}\{-\theta^2(1+\theta)(1-\theta)^3\}\\ &\quad-\{u^{14}+6u^6\theta(1-\theta^2)-v^2\theta^2\}\{1-v^8\},\\ &=-\theta^2(1+\theta)(1-\theta)^3\{v^2(-3+4\theta)+u^6(-4\theta+3\theta^2)\}\\ &\quad-\{u^{14}+6u^6(\theta-\theta^3)-v^2\theta^2\}(1-v^8)\,; \end{aligned}$$

but instead of $\theta^2v^2$ writing $u^2v^4$, the expression on the right hand side becomes divisible by $u^2$; and we find

$$\begin{aligned} \frac{v^2}{u^2}(ca'-c'a)&=-(1+\theta)(1-\theta)^3\{v^4(-3+4\theta)+u^4(-4\theta^3+3\theta^4)\}\\ &\quad-\{u^{12}+6u^4(\theta-\theta^3)-v^4\}(1-v^8), \end{aligned}$$

and thence

$$\begin{aligned} -\frac{v^2}{u^2}(ca'-c'a)=u^{12} \qquad &+u^4(6\theta-10\theta^3+11\theta^4-6\theta^5-8\theta^6+10\theta^7-4\theta^8)\\ &+v^4(-4+10\theta-8\theta^2-6\theta^3+11\theta^4-10\theta^5+6\theta^7)+v^{12}, \end{aligned}$$

and similarly we have

$$\begin{aligned} \frac{v^2}{u^2}(bc'-b'c)&=u^{12}(5-5\theta+4\theta^2-5\theta^3+2\theta^4)+u^4(9\theta-16\theta^2+\theta^3+10\theta^4+\theta^5-16\theta^6+9\theta^7)\\ &\quad+v^4(2-5\theta+4\theta^2-5\theta^3+5\theta^4),\\ \frac{v^2}{u^2}(ab'-a'b)&=u^4(\theta+\theta^3-\theta^4) \qquad +v^4(2-5\theta+4\theta^2+3\theta^3-10\theta^4+3\theta^5+4\theta^6-5\theta^7+2\theta^8)\\ &\quad+v^{12}(-1+\theta+\theta^3)\,; \end{aligned}$$

say these values are

$$u^{12}+pu^4+qv^4+v^{12},\ \lambda u^{12}+\mu u^4+\nu v^4,\ \rho u^4+\sigma v^4+\tau v^{12}.$$

The required equation is thus

$$0=(u^{12}+pu^4+qv^4+v^{12})^2-4(\lambda u^{12}+\mu u^4+\nu v^4)(\rho u^4+\sigma v^4+\tau v^{12}),$$

viz., the function is

$$\begin{aligned}&u^{24}\\ &+u^{16}(2p-4\lambda\rho)\\ &+u^8\ (2q\theta^4+p^2-4\lambda\sigma\theta^4-4\mu\rho)\\ &+\quad (2\theta^{12}+2pq\theta^4-4\lambda\tau\theta^{12}-4\mu\sigma\theta^4-4\nu\rho\theta^4)\\ &+v^8\ (2p\theta^4+q^2-4\mu\tau\theta^4-4\nu\sigma)\\ &+v^{16}(2q-4\nu\tau)\\ &+v^{24},\end{aligned}$$

or say it is

$$=(1,\ b,\ c,\ d,\ e, f,\ 1\between u^{24},\ u^{16},\ u^8,\ 1,\ v^8,\ v^{16},\ v^{24}).$$

Supposing that this has a factor $u^8-\Theta+v^8$, the form is

$$(u^{16}+\mathrm{B}u^8+\mathrm{C}+\mathrm{D}v^8+v^{16})(u^8-\Theta+v^8);$$

and comparing coefficients we have

$$\begin{aligned}\mathrm{B}-\Theta\quad &=b,\\ \mathrm{C}-\Theta\mathrm{B}+\theta^8\quad &=c,\\ \mathrm{D}\theta^8-\Theta\mathrm{C}+\mathrm{B}\theta^8 &=d,\\ \theta^8-\Theta\mathrm{D}+\mathrm{C}\quad &=e,\\ -\Theta\ \ +\mathrm{D}\quad &=f,\end{aligned}$$

where $\Theta$ has the before-mentioned value

$$=(8,\ -28,\ +56,\ -70,\ +56,\ -28,\ +8\between\theta,\ \theta^2,\ \theta^3,\ \theta^4,\ \theta^5,\ \theta^6,\ \theta^7);$$

from the first, second, and fifth equations, $\mathrm{B}=b+\Theta$, $\mathrm{C}=c+\Theta\mathrm{B}-\theta^8$, $\mathrm{D}=f+\Theta$; and the third and fourth equations should then be verified identically. Writing down the coefficients of the different powers of $\theta$ we find

$$\begin{array}{rl} 2p= & 0+12\quad 0-20+22-12-16+20-8\ (\theta^0\ldots\theta^8)\\ -4\lambda\rho= & 0-20+20-36+60-44+36-28+8\qquad ,,\\ \hline b= & 0-\ 8+20-56+82-56+20-\ 8\quad 0\qquad ,,\\ \Theta= & 0+\ 8-28+56-70+56-28+\ 8\quad 0\qquad ,,\\ \hline \therefore\ \mathrm{B}= & 0\quad 0-\ 8\quad 0+12\quad 0-\ 8\quad 0\quad 0\qquad ,,\end{array}$$

that is

$$B=-8\theta^2+12\theta^4-8\theta^6;$$

and in precisely the same way the fifth equation gives

$$D=-8\theta^2+12\theta^4-8\theta^6.$$

We find similarly C from the second equation: writing down first the coefficients of $p^2$, $2q\theta^4$, $-4\lambda\sigma\theta^4$, and $-4\mu\rho$, the sum of these gives the coefficients of $c$; and then writing underneath these the coefficients of $B\Theta$ and of $-\theta^8$, the final sum gives the coefficients of C: the coefficients of each line belong to $(\theta^0, \theta^1, \ldots \theta^{16})$.

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|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 36 | 0 | −120 | +132 | +28 | −316 | +361 | −20 | −340 | +396 | −144 | −112 | +164 | −80 | +16 |
| | | | | −8 | +20 | −16 | −12 | +22 | −20 | 0 | +12 | | | | | |
| | | | | −40 | +140 | −212 | +140 | +80 | −188 | +168 | −92 | −64 | +176 | −164 | +80 | −16 |
| | | −36 | +64 | −40 | +60 | −72 | +28 | 0 | +68 | −100 | +36 | | | | | |
| 0 | 0 | 0 | +64 | −208 | +352 | −272 | −160 | +463 | −160 | −272 | +352 | −208 | +64 | 0 | 0 | 0 |
| 0 | 0 | 0 | −64 | +224 | −352 | +224 | +160 | −392 | +160 | +224 | −352 | +224 | −64 | 0 | 0 | 0 |
| | | | | | | | | −1 | | | | | | | | |
| 0 | 0 | 0 | 0 | +16 | 0 | −48 | 0 | +70 | 0 | −48 | 0 | +16 | 0 | 0 | 0 | 0 |

that is

$$C=16\theta^4-48\theta^6+70\theta^8-48\theta^{10}+16\theta^{12},$$

and in precisely the same way this value of C would be found from the fourth equation. There remains to be verified only the fourth equation $(D+B)\theta^8-\Theta C=d$, that is

$$2\theta^8(-8\theta^2+12\theta^4-8\theta^6)-\Theta C=(2-4\lambda\tau)\theta^{12}+(2pq-4\mu\sigma-4\nu\rho)\theta^4,$$

and this can be effected without difficulty.

The factor of the modular equation thus is

$$u^{16}+v^{16}+(-8\theta^2+12\theta^4-8\theta^6)(u^8+v^8)+16\theta^4-48\theta^6+70\theta^8-48\theta^{10}+16\theta^{12},$$

viz., this is

$$\begin{aligned}&(u^8+v^8)^2+(-4\theta^2+6\theta^4-4\theta^6)2(u^8+v^8)+16\theta^4-48\theta^6+68\theta^8-48\theta^{10}+16\theta^{12},\\ &=(u^8+v^8-4\theta^2+6\theta^4-4\theta^6)^2,\\ &=\{(u^4-v^4)^2-4\theta^2(1-\theta^2)^2\}^2\end{aligned}$$

that is

$$\{u^4-v^4-2\theta(1-\theta^2)\}^2\{u^4-v^4+2\theta(1-\theta^2)\}^2;$$

or the modular equation is

$$\{u^4-v^4-2\theta(1-\theta^2)\}^2\{u^4-v^4+2\theta(1-\theta^2)\}^2(u^8+v^8-\Theta)=0\,;$$

viz., the first and second factors belong to the cubic transformation; and we have for the proper modular equation in the septic transformation $u^8+v^8-\Theta=0$, or what is the same thing $(1-u^8)(1-v^8)-(1-\theta)^8=0$, that is $(1-u^8)(1-v^8)-(1-uv)^8=0$, the known result; or as it may also be written $(\theta-u^8)(\theta-v^8)+7\theta^2(1-\theta)^2(1-\theta+\theta^2)^2=0$.

The value of M is given by the foregoing relations

$$\frac{1}{\mathrm{M}^2}:\frac{2}{\mathrm{M}}:1=\lambda u^{12}+\mu u^4+\nu v^4 : -(u^{12}+pu^4+qv^4+v^{12}) : \rho u^4+\sigma v^4+\tau v^{12}\,;$$

but these can be, by virtue of the proper modular equation, $u^8+v^8-\Theta=0$, reduced into the form

$$\frac{1}{\mathrm{M}^2}:\frac{2}{\mathrm{M}}:1=7(\theta-u^8) : 14(\theta-2\theta^2+2\theta^3-\theta^4) : -\theta+v^8,$$

viz., the equality of these two sets of ratios depends upon the following identities,

$$\begin{aligned}&(-\theta+v^8)(u^{12}+pu^4+qv^4+v^{12})+14(\theta-2\theta^2+2\theta^3-\theta^4)(\rho u^4+\sigma v^4+\tau v^{12})\\&\qquad=\{-\theta u^4+(1-\theta)(-4-\theta+5\theta^2-\theta^3-4\theta^4)v^4+v^{12}\}(u^8-\Theta+v^8),\\&-7(\theta-u^8)(\rho u^4+\sigma v^4+\tau v^{12})-(\theta-v^8)(\lambda^{12}+\mu u^4+\nu u^4)\\&\qquad=\{(2\theta+5\theta^2+3\theta^3-2\theta^4-2\theta^5)u^4+(2+2\theta-3\theta^2-5\theta^3-2\theta^4)v^4\}(u^8-\Theta+v^8),\\&-2(\theta-2\theta^2+2\theta^3-\theta^4)(\lambda u^{12}+\mu u^4+\nu u^4)+(u^8-\theta)(u^{12}+pu^4+qv^4+v^{12})\\&\qquad=\{u^{12}+\theta(1-\theta)(3+5\theta+3\theta^2)u^4-\theta v^4\}(u^8-\Theta+v^8),\end{aligned}$$

which can be verified without difficulty: from the last-mentioned system of values, replacing $\theta$ by its value $uv$, we then have

$$\frac{1}{\mathrm{M}^2}:\frac{2}{\mathrm{M}}:1=7u(v-u^7) : 14uv(1-uv)(1-uv+u^2v^2) : -v(u-v^7),$$

which agree with the values given p. 482 of the 'Memoir,' and the analytical theory is thus completed.