IX. *On the Vibrations of a Vortex Ring, and the Action upon each other of Two Vortices in a Perfect Fluid.*

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The following paper contains (1) a discussion of the vibrations which take place in the axis of the core of a vortex ring whose section is very small in comparison with its aperture when the axis is made to deviate slightly from the circular form; and (2) a discussion of the action upon each other of two vortex rings which move in such a way that they never approach nearer than a large multiple of the diameter of either.

The fluid in which these vortices exist is supposed to be frictionless and incompressible.

The method which I have employed is the same in both cases, and is purely kinematical. It is merely the application of the fact that if $F(x, y, z, t)=0$ be any equation to a surface which always consists of the same particles then

$$\frac{dF}{dt}+u\frac{dF}{dx}+v\frac{dF}{dy}+w\frac{dF}{dz}=0$$

where u, v, w are the velocities of the particle at (x, y, z) along the axis of x, y, z respectively, and where the differential coefficients are partial.

The surface of a vortex ring is evidently a surface of this kind, and the equation just written is the condition that $F(x, y, z, t)=0$ should be the equation to the surface of a vortex ring. I have found that this condition, joined to the ordinary expressions for the velocity due to a vortex element, is sufficient to solve the problems discussed in this paper. This is an instance of the large number of problems in vortex motion which are capable of purely kinematical solution; indeed, a vortex theory of gases would be entirely kinematical so long as we only considered the molecules of gas themselves and not their effects upon the containing vessel, &c. For example, in this theory when two atoms clash, the problem of finding their subsequent motion must be capable of solution by purely kinematical considerations, but in the ordinary theory of

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gases the "clash of atoms" involves dynamical considerations of very considerable complexity. This is a consequence of the vortex theory being of a much more fundamental character than the ordinary one that atoms consist of small pieces of solid matter.

Problem I. To find the vibrations of the circular axis of a vortex ring.

Using cylindrical coordinates let the equations to the axis of the core be

$$\rho = a + \Sigma \alpha_n \cos n\theta$$
$$z = \mathfrak{z} + \Sigma \beta_n \cos n\theta$$

where α_n and β_n are small compared with a the radius of the core when undisturbed; the summation over all integer values of n between zero and infinity. The axis of z is perpendicular to the plane of the vortex, and θ is measured from the axis of x as initial line.

The velocity due to a distribution of vortices is proportional to the magnetic force produced by a system of currents arranged in exactly the same way as the vortices and of the same strength.

Now the vortex filaments we are considering are distributed uniformly (or very approximately so)* in a ring the radius of whose transverse section is very small in comparison with the radius of the aperture. Now if electric currents flow uniformly through a conductor of such a shape the magnetic action at a point outside or on the surface of the conductor is the same as if all the currents were condensed into one flowing along the axis.† Hence when finding the velocities outside the vortex ring we may suppose the vortices condensed into one at the axis of the core. If ω be the angular velocity of molecular rotation, e the radius of the transverse section of the core, then $\pi e^2\omega$ is the strength of the vortex which we must suppose placed at the axis of the core. We shall for brevity denote $\pi e^2\omega$ by m.

The components (u, v, w) of the velocity at the point (x, y, z) are given by

$$u = \frac{m}{2\pi}\int_0^{s'} \frac{1}{r^3}\left\{\frac{dz'}{ds'}(y'-y) - \frac{dy'}{ds'}(z'-z)\right\} ds'$$
$$v = \frac{m}{2\pi}\int_0^{s'} \frac{1}{r^3}\left\{\frac{dx'}{ds'}(z'-z) - \frac{dz'}{ds'}(x'-x)\right\} ds'$$
$$w = \frac{m}{2\pi}\int_0^{s'} \frac{1}{r^3}\left\{\frac{dy'}{ds'}(x'-x) - \frac{dx'}{ds'}(y'-y)\right\} ds'$$

where r is the distance of the point (x, y, z) from the point (x', y', z'), a point on the vortex whose polar coordinates are (ρ', θ); s' is an arc of the vortex ring.

* See a note by Sir W. THOMSON at the end of HELMHOLTZ's paper on "Vortex Motion," Phil. Mag., 1867.

† MAXWELL's 'Electricity and Magnetism,' 2nd edition, §. 683.

Now from the equation to the axis of the vortex we have

$$dx'=d(\rho'\cos\theta)=d\theta\Sigma(-n\alpha_n\sin n\theta\cos\theta-a\sin\theta-\alpha_n\cos n\theta\sin\theta)$$
$$dy'=d(\rho'\sin\theta)=d\theta\Sigma(-n\alpha_n\sin n\theta\cos\theta+a\cos\theta+\alpha_n\cos n\theta\cos\theta)$$
$$dz'=-d\theta\Sigma(n\beta_n\sin n\theta)$$

neglecting α_n^2 and β_n^2

$$ds'=d\theta(a+\Sigma\alpha_n\cos n\theta)$$

and

$$\frac{dx'}{ds'}=-\sin\theta-\frac{1}{a}\Sigma n\alpha_n\cos\theta\sin n\theta$$
$$\frac{dy'}{ds'}=\cos\theta-\frac{1}{a}\Sigma n\alpha_n\sin\theta\sin n\theta$$
$$\frac{dz'}{ds'}=-\frac{1}{a}\Sigma n\beta_n\sin n\theta$$

If ρ, ψ, $\mathfrak{z}+\zeta$ be the cylindrical coordinates of the point x, y, z

$$r^2=\rho^2+\rho'^2+(\zeta-\Sigma\beta_n\cos n\theta)^2-2\rho\rho'\cos(\theta-\psi)$$

say

$$r^2=\rho^2+\rho'^2+\zeta'^2-2\rho\rho'\cos(\theta-\psi)$$

Let

$$\frac{1}{\{\rho^2+\rho'^2+\zeta'-2\rho\rho'\cos(\theta-\psi)\}^{\frac{3}{2}}}=C_0+C_1\cos(\theta-\psi)+\ldots C_n\cos n(\theta-\psi)$$

where the C's are functions of ρ, ρ', and ζ'.

Since ρ' and ζ' are functions of θ, C_0, C_1, ... C_n will be functions of θ, but since θ only enters into ρ' and ζ' in the form $\alpha_n\cos n\theta$, $\beta_n\cos n\theta$, the terms in the C's which involve θ will be multiplied by α_n or β_n and so will be small.

If

$$\frac{1}{\{a^2+\rho^2+\zeta^2-2a\rho\cos(\theta-\psi)\}^{\frac{3}{2}}}=A_0+A_1\cos(\theta-\psi)+\ldots A_n\cos n(\theta-\psi)$$

then

$$C_m=A_m+\Sigma\alpha_n\cos n\theta\frac{dA_m}{da}$$
$$+2\zeta\Sigma\beta_n\cos n\theta\frac{dA_n}{d\zeta^2}$$

$+$ terms of higher dimensions in α_n and β_n.

We shall only require those expressions for a point nearly in the plane of the vortex where ζ is very small, so that in this case

$$C_m=A_m+\Sigma\alpha_n\cos n\theta\frac{dA_m}{da}$$

We can determine the components of the velocity in terms of the quantities we have denoted by $A_0 \ldots A_n$, we shall for the sake of clearness divide the determination up into several steps.

To determine the parts of u, v, w independent of α_n and β_n, in which we shall not suppose ζ small,

$$u=\frac{m}{2\pi}\int_0^{2\pi}\cos\theta\zeta\{A_0+A_1\cos(\theta-\psi)+\ldots A_n\cos n(\theta-\psi)\}ad\theta$$

$$=\frac{ma}{2\pi}\zeta\int_0^{2\pi}A_1\cos^2\theta\cos\psi d\theta=\tfrac{1}{2}ma\zeta A_1\cos\psi \quad \ldots \quad (1)$$

$$v=\frac{m}{2\pi}\int_0^{2\pi}\sin\theta\zeta(A_0+A_1\cos(\theta-\psi)+\ldots)ad\theta=\tfrac{1}{2}ma\zeta A_1\sin\psi \quad \ldots \quad (2)$$

$$w=\frac{m}{2\pi}\int_0^{2\pi}\{\cos\theta(a\cos\theta-\rho\cos\psi)+\sin\theta(a\sin\theta-\rho\sin\psi)\}(A_0+A_1\cos\overline{\theta-\psi}+\ldots)ad\theta$$

$$=\frac{ma}{2\pi}\int_0^{2\pi}\{a-\rho\cos(\theta-\psi)\}(A_0+A_1\cos(\theta-\psi)+\ldots)d\theta$$

$$=\frac{ma}{2\pi}(A_0a2\pi-A_1\rho\pi)$$

$$=\tfrac{1}{2}ma\{2A_0a-A_1\rho\} \quad \ldots \quad (3)$$

These are the velocities due to the undisturbed vortex, and in using them in the second half of the paper we require A_0, A_1 determined without supposing ζ to be small.

2nd. The values of u, v, w arising from small terms in ds'.

As far as now concerned,

$$ds'=d\theta\alpha_n\cos n\theta$$

$u=0$, $v=0$ because they involve $\zeta\alpha_n$.

$$w=\frac{m}{2\pi}\int_0^{2\pi}(a+\rho\cos(\theta-\psi))[A_0+A_1\cos(\theta-\psi)+\ldots]\alpha_n\cos n\theta d\theta$$

$$=\frac{m}{2\pi}\int_0^{2\pi}[aA_n\cos n(\theta-\psi)-\tfrac{1}{2}\rho A_{n+1}\cos n(\theta-\psi)-\tfrac{1}{2}\rho A_{n-1}\cos n(\theta-\psi)]\alpha_n\cos n\theta d\theta$$

$$=\tfrac{1}{2}m\alpha_n[aA_n-\tfrac{1}{2}\rho(A_{n+1}+A_{n-1})]\cos n\psi$$

3rd. Small terms in $\frac{dx'}{ds'}, \frac{dy'}{ds'}, \frac{dz'}{ds'}$.

$$u=\frac{m}{2\pi}\int_0^{2\pi}-\frac{n\beta_n}{a}\sin n\theta(a\sin\theta-\rho\sin\psi)[\mathrm{A}_0+\mathrm{A}_1\cos(\theta-\psi)+\ldots]ad\theta$$
$$=\frac{ma}{2\pi}\int_0^{2\pi}\tfrac{1}{2}u\beta_n(\cos(n+1)\theta-\cos(n-1)\theta)+\frac{n\beta_n}{a}\rho\sin\psi\sin n\theta[\mathrm{A}_0+\mathrm{A}_1\cos(\theta-\psi)]d\theta$$
$$=\tfrac{1}{2}ma\left\{n\beta_n(\mathrm{A}_{n+1}\cos(n+1)\psi-\mathrm{A}_{n-1}\cos(n-1)\psi)+\frac{n\beta_n}{a}\rho\sin\psi\sin n\psi\mathrm{A}_n\right\}$$
$$=\tfrac{1}{4}man\beta_n\left\{\cos(n+1)\psi\left(\mathrm{A}_{n+1}-\frac{\rho}{a}\mathrm{A}_n\right)-\cos(n-1)\psi\left(\mathrm{A}_{n-1}-\frac{\rho}{a}\mathrm{A}_n\right)\right\}$$

$$v=\frac{m}{2\pi}\int_0^{2\pi}n\beta_n\sin n\theta(a\cos\theta-\rho\cos\psi)[\mathrm{A}_0+\mathrm{A}_1\cos(\theta-\psi)-]ad\theta$$
$$=\tfrac{1}{4}man\beta_n\left\{\sin(n+1)\psi\left(\mathrm{A}_{n+1}-\frac{\rho}{a}\mathrm{A}_n\right)+\sin(n-1)\psi\left(\mathrm{A}_{n-1}-\frac{\rho}{a}\mathrm{A}_n\right)\right\}$$

$$w=\frac{m}{2\pi}\int_0^{2\pi}\left\{-\frac{n\alpha_n}{a}\sin\theta\sin n\theta(a\cos\theta-\rho\cos\psi)\right.$$
$$\left.+\frac{n\alpha_n}{a}\cos\theta\sin n\theta(a\sin\theta-\rho\sin\psi)\right\}(\mathrm{A}_0+\mathrm{A}_1\cos(\theta-\psi)-)ad\theta$$
$$=\frac{m}{2\pi}n\alpha_n\int_0^{2\pi}\{\tfrac{1}{2}\rho\cos\psi(\cos(n-1)\theta-\cos(n+1)\theta)$$
$$-\tfrac{1}{2}\rho\sin\psi(\sin(n+1)\theta+\sin(n-1)\theta)(\mathrm{A}_0+\mathrm{A}_1\cos(\theta-\psi))ad\theta\}$$
$$=\tfrac{1}{2}mn\alpha_n[\tfrac{1}{2}\rho\cos\psi(\mathrm{A}_{n-1}\cos(n-1)\psi-\mathrm{A}_{n+1}\cos(n+1)\psi)$$
$$-\tfrac{1}{2}\rho\sin\psi(\mathrm{A}_{n+1}\sin(n+1)\psi+\mathrm{A}_{n-1}\sin(n-1)\psi)]$$
$$=\tfrac{1}{4}mn\alpha_n\rho(\mathrm{A}_{n-1}-\mathrm{A}_{n+1})\cos n\psi.$$

4th. Small terms in $x'-x$, $y'-y$.

$$u=-\frac{m}{2\pi}\int_0^{2\pi}a\cos\theta\cos n\theta\beta_n(\mathrm{A}_0+\mathrm{A}_1\cos(\theta-\psi)+\ldots)d\theta$$
$$=-\frac{ma}{2\pi}\int_0^{2\pi}\tfrac{1}{2}\beta_n\{\cos(n-1)\theta+\cos(n+1)\theta\}(\mathrm{A}_0+\mathrm{A}_1\cos(\theta-\psi)+\ldots)d\theta$$
$$=-\tfrac{1}{4}ma\beta_n(\mathrm{A}_{n-1}\cos(n-1)\psi+\mathrm{A}_{n+1}\cos(n+1)\psi)$$

$$v=\frac{m}{2\pi}\int_0^{2\pi}-a\sin\theta\cos n\theta\beta_n(\mathrm{A}_0+\mathrm{A}_1\cos(\theta-\psi)+\ldots)d\theta$$
$$=\tfrac{1}{4}ma\beta_n(\mathrm{A}_{n-1}\sin(n-1)\psi-\mathrm{A}_{n+1}\sin(n+1)\psi)$$

$$w=\frac{m}{2\pi}\int_0^{2\pi}(\cos\theta(\alpha_n\cos n\theta\cos\theta)+\sin\theta(\alpha_n\cos n\theta\sin\theta)\}(\mathrm{A}_0+\mathrm{A}_1\cos(\theta-\psi))ad\theta$$
$$=\frac{m}{2\pi}\int_0^{2\pi}\alpha_n\cos n\theta(\mathrm{A}_0+\mathrm{A}_1\cos(\theta-\psi)+\ldots)ad\theta$$
$$=\tfrac{1}{2}ma\alpha_n\mathrm{A}_n\cos n\psi.$$

5th. Small terms arising from C_n containing $\cos n\theta$. These are

$$u=0,\ v=0$$

$$w=\frac{m}{2\pi}\int_0^{2\pi}(a-\rho\cos(\theta-\psi))\left(\alpha_n\cos n\theta\frac{dA_0}{da}+-\alpha_n\cos n\theta\frac{dA_m}{da}\cos m(\theta-\psi)\ldots\right)ad\theta$$

$$=\tfrac{1}{2}m\left\{a^2\alpha_n\frac{dA_n}{da}\cos n\psi-\frac{\rho a\alpha_n}{2}\left(\frac{dA_{n+1}}{da}+\frac{dA_{n-1}}{da}\right)\cos n\psi\right\}$$

$$=\tfrac{1}{2}m\alpha_n\left\{a^2\frac{dA_n}{da}-\tfrac{1}{2}\rho a\left(\frac{dA_{n+1}}{da}+\frac{dA_{n-1}}{da}\right)\right\}\cos n\psi.$$

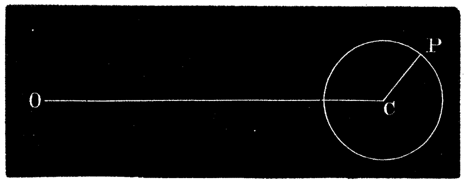
Collecting the terms we find

$$u=\tfrac{1}{2}ma\Big\{\zeta A_1\cos\psi+\tfrac{1}{2}\beta_n(n-1)A_{n+1}\cos(n+1)\psi-(n+1)A_{n-1}\cos(n-1)\psi)$$
$$+\tfrac{1}{2}n\beta_nA_n\frac{\rho}{a}(\cos(n-1)\psi-\cos(n+1)\psi)\Big\}\ . \quad (4)$$

$$v=\tfrac{1}{2}ma\Big\{\zeta A_1\sin\psi+\tfrac{1}{2}\beta_n((n-1)A_{n+1}\sin(n+1)\psi+(n+1)A_{n-1}\sin(n-1)\psi)$$
$$-\tfrac{1}{2}n\beta_nA_n\frac{\rho}{a}(\sin(n+1)\psi+\sin(n-1)\psi)\Big\}\ . \quad (5)$$

$$w=\tfrac{1}{2}ma\Big\{2aA_0-\rho A_1+\left(2\alpha_nA_n+\tfrac{1}{2}\alpha_n\frac{\rho}{a}((n-1)A_{n-1}-(n+1)A_{n+1})\right)\cos n\psi$$
$$+\alpha_n\left[a\frac{dA_n}{da}-\rho\left(\frac{dA_{n+1}}{da}+\frac{dA_{n-1}}{da}\right)\right]\cos n\psi\Big\}\ . \quad (6)$$

Fig. 1.



Let the figure represent a section of the vortex ring by a plane through its straight axis. Let ϕ be the angle which the radius vector drawn from C the centre of the section of the core to any point P on the surface of the ring makes with the straight axis of the ring. Let C P$=e$.

Then the equations to the surface of the core are

$$\rho=a+\Sigma\alpha_n\cos n\psi+e\sin\phi$$
$$z=\mathfrak{z}+\Sigma\beta_n\cos n\psi+e\cos\phi$$

Since the vortex rings always consists of the same particles if $F(\rho, \theta, \phi)=0$ be an equation to its surface, we must have

$$\frac{dF}{dt}+\frac{dR}{d\rho}.R+\frac{dF}{d\psi}.\Psi+\frac{dF}{d\phi}.\Phi=0$$

when the differential coefficients are partial.

R is the velocity in the direction in which ρ is measured, Ψ the angular velocity round the axis of z, and Φ the angular velocity of C P round a normal to the plane containing the axis of z and O C.

Applying this equation to the first of the equations to the core, we get

$$\Sigma\dot{\alpha}_n \cos n\psi-R-\Sigma n\alpha_n \sin n\psi.\Psi+e \cos \phi.\Phi=0$$

or

$$R=\Sigma\dot{\alpha}_n \cos n\psi-\Sigma n\alpha_n \sin n\psi.\Psi+e \cos \phi.\Phi$$

and in a similar way we find

$$w=\dot{\mathfrak{z}}+\Sigma(\dot{\beta}_n \cos n\psi-n\beta_n \sin n\psi\Psi)-e \sin \phi.\Phi$$

where w is the velocity of a point on the surface of the core parallel to the axis of z. Now Ψ is zero when α_n and β_n are both zero, and it will be small in this case since α_n and β_n are both small, hence neglecting the squares of small quantities, these equations become

$$R=\Sigma\dot{\alpha}_n \cos n\psi+e \cos \phi.\Phi \quad \text{. . . . .} \quad (7)$$

$$w=\dot{\mathfrak{z}}+\Sigma\dot{\beta}_n \cos n\psi-e \sin \phi.\Phi \quad \text{. . . . .} \quad (8)$$

But $R=u \cos \psi+v \sin \psi$.

Substituting for u and v the values given in equations (4) and (5), we get

$$R=\tfrac{1}{2}ma\{\zeta A_1+\tfrac{1}{2}\Sigma\beta_n \cos n\psi((n-1)A_{n+1}-(n+1)A_{n-1})\}$$

Since the A's are multiplied by the small quantities ζ, α_n, β_n, we may suppose since we neglect quantities of the order α_n^2 that the A's are found on the supposition that α_n and β_n are zero, or that the A's are the same as if the vortex was undisturbed.

Let us denote the value of the A's for the undisturbed vortex by German letters.

Equating the two expressions for R and putting

$$\zeta=\Sigma\beta_n \cos n\psi=e \cos \phi$$

we get

$$\tfrac{1}{2}ma\Sigma\{(\beta_n \cos n\psi+e \cos \phi)\mathfrak{A}_1+\tfrac{1}{2}\beta_n \cos n\psi((n-1)\mathfrak{A}_{n+1}-(n+1)\mathfrak{A}_{n-1})\}$$
$$=\Sigma\dot{\alpha}_n \cos n\psi+e \cos \phi.\Phi$$

equating the coefficients of $\cos \phi$ and $\cos n\psi$ we get

$$\tfrac{1}{2}ma\mathfrak{A}_1=\Phi \quad \text{. . . . .} \quad (9)$$

$$\tfrac{1}{2}ma\beta_n\{\mathfrak{A}_1+\tfrac{1}{2}((n-1)\mathfrak{A}_{n+1}-(n+1)\mathfrak{A}_{n-1})\}=\dot{\alpha}_n \quad \text{. . . . . .} \quad (10)$$

If we equate the value of w from equation (6) to that given by equation (7) we get

$$\tfrac{1}{2}ma\Big(2a\mathrm{A}_0-\rho\mathrm{A}_1+\Sigma\tfrac{1}{2}\tfrac{\rho}{a}.\alpha_n[(n-1)\mathrm{A}_{n-1}-(n+1)\mathrm{A}_{n+1}]\cos n\psi$$
$$+\left\{2\Sigma\alpha_n\mathrm{A}_n+\Sigma\alpha_n\left[\frac{ad\mathrm{A}_n}{da}-\rho\left(\frac{d\mathrm{A}_{n+1}}{da}+\frac{d\mathrm{A}_{n-1}}{da}\right)\right]\right\}\cos n\psi\Big)$$
$$=\dot{\mathfrak{z}}+\Sigma\dot{\beta}_n\cos n\psi-e\sin\phi.\Phi$$

Since the term $2a\mathrm{A}_0-\rho\mathrm{A}_1$ is not multiplied by any small quantity we cannot suppose the A's to have the same value as for the undisturbed vortex, we must substitute for $2a\mathrm{A}_0-\rho\mathrm{A}_1$

$$2a\mathfrak{A}_0-a\mathfrak{A}_1-\alpha_n\cos n\psi\mathfrak{A}_1+\alpha_n\cos n\psi\frac{d}{d\rho}(2a\mathfrak{A}_0-a\mathfrak{A}_1)-e\sin\phi\mathfrak{A}_1$$

Since the other terms are multiplied by small quantities, we may substitute for the A's their undisturbed values. Making these substitutions we get

$$\tfrac{1}{2}ma\Big\{2a\mathfrak{A}_0-a\mathfrak{A}_1-e\sin\phi\mathfrak{A}_1+\Sigma\alpha_n\cos n\psi\Big[\frac{d}{d\rho}(2a\mathfrak{A}_0-a\mathfrak{A}_1)$$
$$+2\mathfrak{A}_n-\mathfrak{A}_1+\tfrac{1}{2}((n-1)\mathfrak{A}_{n-1}-(n+1)\mathfrak{A}_{n+1})+\frac{ad\mathfrak{A}_n}{da}-\frac{a}{2}\left(\frac{d\mathfrak{A}_{n+1}}{da}+\frac{d\mathfrak{A}_{n-1}}{da}\right)\Big]\Big\}$$
$$=\dot{\mathfrak{z}}-e\sin\phi\Phi+\Sigma\dot{\beta}_n\cos n\psi$$

Equating constant terms and the coefficients of $\sin\phi$ and $\cos n\psi$, we get

$$\tfrac{1}{2}ma^2(2\mathfrak{A}_0-\mathfrak{A}_1)=\dot{\mathfrak{z}}\quad\ldots\ldots\ldots\ldots\quad(11)$$

$$\tfrac{1}{2}ma\mathfrak{A}_1=\Phi$$
$$\tfrac{1}{2}ma\alpha_n\Big\{\frac{d}{d\rho}(2a\mathfrak{A}_0-a\mathfrak{A}_1)+2\mathfrak{A}_n-\mathfrak{A}_1+\tfrac{1}{2}((n-1)\mathfrak{A}_{n-1}-(n+1)\mathfrak{A}_{n+1}$$
$$+a\frac{d\mathfrak{A}_n}{da}-\tfrac{1}{2}a\left(\frac{d\mathfrak{A}_{n+1}}{da}+\frac{d\mathfrak{A}_{n-1}}{da}\right)\Big\}=\dot{\beta}_n\,.\quad\ldots\ldots\ldots\quad(12)$$

The first of these equations gives the velocity of translation of an undisturbed circular vortex ring, the second is the same as the one we previously obtained for Φ.

We must now proceed to find the values of the $\mathfrak{A}$'s supposing the transverse section of the vortex core to be small compared with its aperture.

Since

$$\frac{1}{\{a^2+\rho+\zeta^2-2\rho a\cos(\theta-\psi)\}^{\frac{3}{2}}}=\mathfrak{A}_0+\mathfrak{A}_1\cos(\theta-\psi)+\ldots\mathfrak{A}_n\cos n(\theta-\psi)$$

$$\mathfrak{A}_n=\frac{1}{\pi}\int_0^{2\pi}\frac{\cos n\chi.d\chi}{(a^2+\rho^2+\zeta^2-2\rho a\cos\chi)^{\frac{3}{2}}}$$

except when $n=0$ when

$$\mathfrak{A}_0=\frac{1}{2\pi}\int_0^{2\pi}\frac{d\chi}{(a^2+\rho^2+\zeta^2-2\rho a\cos\chi)^{\frac{3}{2}}}$$

Now

$$a^2+\rho^2+\zeta^2-2\rho a\cos\chi=(a+\rho)^2+\zeta^2-4\rho a\cos^2\frac{\chi}{2}=((a+\rho)^2+\zeta^2)\left(1-\kappa^2\cos^2\frac{\chi}{2}\right)$$

where

$$\kappa^2=\frac{4\rho a}{(a+\rho)^2+\zeta^2}$$

Now in the case we are considering ρ is very nearly equal to a and ζ is very small, hence κ is very nearly equal to unity

$$1-\kappa^2\cos^2\frac{\chi}{2}=\kappa_1^{\,2}+\kappa^2\sin^2\frac{\chi}{2}$$

where

$$\kappa_1^{\,2}=1-\kappa^2=\frac{(\rho-a)^2+\zeta^2}{(\rho+a)^2+\zeta^2}$$

and is very small

Therefore
$$\mathfrak{A}_n=\frac{1}{\pi}\int_0^{2\pi}\frac{\cos n\chi d\chi}{((a+\rho)^2+\zeta^2)^{\frac{3}{2}}\left(\kappa_1^{\,2}+\kappa^2\sin^2\frac{\chi}{2}\right)^{\frac{3}{2}}}$$

Now since κ_1 is very small

$$\int_0^{2\pi}\frac{\cos n\chi d\chi}{\left(\kappa_1^{\,2}+\kappa^2\sin^2\frac{\chi}{2}\right)^{\frac{3}{2}}}$$

will be very large, and the large part will arise from very small values of χ, or from values of χ very nearly equal to 2π, the parts arising from small values of χ and from values near to 2π will evidently be equal; hence the integral will be approximately

$$\int_0^{\epsilon}\frac{\left(1-\frac{n^2\chi^2}{2}\right)d\chi}{\left(\kappa_1^{\,2}+\frac{\chi^2}{4}\right)^{\frac{3}{2}}}$$

where ϵ is large compared with κ_1, this integral

$$=\frac{4(1+2n^2\kappa_1^{\,2})}{\kappa_1^{\,2}}\left[\frac{\chi}{(\kappa_1^{\,2}+\chi^2)^{\frac{1}{2}}}\right]_0^{\epsilon}-8n^2\left[\log\left(\chi+\sqrt{\kappa_1^{\,2}+\chi^2}\right)\right]_0^{\epsilon}$$

or, since ϵ is large compared with κ_1, this is approximately

$$\frac{4(1+2n^2\kappa_1^{\,2})}{\kappa_1^{\,2}}-8n^2\log\frac{2\epsilon}{\kappa_1}$$

or retaining only the more important terms

$$=\frac{4}{\kappa_1^{\,2}}+8n^2\log\kappa_1$$

hence

$$\mathfrak{A}_n=\left(\frac{4}{\pi\kappa_1^2}+\frac{8n^2}{\pi}\log\kappa_1\right)\div((a+\rho)^2+\zeta^2)^{\frac{3}{2}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

except when $n=0$ and

$$\mathfrak{A}_0=\frac{2}{\pi\kappa_1^2}\div((a+\rho)^2+\zeta^2)^{\frac{3}{2}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

If we substitute these values for the $\mathfrak{A}$'s the equation

$$\tfrac{1}{2}ma\mathfrak{A}_1=\Phi$$

gives

$$\tfrac{1}{2}ma\left(\frac{4}{\pi\kappa_1^2}+\frac{8}{\pi}\log\kappa_1\right)\div 8a^3=\Phi$$

or, since κ_1 is approximately $e^2/4a^2$, we get if we substitue this value for κ_1 and $\pi e^2\omega$ for m

$$\omega-\tfrac{1}{2}\omega\frac{e^2}{a^2}\log\frac{2a}{e}=\Phi \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

The second term on the left-hand side of this equation being small compared with the first, we get as a rougher approximation

$$\omega=\Phi \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

The equation

$$\tfrac{1}{2}ma\beta_n\{\mathfrak{A}_1+\tfrac{1}{2}((n-1)\mathfrak{A}_{n+1}-(n+1)\mathfrak{A}_{n-1})\}=\dot{\alpha}_n$$

gives on substitution

$$-\frac{\omega e^2}{2a^2}n^2\log\frac{2a}{e}\beta_n=\dot{\alpha}_n \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

Substituting for $\mathfrak{A}_0$, $\mathfrak{A}_1$ in equation (11) we find

$$\dot{\mathfrak{z}}=\frac{\omega e^2}{2a}\log\frac{2a}{e} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

This agrees to the degree of approximation we are working to with the value for the velocity of translation of a circular vortex found by Sir W. THOMSON and given in Professor TAIT's translation of HELMHOLTZ's paper on "Vortex Motion" (Phil. Mag., June, 1867).

The value of ϕ given by equation (16) is also the same as that obtained by Sir W. THOMSON.

Substituting for the $\mathfrak{A}$'s in equation (9) we find

$$\tfrac{1}{2}ma\alpha_n\left\{-\left(\frac{d}{da}+\frac{d}{d\rho}\right)\left[\frac{4}{\pi}\log\frac{(a-\rho)^2+\zeta^2}{(a+\rho)^2+\zeta^2}\Big/((a+\rho)^2+\zeta^2)^{\frac{3}{2}}\right]\right.$$
$$\left.-\frac{4}{\pi}(n^2+2)\log\frac{(a-\rho)^2+\zeta^2}{(a+\rho)^2+\zeta^2}\Big/((a+\rho)^2+\zeta^2)^{\frac{3}{2}}\right\}=\dot{\beta}_n$$

or neglecting terms on the left-hand side which are not multiplied by the large quantity $\log \frac{\overline{a-\rho}^2+\zeta^2}{(a+\rho)^2+\zeta^2}=\log \frac{e^2}{4a^2}$ we find

$$\tfrac{1}{2}ma\alpha_n \frac{8}{\pi} \frac{(n^2-1)}{8a^2} \log \frac{2a}{e}=\dot{\beta}_n$$

or

$$\tfrac{1}{2}\omega \frac{e^2}{a^2}(n^2-1) \log \frac{2a}{e}\cdot\alpha_n=\dot{\beta}_n \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

Differentiating equation (15) with respect to the time and substituting for $\dot{\beta}_n$ from (16) we get

$$-\left(\tfrac{1}{2}\omega \frac{e^2}{a^2} \log \frac{2a}{e}\right)^2(n^2-1)n^2\alpha_n=\alpha''_n \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

or

$$\alpha_n=\mathrm{A} \cos \left\{\frac{\omega e^2}{2a^2} \log \frac{2a}{e}\, n.\sqrt{n^2-1}t+\beta\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

$$\beta_n=\mathrm{A}\frac{\sqrt{n^2-1}}{n} \sin \left\{\frac{\omega e^2}{2a^2} \log \frac{2a}{e}\cdot n\sqrt{n^2-1}.t+\beta\right\} \quad . \quad . \quad . \quad . \quad . \quad (22)$$

where A and β are arbitrary constants.

These equations show that the circular vortex ring of indefinitely small section is stable for all displacements of its circular axis, and that the time of vibration for a displacement expressed by

$$\rho=a+\alpha_n \cos n\theta$$

is

$$2\pi\Big/\frac{\omega e^2}{2a^2} \log \frac{2a}{e}.n.\sqrt{n^2-1} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (23)$$

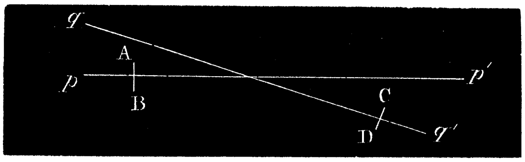
If V be the velocity of translation of the vortex, viz.: $\omega e^2 \log \frac{2a}{e}\Big/2a$, the time of vibration is $2\pi a/\mathrm{V}.n\sqrt{n^2-1}$.

Sir W. Thomson has proved that the circular vortex ring is stable for all alterations in the shape of the cross section. If we combine this with the result just obtained we see that the circular vortex ring is stable for all possible displacements. Sir W. Thomson has also proved that for a displacement of the n^{th} order in the shape of the cross section of the vortex arc the time of vibration $=2\pi/(n-1)\omega$; hence these vibrations begin by being much quicker than those we have been considering, but since for large values of n the latter are proportional to n^2 whilst the former are only proportional to n, the vibrations of a higher order will be quicker for the circular axis than for the core. When n is very great $\sqrt{n^2-1}=n$, thus the amplitude of α_n is equal to the amplitude of β_n and $\alpha_n^2+\beta_n^2=\mathrm{A}^2$ a constant quantity; thus each point on the arc describes a circle about its mean position with an angular velocity $\omega e^2 n^2 \log 2a/e\Big/a^2$.

Problem II. To find the action upon each other of two vortex rings which move so as never to approach nearer than a large multiple of the diameter of either.

For the sake of simplicity we shall suppose that the normals to the planes of the vortices intersect.

Fig. 2.



Let the plane of the paper contain $p\,p'$ and $q\,q'$ the normals to the two vortices, let A B be the vortex moving along $p\,p'$, C D the vortex moving along $q\,q'$.

Let the figure of the circular axis of the vortex A B be given by

$$\rho'=a'+\Sigma\alpha_n' \cos n\theta'$$
$$z'=\mathfrak{z}' +\Sigma\beta_n' \cos n\theta'$$

where z' is measured along and ρ' perpendicular to $p\ p'$. Since the vortices never approach near to one another α_n' and β_n' will be small compared with a'; they will be functions of the time which we shall have to find.

Let the figure of the circular axis of C D be given by

$$\rho=a+\Sigma\alpha_n \cos n\theta$$
$$z=\mathfrak{z} +\Sigma\beta_n \cos n\theta$$

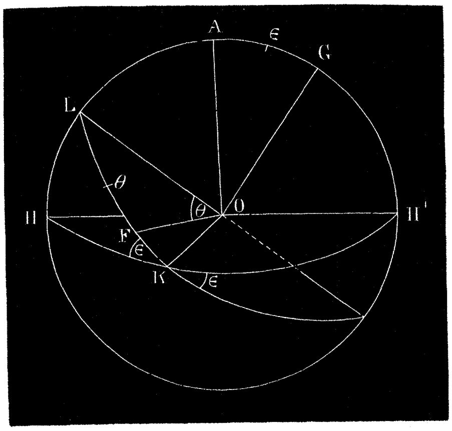
where z and ρ are measured respectively along and perpendicular to $q\ q'$. For the same reason as before α_n and β_n will be small compared with a. To find how the vortex C D is affected by the vortex A B we shall have to find the velocities of the fluid along z and ρ due to the vortex A B; in doing this we may as a first approximation assume that the axis of A B is circular and in one plane, *i.e.*, we may calculate the velocities as if α_n' and β_n' were both zero.

Let ϵ denote the angle between $p\ p'$ and $q\ q'$. Let $p\ p'$ be taken as the axis of z', the perpendicular to $p\ p'$ drawn upwards through the centre of the vortex A B, being the axis of x'.

Let l, m, n be the direction cosines referred to these axes of a radius vector in the plane of the vortex ring C D, drawn from the centre of the vortex ring and making an angle θ with C D the intersection of the plane of the vortex ring with the plane of the paper.

To find l, m, n through the centre of a sphere draw planes parallel to the two vortex rings and let these be H K H′, L K H′, the former being parallel to the ring A B and the latter to C D. Let A G be the poles of these great circles.

Fig. 3.



Then O H, K O, O A are parallel to our axes of x', y', z' respectively. The angle A G or K is ϵ, and if F is parallel to the radius vector above referred to, F L is equal to θ. Then

$$
\begin{aligned}
l &= \cos \mathrm{HF} = \cos\theta \cos\epsilon \\
m &= -\cos \mathrm{FK} = -\sin\theta \\
n &= \cos \mathrm{FA} = \cos\theta \sin\epsilon
\end{aligned}
$$

The velocity γ along the axis of z' due to the vortex A B is by formula 3 given by

$$\gamma = \tfrac{1}{2} m'(2a'^2 \mathrm{A}_0 - a'\rho' \mathrm{A}_1)$$

where m' is the strength of the vortex A B

$$\mathrm{A}_0 = \frac{1}{2\pi}\int_0^{2\pi} \frac{d\theta}{(a'^2+\rho'^2+\zeta'^2-2\rho' a' \cos\theta)^{\frac{3}{2}}}$$

$$\mathrm{A}_1 = \frac{1}{2\pi}\int_0^{2\pi} \frac{\cos\theta . d\theta}{(a'^2+\rho'^2+\zeta'^2-2\rho' a' \cos\theta)^{\frac{3}{2}}}$$

Now since the vortex rings never approach nearer than a large multiple of their diameter, a'^2 will be small compared with $\rho'^2+\zeta'^2$, if we neglect small quantities of a higher order than $a'^2/\rho'^2+\zeta'^2$, we find

$$\mathrm{A}_0 = \frac{1}{(\rho'^2+\zeta'^2)^{\frac{3}{2}}} + \frac{3}{4}\frac{a'^2(3\rho'^2-2\zeta'^2)}{(\rho'^2+\zeta'^2)^{\frac{7}{2}}}$$

$$\mathrm{A}_1 = \frac{3a'\rho'}{(\rho'^2+\zeta'^2)^{\frac{5}{2}}}$$

Let the coordinates of the centre of the vortex ring C D be f, 0, h, then for a point on the vortex ring

$$
\begin{aligned}
x &= f + al = f + a\cos\epsilon\cos\theta \\
y &= am = -a\sin\theta \\
z &= h + an = h + a\sin\epsilon\cos\theta
\end{aligned}
$$

Substituting in the expressions for A_0 and A_1, we find on neglecting small quantities of an order higher than a'^2/f^2+h^2

$$A_0=\frac{1}{(f^2+h^2)^{\frac{3}{2}}}-\frac{3}{2}\frac{a^2}{(f^2+h^2)^{\frac{5}{2}}}+\frac{15}{4}\frac{a^2(h\sin\epsilon+f\cos\epsilon)^2}{(f^2+h^2)^{\frac{7}{2}}}+\frac{3}{4}\frac{a'^2(3f^2-2h^2)}{(f^2+h^2)^{\frac{7}{2}}}$$

$$-\cos\theta\,\frac{3a(h\sin\epsilon+f\cos\epsilon)}{(f^2+h^2)^{\frac{5}{2}}}+\cos 2\theta\,\frac{15}{4}\frac{a^2(h\sin\epsilon+f\cos\epsilon)^2}{(f^2+h^2)^{\frac{7}{2}}}$$

$$\rho'A_1=\frac{3a'\rho'^2}{(\rho'^2+\zeta'^2)^{\frac{5}{2}}}$$

$$=\frac{3a'f^2}{(f^2+h^2)^{\frac{5}{2}}}-\frac{15}{2}\frac{a'a^2f^2}{(f^2+h^2)^{\frac{7}{2}}}+\frac{105}{4}\frac{a'a^2f^2(h\sin\epsilon+f\cos\epsilon)^2}{(f^2+h^2)^{\frac{9}{2}}}+\frac{3a'a^2(1-\frac{1}{2}\sin^2\epsilon)}{(f^2+h^2)^{\frac{5}{2}}}$$

$$-\frac{15a'a^2f(h\sin\epsilon+f\cos\epsilon)}{(f^2+h^2)^{\frac{7}{2}}}+\cos\theta\left\{\frac{6aa'f\cos\epsilon}{(f^2+h^2)^{\frac{5}{2}}}-\frac{15aa'f^2(h\sin\epsilon+f\cos\epsilon)}{(f^2+h^2)^{\frac{7}{2}}}\right\}$$

$$+\cos 2\theta\left\{\frac{105}{4}\frac{a'a^2f^2(h\sin\epsilon+f\cos\epsilon)^2}{(f^2+h^2)^{\frac{9}{2}}}-\frac{15a'a^2f\cos\epsilon(h\sin\epsilon+f\cos\epsilon)}{(f^2+h^2)^{\frac{7}{2}}}-\frac{3}{2}\frac{a'a^2\sin^2\epsilon}{(f^2+h^2)^{\frac{5}{2}}}\right\}$$

Although for reference we give the complete values of A_0 and A_1 to the order of approximation we are working to, yet when we have in the expressions for the velocities a coefficient consisting of terms of differerent orders, we shall only retain the largest term.

If we do this we find

$$\gamma=\frac{1}{2}\frac{m'a'^2}{(f^2+h^2)^{\frac{5}{2}}}(2h^2-f^2)$$

$$+\frac{3}{2}\cos\theta\frac{m'a'^2}{(f^2+h^2)^{\frac{7}{2}}}(f^2(f\cos\epsilon+3h\sin\epsilon)-2h^2(2f\cos\epsilon+h\sin\epsilon))$$

$$+\frac{3}{2}\cos 2\theta\frac{m'a'^2a^2}{(f^2+h^2)^{\frac{7}{2}}}\left\{\frac{5}{2}(f\cos\epsilon+h\sin\epsilon)^2-\frac{35}{4}\frac{f^2(f\cos\epsilon+h\sin\epsilon)^2}{(f^2+h^2)}+5f\cos\epsilon(f\cos\epsilon+h\sin\epsilon)\right.$$
$$\left.+\frac{1}{2}\sin^2\epsilon(f^2+h^2)\right\}$$

From the formulæ (1) and (3) we find the velocity along ρ'

$$=\tfrac{1}{2}m'a'\zeta'A_1$$
$$=\frac{3}{2}\frac{m'a'^2\rho\zeta'}{(\rho'^2+\zeta'^2)^{\frac{5}{2}}}$$

Hence α, the velocity along x' at the vortex C D due to the vortex A B,

$$=\frac{3}{2}\frac{m'a'^2\zeta'(f+a\cos\epsilon\cos\theta)}{(\rho'^2+\zeta'^2)^{\frac{5}{2}}}$$

$$=\frac{3m'a'^2}{2}\frac{fh}{(f^2+h^2)^{\frac{5}{2}}}$$

$$+\cos\theta.\frac{3}{2}\frac{m'a'^2a}{(f^2+h^2)^{\frac{5}{2}}}.\left\{(h\cos\epsilon+f\sin\epsilon)-\frac{5fh(h\sin\epsilon+f\cos\epsilon)}{f^2+h^2}\right\}$$

$$+\cos 2\theta.\frac{3}{2}\frac{m'a'^2a^2}{(f^2+h^2)^{\frac{5}{2}}}\left\{\frac{35}{4}\frac{(h\sin\epsilon+f\cos\epsilon)^2fh}{(f^2+h^2)^2}-\frac{5}{4}\frac{(h\cos\epsilon+f\sin\epsilon)(h\sin\epsilon+f\cos\epsilon)}{(f^2+h^2)}+\frac{1}{2}\sin\epsilon\cos\epsilon\right\}$$

In this expression for α we have only retained the highest terms in each coefficient. If β be the velocity parallel to the axis of y', we have

$$\beta=-\tfrac{3}{2}\frac{m'a'^2a\zeta'\sin\theta}{(\rho'^2+\zeta^2)^{\frac{5}{2}}}$$
$$=-\tfrac{3}{2}\frac{m'a'^2a}{(f^2+h^2)^{\frac{5}{2}}}\left(h\sin\theta-\frac{5ah\sin\theta\cos\theta(h\sin\epsilon+f\cos\epsilon)}{f^2+h^2}+a\sin\theta\cos\theta\sin\epsilon\right)$$

The velocity perpendicular to the plane of the vortex C D

$$=\gamma\cos\epsilon-\alpha\sin\epsilon$$

$$\left.\begin{aligned}&=\tfrac{1}{2}\frac{m'a'^2}{(f^2+h^2)^{\frac{5}{2}}}\{(2h^2-f^2)\cos\epsilon-3fh\sin\epsilon\}\\&+\cos\theta.\tfrac{3}{2}\frac{m'a'^2a}{(f^2+h^2)^{\frac{7}{2}}}\{\cos 2\epsilon(f^3-4fh^2)+\tfrac{1}{2}\sin 2\epsilon(7f^2h-3h^3)\}\\&+\cos 2\theta.\tfrac{3}{2}\frac{m'a'^2a^2}{(f^2+h^2)^{\frac{7}{2}}}\left\{\tfrac{5}{2}(h\sin\epsilon+f\cos\epsilon)(h\sin 2\epsilon+f\cos 2\epsilon+2f)-\tfrac{3}{4}\tfrac{5}{}\frac{f^2(h\sin\epsilon+f\cos\epsilon)^2}{f^2+h^2}\right\}\\&+\ldots\end{aligned}\right\}\quad(24)$$

The velocity along the radius vector of the vortex ring C D due to the vortex A B

$$=\alpha l+\beta m+\gamma n$$
$$=\alpha\cos\epsilon\cos\theta-\beta\sin\theta+\gamma\sin\epsilon\cos\theta$$

or substituting for α, β, γ their values

$$\left.\begin{aligned}&=\tfrac{3}{4}\frac{m'a'^2a}{(f^2+h^2)^{\frac{5}{2}}}\left\{\tfrac{1}{2}h+\tfrac{1}{2}\cos 2\epsilon\frac{(3h^3-7hf^2)}{f^2+h^2}+\sin 2\epsilon\frac{(f^3-4fh^2)}{f^2+h^2}\right\}\\&+\tfrac{1}{2}\cos\theta\frac{m'a'^2}{(f^2+h^2)^{\frac{5}{2}}}\{(2h^2-f^2)\sin\epsilon+3fh\cos\epsilon\}\\&+\tfrac{3}{4}\cos 2\theta\frac{m'a'^2a}{(f^2+h^2)^{\frac{5}{2}}}\left\{-\tfrac{3}{2}h+\tfrac{1}{2}\cos 2\epsilon\frac{(3h^3-7f^2h)}{f^2+h^2}+\sin 2\epsilon\frac{(f^3-4fh^2)}{f^2+h^2}\right\}\end{aligned}\right\}.\quad(25)$$

These expressions will enable us to find the effect of one vortex on another. We have, for example, expressed the velocity perpendicular to the plane of C D due to the vortex ring A B in the form A+B cos θ+C cos 2θ+ . . . , in Problem I. we expressed the velocity in the same direction due to the vortex C D itself in the same form, hence the total velocity perpendicular to C D can be expressed in this form, but by formula (8) the velocity perpendicular to the plane of C D is

$$\Sigma\frac{d\beta_n}{dt}\cos n\theta$$

hence $\frac{d\beta_n}{dt}$= coefficient of $\cos n\theta$ in the expression for the velocity perpendicular to the plane of the vortex C D. To find β_n we must therefore express the velocity due to the vortex A B as a function of the time. In order to make the work as simple as possible we shall suppose that the vortices in their undisturbed states had equal strengths and radii. In the small terms which express the velocity at the vortex C D due to the vortex A B we may as a first approximation calculate the quantities on the supposition that the motion is undisturbed. In order to make the expressions as simple as possible, let us measure the time from the instant when the distance between the centres of the vortices has its least value (it is easy to see that this will be when the line joining the centres of the vortices is parallel to the line bisecting the angle between their directions of motion), then the square of the distance between their centres will be expressible in the form $c^2+\beta t^2$ when c is the least distance between the centres, t the time that has elapsed since the centres were this distance apart; let v be the velocity of translation of either vortex when undisturbed, then

$$\begin{aligned} f &= -c\sin\tfrac{1}{2}\epsilon - v\sin\epsilon . t \\ h &= \quad c\cos\tfrac{1}{2}\epsilon - v(1-\cos\epsilon)t \end{aligned}$$

Therefore

$$f^2+h^2=c^2+4v^2\sin^2\tfrac{1}{2}\epsilon . t^2$$

Making these substitutions we find that the velocity perpendicular to the plane of the vortex C D

$$\begin{aligned} &=\tfrac{1}{2}\frac{m'a'^2}{(c^2+4v^2\sin^2\frac{1}{2}\epsilon . t^2)^{\frac{5}{2}}}\{\tfrac{1}{2}c^2(3+\cos\epsilon)+2v^2\sin^2\tfrac{1}{2}\epsilon(1-3\cos\epsilon)t^2\} \\ &+\cos\theta\tfrac{3}{2}\frac{m'a'^2a}{(c^2+4v^2\sin^2\frac{1}{2}\epsilon . t^2)^{\frac{7}{2}}}\{At^3+Bt^2+Ct+D\} \\ &+\cos 2\theta\tfrac{3}{2}\frac{m'a'^2a^2}{(c^2+4v^2\sin^2\frac{1}{2}\epsilon . t^2)^{\frac{9}{2}}}\{A't^4+B't^3+C't^2+D't+E'\} \end{aligned}$$

where

$$\left.\begin{aligned} A &= 8v^3\sin^3\tfrac{1}{2}\epsilon(\cos\tfrac{3}{2}\epsilon-5\cos\tfrac{1}{2}\epsilon) \\ B &= cv^2\sin^2\tfrac{1}{2}\epsilon(15\sin\tfrac{1}{2}\epsilon-\sin\tfrac{3}{2}\epsilon) \\ C &= c^2v\sin\tfrac{1}{2}\epsilon(15\cos\tfrac{1}{2}\epsilon+\cos\tfrac{3}{2}\epsilon) \\ D &= -\tfrac{1}{4}c^3(5\sin\tfrac{1}{2}\epsilon+\sin\tfrac{3}{2}\epsilon) \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (26)$$

$$\left.\begin{aligned} A' &= -10v^4\sin^4\tfrac{1}{2}\epsilon(2\cos 3\epsilon+\tfrac{30}{8}\cos 2\epsilon+7\cos\epsilon+\tfrac{42}{8}) \\ B' &= \quad 10cv^3\sin^3\tfrac{1}{2}\epsilon(3\sin\epsilon-\tfrac{1}{2}\sin 2\epsilon) \\ C' &= -10c^2v^2\sin^2\tfrac{1}{2}\epsilon\cos 3\epsilon \\ D' &= -\tfrac{5}{8}c^3v\sin\epsilon(14\sin\epsilon+\sin 2\epsilon) \\ E' &= \quad \tfrac{5}{8}c^4\sin^2\tfrac{1}{2}\epsilon(11+\cos\epsilon+8\cos 2\epsilon) \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (27)$$

The velocity along the radius vector

$$=\tfrac{3}{4}\frac{m'a'^2a}{(c^2+4v^2\sin^2\frac{1}{2}\epsilon.t^2)^{\frac{7}{2}}}(\mathrm{F}t^3+\mathrm{G}t^2+\mathrm{H}t+\mathrm{K})$$

$$+\cos\theta\tfrac{1}{2}\frac{m'a'^2}{(c^2+4v^2\sin^2\frac{1}{2}\epsilon.t^2)^{\frac{5}{2}}}\{\tfrac{1}{2}c^2\sin\epsilon-6cv\sin\tfrac{1}{2}\epsilon.t+2\sin^2\tfrac{1}{2}\epsilon\sin\epsilon.v^2t^2\}$$

$$+\cos 2\theta\tfrac{3}{4}\frac{m'a'^2a}{(c^2+4v^2\sin^2\frac{1}{2}\epsilon.t^2)^{\frac{7}{2}}}(\mathrm{F}'t^3+\mathrm{G}'t^2+\mathrm{H}'t+\mathrm{K}')$$

where

$$\left.\begin{aligned}\mathrm{F}&=v^3\sin^3\tfrac{1}{2}\epsilon(2\sin\tfrac{3}{2}\epsilon-14\sin\tfrac{1}{2}\epsilon)\\ \mathrm{G}&=cv^2\sin^2\tfrac{1}{2}\epsilon(\cos\tfrac{3}{2}\epsilon-13\cos\tfrac{1}{2}\epsilon)\\ \mathrm{H}&=\tfrac{1}{2}c^2v\sin\tfrac{1}{2}\epsilon(\sin\tfrac{3}{2}\epsilon+13\sin\tfrac{1}{2}\epsilon)\\ \mathrm{K}&=\tfrac{1}{4}c^3(\cos\tfrac{3}{2}\epsilon+7\cos\tfrac{1}{2}\epsilon)\end{aligned}\right\}\quad\ldots\ldots\quad(28)$$

$$\left.\begin{aligned}\mathrm{F}'&=2v^3\sin^3\tfrac{1}{2}\epsilon(\sin\tfrac{3}{2}\epsilon+\sin\tfrac{1}{2}\epsilon)\\ \mathrm{G}'&=cv^2\sin^2\tfrac{1}{2}\epsilon(\cos\tfrac{3}{2}\epsilon-21\cos\tfrac{1}{2}\epsilon)\\ \mathrm{H}'&=\tfrac{1}{2}c^2v\sin\tfrac{1}{2}\epsilon(\sin\tfrac{3}{2}\epsilon+21\sin\tfrac{1}{2}\epsilon)\\ \mathrm{K}'&=\tfrac{1}{4}c^3(\cos\tfrac{3}{2}\epsilon-\cos\tfrac{5}{2}\epsilon)\end{aligned}\right\}\quad\ldots\ldots\quad(29)$$

We can now write down the differential equations giving α_n and β_n. We shall begin with that giving $\beta_{,}$ as the simplest, as a reference to equation (19) will show that the vortex ring C D contributes nothing to this term, so that

$$\frac{d\beta_{,}}{dt}=\tfrac{3}{2}\frac{m'a'^2a}{(c^2+4v^2\sin^2\frac{1}{2}\epsilon.t^2)^{\frac{7}{2}}}(\mathrm{A}t^3+\mathrm{B}t^2+\mathrm{C}t+\mathrm{D})$$

integrating and writing for brevity κ instead of $4v^2\sin^2\frac{1}{2}\epsilon$ we find

$$\beta_{,}=\tfrac{3}{2}m'a'^2a\left\{\tfrac{1}{5}\frac{\frac{\mathrm{A}c^2}{\kappa^4}-\frac{\mathrm{C}}{\kappa^2}}{(c^2+\kappa^2t^2)^{\frac{5}{2}}}-\tfrac{1}{3}\frac{\frac{\mathrm{A}}{\kappa^4}}{(c^2+\kappa^2t^2)^{\frac{3}{2}}}+\tfrac{1}{5}\frac{\mathrm{D}-\frac{\mathrm{B}c^2}{\kappa^2}}{c^2}\frac{t}{(c^2+\kappa^2t^2)^{\frac{5}{2}}}+\tfrac{1}{15}\frac{\left(\frac{4\mathrm{D}}{c^4}+\frac{\mathrm{B}}{c^2\kappa^2}\right)}{(c^2+\kappa^2t^2)^{\frac{3}{2}}}.t\right.$$

$$\left.+\tfrac{1}{15}\left(\frac{8\mathrm{D}}{c^6}+\frac{2\mathrm{B}}{c^4\kappa^2}\right)\left(\frac{t}{(c^2+\kappa^2t^2)^{\frac{1}{2}}}+\frac{1}{2v\sin\frac{1}{2}\epsilon}\right)\right\}$$

where the arbitrary constant arising from the integration has been determined so as to make $\beta_{,}=0$ when $t=-\infty$.

Substituting for A, B, C, D their values we find

$$\beta_{,}=\tfrac{3}{2}m'a'^2a\left\{\tfrac{1}{20}\frac{c^2(\cos\frac{3}{2}\epsilon-\frac{35}{2}\cos\frac{1}{2}\epsilon)}{v\sin\frac{1}{2}\epsilon}\;\frac{1}{(c^2+\kappa^2t^2)^{\frac{5}{2}}}-\frac{\cos\frac{3}{2}\epsilon-5\cos\frac{1}{2}\epsilon}{6v\sin\frac{1}{2}\epsilon}\;\frac{1}{(c^2+\kappa^2t^2)^{\frac{3}{2}}}\right.$$

$$\left.-\frac{c\sin\frac{1}{2}\epsilon}{(c^2+\kappa^2t^2)^{\frac{5}{2}}}t-\frac{(\sin\frac{1}{2}\epsilon+\sin\frac{3}{2}\epsilon)\epsilon}{12c(c^2+\kappa^2t^2)^{\frac{3}{2}}}-\frac{(\sin\frac{1}{2}\epsilon+\sin\frac{3}{2}\epsilon)}{6c^3}\left(\frac{\epsilon}{(c^2+\kappa^2t^2)^{\frac{1}{2}}}+\frac{1}{2v\sin\epsilon}\right)\right\}\quad(30)$$

This expression vanishes when $t=-\infty$; it begins by being positive but soon changes sign; it is negative when $t=0$ and remains negative for all greater values of t.

The last terms are the only ones that do not vanish when $t=\infty$. Putting $t=\infty$ we find

$$\beta_{,}=-\tfrac{1}{4}\frac{m'a'^2a}{c^3v\sin\frac{1}{2}\epsilon}(\sin\tfrac{1}{2}\epsilon+\sin\tfrac{3}{2}\epsilon)$$

$$=-\frac{m'a'^2a}{c^3v}\cos^2\tfrac{1}{2}\epsilon$$

Now $\beta_{,}/a$ is the angle through which the plane of the vortex is turned, and since the vortex moves at right angles to its plane this will be the angle through which the direction of motion of the vortex ring is turned.

Since $\beta_{,}$ is negative, the part of the vortex ring C D where $\cos\theta$ is positive is tilted backwards; now as we have taken it, $\cos\theta$ is positive for the upper part of the vortex, hence this part of the vortex is tilted backwards, and the normal to its plane, which is the direction in which the vortex ring moves, is bent towards pp', the direction of motion of the vortex ring A B through an angle whose circular measure

$$=\frac{m'a^3}{vc^3}\cos^2\tfrac{1}{2}\epsilon.$$

Thus the deflection, other things being the same, varies inversely as the cube of the least distance between the vortices.

Let us now consider the effect of the vortex ring C D on the vortex A B. Let us take the perpendicular to the plane of C D as the new axis of z, the perpendicular to this drawn upwards in the plane of the paper as the new axis of x. The work we went through before consisted in finding expressions for the velocities along the axes of coordinates due to one vortex at a point on the other in terms of f, h, l, m, n, and then finding the velocities perpendicular to the plane of the vortex and along its radius vector in terms of the time by substituting from the equations

$$\left.\begin{aligned} f&=-c\sin\tfrac{1}{2}\epsilon-v\sin\epsilon.t \\ h&=c\cos\tfrac{1}{2}\epsilon-v(1-\cos\epsilon)t \end{aligned}\right\}$$

$$\left.\begin{aligned} l&=\cos\epsilon\cos\theta \\ m&=-\sin\theta \\ n&=\sin\epsilon\sin\theta \end{aligned}\right\}$$

velocity perpendicular to the plane of the vortex $=\gamma\cos\epsilon-\alpha\sin\epsilon$, velocity along the radius vector $=\alpha\cos\epsilon\cos\theta-\beta\sin\theta+\gamma\sin\epsilon\cos\theta$.

Now in finding the effect of the vortex ring C D on the vortex A B, the general expressions giving the velocities will be the same as before, as we have taken corre-

sponding axes of coordinates. The difference in the work will come in when we substitute for the quantities involved their expressions in terms of the time; if we denote corresponding quantities in this case by affixing dashes to the symbols, denoting them in the previous one, we easily find

$$\left.\begin{aligned} f' &= -c \sin \tfrac{1}{2}\epsilon + v \sin \epsilon . t \\ h' &= -c \cos \tfrac{1}{2}\epsilon - v(1-\cos \epsilon)t \end{aligned}\right\}$$

$$\left.\begin{aligned} l' &= \cos \epsilon \cos \theta' \\ m' &= -\sin \theta' \\ n' &= -\sin \epsilon \cos \theta' \end{aligned}\right\}$$

velocity perpendicular to the plane of the vortex A B

$$=\gamma \cos \epsilon + \alpha \sin \epsilon$$

velocity along the radius vector $=\alpha \cos \epsilon \cos \theta' - \beta \sin \theta' - \gamma \sin \epsilon \cos \theta'$.

It will be seen that we can get the expressions for the quantities denoted by the accented letters from those for the quantities denoted by the unaccented letters by writing $2\pi-\epsilon$ instead of ϵ, hence the value of $\beta_{,}'$ will be got by writing in the expression for $\beta_{,}$ (equation (30)) $2\pi-\epsilon$ instead of ϵ, and interchanging a and a'.

Hence the value of $\beta_{,}'$ when $t=\infty$

$$=-\frac{ma^2a'}{vc^3}\cos^2\frac{2\pi-\epsilon}{2}=-\frac{ma^2a'}{vc^3}\cos^2 \tfrac{1}{2}\epsilon$$

Now this being negative shows that the parts of the vortex ring A B where $\cos \theta'$ is positive are tilted backwards, now $\cos \theta'$ is positive in the upper half of the vortex ring A B, therefore the direction of motion of the vortex A B, which is perpendicular to the plane of the vortex, is turned away from the direction of motion of the vortex C D through an angle whose circular measure is $ma^2 \cos^2 \frac{1}{2}\epsilon/vc^3$; but since in the case we are considering $a=a'$, $m=m'$, this angle is the same as that through which the path of the vortex C D is turned towards the path of the vortex A B. We may express the results we have obtained by saying that the direction of motion of the vortex which is in front when the vortices are nearest together, is bent towards the direction of motion of the one which is behind, that the direction of motion of the latter is bent through an equal amount in the same direction, and that the amount of this bending is $ma^2 \cos^2 \frac{1}{2}\epsilon/vc^3$.

Let us now consider the effect the collision has on the size of the vortices.

The equation giving the increase in radius is

$\frac{d\alpha_0}{dt}$ = the part independent of θ in the expression for the velocity along the radius vector of C D.

A reference to equation (17) will show that the vortex ring C D itself contributes nothing to this term, therefore

$\frac{d\alpha_0}{dt}$= the part independent of θ in the expression for the velocity due to the vortex ring A B along the radius vector of C D.

Hence from equation (28) we have

$$\frac{d\alpha_0}{dt}=\frac{3}{4}\frac{m'a'^2a}{(c^2+4v^2\sin^2\frac{1}{2}\epsilon.t^2)^{\frac{7}{2}}}\{\mathrm{F}t^3+\mathrm{G}t^2+\mathrm{H}t+\mathrm{K}\}$$

If we integrate this equation, substitute for F, G, H, K their values as given in equation (28) and determine the arbitrary constant introduced by the integration, so that $\alpha_0=0$ when $t=-\infty$, we find

$$\alpha_0=\frac{3}{4}m'a'^2a\left\{\frac{-c^2}{2v(c^2+4v^2\sin^2\frac{1}{2}\epsilon.t^2)^{\frac{5}{2}}}+\frac{(1+\sin^2\frac{1}{2}\epsilon)}{6v(c^2+4v^2\sin^2\frac{1}{2}\epsilon.t^2)^{\frac{3}{2}}}+\frac{c\cos\frac{1}{2}\epsilon.t}{(c^2+4v^2\sin^2\frac{1}{2}\epsilon.t^2)^{\frac{5}{2}}}\right.$$
$$\left.+\frac{(\cos\frac{3}{2}\epsilon+3\cos\frac{1}{2}\epsilon)t}{12c(c^2+4v^2\sin^2\frac{1}{2}\epsilon.t^2)^{\frac{3}{2}}}+\frac{3\cos\frac{1}{2}\epsilon+\cos\frac{3}{2}\epsilon}{6c^3}\left(\frac{\epsilon}{(c^3+4v^2\sin^2\frac{1}{2}\epsilon.t^2)^{\frac{1}{2}}}+\frac{1}{2v\sin\frac{1}{2}\epsilon}\right)\right\}$$

This expression vanishes when $t=-\infty$; it begins by being negative, so that the radius of C D is diminished at first when $t=0$, the sign of α_0 depends upon the value of ϵ, if ϵ be less than 60° it is certainly positive when $t=0$; when $t=\infty$ α_0 is positive, and its value is

$$\frac{m'a'^2a}{8vc^3\sin\frac{1}{2}\epsilon}(3\cos\frac{1}{2}\epsilon+\cos\frac{3}{2}\epsilon)$$
$$=\frac{1}{2}\frac{m'a'^2a}{vc^3\sin\frac{1}{2}\epsilon}\cos^3\frac{1}{2}\epsilon.$$

As this is positive, the vortex ring C D is bigger after the collision. The effect of the vortex ring C D on the vortex A B can be got as we saw before by writing $2\pi-\epsilon$ for ϵ in the formula given above. We have thus the ultimate increase α_0' in the radius of A B given by

$$\alpha_0'=\frac{1}{2}\frac{ma^2a'\cos^3\frac{1}{2}(2\pi-\epsilon)}{c^3\sin\frac{1}{2}(2\pi-\epsilon)}$$

or since $a=a'$, $m=m'$

$$\alpha_0=-\frac{1}{2}\frac{m'a'^2a}{c^3\sin\frac{1}{2}\epsilon}\cos^3\frac{1}{2}\epsilon$$

Hence the radius of the vortex ring A B is diminished by the collision. The effect of the collision on the size of the vortices is thus to increase the radius of the one which is in front when the vortex rings are nearest together, and decrease that of the one in the rear by $m'a^3\cos^3\frac{1}{2}\epsilon/2vc^3\sin\frac{1}{2}\epsilon$. Hence the alteration in the radius is, *cœteris paribus*, inversely proportional to the cube of the shortest distance between the vortices.

We can now find the force resultant of the impulse after collision. The impulse for a vortex ring with a very fine core equals the strength multiplied by the area. Let ω be the small angle through which the direction of motion of the vortices is deflected; let δa be the alteration in the radius of either, for the vortex ring in front δa will be positive, for the one in the rear it will be negative but of equal numerical value.

The resolved part of the force resultant of the impulse along the line bisecting their original direction of motion after collision

$$=\pi m(a+\delta a)^2 \cos\left(\frac{\epsilon}{2}-\omega\right)+\pi m(a-\delta a)^2 \cos\left(\frac{\epsilon}{2}+\omega\right)$$
$$=2\pi m a^2 \cos \tfrac{1}{2}\epsilon$$

the same as before collision.

The component perpendicular to the bisector of the angle between their directions of motion

$$=\pi m(a+\delta a)^2 \sin(\tfrac{1}{2}\epsilon-\omega)-\pi m(a-\delta a)^2 \sin(\tfrac{1}{2}\epsilon+\omega)$$
$$=\pi m(4a\delta a \sin \tfrac{1}{2}\epsilon-2\omega a^2 \cos \tfrac{1}{2}\epsilon)$$

Substituting for δa and ω the values $ma^3 \cos^3 \frac{1}{2}\epsilon/2vc^3 \sin \frac{1}{2}\epsilon$, and $ma^2 \cos^2 \frac{1}{2}\epsilon/vc^3$ respectively, we find that the component of the impulse perpendicular to the bisector of the angle between the directions of motion vanishes, as it did before the collision; hence we see that the force resultant of the impulse is not altered by the collision, a result which we know is true.

We pass on to consider the terms α_2 and β_2. We know that

$\frac{d\alpha_2}{dt}=$ coefficient of $\cos 2\theta$ in the expression for the velocity along the radius vector of the vortex ring C D.

Now the vortex ring C D itself, as we see from equation (17), contributes to the expression for the velocity along its radius vector the term

$$-\cos 2\theta.\frac{2m'}{a^2}\log\frac{2a}{e}.\beta_2$$

The vortex ring A B contributes as we see from equation (29) the term

$$-\cos 2\theta.\tfrac{3}{4}\frac{m'a'^2a}{(c^2+4v^2\sin^2\frac{1}{2}\epsilon.t^2)^{\frac{5}{2}}}(\mathrm{F}'t^3+\mathrm{G}'t^2+\mathrm{H}'t+\mathrm{K}')$$

say

$$\cos 2\theta f(t)$$

Thus

$$\frac{d\alpha_2}{dt}=-\frac{2m}{a^2}\log\frac{2a}{e}.\beta_2+f(t)$$

Now

$\frac{d\beta_2}{dt}=$ the coefficient of $\cos 2\theta$ in the expression for the velocity perpendicular to the plane of the vortex C D.

The vortex C D itself contributes to this coefficient the term

$$\tfrac{3}{2}\frac{m}{a^2}\log\frac{2a}{e}.\alpha_2$$

The vortex A B contributes the term

$$\tfrac{3}{2}\frac{m'a^2a'^2}{(c^2+4v^2\sin^2\frac{1}{2}\epsilon.t^2)^{\frac{5}{2}}}(A't^4+B't^3+C't^2+D't+E')$$

say

$$F(t)$$

Thus

$$\frac{d\beta_2}{dt}=\tfrac{3}{2}\frac{m}{a^2}\log\frac{2a}{e}\alpha_2+F(t)$$

Eliminating β_2 we find

$$\frac{d^2\alpha_2}{dt^2}+\frac{3m^2}{a^4}\left(\log\frac{2a}{e}\right)^2\alpha_2=f'(t)-\frac{2m}{a^2}\log\frac{2a}{e}.F(t)$$

say

$$=\chi(t)$$

or writing n^2 for

$$\frac{3m^2}{a^4}\left(\log\frac{2a}{e}\right)^2$$

the equation takes the form

$$\frac{d^2\alpha_2}{dt^2}+n^2\alpha_2=\chi(t)$$

The solution of this differential equation is

$$\alpha_2=A\cos nt+B\sin nt+\frac{\cos nt}{n}\int^t\chi(t')\sin nt'dt'$$
$$-\frac{\sin nt}{n}\int^t\chi(t')\cos nt'dt'$$

or choosing the arbitrary constants so that α_2 and $\frac{d\alpha_2}{dt}$ both vanish when $t=-\infty$ we find

$$\alpha_2=\frac{\cos nt}{n}\int_{-\infty}^t\chi(t')\sin nt'dt'-\frac{\sin nt}{n}\int_{-\infty}^t\chi(t')\cos nt'dt'$$

The complete value of $\chi(t)$ is given by the equation

$$\chi(t)=\tfrac{3}{4}\frac{m'a'^2(3F't^2+2G't+H')}{(c^2+4v^2\sin^2\frac{1}{2}\epsilon.t^2)^{\frac{7}{2}}}$$
$$-\frac{21m'a'^2av^2\sin^2\frac{1}{2}\epsilon(F't^4+G't^3+H't^2+K't)}{(c^2+4v^2\sin^2\frac{1}{2}\epsilon.t^2)^{\frac{9}{2}}}$$
$$-\frac{2m}{a^2}\log\frac{2a}{e}.\tfrac{3}{2}\frac{m'a'^2a^2(A't^4+B't^3+C't^2+D't+E')}{(c^2+4v^2\sin^2\frac{1}{2}\epsilon.t^2)^{\frac{9}{2}}}$$

Thus the coefficients of $\cos nt$ and $\sin nt$ in this expression for α_2 will involve integrals of the type

$$\int_{-\infty}^{t} \frac{\cos nt dt}{(a^2+t^2)^{\frac{2P+1}{2}}}$$

I have not succeeded in evaluating this integral; it is evident however that the more important part of these integrals will be produced during the time the vortices are nearly at their minimum distance apart. During the time they are far apart they will not contribute anything appreciable to this integral, so that soon after the vortex rings have passed their minimum distance the equation may without sensible error be written

$$\alpha_2 = P\frac{\cos nt}{n} - Q\frac{\sin nt}{n}$$

where P and Q are constants and

$$P = \int_{-\infty}^{+\infty} \chi(t) \cos nt.dt$$

$$Q = \int_{-\infty}^{+\infty} \chi(t) \sin nt.dt$$

There will be a similar expression for β_2. Thus the vortex rings are thrown by the collision into a state of vibration about their circular form.

We can find the action of two unequal vortices on each other by means of work of a very similar character to that just given. The only difference is that instead of the former values for f and h we must substitute the values

$$f = -c \sin \alpha - v \sin \epsilon . t$$

$$h = c \cos \alpha + (v \cos \epsilon - w)t$$

where v is the velocity of the vortex which is in front when they are nearest together, w the velocity of the one in the rear; α is the angle between the line joining their centres when they are nearest together, and the direction of motion of the vortex in the rear, β is the angle between this line and the direction of motion of the vortex in front, ϵ is the angle between the direction of motion of the vortices; α and β are given by the equations

$$w \cos \alpha = v \cos \beta$$

$$\alpha + \beta = \epsilon$$

I shall not trouble the reader with the expressions for the velocities perpendicular to the plane of either vortex and along the radius vector, but confine myself to quoting the most important consequences to be got from these expressions.

I find that after the collision the direction of motion of C D (the vortex which is in

front when they are nearest together) is deflected towards the direction of motion of the other vortex A B through an angle whose circular measure is

$$\frac{m'a'^2 \cos\alpha \sin 2\beta}{\kappa c^2} = \frac{2m'a'^2 \sin^2 \epsilon vw(v - w\cos\epsilon)}{\kappa^4 c^3}$$

where m', m are the strengths of the vortices A B and C D respectively, and a' and a their radii, κ is the relative velocity of the two vortices, viz.:

$$(v^2 + w^2 - 2vw\cos\epsilon)^{\frac{1}{2}}$$

The direction of motion of A B is deflected from that of C D through an angle whose circular measure is

$$\frac{ma^2 \cos\beta \sin 2\alpha}{\kappa c^3} = \frac{2ma^2 \sin^2 \epsilon vw(w - v\cos\epsilon)}{\kappa^4 c^3}$$

The radius of the vortex C D is increased by

$$\frac{m'a'^2 a \cos\alpha \cos^2\beta}{\kappa c^3} = \frac{m'a'^2 a \sin^3 \epsilon vw^2}{\kappa^4 c^3}$$

The radius of the vortex A B is diminished by

$$\frac{ma^2 a' \cos\beta \cos^2\alpha}{\kappa c^3} = \frac{ma^2 a' \sin^3 \epsilon v^2 w}{\kappa^4 c^3}$$

The velocity of the vortex C D is diminished by

$$\frac{mm'}{2\pi a} \frac{\cos\alpha \cos^2\beta}{\kappa c^3} a'^2 \log \frac{2a}{e}$$

$$= \frac{mm'a'^2 \sin^3 \epsilon vw^2}{2\pi a \kappa^4 c^3} \log \frac{2a}{e}$$

The velocity of the vortex A B is increased by

$$\frac{mm'a^2 \sin^3 \epsilon wv^2}{2\pi a' \kappa^4 c^3} \log \frac{2a'}{e'}$$

where e, e' are the radii of the cross sections of the vortices C D, A B respectively.

The kinetic energy of the vortex C D is increased by

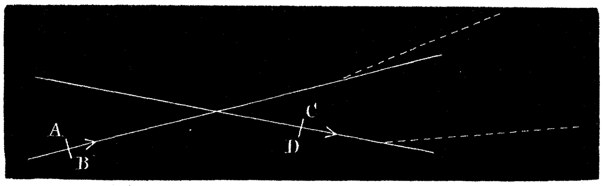
$$\frac{2\pi\rho mm'a'^2 a^2 v \cos\alpha \cos^2\beta}{\kappa c^3} = \frac{2\pi\rho mm'a'^2 a^2 \sin^3 \epsilon . v^2 w^2}{\kappa^4 c^3}$$

The kinetic energy of A B is diminished by the same amount.

With the help of these results we may find the way in which two vortices affect each other in all cases.

Case I.—Vortices moving in the same direction.

Fig 4.



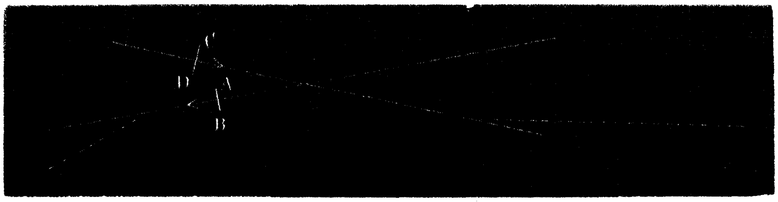
1. When the positions of the vortices when nearest together are as represented in fig. 4.

The way in which their paths are deflected is indicated by the dotted lines.

The vortex C D increases in radius and energy and its velocity is decreased.

Case II.—Vortices moving in opposite directions.

Fig 5.



The position of the vortices when nearest together is represented in fig. 5.

The way their paths are deflected is indicated by the dotted lines in the figure.

The vortex C D decreases in radius and energy and its velocity is increased.

The vortex A B increases in radius and energy and its velocity is decreased.

These results may be summed up in the following rule. The vortex which first passes through the point of intersection of the directions of motion of the vortices is deflected towards the direction of motion of the other: it increases in radius and energy and its velocity is decreased; the other vortex is deflected in the same direction: it decreases in radius and energy and its velocity is increased.

(Note added March 21st, 1882.)

Sir WILLIAM THOMSON has pointed out to me that when n is very great the time of vibration of the single vortex which for this case is (equation 23)

$$2\pi\sqrt{\frac{\varpi 2\pi^2 e^2}{l^2}}\log\frac{nl}{\pi e}$$

when l is the wave-length $2\pi a/n$, does not agree infinitely nearly as it ought with the value obtained by him for the rapidity of the transverse vibrations of a straight

columnar vortex, which by formula 61 of his paper "On the Vibrations of a Columnar Vortex," is

$$2\pi\Big/\frac{\omega 2\pi^2 e^2}{l^2}\left\{\log\frac{l}{2\pi e}+\cdot 1159\right\}$$

The results would agree approximately if $l/2e$ were indefinitely great compared with n, but it obliges us to neglect the factor $\log n$, and thus the agreement instead of getting better as it ought gets worse as n increases. The way I determined A_n is only suitable when $n\kappa_{,}$ is small, as it is only allowable to expand $\cos n\phi$ as $1-\frac{n^2\phi^2}{2}$ when $n\phi$ remains small within the limits of integration. I have therefore endeavoured to determine A_n in a way which shall not be open to these objections.

With the notation of the paper since

$$A_n=\frac{1}{\pi}\int_0^{2\pi}\frac{\cos n\phi . d\phi}{(a^2+\rho^2+\zeta^2-2a\rho\cos\phi)^{\frac{3}{2}}}$$

we may easily prove that if $a-\rho$ be small compared with a, that

$$a\left(\frac{dA_{n-1}}{da}-\frac{dA_n}{da}\right)=(n-1)A_{n-1}-(n-3)A_n$$

and

$$a\left(\frac{dA_{n+1}}{da}-\frac{dA_n}{da}\right)=(n+3)A_n-(n+1)A_{n+1}$$

If we make these substitutions we find from equations (10) and (12)

$$\frac{d\alpha_n}{dt}=\tfrac{1}{2}ma\beta_n\{\mathfrak{A}_1+\tfrac{1}{2}((n-1)\mathfrak{A}_{n+1}-(n+1)\mathfrak{A}_{n-1})\}$$

$$\frac{d\beta_n}{dt}=\tfrac{1}{2}ma\alpha_n\{\mathfrak{A}_1-\mathfrak{A}_n\}$$

Hence we have only to find an expression for $\mathfrak{A}_n$

$$\mathfrak{A}_n=\frac{1}{\pi((a+\rho)^2+\zeta^2)^{\frac{3}{2}}}\int_0^{2\pi}\frac{\cos n\phi}{\left(\kappa_1^2+\kappa^2\sin^2\frac{\phi}{2}\right)^{\frac{3}{2}}}$$

Now as κ_1^2 is very small this integral will be very large, and as the large part arises when ϕ is small or nearly 2π, we may write as the approximate value of $\mathfrak{A}_n$

$$\frac{2}{\pi\{(a+\rho)^2+\zeta^2\}^{\frac{3}{2}}}\int_0^{\infty}\frac{\cos n\phi d\phi}{\left(\kappa_1^2+\frac{\phi^2}{4}\right)^{\frac{3}{2}}}$$

$$=\frac{4}{\pi\{(a+\rho)^2+\zeta^2\}^{\frac{3}{2}}}\int_0^{\infty}\frac{\cos 2n\phi . d\phi}{(\kappa_1^2+\phi^2)^{\frac{3}{2}}}$$

Let us consider the more general integral

$$u=\int_0^\infty \frac{\cos s\phi d\phi}{(a^2+\phi^2)^{\frac{2p+1}{2}}}$$

Considered as a function of s it is easy to prove that the integral satisfies the differential equation

$$\frac{d^2u}{ds^2}-\frac{2p-1}{2s}\frac{du}{ds}-a^2u=0$$

or if $u=s^pv$

$$\frac{d^2v}{ds^2}+\frac{1}{s}\frac{dv}{ds}-v\left(\frac{p^2}{s^2}+a^2\right)=0$$

The solution of this differential equation may be expressed in terms of BESSEL'S functions of the first and second kinds; we shall, however, only solve here the special case which we require.

It is easily proved that

$$\int_0^\infty \frac{\cos s\phi}{(a^2+\phi^2)^{\frac{3}{2}}}d\phi=-\frac{s}{a^2}\frac{d}{ds}\int_0^\infty \frac{\cos s\phi}{(a^2+\phi^2)^{\frac{1}{2}}}d\phi$$

Since $p=0$ for $\int_0^\infty \frac{\cos s\phi}{(a^2+\phi^2)^{\frac{1}{2}}}d\phi$ it satisfies the differential equation

$$\frac{d^2u}{ds^2}+\frac{1}{s}\frac{du}{ds}-a^2u=0$$

This is the equation solved by Professor STOKES in his paper "On the effect of Internal Friction on the Motion of Pendulums," Camb. Phil. Trans., 1850, and quoted by Sir WILLIAM THOMSON, Phil. Mag., September, 1880.

The solution is there shown to be

$$u=\left(\mathrm{E}+\mathrm{D}\log\frac{1}{as}\right)\left(1+\frac{a^2s^2}{2^2}+\frac{a^4s^4}{2^24^2}+\ldots\right)$$

$$+\mathrm{D}\left(\frac{a^2s^2}{2^2}\mathrm{S}_1+\frac{a^4s^4}{2^24^2}\mathrm{S}_2+\ldots\right)$$

where

$$\mathrm{S}_n=1^{-1}+2^{-1}+\ldots n^{-1}$$

and if as for $\int_0^\infty \frac{\cos s\phi}{(a^2+\phi^2)^{\frac{1}{2}}}d\phi$, $u=0$ when s is infinite E/D is shown to be

$$\log 8+\pi^{-\frac{1}{2}}\Gamma'\tfrac{1}{2}=\cdot 11593$$

so that

$$\int_0^\infty \frac{\cos s\phi}{(a^2+\phi^2)^{\frac{1}{2}}}d\phi = \mathrm{D}\left[\left(\cdot 11593+\log\frac{1}{as}\right)\left(1+\frac{a^2s^2}{2^2}+\frac{a^4s^4}{2^24^2}+\ldots\right.\right.$$
$$\left.\left.+\left(\frac{a^2s^2}{2^2}\mathrm{S}_1+\frac{a^4s^4}{2^24^2}\mathrm{S}_2+\ldots\right)\right)\right]$$

Therefore

$$\int_0^\infty \frac{\cos s\phi}{(a^2+\phi^2)^{\frac{3}{2}}}d\phi = -\frac{s}{a^2}\mathrm{D}\left[\left(\cdot 11593+\log\frac{1}{as}\right)\left(\frac{a^2s}{2}+\frac{a^4s^3}{2^24}+\ldots\right)\right.$$
$$+\frac{1}{s}\left(1+\frac{a^2s^2}{2^2}+\frac{a^4s^4}{2^24^2}+\ldots\right)$$
$$\left.+\left(\frac{a^2s^2}{2}\mathrm{S}_1+\frac{a^4s^3}{2^24}\mathrm{S}_2+\ldots\right)\right]$$

Now since

$$\int_0^\infty \frac{d\phi}{(a^2+\phi^2)^{\frac{3}{2}}} = \frac{1}{a^2}$$

putting $s=0$ we find $\mathrm{D}=-1$.

Therefore

$$\int_0^\infty \frac{\cos s\phi . d\phi}{(a^2+\phi^2)^{\frac{3}{2}}} = \frac{1}{a^2}\left\{1+\frac{a^2s^2}{2^2}+\frac{a^4s^4}{2^24^2}+\ldots\right\}$$
$$+\left(\cdot 11593+\log\frac{1}{as}\right)\left(\frac{s^2}{2}+\frac{a^2s^4}{2^24}+\ldots\right)$$
$$+\left(\frac{s^2}{2}\mathrm{S}_1+\frac{a^2s^4}{2^24}\mathrm{S}_2+\ldots\right)$$

Now approximately

$$\mathfrak{A}_n = \frac{1}{2\pi a^3}\int_0^\infty \frac{\cos 2n\phi d\phi}{(\kappa_1^{\,2}+\phi^2)^{\frac{3}{2}}}$$

hence A_n is found, and if we substitute these values in the expressions for $\frac{d\alpha_n}{dt}$, $\frac{d\beta_n}{dt}$ we shall be able to find the time of vibration in any particular case.

If we suppose $n\kappa_1$ is small, then approximately

$$\mathfrak{A}_n = \frac{1}{2\pi a^3}\left\{\frac{1}{\kappa_1^{\,2}}+n^2\left\{2\left(\log\frac{1}{2n\kappa_1^{\,2}}+\cdot 11593\right)+1\right\}\right\}$$

or if we neglect 1.23 in comparison with $2\log\frac{1}{2n\kappa_1}$

$$\mathfrak{A}_n = \frac{1}{2\pi a^3}\left\{\frac{1}{\kappa^1}+2n^2\log\frac{1}{2n\kappa_1}\right\}$$

Now

$$\frac{d\alpha_n}{dt} = \tfrac{1}{2}ma\beta_n\{\mathfrak{A}_1+\tfrac{1}{2}\}(n-1)\mathfrak{A}_{n+1}-(n+1)\mathfrak{A}_{n-1})$$

$$\frac{d\beta_n}{dt} = \tfrac{1}{2}ma\alpha_n\{\mathfrak{A}_1-\mathfrak{A}_n\}$$

Substituting we find

$$\frac{d\alpha_n}{dt}=-\frac{m\beta_n}{2\pi a^2}(n^2 \log 2\kappa_1+(n^2-1)(n+1) \log n+1-(n-1) \log n-1)$$

$$=-\frac{m}{2\pi a^2}f\beta_n \quad \text{say}$$

$$\frac{d\beta_n}{dt}=\frac{m\alpha_n}{2\pi a^2}(n^2 \log 2n\kappa_1-\log 2\kappa_1)$$

$$=\frac{m}{2\pi a^2}g\alpha_n \quad \text{say}$$

Thus

$$\frac{d^2\beta_n}{dt}+\left(\frac{m}{2\pi a^2}\right)^2 fg\beta_n=0$$

or the time of vibration

$$=2\pi\Big/\frac{\omega e^2}{2a^2}\sqrt{fg}$$

Now if n be not very large

$$f=n^2 \log 2\kappa_1=-n^2 \log \frac{a}{e}$$

$$g=(n^2-1) \log 2\kappa_1=-(n^2-1) \log \frac{a}{e}$$

and the time of vibration

$$=2\pi\Big/\frac{\omega e^2}{2a^2}n\sqrt{n^2-1} \log \frac{a}{e}$$

If n be large

$$f=n^2 \log 2\kappa_1+n^2 \log n=-n^2 \log \frac{a}{ne}$$

$$g=n^2 \log 2n\kappa_1=-n^2 \log \frac{a}{ne}$$

and the time of vibration

$$=2\pi\Big/\frac{\omega e^2}{a^2}n^2 \log \frac{a}{ne}$$

or if l be the wave length $=2\pi a/n$

$$=2\pi\Big/\frac{2\pi^2\omega e^2}{l^2} \log \frac{l}{2\pi e}$$

which agrees approximately with Sir WILLIAM THOMSON's result.