PHILOSOPHICAL TRANSACTIONS.

I. *On the Tides of the Arctic Seas.*

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Part VII. *Tides of Port Kennedy, in Bellot Strait.* (*Final Discussion.*)

Received February 17,—Read March 15, 1877.

[PLATE 1.]

IN Part VI. I have discussed McCLINTOCK's observations on the Tides of Port Kennedy, using only the Heights and Times of High and Low Water, as I wished to follow the same method in comparing all the Tidal observations in the Arctic Seas. Although I adopted this method for the purpose of comparison, I was well aware that I had not exhausted all the information at my disposal, for McCLINTOCK's observations were made hourly during 23 days, and I used of these observations only those in the neighbourhood of H.W. and L.W. of Diurnal and Semidiurnal Tides. I shall now discuss the observations, with the aid of FOURIER's Theorem, so that all the observations made at every hour of each day shall enter into the constants determined for that day. If F denote the height of the tide, observed at every hour of the day, we have by FOURIER's Theorem

$$\begin{aligned} F = A_0 + A_1 \cos s + A_2 \cos 2s + A_3 \cos 3s + \&c. \\ B_1 \sin s + B_2 \sin 2s + B_3 \sin 3s + \&c. \end{aligned} \quad \ldots \ldots \quad (1)$$

where s denotes the sun's hour angle, and where the coefficients A_0, A_1, B_1, A_2, B_2, &c., are found from the following equations, in which F_0, F_1, F_2, &c., denote the values of F at the hours 0, 1, 2, &c., 23.

$$24A_0 = F_0 + F_1 \ldots + F_{23} \quad \ldots \ldots \quad (2)$$

$$
\begin{aligned}
12A_1 = (F_0 - F_{12}). & \\
& + \{(F_1+F_{23})-(7_{11}+F_{13})\} \cos \phi \\
& + \{(F_2+F_{22})-(F_{10}+F_{14})\} \cos 2\phi \\
& + \{(F_3+F_{21})-(F_9+F_{15})\} \cos 3\phi \\
& + \{(F_4+F_{20})-(F_8+F_{16})\} \cos 4\phi \\
& + \{(F_5+F_{19})-(F_7+F_{17})\} \cos 5\phi \quad \ldots \ldots \ldots \quad (3)
\end{aligned}
$$

$$
\begin{aligned}
12B_1 = (F_6 - F_{18}). & \\
& + \{(F_1+F_{11})-(F_{13}+F_{23})\} \sin\phi \\
& + \{(F_2+F_{10})-(F_{14}+F_{22})\} \sin 2\phi \\
& + \{(F_3+F_9)-(F_{15}+F_{21})\} \sin 3\phi \\
& + \{(F_4+F_8)-(F_{16}+F_{20})\} \sin 4\phi \\
& + \{(F_5+F_7)-(F_{17}+F_{19})\} \sin 5\phi \quad \ldots \ldots \ldots \quad (4)
\end{aligned}
$$

$$
\begin{aligned}
12A_2 = (F_0+F_{12})-(F_6+F_{18}). & \\
& + \left\{ \begin{array}{l} (F_1+F_{23})+(F_{11}+F_{13}) \\ -(F_5+F_{19})-(F_7+F_{17}) \end{array} \right\} \cos 2\phi \\
& + \left\{ \begin{array}{l} (F_2+F_{22})+(F_{10}+F_{14}) \\ -(F_4+F_{20})-(F_8+F_{16}) \end{array} \right\} \cos 4\phi \quad \ldots \ldots \ldots \quad (5)
\end{aligned}
$$

$$
\begin{aligned}
12B_2 = (F_3+F_{15})-(F_9+F_{21}). & \\
& + \left\{ \begin{array}{l} (F_1+F_5)+(F_{13}+F_{17}) \\ -(F_7+F_{11})-(F_{19}+F_{23}) \end{array} \right\} \sin 2\phi \\
& + \left\{ \begin{array}{l} (F_2+F_4)+(F_{14}+F_{16}) \\ -(F_8+F_{10})-(F_{20}+F_{22}) \end{array} \right\} \sin 4\phi \quad \ldots \ldots \ldots \quad (6)
\end{aligned}
$$

$$
\begin{aligned}
12A_3 = (F_0+F_8+F_{16})-(F_4+F_{12}+F_{20}). & \\
& \left\{ \begin{array}{l} (F_1+F_{23})+(F_7+F_{17})+(F_9+F_{15}) \\ -(F_3+F_{21})-(F_5+F_{19})-(F_{11}+F_{13}) \end{array} \right\} \cos 3\phi \quad \ldots \quad (7)
\end{aligned}
$$

$$
\begin{aligned}
12B_3 = (F_2+F_{10}+F_{18})-(F_6+F_{14}+F_{22}). & \\
& \left\{ \begin{array}{l} (F_1+F_{11})+(F_3+F_9)+(F_{17}+F_{19}) \\ -(F_5+F_7)-(F_{13}+F_{23})-(F_{15}+F_{21}) \end{array} \right\} \sin 3\phi \quad \ldots \quad (8)
\end{aligned}
$$

In these equations $\phi = \frac{360}{24} = 15^\circ$.

Applying the foregoing equations to the hourly observations at Port Kennedy already published in Part VI., we find the following values of the Coefficients, which contain implicitly the 24 observations made on each day :—

TABLE I.—Tidal Coefficients for July, 1859, at Port Kennedy.

			A_0	A_1	B_1	A_2	B_2	A_3	B_3
1859.—July	6	0	75·6	−3·8	−12·7	−19·9	+11·2	+0·1	+1·7
”	7	1	79·6	−5·9	−12·6	−18·3	−0·9	−2·2	−1·3
”	8	2	82·1	−9·0	−6·8	−14·2	−8·7	−1·3	−1·6
”	9	3	92·2	−12·8	−4·6	−8·8	−16·1	+0·0	+0·6
”	10	4	95·8	−15·4	−4·9	−0·8	−11·4	+1·8	−1·6
”	11	5	98·1	−14·1	−6·5	+5·4	−3·7	−3·6	+3·2
”	12	6	88·3	−17·4	−12·4	+15·6	−6·2	+0·4	+0·6
”	13	7	92·9	−23·9	−15·8	+18·7	−1·3	−0·4	−1·1
”	14	8	92·9	−21·7	−20·3	+20·8	+5·2	+0·9	−0·5
”	15	9	89·7	−11·5	−17·3	+18·1	+5·9	+0·5	+2·4
”	16	10	93·5	−13·5	−22·5	+14·2	+15·6	+1·4	+1·1
”	17	11	93·4	−9·9	−21·1	+8·5	+20·5	+1·3	+1·4
”	18	12	91·2	−5·2	−15·3	+4·5	+19·9	−0·3	+2·1
”	19	13	89·5	−4·1	−15·1	−2·7	+20·4	−0·9	+1·4
”	20	0	89·8	−4·7	−14·3	−8·4	+18·7	−0·5	+1·6
”	21	1	85·5	−3·8	−8·5	−11·7	+12·6	+0·0	+2·4
”	22	2	83·5	−5·5	−7·2	−11·1	+7·9	+2·7	+2·0
”	23	3	82·0	−8·8	−4·5	−13·9	−3·0	+0·2	+0·2
”	24	4	85·0	−12·2	−3·6	−10·6	−6·3	+0·8	−1·6
”	25	5	88·0	−18·6	−4·1	−3·0	−11·1	−0·3	−1·1
”	26	6	94·2	−20·0	−8·6	−1·9	−14·3	−0·3	+1·6
”	27	7	102·6	−26·9	−21·2	+13·0	−10·9	−1·4	−5·2

The seven Coefficients of the preceding Table are drawn to scale in Plate* I., and they ought to show a fortnightly Tide.

I. *The Fortnightly Change of Mean Level.*

This Tide is represented by the Column A_0. .

If we write

$$A_0 = a_0 + a_1 \cos u + a_2 \cos 2u + \&c.$$
$$+ b_1 \sin u + b_2 \sin 2u + \&c.$$

where u is an angle passing through all its changes in 14 days, we have

$$14a_0 = F_0 + F_1 + \&c. \dots + F_{13} \quad \dots \dots \quad (9)$$

$$\begin{aligned} 7a_1 = & (F_0 - F_7) \\ & \{(F_1+F_{13}) - (F_6+F_8)\} \cos \phi \\ & \{(F_2+F_{12}) - (F_5+F_9)\} \cos 2\phi \\ & \{(F_3+F_{11}) - (F_4+F_{10})\} \cos 3\phi \quad \dots \dots \quad (10) \end{aligned}$$

$$\begin{aligned} 7b_1 = & \{(F_1+F_6) - (F_8+F_{13})\} \sin \phi \\ & \{(F_2+F_5) - (F_9+F_{12})\} \sin 2\phi \\ & \{(F_3+F_4) - (F_{10}+F_{11})\} \sin 3\phi \quad \dots \dots \quad (11) \end{aligned}$$

where $\phi = \frac{360^\circ}{14}$.

* In this Plate, the vertical coordinates are inches, and the horizontal coordinates are days of the month.

From these equations we readily obtain—

TABLE II.—A_0.

	a_0. inches.	a_1. inches.	b_1. inches.
	88·9	−5·4	−0·8
	90·6	−3·4	−0·8
	91·0	−2·7	−0·4
	91·2	−2·5	−0·3
	90·4	−2·9	−1·7
	89·7	−2·5	−3·2
	88·9	−1·6	−4·3
	89·4	+2·4	−4·0
	90·1	+3·8	−4·0
Mean	90·02	−1·64	−2·17

These numbers differ so much, that we must conclude it to be impossible, from the limited number of observations made, to deduce the law of the Fortnightly Change in the Mean Level of the water.

II. *The Diurnal Tide.*

The Diurnal Tide is represented by the expression

$$A_1 \cos s + B_1 \sin s,$$

in which the coefficients A_1 and B_1 are to be expanded by means of a fortnightly arc, as in the case of the coefficient A_0.

Let

$$A_1 = a_0 + a_1 \cos u + a_2 \cos 2u + \&c.$$
$$+ b_1 \sin u + b_2 \sin 2u + \&c. \quad \ldots \ldots \quad (12)$$

$$B_1 = \alpha_0 + \alpha_1 \cos u + \alpha_2 \cos 2u + \&c.$$
$$+ \beta_1 \sin u + \beta_2 \sin 2u + \&c. \quad \ldots \ldots \quad (13)$$

We find by means of (9), (10), and (11) the following values :—

TABLE III.—(A_1).

	a_0. inches.	a_1. inches.	b_1. inches.
	$-12{\cdot}0$	$+7{\cdot}8$	$-1{\cdot}2$
	$-12{\cdot}1$	$+7{\cdot}7$	$-1{\cdot}2$
	$-11{\cdot}9$	$+8{\cdot}0$	$-1{\cdot}1$
	$-11{\cdot}7$	$+8{\cdot}3$	$-0{\cdot}7$
	$-11{\cdot}4$	$+8{\cdot}7$	$-0{\cdot}1$
	$-11{\cdot}2$	$+8{\cdot}3$	$+0{\cdot}3$
	$-11{\cdot}5$	$+8{\cdot}7$	$-0{\cdot}2$
	$-11{\cdot}7$	$+9{\cdot}0$	$-0{\cdot}3$
	$-11{\cdot}9$	$+9{\cdot}5$	$-0{\cdot}3$
Mean	$-11{\cdot}7$	$+8{\cdot}4$	$-0{\cdot}5$

TABLE IV.—(B_1).

	α_0. inches.	α_1. inches.	β_1. inches.
	$-13{\cdot}4$	$+1{\cdot}3$	$+7{\cdot}5$
	$-13{\cdot}5$	$+1{\cdot}1$	$+7{\cdot}5$
	$-13{\cdot}2$	$+1{\cdot}6$	$+7{\cdot}8$
	$-13{\cdot}3$	$+1{\cdot}6$	$+7{\cdot}7$
	$-13{\cdot}3$	$+1{\cdot}6$	$+7{\cdot}7$
	$-13{\cdot}2$	$+1{\cdot}5$	$+7{\cdot}9$
	$-13{\cdot}0$	$+1{\cdot}3$	$+8{\cdot}2$
	$-12{\cdot}7$	$+0{\cdot}8$	$+8{\cdot}4$
	$-13{\cdot}1$	$+1{\cdot}7$	$+8{\cdot}4$
Mean	$-13{\cdot}2$	$+1{\cdot}4$	$+8{\cdot}0$

The foregoing Tables show that a very regular Diurnal Tide exists at Port Kennedy. In order to compare the observed Tide with theory, we must compare the observed Tide, viz. :—

$$
\begin{aligned}
&A_1 \cos s + B_1 \sin s \\
&= a_0 + \{a_1 \cos u + b_1 \sin u\} \cos s \\
&+ \alpha_0 + \{\alpha_1 \cos u + \beta_1 \sin u\} \sin s \quad \ldots \ldots \ldots \quad (14)
\end{aligned}
$$

with the theoretical Diurnal Tide, viz. :—

$$
M' \cos (m - i_m) + S' \cos (s - i_s) \quad \ldots \ldots \ldots \quad (15)
$$

where

$$M'=M\left(\frac{p}{p_m}\right)^3 \sin \overline{2\mu}$$

$$S'=S\left(\frac{P}{P_m}\right)^3 \sin \overline{2\sigma}$$

M = Lunar Coefficient,
S = Solar Coefficient,
p, p_m = Lunar parallax, Lunar mean parallax,
P, P_m = Solar parallax, Solar mean parallax,
$\bar{\mu}$ = Lunar declination, for a period preceding the time of observation by an interval called the Age of the Lunar Tide—
$\bar{\sigma}$ = Solar declination, for a period preceding the time of observation by an interval called the Age of the Solar Tide—
s = Sun's hour angle,
m = Moon's hour angle,
$\dot{i}_s$ = Solitidal interval,
$\dot{i}_m$ = Lunitidal interval.

In order to compare (15) with (14), we must transform (15) into a function of s and u. This may be accomplished as follows:—

Writing

$$m=s+m-s$$

we have

$$m-\dot{i}_m=s+\overline{m-s-\dot{i}_m}$$

from which (15) becomes

$$\begin{aligned} &M' \cos (m-\dot{i}_m)+S' (\cos s-\dot{i}_s) \\ &=\{M' \cos \overline{m-s-\dot{i}_m}+S' \cos \dot{i}_s\} \cos s \\ &\{-M' \sin \overline{m-s-\dot{i}_m}+S' \sin \dot{i}_s\} \sin s \quad \ldots \ldots \quad (16) \end{aligned}$$

From which we obtain, by comparison with (14)

$$A_1=M' \cos \overline{m-s-\dot{i}_m}+S' \cos \dot{i}_s \quad \ldots \ldots \quad (17)$$

$$B_1=-M' \sin \overline{m-s-\dot{i}_m}+S' \sin \dot{i}_s \quad \ldots \ldots \quad (18)$$

Having thus got rid of s, we must transform (17) and (18) (which are now functions of $m-s$, and of the Moon's declination, and parallax) into functions of u. This may be thus effected—

1°. We have

$$\frac{p}{p_m}=\frac{a}{r}=\frac{1+e \cos v}{1-e^2} \quad \ldots \ldots \quad (19)$$

where

r = Moon's distance
a = Moon's mean distance
e = Eccentricity of Moon's orbit
v = True anomaly, measured from perigee;

but, by SETH WARD's hypothesis,* we have, also

$$2a-r=\frac{a(1-e^2)}{1-e\cos(nt)}$$

where (nt) is the mean anomaly, measured from perigee. From this we find,

$$\frac{p}{p_m}=\frac{a}{r}=\frac{1-e\cos(nt)}{1-2e\cos(nt)+e^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

hence we find

$$\left(\frac{p}{p_m}\right)^3=\left(1-\frac{e^2}{2}\right)-e\cos(nt)+\frac{e^2}{2}\cos(2nt)+\&c. \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

Substituting for e in this equation, its value $\frac{1}{20}$, we obtain

$$\left(\frac{p}{p_m}\right)^3=0\cdot999-0\cdot05\cos(nt)+0\cdot0012\cos(2nt) \quad . \quad . \quad . \quad . \quad . \quad . \quad (22)$$

2°. We have, also,

$$\sin 2\mu=2\sin\mu\cos\mu$$
$$=2\sin \mathrm{I}\sin(v-n)\sqrt{1-\sin^2 \mathrm{I}\sin^2(v-n)}$$

where v is reckoned from perigee, and n is the interval from perigee to the ascending node of the moon's orbit with the equinoctial, and $\mathrm{I}=27° 40'$, is the inclination of the moon's orbit to the equinoctial.

Hence we find, since $\sin \mathrm{I}=0\cdot464$

$$\sin 2\mu=2\sin \mathrm{I}\sin(v-n)\left[1-\tfrac{1}{2}\sin^2 \mathrm{I}\sin^2(v-n) -\tfrac{1}{8}\sin^4 \mathrm{I}\sin^4(v-n)+\&c.\right] \quad . \quad . \quad . \quad (23)$$

$$\operatorname{Sin} 2\mu=0\cdot89\sin(v-n)+0\cdot012\sin 3(v-n)+\&c. \quad . \quad . \quad . \quad . \quad . \quad (24)$$

The perigee occurred July, $1^d\ 0^h$, and the ascending node, $19^d\ 17^h$; hence $n=18^d\ 17^h=247°$.

Substituting this value of n, we find

$$\sin 2\mu=-0\cdot35\sin v+0\cdot82\cos v+0\cdot01\sin 3v -0\cdot004\cos 3v+\&c. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (24)$$

By the well known expansion of the true anomaly in terms of the mean anomaly, we have

* This hypothesis, according to which the angular motion of the revolving body about the focus in which the central body is not, is uniform, is mentioned by Bishop BRINKLEY ('Astronomy,' p. 155 of 1st edition) under the title of the *Simple Elliptic Hypothesis*, and was much in use between the times of KEPLER and NEWTON.

$$v=nt+\left(2e-\frac{e^3}{4}\right)\sin(nt)+\frac{5e^2}{4}\sin(2nt)$$
$$+\frac{13}{12}e^3\sin(3nt)+\&c. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (25)$$

Substituting this value in (24) and using the common expansions,

$$\cos x=1-\frac{x^2}{1\cdot2}+\frac{x^4}{1\cdot2\cdot3\cdot4}-\&c.$$
$$\sin x=x-\frac{x^3}{1\cdot2\cdot3}+\&c.$$

we find

$$\sin 2\mu=-0\cdot04-0\cdot35\sin(nt)-0\cdot02\sin(2nt)$$
$$+0\cdot82\cos nt+0\cdot04\cos 2nt+\&c.. \quad . \quad . \quad . \quad . \quad (25)$$

Multiplying together (22) and (25) we find

$$\mathrm{M}'=\mathrm{M}\left(\frac{p}{p_m}\right)^3\sin 2\mu$$
$$=\mathrm{M}\,(-0\cdot06+0\cdot82\cos nt+0\cdot02\cos 2nt$$
$$-0\cdot35\sin nt-0\cdot01\sin 2nt+\&c. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (26)$$

3°. We have now to find values in terms of nt for $\cos\overline{m-s-i_m}$ and $\sin\overline{m-s-i_m}$. This may be done as follows:—

$$m=+kt-m'$$
$$s=+kt-s'$$

where kt is the angle due to the rotation of the earth, and m', s', are the proper motions in right ascension of the moon and sun; therefore

$$m-s=s'-m'$$

We may find m' in terms of v, as follows:—If n be the angle between perigee and ascending node, and c be the angle between conjunction and ascending node, we have

$$\tan(m'-c)=\cos\mathrm{I}\tan(v-n)$$
$$\cos(m'-c)=\frac{1}{\sqrt{1+\cos^2\mathrm{I}\tan^2(v-n)}}$$
$$\sin(m'-c)=\frac{\cos\mathrm{I}\tan(v-n)}{\sqrt{1+\cos^2\mathrm{I}\tan^2(v-n)}}$$

or

$$\cos(m'-c)=\frac{\cos(v-n)}{\sqrt{1-\sin^2\mathrm{I}\sin^2(v-n)}}$$
$$\sin(m'-c)=\frac{\cos\mathrm{I}\sin(v-n)}{\sqrt{1-\sin^2\mathrm{I}\sin^2(v-n)}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (27)$$

or

$$\cos(m'-c)=\cos(v-n)\times\left|\begin{array}{l}1+0{\cdot}108\sin^2(v-n)\\ +0{\cdot}017\sin^4(v-n)\\ +\&c.\end{array}\right.$$

$$\sin(m'-c)=0{\cdot}886\sin(v-n)\times\left|\begin{array}{l}1+0{\cdot}108\sin^2(v-n)\\ +0{\cdot}017\sin^4(v-n)\\ +\&c.\end{array}\right. \quad \ldots\ldots \quad (28)$$

From these equations we have, remembering that

$$\begin{aligned}&n=247^\circ\\ &\cos(m'-c)=-0{\cdot}42\cos v-0{\cdot}99\sin v+\&c.\\ &\sin(m'-c)=+0{\cdot}85\cos v-0{\cdot}36\sin v+\&c. \quad \ldots\ldots \quad (29)\end{aligned}$$

We now have—

$$\begin{aligned}&\cos(m-s-i_m)=\cos(s'-m'-i_m)\\ &=\cos\{(-m'+c)+(-c+s'-i_m)\}\\ &=\cos(-m'+c+\phi)\\ &\text{where } \phi=-c+s'-i_m.\end{aligned}$$

Expanding and substituting from (29) we find

$$\begin{aligned}&\cos\overline{(m-s-i_m)}\\ &\{-0{\cdot}42\cos\phi+0{\cdot}85\sin\phi\}\cos v\\ &+\{-0{\cdot}99\cos\phi-0{\cdot}36\sin\phi\}\sin v \quad \ldots\ldots \quad (30)\end{aligned}$$

In like manner we have

$$\begin{aligned}&\sin\overline{(m-s-i_m)}=\\ &\{-0{\cdot}85\cos\phi-0{\cdot}42\sin\phi\}\cos v\\ &+\{+0{\cdot}36\cos\phi-0{\cdot}99\sin\phi\}\sin v \quad \ldots\ldots \quad (31)\end{aligned}$$

Substituting in (30) and (31), for v, its value $v=nt+2e\sin nt$ (25),

we obtain, writing $\left(\begin{array}{l}A=-0{\cdot}42\cos\phi+0{\cdot}85\sin\phi\\ B=-0{\cdot}99\cos\phi-0{\cdot}36\sin\phi\end{array}\right)$

$$\begin{aligned}&\cos\overline{(m-s-i_m)}=\\ &-Ae+A\cos nt+Ae\cos 2nt+\&c.\\ &\quad+B\sin nt+Be\sin 2nt+\&c. \quad \ldots\ldots \quad (32)\end{aligned}$$

$$\begin{aligned}&\sin\overline{(m-s-i_m)}=\\ &-A'e+A'\cos nt+Ae\cos 2nt+\&c.\\ &\quad+B'\sin nt+B'e\sin 2nt+\&c. \quad \ldots\ldots \quad (33)\end{aligned}$$

where

$$\begin{aligned}&A'=-0{\cdot}85\cos\phi-0{\cdot}42\sin\phi\\ &B'=+0{\cdot}36\cos\phi-0{\cdot}99\sin\phi\end{aligned}$$

Multiplying together (26) and (32), we have, as a first approximation

$$M' \cos \overline{m-s-i_m} = (0{\cdot}41A - 0{\cdot}18B) + (0{\cdot}41A + 0{\cdot}18B) \cos 2nt \\ (-0{\cdot}18A + 0{\cdot}41B) \sin 2nt$$

or,

$$M' \cos (m-s-i_m) = +0{\cdot}33M \sin \phi \\ +M\,(-0{\cdot}35 \cos \phi + 0{\cdot}29 \sin \phi) \cos 2nt \\ +M\,(-0{\cdot}34 \cos \phi + 0{\cdot}30 \sin \phi) \sin 2nt \quad . \ . \ . \ . \ . \ . \ . \quad (34)$$

Multiplying together (26) and (33) we find, also,

$$M' \sin (\overline{m-s-i_m}) = (0{\cdot}42A' - 0{\cdot}18B') + (0{\cdot}42A' + 0{\cdot}18B') \cos 2nt \\ +(-0{\cdot}18A' + 0{\cdot}42B') \sin 2nt$$

or,

$$M' \sin (\overline{m-s-i_m}) = -0{\cdot}41M \cos \phi \\ +M\,(-0{\cdot}30 \cos \phi - 0{\cdot}35 \sin \phi) \cos 2nt \\ +M\,(+0{\cdot}30 \cos \phi - 0{\cdot}34 \sin \phi) \sin 2nt \quad . \ . \ . \ . \ . \ . \quad (35)$$

Equations (34) and (35) are now to be converted into functions of u, which may be effected, in general, by means of the following expansions,* in which a is a proper fraction :—

$$\cos a x = \frac{\sin a\pi}{\pi} \left[\frac{1}{a} + 2a \left[\frac{\cos x}{1^2 - a^2} - \frac{\cos 2x}{2^2 - a^2} + \frac{\cos 3x}{3^2 - a^2} - \frac{\cos 4x}{4^2 - a^2} + \&c. \right] \right] \quad . \ . \ . \ . \ . \ . \ . \ . \ . \ . \quad (36)$$

$$\sin a x = \frac{2 \sin a\pi}{\pi} \left[\frac{\sin x}{1^2 - a^2} - \frac{2 \sin 2x}{2^2 - a^2} + \frac{3 \sin 3x}{3^2 - a^2} - \frac{4 \sin 4x}{4^2 - a^2} + \&c. \right] \quad . \ . \ . \ . \ . \ . \ . \ . \ . \ . \quad (37)$$

* I am indebted to BENJAMIN WILLIAMSON, F.T.C.D., for these formulæ.

Let us now proceed to express (20) and (21) in terms of u,

$$\left(\frac{p}{p_m}\right)^3 = 2\cdot35 + 4\cdot57 \cos v + 0\cdot45 \cos 2v \quad (20)$$

neglecting smaller terms, and v is reckoned from Perigee, which occurred at noon, on 1st July.

The angle $u=0$ at midnight of July 12th, and $nt=0$ at Perigee at noon of July 1st.

$$\begin{array}{lrr} u=0, \text{ at July} & 12^{d} & 12^{h} \\ nt=0 \quad ,, & 1 & 0 \\ \hline \text{Diff.} & 11^{d} & 12^{h} \end{array}$$

This is equivalent to 152°, since the periodic time is 27·3 days.

Hence we have

$$nt - 152^\circ = \frac{140}{273}u$$

$$2nt = 304^\circ + \frac{280}{273}u$$

Substituting this value in (34) and (35) we obtain

$$\begin{aligned} M' \cos \overline{(m-s-i_m)} = & +0\cdot41 \text{ M} \sin \phi \\ & + \text{M}(-0\cdot35 \cos \phi + 0\cdot29 \sin \phi) \cos\left(\frac{280}{273}u\right) \\ & + \text{M}(-0\cdot34 \cos \phi - 0\cdot30 \sin \phi) \sin\left(\frac{280}{273}u\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (38) \end{aligned}$$

$$\begin{aligned} M' \sin \overline{(m-s-i_m)} = & -0\cdot41 \text{ M} \cos \phi) \\ & + \text{M}(-0\cdot30 \cos \phi - 0\cdot35 \sin \phi) \cos\left(\frac{280}{273}u\right) \\ & + \text{M}(0\cdot30 \cos \phi - 0\cdot34 \sin \phi) \sin\left(\frac{280}{273}u\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (39) \end{aligned}$$

As a first approximation, $\frac{280}{273}u$ may be made equal to u, and afterwards, if necessary, the expansions (36) and (37) employed.

Equations (17) and (18) give us

$$\begin{aligned} A_1 &= M' \cos(m-s-i_m) + S \cos 2\sigma \cos i_s \\ B_1 &= -M' \sin(m-s-i_m) + S \cos 2\sigma \sin i_s \end{aligned}$$

or, since $\sigma=22^\circ$, when $u=0$, on 12th July, 1859,

$$\begin{aligned} A_1 &= M' \cos(m-s-i_m) + 0\cdot69S \cos i_s \\ B_1 &= -M' \sin(m-s-i_m) + 0\cdot69S \sin i_s \end{aligned}$$

or, from (38) and (39),

$$\begin{aligned}A_1=&\{0\cdot69S\cos i_s+0\cdot41M\sin\phi\}\\&+M\{-0\cdot35\cos\phi+0\cdot29\sin\phi\}\cos u\\&+M\{-0\cdot34\cos\phi-0\cdot30\sin\phi\}\sin u \quad \ldots \ldots \quad (40)\end{aligned}$$

$$\begin{aligned}B_1=&\{0\cdot69S\sin i_s-0\cdot41M\cos\phi\}\\&+M\{0\cdot30\cos\phi+0\cdot35\sin\phi\}\cos u\\&+M\{-0\cdot30\cos\phi+0\cdot34\sin\phi\}\sin u \quad \ldots \ldots \quad (41)\end{aligned}$$

We have, therefore, from Tables III. and IV.,

$$0\cdot69S\cos i_s+0\cdot41M\sin\phi=-11\cdot7 \quad \ldots \ldots \quad (a)$$
$$M(-0\cdot35\cos\phi+0\cdot29\sin\phi)=+8\cdot4 \quad \ldots \ldots \quad (b)$$
$$M(-0\cdot34\cos\phi-0\cdot30\sin\phi)=-0\cdot5 \quad \ldots \ldots \quad (c)$$
$$0\cdot69S\sin i_s-0\cdot41M\cos\phi=-13\cdot2 \quad \ldots \ldots \quad (d)$$
$$+M(0\cdot30\cos\phi+0\cdot35\sin\phi)=+1\cdot4 \quad \ldots \ldots \quad (e)$$
$$+M(-0\cdot30\cos\phi+0\cdot34\sin\phi)=+8\cdot0 \quad \ldots \ldots \quad (f)$$

From (*b*) and (*c*) we find

$$M=19\cdot4 \text{ inches}, \quad \phi=128^\circ\ 5',$$

choosing that value of ϕ which makes M positive in the equations (*b*) and (*c*).

From (*e*) and (*f*) we find

$$M=17\cdot6 \text{ inches}, \quad \phi=129^\circ\ 30'.$$

These values of M and ϕ agree very well, and taking means, we have

$$\begin{aligned}M&=18\cdot5 \text{ inches}\\ \phi&=128^\circ\ 48' \quad \ldots \ldots \quad (42)\end{aligned}$$

but

$$\phi=-c+s'-i_m,$$

and Conjunction happened at June $29^d\ 0^h$
and ascending node . . . July $19^d\ 17^h$

Diff. $20^\circ\cdot7$

Hence

$$c=\frac{20\cdot7\times360}{29\cdot5}=252^\circ\ 30'$$

and s' increases at the rate of a degree per day, or $s'=20^\circ\ 42'$.

Therefore

$$\begin{aligned}i_m&=-c+s'-\phi\\&=-360^\circ\ 36'=-0^\circ\ 36'\end{aligned}$$

or, converting the arc into time

$$i_m=-2^m\cdot48 \quad \ldots \ldots \quad (43)$$

From (a) and (d) we obtain

$$0{\cdot}69\mathrm{S}\cos i_s + 0{\cdot}41\mathrm{M}\sin\phi = -11{\cdot}7 \quad (a)$$
$$0{\cdot}69\mathrm{S}\sin i_s - 0{\cdot}41\mathrm{M}\cos\phi = -13{\cdot}2 \quad (d)$$

Introducing the mean values of M and ϕ, (42) we find

$$\mathrm{S} = 36{\cdot}4 \text{ inches}$$
$$i_s = 45^\circ\ 26' = 3^{\mathrm{h}}\ 2^{\mathrm{m}} \quad \text{. . . . .} \quad (44)$$

We thus find, finally, from the present and former calculations,

Diurnal Tidal Constants.

I. Hourly Observations.

$\mathrm{M} = 18{\cdot}5$ inches
$i_m = -2^{\mathrm{m}}{\cdot}48$
$\mathrm{S} = 36{\cdot}4$ inches
$i_s = +3^{\mathrm{h}}\ 2^{\mathrm{m}}$

II. High and Low Water Observations.

$\mathrm{M} = 20{\cdot}9$ inches
$i_m = +33^{\mathrm{m}}{\cdot}8$
$\mathrm{S} = 23{\cdot}4$ inches
$i_s = +5^{\mathrm{h}}\ 12^{\mathrm{m}}$

From the preceding values, it is evident, that the present more complete investigation fully confirms my former conclusion, as to the unusually great magnitude of the Solar Diurnal Tide at Port Kennedy.

III. *Semidiurnal Tide.*

We must now calculate the values of A_2 and B_2 from the equations (9–14). We thus obtain

TABLE V.—(A_2).

	a_0 inches.	a_1 inches.	b_1 inches.
	+2·9	−16·3	− 9·3
	+3·7	−14·7	− 9·3
	+4·2	−13·8	− 8·9
	+4·4	−13·5	− 8·5
	+4·1	−13·7	− 9·2
	+3·4	−13·4	−10·6
	+2·8	−12·7	−10·5
	+1·5	−11·8	−12·6
	+1·1	− 9·6	−12·6
Mean	+3·1	−13·3	−10·2

TABLE VI.—(B_2).

	a_0 inches.	a_1 inches.	β_1 inches.
	+3·6	+ 5·2	−15·2
	+4·1	+ 6·3	−15·2
	+5·1	+ 8·0	−14·3
	+6·2	+ 9·5	−12·4
	+7·2	+ 9·9	− 9·2
	+7·5	+ 9·8	−10·8
	+7·0	+10·4	−10·7
	+6·4	+11·5	−11·2
	+6·8	+12·9	−11·2
Mean	+6·0	+ 8·2	−12·2

These results are to be compared with the expansion of the formula for the Semidiurnal Tide, viz.

$$M' \cos 2\,(m-i_m) + S' \cos 2\,(s-i_s) \quad \ldots \ldots \quad (45)$$

from which we obtain as before, writing

$$m=s+m-s$$
$$A_2=M' \cos 2(m-s-i_m)+S' \cos 2i_s$$
$$B_2=M' \sin 2(m-s-i_m)+S' \sin 2i_s \quad \ldots \ldots \quad (46)$$

where

$$M'=M\left(\frac{p}{p_m}\right)^3 \cos^2 \bar{\mu}$$
$$S'=S\left(\frac{p}{p_m}\right)^3 \cos^2 \bar{\sigma} \quad \ldots \ldots \quad (47)$$

Neglecting the eccentricity of the Lunar Orbit, for a first approximation, we have

$$M'=M(\cos {}^2\mu=M(1-\sin^2 \iota \sin^2 (v-n))$$
$$=M(1-0{\cdot}215 \sin^2 (v-n))$$
$$=M(0{\cdot}89+0{\cdot}11 \cos 2\,(v-n))$$

or since $n=247°$,

$$M'=M(0{\cdot}89-0{\cdot}76 \cos 2v+0{\cdot}79 \sin 2v) \quad \ldots \ldots \quad (48)$$

We have also

$$2(m-s)-2i_m=-2(m'-s')-2i_m$$
$$=-2(m'-c)+2(s'-c-i_m)$$
$$=-2(m'-c)+2\phi$$

where as before

$$\phi=s'-c-i_m \quad \ldots \ldots \quad (49)$$

From (29) we find

$$\cos(2m'-c) = +0{\cdot}16 - 0{\cdot}71 \cos 2v + 0{\cdot}72 \sin 2v$$

$$\sin 2(m'-c) = -0{\cdot}72 \cos 2v - 0{\cdot}69 \sin 2v \quad \ldots \quad (50)$$

Hence we find

$$\begin{aligned} \cos 2(m-s-i_m) &= \cos 2(-(m'-c)+\phi) \\ &= 0{\cdot}16 \cos 2\phi - \{0{\cdot}71 \cos 2\phi - 0{\cdot}72 \sin 2\phi\} \cos 2v \\ &\quad + \{0{\cdot}72 \cos 2\phi - 0{\cdot}69 \sin 2\phi\} \sin 2v \quad \ldots \quad (51) \end{aligned}$$

$$\begin{aligned} \sin 2(m-s-i_m) &= \sin 2(-(m'-c)+\phi) \\ &= 0{\cdot}16 \sin 2\phi + \{0{\cdot}72 \cos 2\phi - 0{\cdot}71 \sin 2\phi\} \cos 2v \\ &\quad + \{0{\cdot}69 \cos 2\phi + 0{\cdot}72 \sin 2\phi\} \sin 2v \end{aligned}$$

If we make $v=u$ (as an approximation) in (48) and (51), we find writing

$$\begin{aligned} A &= 0{\cdot}71 \cos 2\phi + 0{\cdot}72 \sin 2\phi \\ B &= 0{\cdot}72 \cos 2\phi - 0{\cdot}69 \sin 2\phi \\ A' &= 0{\cdot}72 \cos 2\phi - 0{\cdot}69 \sin 2\phi \\ B' &= 0{\cdot}69 \cos 2\phi + 0{\cdot}72 \sin 2\phi \quad \ldots \quad (52) \end{aligned}$$

$$\begin{aligned} M' \cos 2(m-s-i_m) &= M \begin{pmatrix} 0{\cdot}14 \cos 2\phi \\ +0{\cdot}38A + 0{\cdot}40B \end{pmatrix} \\ &\quad - M\{0{\cdot}89A + 0{\cdot}12 \cos 2\phi\} \cos 2v \\ &\quad + M\{0{\cdot}89B + 0{\cdot}12 \cos 2\phi\} \sin 2v + \&c. \\ M' \sin 2(m-s-i_m) &= M \begin{pmatrix} 0{\cdot}14 \sin 2\phi \\ -0{\cdot}38A' + 0{\cdot}40B' \end{pmatrix} \quad \ldots \quad (52) \\ &\quad + M\{0{\cdot}89A' - 0{\cdot}12 \sin 2\phi\} \cos 2v \\ &\quad + M\{0{\cdot}89B' + 0{\cdot}12 \sin 2\phi\} \sin 2v + \&c. \end{aligned}$$

Hence, from (46), since $S' = S \cos 2\sigma$, and $\sigma = 22°$, we find

$$\begin{aligned} A_2 &= M\{0{\cdot}14 \cos 2\phi + 0{\cdot}38A + 0{\cdot}40B\} + 0{\cdot}86S \cos 2i_s \\ &\quad - M\{0{\cdot}89A + 0{\cdot}12 \cos 2\phi\} \cos 2v \\ &\quad + M\{0{\cdot}89B + 0{\cdot}12 \cos 2\phi\} \sin 2v + \&c. \\ B_2 &= M\{-0{\cdot}14 \sin 2\phi + 0{\cdot}38A' - 0{\cdot}40B'\} + 0{\cdot}86\, S \sin 2i_s \\ &\quad - M\{0{\cdot}89A' - 0{\cdot}12 \sin 2\phi\} \cos 2v \\ &\quad - M\{0{\cdot}89B' + 0{\cdot}12 \sin 2\phi\} \sin 2v + \&c. \end{aligned}$$

Comparing these expansions with Tables V. and VI. we find

$$M\{0{\cdot}14 \cos 2\phi + 0{\cdot}38\, A + 0{\cdot}40\, B\} + 0{\cdot}86\, S \cos 2i_s = 3{\cdot}1 \quad \ldots \quad (a')$$

$$M\{0{\cdot}89\, A + 0{\cdot}12 \cos 2\phi\} = 13{\cdot}3 \quad \ldots \quad (b')$$

$$M\{0{\cdot}89\, B + 0{\cdot}12 \cos 2\phi\} = -10{\cdot}2 \quad \ldots \quad (c')$$

$$M\{-0{\cdot}14 \sin 2\phi + 0{\cdot}38\, A' - 0{\cdot}40\, B'\} + 0{\cdot}86\, S \sin 2i_s = 6{\cdot}0 \quad \ldots \quad (d')$$

$$M\{-0{\cdot}89\, A' - 0{\cdot}12 \sin 2\phi\} = -8{\cdot}2 \quad \ldots \quad (e')$$

$$M\{0{\cdot}89\, B' + 0{\cdot}12 \sin 2\phi\} = 12{\cdot}2 \quad \ldots \quad (f')$$

Substituting, in these equations, the values of A, B, A′, B′, we find

$$
\begin{aligned}
&0{\cdot}60\,\mathrm{M}\cos 2\phi+0{\cdot}86\,\mathrm{S}\cos i_s=3{\cdot}1 \quad \ldots\ldots \quad (a') \\
&\mathrm{M}\,(0{\cdot}75\cos 2\phi+0{\cdot}64\sin 2\phi)=13{\cdot}3. \quad \ldots\ldots \quad (b') \\
&\mathrm{M}\,(0{\cdot}76\cos 2\phi-0{\cdot}61\sin 2\phi)=-10{\cdot}2\,. \quad \ldots\ldots \quad (c') \\
&-0{\cdot}60\,\mathrm{M}\sin 2\phi+0{\cdot}86\,\mathrm{S}\sin i_s=6{\cdot}0\,. \quad \ldots\ldots \quad (d') \\
&\mathrm{M}\,(0{\cdot}64\cos 2\phi-0{\cdot}75\sin 2\phi=-8{\cdot}2 \quad \ldots\ldots \quad (e') \\
&\mathrm{M}\,(0{\cdot}61\cos 2\phi+0{\cdot}76\sin 2\phi)=12{\cdot}2 \quad \ldots\ldots \quad (f') \qquad (53)
\end{aligned}
$$

From (b') and (c') we find, remembering that M must be positive,

$$2\phi=84^\circ\ 48', \quad \mathrm{M}=19{\cdot}0 \text{ inches}$$

and from (d') and (f') we find

$$2\phi=77^\circ\ 10', \quad \mathrm{M}=14{\cdot}0 \text{ inches.}$$

Mean values

$$2\phi=80^\circ\ 59', \quad \mathrm{M}=15{\cdot}5 \text{ inches.} \quad \ldots\ldots \quad (54)$$

Hence we find, since,

$$i_m=s'-c-\phi,$$

using the values of s' and c already given,

$$i_m=87^\circ\ 42'=6^{\mathrm{h}}\ 2^{\mathrm{m}}\ 5^{\mathrm{s}}\,. \quad \ldots\ldots \quad (55)$$

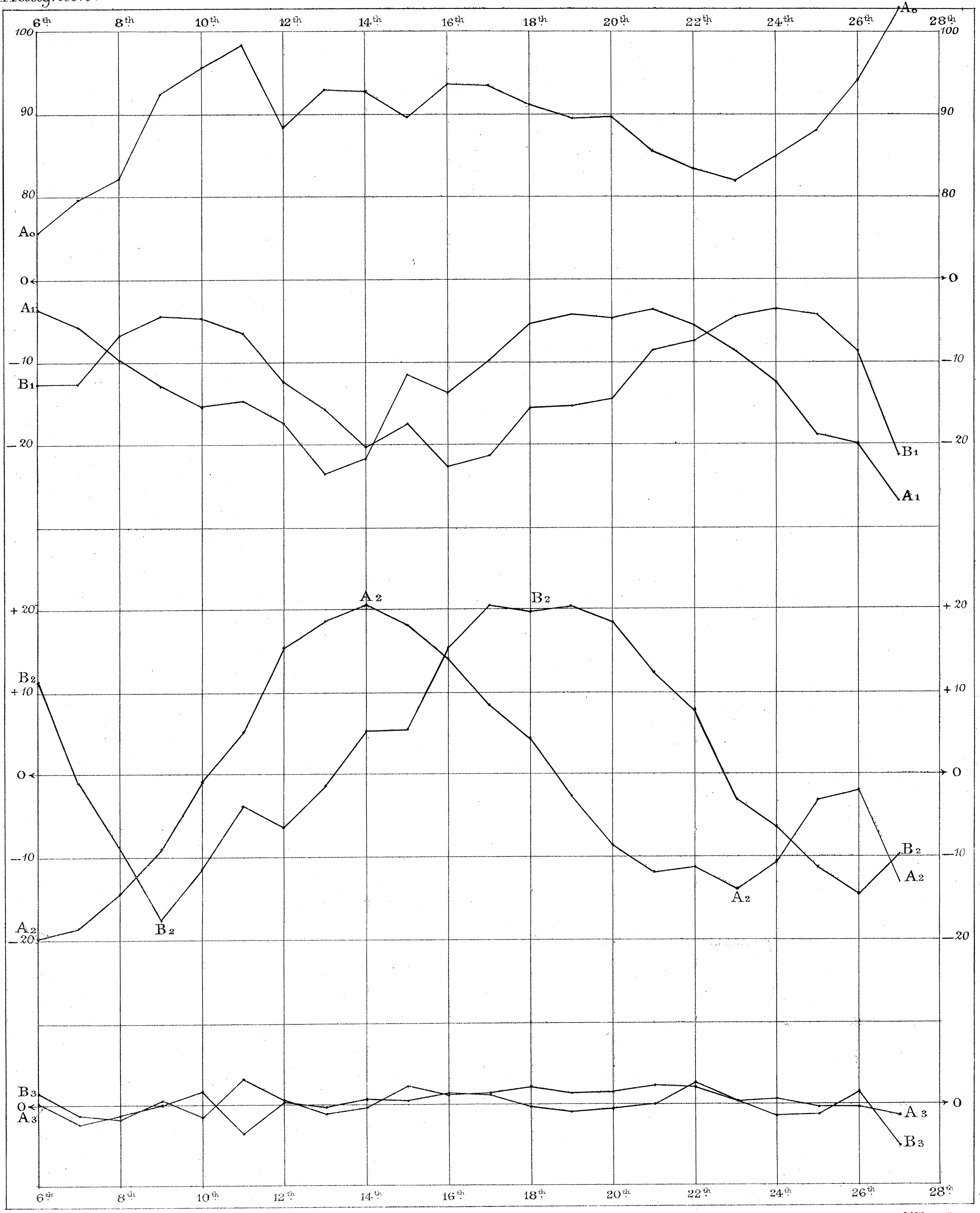
From (a') and (d') we find, using the mean values of M and $2\,\phi$,

$$\mathrm{S}=5{\cdot}9 \text{ inches.} \qquad i_s=41^\circ\ 55'=2^{\mathrm{h}}\ 48^{\mathrm{m}} \quad \ldots\ldots \quad (56)$$

We thus find, finally, from the present and former calculations,

Semidiurnal Tidal Constants.

I.—*Hourly Observations.*	II.—*H. and L. W. Observations.*
$\mathrm{M}=15{\cdot}5$ inches	$\mathrm{M}=17{\cdot}0$ inches
$i_m=6^{\mathrm{h}}\ 2^{\mathrm{m}}\tfrac{1}{2}$	$i_m=23^{\mathrm{h}}\ 48^{\mathrm{m}}$
$\mathrm{S}=5{\cdot}9$ inches	$\mathrm{S}=7{\cdot}0$ inches
$i_s=2^{\mathrm{h}}\ 48^{\mathrm{m}}$	$i^s=$ ——
$\dfrac{\mathrm{S}}{\mathrm{M}}=0{\cdot}39$	$\dfrac{\mathrm{S}}{\mathrm{M}}=0{\cdot}41$



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