PHILOSOPHICAL TRANSACTIONS.

I. *On the Circulation of Air observed in* KUNDT'S *Tubes, and on some Allied Acoustical Problems.*

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Received October 23,—Read November 15, 1883.

EXPERIMENTERS in Acoustics have discovered more than one set of phenomena apparently depending for their explanation upon the existence of regular currents of air, resulting from vibratory motion, of which theory has as yet rendered no account. This is not, perhaps, a matter for surprise, when we consider that such currents, involving as they do *circulation* of the fluid, could not arise in the absence of friction, however great the extent of vibration. And even when we are prepared to include in our investigations the influence of friction, by which the motion of fluid in the neighbourhood of solid bodies may be greatly modified, we have no chance of reaching an explanation, if, as is usual, we limit ourselves to the supposition of infinitely small motion and neglect the squares and higher powers of the mathematical symbols by which it is expressed.

In the present paper three problems of this kind are considered, two of which are illustrative of phenomena observed by FARADAY.* In these problems the fluid may be treated as incompressible. The more important of them relates to the currents generated over a vibrating plate, arranged as in CHLADNI'S experiments. It was discovered by SAVART that very fine powder does not collect itself at the nodal lines, as does sand in the production of CHLADNI'S figures, but gathers itself into a cloud which, after hovering for a time, settles itself over the places of maximum vibration. This was traced by FARADAY to the action of currents of air, rising from the plate at

* "On a Peculiar Class of Acoustical Figures; and on certain Forms assumed by groups of particles upon Vibrating Elastic Surfaces," Phil. Trans., 1831, p. 299.

the places of maximum vibration, and falling back to it at the nodes. In a vacuum the phenomena observed by SAVART do not take place, all kinds of powder collecting at the nodes. In the investigation of this, as of the other problems, the motion is supposed to take place in two dimensions.

It is probable that the colour phenomena observed by SEDLEY TAYLOR* on liquid films under the action of sonorous vibrations are to be referred to the operation of the aerial vortices here investigated. In a memoir on the colours of the soap-bubble,† BREWSTER has described the peculiar arrangements of colour accompanied by whirling motions, caused by the impact of a gentle current of air. In Mr. TAYLOR'S experiments the film probably divides itself into vibrating sections, associated with which will be aerial vortices reacting laterally upon the film.

The third problem relates to the air currents observed by DVORAK in a KUNDT'S tube, to which is apparently due the formation of the dust figures. In this case we are obliged to take into account the compressibility of the fluid.

[My best thanks are due to Mr. W. M. HICKS, who has been good enough to examine the mathematical work of the paper. The results are thus put forward with greater confidence than I could otherwise have felt.]

§ 1. In the usual notation the equations of motion in two dimensions are

$$\left.\begin{aligned}\frac{1}{\rho}\frac{dp}{dx}&=-\frac{du}{dt}+\nu\nabla^2u-u\frac{du}{dx}-v\frac{du}{dy}\\ \frac{1}{\rho}\frac{dp}{dy}&=-\frac{dv}{dt}+\nu\nabla^2v-u\frac{dv}{dx}-v\frac{dv}{dy}\end{aligned}\right\}\quad . \; . \; . \; . \; . \; . \; . \; . \quad (1),$$

and since the fluid is incompressible,

$$\frac{du}{dx}+\frac{dv}{dy}=0\quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (2).$$

In virtue of (2) we may write

$$u=\frac{d\psi}{dy},\quad v=-\frac{d\psi}{dx}\quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (3).$$

Eliminating p between equations (1), we get

$$\nu\nabla^2\left(\frac{du}{dy}-\frac{dv}{dx}\right)-\frac{d}{dt}\left(\frac{du}{dy}-\frac{dv}{dx}\right)=\frac{d}{dy}\left(u\frac{du}{dx}+v\frac{du}{dy}\right)-\frac{d}{dx}\left(u\frac{dv}{dx}+v\frac{dv}{dy}\right).$$

Now

$$u\frac{du}{dx}+v\frac{du}{dy}=\tfrac{1}{2}\frac{d(u^2+v^2)}{dx}+v\left(\frac{du}{dy}-\frac{dv}{dx}\right)$$

$$u\frac{dv}{dx}+v\frac{dv}{dy}=\tfrac{1}{2}\frac{d(u^2+v^2)}{dy}-u\left(\frac{du}{dy}-\frac{dv}{dx}\right),$$

* Proc. Roy. Soc., 1878.

† Edinburgh Transactions, 1866–67.

and

$$\frac{du}{dy}-\frac{dv}{dx}=\nabla^2\psi,$$

so that

$$\nabla^4\psi-\frac{1}{\nu}\frac{d}{dt}\nabla^2\psi=\frac{u}{\nu}\frac{d\nabla^2\psi}{dx}+\frac{v}{\nu}\frac{d\nabla^2\psi}{dy} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4).$$

For the first approximation we neglect the right-hand member of (4), as being of the second order in the velocities, and take simply

$$\nabla^2\left(\nabla^2-\frac{1}{\nu}\frac{d}{dt}\right)\psi=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5).$$

The solution of (5) may be written*

$$\psi=\psi_1+\psi_2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6),$$

where

$$\nabla^2\psi_1=0, \quad \left(\nabla^2-\frac{1}{\nu}\frac{d}{dt}\right)\psi_2=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7).$$

We will now introduce the suppositions that the motion is periodic with respect to x, and also (to a first approximation) with respect to t. We thus assume that ψ_1 and ψ_2 are proportional to $\cos kx$, and also to e^{int}. The wave-length (λ) along x is $2\pi/k$, and the period τ is $2\pi/n$. The equations (7) now become

$$\left(\frac{d^2}{dy^2}-k^2\right)\psi_1=0, \quad \left(\frac{d^2}{dy^2}-k^2-\frac{in}{\nu}\right)\psi_2=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8),$$

by which ψ_1 and ψ_2 are to be determined as functions of y. If we write

$$k'^2=k^2+\frac{in}{\nu} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9),$$

we have as the most general solutions of (8)

$$\psi_1=\mathrm{A}e^{-ky}+\mathrm{B}e^{+ky} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10),$$

$$\psi_2=\mathrm{C}e^{-k'y}+\mathrm{D}e^{+k'y} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11).$$

With respect to the value of k', we see from (9) that it is complex. If we write

$$k^2=\mathrm{P}^2\cos 2\alpha, \quad \frac{n}{\nu}=\mathrm{P}^2\sin 2\alpha,$$

then

$$k'=\mathrm{P}\cos\alpha+i\,\mathrm{P}\sin\alpha.$$

* STOKES "On Pendulums," Camb. Phil. Trans., vol. ix., 1850.

In all the applications that we shall have occasion to make, an approximate value of k' is admissible. On account of the smallness of ν, n/ν is very large in comparison with k^2, that is to say, the thickness of the stratum through which the tangential motion can be propagated in time τ is very small relatively to the wave-length λ. We may therefore neglect k^4 in the equation

$$P^4=k^4+\frac{n^2}{\nu^2},$$

and take simply

$$P^2=n/\nu.$$

Again

$$(\sin\alpha-\cos\alpha)^2=1-\sin 2\alpha=\tfrac{1}{2}\frac{k^4\nu^2}{n^2},$$

so that the difference between $\cos\alpha$ and $\sin\alpha$ may be neglected. We will therefore write

$$k'=\beta(1+i) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12),$$

where

$$\beta=\sqrt{\frac{n}{2\nu}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13).$$

We must now distinguish the cases which we have to investigate. In the first we suppose that a wave motion is in progress in a vessel whose horizontal bottom occupies a fixed plane $y=0$. We may conceive the fluid to be water vibrating in stationary waves under the action of gravity, the question being to examine the influence of the bottom upon the motion. If there are no other solids in the neighbourhood of the bottom, we may put $D=0$, y being measured upwards, and β being taken positive.

The conditions to be satisfied at $y=0$ are that u and v should there vanish. Thus

$$A+B+C=0, \quad -kA+kB-k'C=0,$$

so that

$$\psi=C\left\{-\cosh ky+\frac{k'}{k}\sinh ky+e^{-k'y}\right\},$$

and

$$u=C\{-k\sinh ky+k'\cosh ky-k'e^{-k'y}\}.$$

At a short distance from the bottom, $u=k'C$. If we denote by u_0 the maximum value of u near the bottom, we have

$$k'C=u_0\, e^{int}\cos kx,$$

and then

$$\psi=u_0\, e^{int}\cos kx\left\{-\frac{\cosh ky}{k'}+\frac{\sinh ky}{k}+\frac{e^{-k'y}}{k'}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14),$$

$$u=u_0\, e^{int} \cos kx\left\{-\frac{k}{k'} \sinh ky+\cosh ky-e^{-k'y}\right\} \quad . \quad . \quad . \quad . \quad . \quad (15),$$

$$v=u_0\, e^{int} \sin kx\left\{-\frac{k}{k'} \cosh ky+\sinh ky+\frac{k}{k'}e^{-k'y}\right\} \quad . \quad . \quad . \quad . \quad (16).$$

These are the symbolical values. If we throw away the imaginary parts, we have as the solution in real quantities by (12),

$$\psi=u_0 \cos kx\left\{-\frac{\cosh ky}{\beta\sqrt{2}} \cos\left(nt-\tfrac{1}{4}\pi\right)+\frac{\sinh ky}{k} \cos nt+\frac{e^{-\beta y}}{\beta\sqrt{2}} \cos\left(nt-\tfrac{1}{4}\pi-\beta y\right)\right\} \quad . \quad (17),$$

$$u=u_0 \cos kx\left\{-\frac{k \sinh ky}{\beta\sqrt{2}} \cos\left(nt-\tfrac{1}{4}\pi\right)+\cosh ky \cos nt-e^{-\beta y} \cos\left(nt-\beta y\right)\right\} \quad . \quad . \quad (18),$$

$$v=u_0 \sin kx\left\{-\frac{k \cosh ky}{\beta\sqrt{2}} \cos\left(nt-\tfrac{1}{4}\pi\right)+\sinh ky \cos nt+\frac{k\, e^{-\beta y}}{\beta\sqrt{2}} \cos\left(nt-\tfrac{1}{4}\pi-\beta y\right)\right\} \quad (19).$$

This is the solution to a first approximation. At a very small distance from the bottom the terms in $e^{-\beta y}$ become insensible.

Although the values of u and v in (18) and (19) are strictly periodic, it is proper to notice that the same property does not attach to the motions thereby defined of the particles of the fluid. In our notation u is not the velocity of any particular particle of the fluid, but of the particle, whichever it may be, that *at the moment under consideration* occupies the point x, y. If $x+\xi$, $y+\eta$ be the actual position at time t of the particle whose mean position during several vibrations is x, y, then the real velocities of the particle at time t are not u, v, but

$$u+\frac{du}{dx}\xi+\frac{du}{dy}\eta, \quad v+\frac{dv}{dx}\xi+\frac{dv}{dy}\eta\,;$$

and thus the mean velocity parallel to x is not necessarily zero, but is equal to the mean value of

$$\frac{du}{dx}\xi+\frac{du}{dy}\eta,$$

in which again

$$\xi=\int u\, dt, \quad \eta=\int v\, dt.$$

From the general form of u, viz., $\cos kx\, \mathrm{F}(y, t)$, it follows readily that $\int \frac{du}{dx}\xi\, dt=0$. For the second term we must calculate from the actual values as given in (18), (19). Thus

$$\eta=\frac{u_0 \sin kx}{n}\left\{-\frac{k \cosh ky}{\beta\sqrt{2}}\sin(nt-\tfrac{1}{4}\pi)+\sinh ky \sin nt+\frac{k\,e^{-\beta y}}{\beta\sqrt{2}}\sin(nt-\tfrac{1}{4}\pi-\beta y)\right\},$$

$$\frac{du}{dy}=u_0 \cos kx\left\{-\frac{k^2 \cosh ky}{\beta\sqrt{2}}\cos(nt-\tfrac{1}{4}\pi)+k \sinh ky \cos nt+\sqrt{2}.\beta\, e^{-\beta y}\cos(nt+\tfrac{1}{4}\pi-\beta y)\right\},$$

of which the two first terms may be neglected relatively to the third (containing the large factor β). The product of η and $\frac{du}{dy}$ will consist of two parts, the first independent of t, and the second harmonic functions of $2nt$. It is with the first only that we are here concerned. The mean value of the velocity parallel to x is thus

$$\frac{u_0^2 \sin 2kx\, e^{-\beta y}}{4n}\left\{k \cosh ky \cos \beta y+\sqrt{2}.\beta \sinh ky \quad \sin(\beta y-\tfrac{1}{4}\pi)-k\,e^{-\beta y}\right\}.$$

On account of the factor $e^{-\beta y}$, this quantity is insensible except when ky is extremely small. We may therefore write it

$$\frac{u_0^2 \sin 2kx\, e^{-\beta y}}{4\mathrm{V}}\left\{\cos \beta y+\beta y(\sin \beta y-\cos \beta y)-e^{-\beta y}\right\} \quad . \quad . \quad . \quad . \quad (20),$$

V (equal to k/n) being the velocity of propagation of waves corresponding to k and n.

The only approximation employed in the derivation of (15) and (16) is the neglect of the right hand member of (4), and the corresponding real values of u and v could if necessary be readily exhibited without the use of a merely approximate value of k'. To proceed further we must calculate the value of

$$\frac{u}{\nu}\frac{d\nabla^2\psi}{dx}+\frac{v}{\nu}\frac{d\nabla^2\psi}{dy} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

in (4), for which it will be sufficient to take the values given by the first approximation. Thus

$$\nabla^2\psi=\nabla^2\psi_2=\frac{1}{\nu}\frac{d\psi_2}{dt},$$

and by (17)

$$\frac{d\psi_2}{dt}=-\frac{nu_0 \cos kx\, e^{-\beta y}}{\beta\sqrt{2}}\sin(nt-\tfrac{1}{4}\pi-\beta y),$$

from which we find as the value of (21),

$$\frac{nku_0^2 \sin 2kx\, e^{-\beta y}}{4\nu^2\beta\sqrt{2}}\left\{\left(\frac{k}{\beta\sqrt{2}}-\frac{\beta\sqrt{2}}{k}\right)\sinh ky \sin \beta y-\sqrt{2}\cosh ky \cos \beta y+\sqrt{2}\,e^{-\beta y}\right\}$$

$$+ \text{ terms in } 2nt.$$

On account of the factor $e^{-\beta y}$ this quantity is sensible only when y is very small. We may write it with sufficient approximation

$$\frac{nku_0^2 \sin 2kx\, e^{-\beta y}}{4\nu^2\beta}\left\{-\beta y \sin \beta y - \cos \beta y + e^{-\beta y}\right\} \quad . \quad . \quad . \quad . \quad . \quad (22).$$

The terms in $2nt$, corresponding to motions of half the original period, are not required for our purpose, which is to investigate the non-periodic motion of the second order. The equation with which we have to proceed is found by equating (22) to $\nabla^4\psi$. The solution will consist of two parts, one resulting from the direct integration of (22) and involving the factor $e^{-\beta y}$, the second a complementary function with arbitrary coefficients satisfying $\nabla^4\psi=0$. In the calculation of the first part we may identify ∇^4 with d^4/dy^4, on account of the smallness of k relatively to β. In this way our equation becomes

$$\frac{d^4\psi}{d(\beta y)^4}=\frac{nku_0^2 \sin 2kx\, e^{-\beta y}}{4\nu^2\beta^5}\left\{-\beta y \sin \beta y - \cos \beta y + e^{-\beta y}\right\} \quad . \quad . \quad . \quad . \quad (23),$$

of which the solution is

$$\psi=\frac{nku_0^2 \sin 2kx\, e^{-\beta y}}{4\nu^2\beta^5}\left\{\tfrac{3}{4} \cos \beta y + \tfrac{1}{2} \sin \beta y + \tfrac{1}{4}\beta y \sin \beta y + \tfrac{1}{16} e^{-\beta y}\right\} \quad . \quad . \quad (24).$$

The complementary function, being proportional to $\sin 2kx$, may be written

$$\frac{nku_0^2 \sin 2kx}{4\nu^2\beta^5}\{(\mathrm{A}+\mathrm{B}y)e^{-2ky}+(\mathrm{A}'+\mathrm{B}'y)e^{+2ky}\}.$$

If the fluid be uninterrupted by a free surface, or otherwise, within distances for which ky is sensible, we must suppose $(\mathrm{A}'+\mathrm{B}'y)=0$, so that by (13) the complementary function may be written

$$\frac{u_0^2 \sin 2kx}{\beta \mathrm{V}}(\mathrm{A}+\mathrm{B}y)e^{-2ky}.$$

The condition that v (equal to $-d\psi/dx$) must vanish when $y=0$, gives $\mathrm{A}=-\frac{13}{16}$. For the velocity parallel to x we have

$$u=\frac{u_0^2 \sin 2kx}{\mathrm{V}}\Big[e^{-\beta y}\{-\sin \beta y - \tfrac{1}{4} \cos \beta y + \tfrac{1}{4}\beta y \cos \beta y - \tfrac{1}{4}\beta y \sin \beta y - \tfrac{1}{8}e^{-\beta y}\}$$
$$+\beta^{-1}e^{-2ky}\{\mathrm{B}-2k(\mathrm{A}+\mathrm{B}y)\}\Big].$$

In order that u should vanish when $y=0$, we must have

$$\mathrm{B}=2k\mathrm{A}+\tfrac{3}{8}\beta=\tfrac{3}{8}\beta-\tfrac{13}{8}k=\tfrac{3}{8}\beta,$$

approximately. Thus

$$u=\frac{u_0^2\sin 2kx}{V}\left[e^{-\beta y}\{-\sin\beta y-\tfrac{1}{4}\cos\beta y+\tfrac{1}{4}\beta y\cos\beta y-\tfrac{1}{4}\beta y\sin\beta y-\tfrac{1}{8}e^{-\beta y}\}+\tfrac{3}{8}e^{-2ky}\{1-2ky\}\right] \quad . \quad (25),$$

and

$$v=-\frac{2ku_0^2\cos 2kx}{\beta V}\left[e^{-\beta y}\{\tfrac{3}{4}\cos\beta y+\tfrac{1}{2}\sin\beta y+\tfrac{1}{4}\beta y\sin\beta y+\tfrac{1}{16}e^{-\beta y}\}+e^{-2ky}\{-\tfrac{13}{16}+\tfrac{3}{8}\beta y\}\right] \quad . \quad (26).$$

To obtain the mean velocity parallel to x of a particle, we must add to (25), the terms previously investigated and expressed by (20). If we call the total u', we have

$$u'=\frac{u_0^2\sin 2kx}{V}\left[e^{-\beta y}\{-\sin\beta y-\tfrac{3}{8}e^{-\beta y}\}+\tfrac{3}{8}e^{-2ky}\{1-2ky\}\right] \quad . \quad . \quad . \quad (27).$$

At a short distance from the bottom $e^{-\beta y}$ becomes insensible, and we have simply

$$u'=\tfrac{3}{8}\frac{u_0^2\sin 2kx}{V}e^{-2ky}(1-2ky). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (28),$$

$$v'=-\frac{2ku_0^2\cos 2kx}{\beta V}e^{-2ky}(-\tfrac{13}{16}+\tfrac{3}{8}\beta y) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (29).$$

The steady motion expressed by (28) and (29) is of a very simple character. It consists of a series of vortices periodic with respect to x in a distance $\frac{1}{2}\lambda$. For a given x the horizontal motion is of one sign near the bottom, and of the opposite sign at a distance from it, the place of transition being at $y=(2k)^{-1}=\lambda/4\pi$. The horizontal motion of the first order near the bottom being by (18) $u=u_0\cos kx\cos nt$, we see that it is a maximum when $kx=0$, π, 2π, . . . If we call these places loops, and the places of minimum velocity nodes, (29) shows that v' is negative and a maximum at the loops, positive and a maximum at the nodes. The fluid therefore rises from the bottom over the nodes and falls back again over the loops, the horizontal motion near the bottom being thus directed towards the nodes and from the loops. The maximum horizontal motion is simply $\frac{3}{8}u_0^2/V$, and is *independent of the value of* ν. We cannot, therefore, avoid considering this motion by supposing the coefficient of viscosity to be very small, the maintenance of the vortices becoming easier in the same proportion as the forces tending to produce the vortical motion diminish.

To ascertain the character of the motion quite close to the bottom, we must include the terms in $e^{-\beta y}$. When y is extremely small

$$u'=u_0^2V^{-1}\sin 2kx\{-\tfrac{1}{4}\beta y+\ldots\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (30),$$

so that the motion is here in the opposite direction to that which prevails when $e^{-\beta y}$ can be neglected.

A few corresponding values of βy and of $-(\sin \beta y+\frac{3}{8}e^{-\beta y})e^{-\beta y}+\frac{3}{8}$ are annexed, in order to show the distribution of velocities within the thin frictional layer.

βy.		βy.	
$\frac{\pi}{16}$	$-\cdot 038$	$\frac{3\pi}{8}$	$+\cdot 055$
$\frac{\pi}{8}$	$-\cdot 054$	$\frac{\pi}{2}$	$+\cdot 151$
$\frac{3\pi}{16}$	$-\cdot 049$	π	$+\cdot 374$
$\frac{\pi}{4}$	$-\cdot 025$	$\frac{3\pi}{2}$	$+\cdot 384$

It appears that ($\sin 2kx$ being positive) the velocity is negative from the plate outwards until βy somewhat exceeds $\frac{1}{4}\pi$, after which it is positive, until reversed by the factor $(1-2ky)$. The greatest negative velocity in the layer is about $\frac{1}{7}$ of that which is found at a little distance outside the layer.

FARADAY found that fine sand, scattered over the bottom, tended to collect at the loops. This is in agreement with what the present calculation would lead us to expect, provided that we can suppose that the sand is controlled by the layer at the bottom whose motion is negative. The exceeding thinness of the layer, however, presents itself as a difficulty. The subject requires further experimental investigation; but in the meantime the following data may be worth notice, though in some respects, *e.g.*, the shallowness of the liquid in relation to the wave-length, the circumstances differed materially from those assumed in the theoretical investigation.

The liquid was water ($\nu=\cdot 014$ C.G.S.), and the period of vibration was $\frac{1}{15}$, so that $n=2\pi\times 15$. The thickness of the layer

$$=\frac{\pi}{4}\sqrt{\frac{2\nu}{n}}=\cdot 0135 \text{ centim.}$$

Measurements of the diameters of the particles of sand gave about $\cdot 02$ centim., so that the grains would be almost wholly immersed in the negative layer, even if isolated. It seems therefore that the observed motion to the loops gives rise in this case to no difficulty. But it is possible that the behaviour of the sand is materially influenced by the vertical motion of the vessel by which in these experiments the liquid vibrations are maintained.*

§ 2. In the problem to which we now proceed the motion will be supposed to have its origin in the assumed motion of a flexible plate situated when in equilibrium at $y=0$. Thus for a first approximation we take $u=0$, $v=v_0 \sin kx\, e^{int}$, when $y=0$, and the question is to investigate the resulting motion of the fluid in contact with the plate.

* See a paper "On the Crispations of Fluid resting upon a Vibrating Support," Phil. Mag., July, 1883.

The solution to a first approximation is readily obtained. As in (10), (11), we have

$$\psi=\psi_1+\psi_2=e^{int}\cos kx\,(\mathrm{A}e^{-ky}+\mathrm{C}e^{-k'y}) \quad . \quad . \quad . \quad . \quad . \quad . \quad (31),$$

in which we may take as before

$$k'=\sqrt{\frac{n}{2\nu}}\cdot(1+i)=\beta(1+i) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (32).$$

By the condition at $y=0$,

$$\mathrm{A}=-\frac{k'}{k}\,\mathrm{C}, \qquad \mathrm{C}=\frac{v_0}{k'-k},$$

so that

$$\psi=\frac{v_0e^{int}\cos kx}{k-k'}\left\{-\frac{k'}{k}\,e^{-ky}+e^{-k'y}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (33),$$

$$u=\frac{v_0e^{int}\cos kx}{k-k'}\left\{k'e^{-ky}-k'e^{-k'y}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (34).$$

In passing to real quantities it will be convenient to write

$$\frac{v_0}{k-k'}=\mathrm{H}e^{i\epsilon} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (35).$$

Thus throwing away the imaginary parts of (33), (34), we get

$$\psi=\cos kx\left\{-\frac{\beta\sqrt{2}}{k}e^{-ky}\cos\left(nt+\epsilon+\tfrac{1}{4}\pi\right)+e^{-\beta y}\cos\left(nt+\epsilon-\beta y\right)\right\} \quad . \quad . \quad . \quad (36),$$

$$u=\sqrt{2}.\beta\,\mathrm{H}\cos kx\left\{e^{-ky}\cos\left(nt+\epsilon+\tfrac{1}{4}\pi\right)-e^{-\beta y}\cos\left(nt+\epsilon+\tfrac{1}{4}\pi-\beta y\right)\right\} \quad . \quad (37),$$

$$v=\mathrm{H}\sin kx\left\{-\beta\sqrt{2}\,e^{-ky}\cos\left(nt+\epsilon+\tfrac{1}{4}\pi\right)+ke^{-\beta y}\cos\left(nt+\epsilon-\beta y\right)\right\} \quad . \quad . \quad (38).$$

From (32), (35), the approximate value of H is $-v_0/\beta\sqrt{2}$, and that of ϵ is $-\frac{1}{4}\pi$. More exact values will however be required later. We find

$$\mathrm{H}=-\frac{v_0}{\sqrt{\{(\beta-k)^2+\beta^2\}}}=-\frac{v_0}{\beta\sqrt{2}}\left(1+\frac{k}{2\beta}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad (39),$$

$$\cos\epsilon=\frac{\beta-k}{\sqrt{\{(\beta-k)^2+\beta^2\}}}=\frac{1}{\sqrt{2}}\left(1-\frac{k}{2\beta}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (40).$$

The values of u and v above expressed give $u=0$, $v=v_0\sin kx\cos nt$, when $y=0$. This is sufficient for a first approximation, but in proceeding further we must remember

that these prescribed velocities apply in strictness not to $y=0$, but to $y=\frac{v_0}{n}\sin kx \sin nt$. Substituting the latter value of y in the expressions (37), and (38), we find

$$u=\sqrt{2}.\beta\,\mathrm{H}\cos kx\left\{-ky\cos\left(nt+\epsilon+\tfrac{1}{4}\pi\right)+\sqrt{2}.\beta y\cos\left(nt+\epsilon+\tfrac{1}{2}\pi\right)\right\}$$

$$=\frac{\beta^2 v_0 \mathrm{H}}{n}\sin 2kx\sin nt\left\{-\frac{k}{\beta\sqrt{2}}\cos\left(nt+\epsilon+\tfrac{1}{4}\pi\right)+\cos\left(nt+\epsilon+\tfrac{1}{2}\pi\right)\right\}$$

$$=\frac{\beta^2 v_0 \mathrm{H}}{2n}\sin 2kx\left\{\frac{k}{\beta\sqrt{2}}\sin\left(\epsilon+\tfrac{1}{4}\pi\right)-\sin\left(\epsilon+\tfrac{1}{2}\pi\right)\right\}+\text{terms in } 2nt.$$

The first term within the bracket is of the *second* order in k/β relatively to the latter term, and may be omitted. Thus

$$u=-\frac{\beta^2 v_0 \mathrm{H}}{2n}\sin 2kx\cos\epsilon.$$

The terms in $2nt$ we need not further examine. From (39), (40), $\mathrm{H}\cos\epsilon=-v_0/2\beta$, very approximately, so that we may write

$$u=\frac{\beta v_0^2}{4n}\sin 2kx \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (41).$$

To the same degree of approximation, $v=v_0\sin kx\cos nt$, simply.

We have next, as in the first problem, to consider the complete equation

$$\nabla^4\psi=\frac{u}{\nu^2}\frac{d^2\psi_2}{dx\,dt}+\frac{v}{\nu^2}\frac{d^2\psi_2}{dy\,dt} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (42)$$

in the right hand member of which we use the approximate values given by (36), (37), (38). Thus

$$\frac{d\psi_2}{dt}=-n\mathrm{H}\cos kx\,e^{-\beta y}\sin\left(nt+\epsilon-\beta y\right),$$

and (42) becomes

$$\nabla^4\psi=\frac{nk\beta\mathrm{H}^2\sin 2kx\,e^{-\beta y}}{4\nu^2}\left\{e^{-ky}\left(\frac{2\beta}{k}\sin\beta y-\sin\beta y-\cos\beta y\right)+2e^{-\beta y}\right\} \quad . \quad . \quad (43).$$

It will be found presently that the term divided by k disappears from the final result, and thus we have to pursue the approximation further than might at first appear necessary. We may however neglect terms of order k^2/β^2, in comparison with the principal term. Thus ∇^4 may be identified with $\frac{d^4}{dy^4}$, and the equation becomes

$$\frac{d^4\psi}{d(\beta y)^4}=\frac{nk\mathrm{H}^2 \sin 2kx\, e^{-\beta y}}{4\nu^2\beta^3}\left\{\left(\frac{2\beta}{k}-1\right)\sin \beta y-\cos \beta y-2\beta y \sin \beta y+2e^{-\beta y}\right\} \quad . \quad (44),$$

whence

$$\psi=\frac{nk\mathrm{H}^2 \sin 2kx\, e^{-\beta y}}{4\nu^2\beta^3}\left\{\left(-\frac{\beta}{2k}+\tfrac{5}{4}\right)\sin \beta y+\tfrac{5}{4}\cos \beta y+\tfrac{1}{2}\beta y \sin \beta y+\tfrac{1}{8}e^{-\beta y}\right\} \quad . \quad (45).$$

And

$$u=\frac{d\psi}{dy}=\frac{nk\mathrm{H}^2 \sin 2kx\, e^{-\beta y}}{4\nu^2\beta^2}\left\{\left(\frac{\beta}{2k}-2\right)\sin\beta y-\frac{\beta}{2k}\cos\beta y-\tfrac{1}{2}\beta y \sin\beta y+\tfrac{1}{2}\beta y\cos \beta y-\tfrac{1}{4}e^{-\beta y}\right\} (46).$$

To obtain the value of u at the surface of the plate it will be sufficient to put $y=0$ in (46). Thus

$$u=\frac{nk\mathrm{H}^2 \sin 2kx}{4\nu^2\beta^2}\left\{-\frac{\beta}{2k}-\tfrac{1}{4}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (47).$$

By (32), (39)

$$\frac{nk\mathrm{H}^2}{4\nu^2\beta^2}=\frac{kv_0^2}{2n}\left(1+\frac{k}{\beta}\right)=\frac{v_0^2}{2\mathrm{V}}\left(1+\frac{k}{\beta}\right),$$

if as before we put V for k/n. Thus in (47)

$$u=\frac{v_0^2}{4\mathrm{V}}\left(-\frac{\beta}{k}-\tfrac{3}{2}\right)\sin 2kx \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (48).$$

To obtain the complete value of u at the surface of the plate, corresponding to (37), (46), we have to add to (48) that given in (41). The term of lowest order disappears, and we are left simply with

$$u=-\frac{3v_0^2}{8\mathrm{V}}\sin 2kx. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (49).$$

In like manner we find for the complete value of v at the surface of the plate corresponding to (38), (45),

$$v=v_0 \sin kx \cos nt-\frac{11v_0^2k \cos 2kx}{8\beta\mathrm{V}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (50).$$

The values of u and v expressed in (49) and the second part of (50) must be cancelled by a suitable choice of the complementary function, satisfying $\nabla^4\psi=0$, so that to the second order of approximation the fluid in contact with the plate may have no relative motion.

The complementary function is

$$\psi=(\mathrm{A}+\mathrm{B}y)e^{-2ky}\sin 2kx,$$

whence

$$u=\{B-2k(A+By)\}e^{-2ky}\sin 2kx,$$

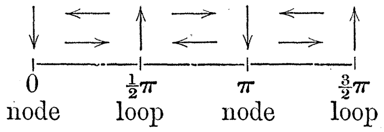
$$v=-2k(A+By)e^{-2ky}\cos 2kx.$$

Determining the constants as indicated above we get

$$u=\frac{3v_0^2}{8V}(1-2ky)e^{-2ky}\sin 2kx \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (51),$$

$$v=-\frac{kv_0^2}{8\beta V}(11+6\beta y)e^{-2ky}\cos 2kx \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (52).$$

The velocities given by (51), (52) are the only part of the motion of the second order which is sensible beyond a very small distance from the vibrating plate. The nodes of the plate (where sand would collect) are at the points given by $kx=0, \pi, 2\pi\ldots$, and the loops at the points $kx=\frac{1}{2}\pi, \frac{3}{2}\pi\ldots$ At the former points v is negative, and at the latter positive. For $kx=\frac{1}{4}\pi$, u is positive, and for $kx=\frac{3}{4}\pi$, u is negative.



The magnitude of the vortical motion is independent of the coefficient of friction.

The complete value of u to the second order of approximation (except the terms in $2nt$) is obtained by adding together (37), (46), and (51), and it will contain the term divided by k in (46), whose appearance, however, is misleading. The objectionable term will be got rid of, if we express the mean velocity of a particle, instead of as in (46), the mean velocity *at a point.* For this purpose we are to add to (46), (51), the mean value of

$$\xi\frac{du}{dx}+\eta\frac{du}{dy},$$

as calculated from the first approximation where

$$\xi=\int udt, \quad \eta=\int vdt.$$

As in the former problem the mean value of $\xi\dfrac{du}{dx}$ is zero.

Multiplying together $\dfrac{du}{dy}$, and $\int vdt$ as found from (37), (38), and rejecting the terms in $2nt$, we get with omission of k^2,

$$-\frac{k\beta^2\mathrm{H}^2e^{-\beta y}\sin 2kx}{n}\left\{\frac{\beta}{2k}(1-ky)(\sin\beta y-\cos\beta y)+\tfrac{1}{2}e^{-\beta y}\right\} \quad . \quad . \quad . \quad . \quad (53),$$

in which we may write

$$-\frac{k\beta^2\mathrm{H}^2}{n}=-\frac{k\beta^4\mathrm{H}^2}{n\beta^2}=-\frac{nk\mathrm{H}^2}{4\nu^2\beta^2}.$$

Combining (53), (46), and (51), we get finally

$$u'=\frac{nk\mathrm{H}^2e^{-\beta y}\sin 2kx}{4\nu^2\beta^2}\left\{-2\sin\beta y-\tfrac{3}{4}e^{-\beta y}\right\}+\frac{3v_0^2}{8\mathrm{V}}(1-2ky)e^{-2ky}\sin 2kx$$

$$=\frac{v_0^2\sin 2kx}{2\mathrm{V}}\left\{-e^{-\beta y}(2\sin\beta y+\tfrac{3}{4}e^{-\beta y})+\tfrac{3}{4}(1-2ky)e^{-2ky}\right\} \quad . \quad . \quad . \quad . \quad . \quad (54),$$

which expresses the mean particle velocity.

When βy is very small, (54) gives

$$u'=\frac{v_0^2\sin 2kx}{2\mathrm{V}}(-\tfrac{1}{2}\beta y+\ldots) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (55).$$

from which it appears that quite close to the plate the mean velocity is in the opposite direction to that which is found outside the frictional layer.

§ 3. In the third problem, relating to KUNDT'S tubes, the fluid must be treated as compressible, as the motion is supposed to be approximately in one dimension, parallel (say) to x. The solution to a first approximation is merely an adaptation to two dimensions of the corresponding solution for a tube of revolution by KIRCHHOFF,* simplified by the neglect of the terms relating to the development and conduction of heat. It is probable that the solution to the second order would be practicable also for a tube of revolution, but for the sake of simplicity I have adhered to the case of two dimensions. The most important point in which the two problems are likely to differ can be investigated very simply, without a complete solution.

If we suppose $p=a^2\rho$, and write σ for $\log\rho-\log\rho_0$, the fundamental equations are

$$a^2\frac{d\sigma}{dx}=-\frac{du}{dt}-u\frac{du}{dx}-v\frac{du}{dy}+\nu\nabla^2u+\nu'\frac{d}{dx}\left(\frac{du}{dx}+\frac{dv}{dy}\right) \quad . \quad . \quad . \quad . \quad (56),$$

with a corresponding equation for v, and the equation of continuity,

$$\frac{du}{dx}+\frac{dv}{dy}+\frac{d\sigma}{dt}+u\frac{d\sigma}{dx}+v\frac{d\sigma}{dy}=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (57).$$

* POGG. Ann., t. cxxxiv., 1868.

Whatever may be the actual values of u and v, we may write

$$u=\frac{d\phi}{dx}+\frac{d\psi}{dy}, \quad v=\frac{d\phi}{dy}-\frac{d\psi}{dx} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (58),$$

in which

$$\nabla^2\phi=\frac{du}{dx}+\frac{dv}{dy}, \quad \nabla^2\psi=\frac{du}{dy}-\frac{dv}{dx} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (59).$$

From (56), (57),

$$\left(a^2+\nu'\frac{d}{dt}\right)\frac{d\sigma}{dx}=-\frac{du}{dt}+\nu\,\nabla^2 u-u\frac{du}{dx}-v\frac{du}{dy}-\nu'\frac{d}{dx}\left(u\frac{d\sigma}{dx}+v\frac{d\sigma}{dy}\right) \quad . \quad . \quad . \quad (60),$$

$$\left(a^2+\nu'\frac{d}{dt}\right)\frac{d\sigma}{dy}=-\frac{dv}{dt}+\nu\,\nabla^2 v-u\frac{dv}{dx}-v\frac{dv}{dy}-\nu'\frac{d}{dy}\left(u\frac{d\sigma}{dx}+v\frac{d\sigma}{dy}\right) \quad . \quad . \quad . \quad (61).$$

Again from (60), (61),

$$\left(a^2+\nu\frac{d}{dt}+\nu'\frac{d}{dt}\right)\nabla^2\sigma-\frac{d^2\sigma}{dt^2}=\frac{d}{dt}\left(u\frac{d\sigma}{dx}+v\frac{d\sigma}{dy}\right)-(\nu+\nu')\nabla^2\left(u\frac{d\sigma}{dx}+v\frac{d\sigma}{dy}\right)$$
$$-\frac{d}{dx}\left(u\frac{du}{dx}+v\frac{du}{dy}\right)-\frac{d}{dy}\left(u\frac{dv}{dx}+v\frac{dv}{dy}\right). \quad (62).$$

For the first approximation the terms of the second order in u, v, and σ are to be omitted. If we assume that as functions of t, all the periodic quantities are proportional to e^{int}, and write q for $a^2+in\nu+in\nu'$, (62) becomes

$$q\nabla^2\sigma+n^2\sigma=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (63).$$

Now by (57), (59),

$$\nabla^2\phi=-in\sigma=i\frac{q}{n}\nabla^2\sigma,$$

so that

$$\phi=i\frac{q}{n}\sigma,\text{*}$$

and

$$u=\frac{iq}{n}\frac{d\sigma}{dx}+\frac{d\psi}{dy}, \quad v=\frac{iq}{n}\frac{d\sigma}{dy}-\frac{d\psi}{dx} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (64).$$

Substituting in (60), (61), with omission of terms of the second order, we get in view of (63),

$$(\nu\nabla^2-in)\frac{d\psi}{dy}=0, \qquad\qquad (\nu\nabla^2-in)\frac{d\psi}{dx}=0,$$

* It is unnecessary to add a complementary function ϕ', satisfying $\nabla^2\phi'=0$, as the motion corresponding thereto may be regarded as covered by ψ.

whence

$$(\nu\nabla^2 - in)\psi = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (65).$$

If we eliminate σ directly from the fundamental equations (56), we get

$$\left(\nu\nabla^4 - \frac{d}{dt}\nabla^2\right)\psi = \frac{d}{dy}\left(u\frac{du}{dx} + v\frac{du}{dy}\right) - \frac{d}{dx}\left(u\frac{dv}{dx} + v\frac{dv}{dy}\right) = \frac{d}{dy}(v\nabla^2\psi) + \frac{d}{dx}(u\nabla^2\psi)$$

$$= \left(\frac{du}{dx} + \frac{dv}{dy}\right)\nabla^2\psi + u\frac{d\nabla^2\psi}{dx} + v\frac{d\nabla^2\psi}{dy} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (66).$$

If we now assume that as functions of x the quantities σ, ψ, &c., are proportional to e^{ikx}, equations (63), (65) may be written

$$\left(\frac{d^2}{dy^2} - k''^2\right)\sigma = 0, \qquad \text{where } k''^2 = k^2 - \frac{n^2}{q} \quad . \quad . \quad . \quad . \quad . \quad . \quad (67),$$

$$\left(\frac{d^2}{dy^2} - k'^2\right)\psi = 0, \qquad \text{where } k'^2 = k^2 + \frac{in}{\nu} \quad . \quad . \quad . \quad . \quad . \quad . \quad (68).$$

If the origin for y be in the middle between the two parallel boundaries, σ must be an even function of y, and ψ must be an odd function. Thus we may write

$$\sigma = \mathrm{A} \cosh k''y \, . \, e^{int} e^{ikx}, \qquad \psi = \mathrm{B} \sinh k'y \, . \, e^{int} e^{ikx} \quad . \quad . \quad . \quad . \quad (69),$$

$$\left.\begin{aligned} u &= \left(-\frac{kq}{n}\mathrm{A}\cosh k''y + k'\mathrm{B}\sinh k'y\right)e^{int}e^{ikx} \\ v &= \left(\frac{ik''q}{n}\mathrm{A}\sinh k''y - ik\mathrm{B}\sinh k'y\right)e^{int}e^{ikx} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (70).$$

If the fixed walls are situated at $y = \pm y_1$, u and v must vanish for these values of y. Eliminating from (70) the ratio of A to B, we get as the equation for determining k,

$$k^2 \tanh k'y_1 = k'k'' \tanh k''y_1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (71),$$

in which k', k'' are given as functions of k by (67), (68). We now introduce further approximations dependent upon the assumption that the direct influence of friction extends through a layer whose thickness is a small fraction only of y_1. On this supposition k' is large, and k'' is small, so that we may put $\tanh k'y_1 = \pm 1$, $\tanh k''y_1 = \pm k''y_1$. Equation (71) then becomes

$$k^2 = k'k''^2 y_1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (72),$$

or if we introduce the values of k', k'' from (67), (68),

$$k^2=\left(k^2-\frac{n^2}{q}\right) y_1\sqrt{\left(k^2+\frac{in}{\nu}\right)}.$$

Since in/ν is great, $k^2=\frac{n^2}{q}=\frac{n^2}{a^2}$ approximately.

Thus

$$k^2=\frac{n^2}{q}+\frac{k^2}{y_1\sqrt{\left(k^2+\frac{in}{\nu}\right)}}=\frac{n^2}{a^2}\left\{1+\frac{1}{y_1\sqrt{\frac{in}{\nu}}}\right\},$$

and

$$k=\pm\frac{n}{a}\left\{1+\frac{1-i}{2y_1\sqrt{\frac{2n}{\nu}}}\right\} \quad \ldots \ldots \ldots \ldots \quad (73).$$

If we write $k=k_1+ik_2$,

$$k_1=\pm\frac{n}{a}\left\{1+\frac{\sqrt{\frac{\nu}{2n}}}{2y_1}\right\}, \quad k_2=\mp\frac{n}{a}\,\frac{\sqrt{\frac{\nu}{2n}}}{2y_1} \quad \ldots \ldots \quad (74),$$

which agrees with the result given in § 347 (11) of my book on the Theory of Sound.

In taking approximate forms for (70), we must distinguish which half of the symmetrical motion we contemplate. If we choose that for which y is *negative*, we replace $\cosh k'y$ and $\sinh k'y$ by $\frac{1}{2}e^{-k'y}$. For $\cosh k''y$ we may write unity, and for $\sinh k''y$ simply $k''y$. If we change the arbitrary multiplier so that the maximum value of u is unity, we have

$$\left.\begin{aligned} u&=(-1+e^{-k'(y+y_1)})e^{ikx}e^{int}\\ v&=\frac{ik}{k'}\left(\frac{y}{y_1}+e^{-k(y+y_1)}\right)e^{ikx}e^{int}\end{aligned}\right\} \quad \ldots \ldots \ldots \ldots \quad (75),$$

in which, of course, u and v vanish when $y=-y_1$.

If in (75) we change k into $-k$, and then take the mean, we obtain

$$\left.\begin{aligned} u&=(-1+e^{-k'(y+y_1)})\cos kx\, e^{int}\\ v&=-\frac{k}{k'}\left(\frac{y}{y_1}+e^{-k'(y+y_1)}\right)\sin kx\, e^{int}\end{aligned}\right\} \quad \ldots \ldots \ldots \quad (76).$$

Although k is not absolutely a real quantity, we may consider it to be so with sufficient approximation for our purpose. If we write as before

$$k'=\sqrt{\left(\frac{n}{2\nu}\right)}.(1+i)=\beta(1+i),$$

we get from (76) in terms of real quantities

$$\left.\begin{aligned} u &= \cos kx[-\cos nt+e^{-\beta(y+y_1)}\cos\{nt-\beta(y+y_1)\}] \\ v &= -\frac{k}{\beta\sqrt{2}}\sin kx\left[\frac{y}{y_1}\cos(nt-\tfrac{1}{4}\pi)+e^{-\beta(y+y_1)}\cos\{nt-\tfrac{1}{4}\pi-\beta(y+y_1)\}\right] \end{aligned}\right\} \quad . \quad (77).$$

It will shorten the expressions with which we have to deal if we measure y from the wall (on the negative side) instead of as hitherto from the plane of symmetry, for which purpose we must write y for $y+y_1$. Thus

$$\left.\begin{aligned} u &= \cos kx\{-\cos nt+e^{-\beta y}\cos(nt-\beta y)\} \\ v &= \frac{k\sin kx}{\beta\sqrt{2}}\left\{\frac{y_1-y}{y_1}\cos(nt-\tfrac{1}{4}\pi)-e^{-\beta y}\cos(nt-\tfrac{1}{4}\pi-\beta y)\right\} \end{aligned}\right\} \quad . \quad . \quad . \quad (78).$$

From (78) approximately

$$\nabla^2\psi=\beta\sqrt{2}.\cos kx\, e^{-\beta y}\sin(nt-\tfrac{1}{4}\pi-\beta y) \quad . \quad . \quad . \quad . \quad . \quad . \quad (79),$$

$$\frac{du}{dx}+\frac{dv}{dy}=k\sin kx\cos nt \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (80),$$

$$u\frac{d\nabla^2\psi}{dx}+v\frac{d\nabla^2\psi}{dy}=\tfrac{1}{2}k\beta\sin 2kx\, e^{-\beta y}(-\cos\beta y+e^{-\beta y})+\text{ terms in } 2nt \quad . \quad . \quad (81),$$

$$\left(\frac{du}{dx}+\frac{dv}{dy}\right)\nabla^2\psi=-\tfrac{1}{4}k\beta\sin 2kx\, e^{-\beta y}(\sin\beta y+\cos\beta y)+\text{terms in } 2nt \quad . \quad . \quad (82).$$

As in former problems the periodic terms in $2nt$ will be omitted. For the non-periodic part of ψ of the second order, we have from (66)

$$\nabla^4\psi=-\frac{k\beta}{4\nu}\sin 2kx\, e^{-\beta y}\{\sin\beta y+3\cos\beta y-2e^{-\beta y}\} \quad . \quad . \quad . \quad . \quad (83).$$

In this we identify ∇^4 with $\frac{d^4}{dy^4}$, so that

$$\psi=\frac{k\sin 2kx\, e^{-\beta y}}{16\,\nu\beta^3}\{\sin\beta y+3\cos\beta y+\tfrac{1}{2}e^{-\beta y}\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (84),$$

to which must be added a complementary function, satisfying $\nabla^4\psi=0$, of the form

$$\psi=\frac{\sin 2kx}{16\,\nu\beta^3}\{\mathrm{A}\sinh 2k(y_1-y)+\mathrm{B}(y_1-y)\cosh 2k(y_1-y)\} \quad . \quad . \quad . \quad (85),$$

or as we may take it approximately, if y_1 be small compared with the wave-length λ,

$$\psi=\frac{k\sin 2kx}{16\,\nu\beta^3}\{A'(y_1-y)+B'(y_1-y)^3\} \quad . \quad . \quad . \quad . \quad (86).$$

The value of σ to a second approximation would have to be investigated by means of (62). It will be composed of two parts, the first independent of t, the second a harmonic function of $2nt$. In calculating the part of $d\phi/dx$ independent of t from

$$\nabla^2\phi=-\frac{d\sigma}{dt}-u\frac{d\sigma}{dx}-v\frac{d\sigma}{dy},$$

we shall obtain nothing from $d\sigma/dt$. In the remaining terms on the right-hand side it will be sufficient to employ the values of u, v, σ of the first approximation. From

$$\frac{d\sigma}{dt}=-\frac{du}{dx}-\frac{dv}{dy},$$

in conjunction with (80), we get

$$\sigma=-\frac{u_0}{a}\sin kx\sin nt,$$

whence

$$\frac{d^2\phi}{d(\beta y)^2}=\frac{ku_0^{\,2}}{2a\beta^2}\cos^2 kx\, e^{-\beta y}\sin\beta y.$$

It is easily seen from this that the part of u resulting from $d\phi/dx$ is of order $k^2\beta^2$ in comparison with the part (87) resulting from ψ, and may be omitted.

Accordingly by (84), with introduction of the value of β and (in order to restore homogeneity) of $u_0^{\,2}$

$$u=-\frac{u_0^{\,2}\sin 2kx\, e^{-\beta y}}{8a}\{4\sin\beta y+2\cos\beta y+e^{-\beta y}\} \quad . \quad . \quad . \quad . \quad (87),$$

$$v=-\frac{2ku_0^{\,2}\cos 2kx\, e^{-\beta y}}{8\beta a}\{\sin\beta y+3\cos\beta y+\tfrac{1}{2}e^{-\beta y}\} \quad . \quad . \quad . \quad . \quad (88);$$

and from (86)

$$u=-\frac{u_0^{\,2}\sin 2kx}{8\beta a}\{A'+3B'(y_1-y)^2\} \quad . \quad . \quad . \quad . \quad (89),$$

$$v=-\frac{2ku_0^{\,2}\cos 2kx}{8\beta a}\{A'(y_1-y)+B'(y_1-y)^3\} \quad . \quad . \quad . \quad . \quad (90).$$

When $y=0$, the complete values of u and v, as given by the four last equations, must vanish. Determining in this way the arbitrary constants A′ and B′, we get as the complete values at any point,

$$u=-\frac{u_0^2\sin 2kx}{8a}\left\{e^{-\beta y}(4\sin\beta y+2\cos\beta y+e^{-\beta y})+\tfrac{3}{2}-\tfrac{9}{2}\frac{(y_1-y)^2}{y_1^2}\right\} \quad . \quad . \quad (91),$$

$$v=-\frac{2ku_0^2\cos 2kx}{8\beta a}\left\{e^{-\beta y}(\sin\beta y+3\cos\beta y+\tfrac{1}{2}e^{-\beta y})+\tfrac{3}{2}\beta(y_1-y)-\tfrac{3}{2}\beta\frac{(y_1-y)^3}{y_1^2}\right\} \quad (92).$$

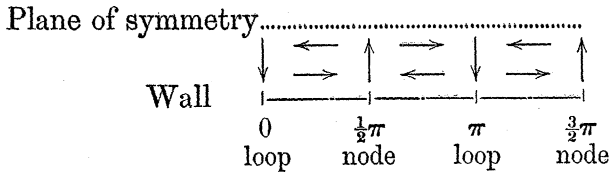
Outside the thin film of air immediately influenced by the friction we may put $e^{-\beta y}=0$, and then

$$u=-\frac{3u_0^2\sin 2kx}{16a}\left\{1-3\frac{(y_1-y)^2}{y_1^2}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (93),$$

$$v=-\frac{3u_0^2 2k\cos 2kx}{16a}\left\{y_1-y-\frac{(y_1-y)^3}{y_1^2}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (94).$$

From (93) we see that u changes sign as we pass from the boundary $y=0$ to the plane of symmetry $y=y_1$, the critical value of y being $y_1(1-\sqrt{\tfrac{1}{3}})$, or $\cdot 423\ y_1$.

The principal motion being $u=-u_0\cos kx\cos nt$, the loops correspond to $kx=0$, π, 2π, . . . , and the nodes correspond to $\tfrac{1}{2}\pi$, $\tfrac{3}{2}\pi$, . . . Thus v is positive at the nodes and negative at the loops, vanishing of course in either case both at the wall $y=0$, and at the plane of symmetry $y=y_1$.



To obtain the mean velocities of the *particles* parallel to x, we must make an addition to u, as in the former problems.

In the present case the mean value of

$$\frac{du}{dx}\xi+\frac{du}{dy}\eta=-\frac{u_0^2\sin 2kx\,e^{-\beta y}}{4a}\left\{e^{-\beta y}-\cos\beta y\right\},$$

so that

$$u'=-\frac{u_0^2\sin 2kx}{8a}\left\{e^{-\beta y}(4\sin\beta y+3e^{-\beta y})+\tfrac{3}{2}-\tfrac{9}{2}\frac{(y_1-y)^2}{y_1^2}\right\} \quad . \quad . \quad . \quad (95).$$

When βy is small,

$$u'=-\frac{u_0^2\sin 2kx}{8a}\left\{-2\beta y+\ldots\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (96).$$

Inside the frictional layer the motion is in the same direction as just beyond it.

We have seen that the width of the direct current along the wall is $\cdot 423\ y_1$, and that of the return current (measured up to the plane of symmetry) is $\cdot 577\ y_1$, so that the direct current is distinctly narrower than the return current. This will be still more the case in a tube of circular section. The point under consideration depends only upon a complementary function analogous to (86), and is so simple that it may be worth while to investigate it.

The equation for ψ is

$$\left(\frac{d^2}{dr^2}-\frac{1}{r}\frac{d}{dr}-4k^2\right)^2\psi=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (97),$$

but if we suppose that the radius of the tube is small in comparison with λ, k^2 may be omitted. The general solution is

$$\psi=\{A+Br^2+B'r^2\log r+Cr^4\}\sin 2kx \quad . \quad . \quad . \quad . \quad . \quad . \quad (98),$$

so that

$$u=\frac{1}{r}\frac{d\psi}{dr}=\{2B+B'(2\log r+1)+4Cr^2\}\sin 2kx,$$

whence $B'=0$, by the condition at $r=0$. Again

$$v=-\frac{1}{r}\frac{d\psi}{dx}=-2k\{Ar^{-1}+Br+Cr^3\}\cos 2kx,$$

whence $A=0$.

We may take therefore

$$\left.\begin{array}{l} u=\{2B+4Cr^2\}\sin 2kx \\ v=-2k\{Br+Cr^3\}\cos 2kx \end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (99).$$

If $v=0$, when $r=R$, $B+CR^2=0$, and

$$u=2C(2r^2-R^2)\sin 2kx \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (100).$$

Thus u vanishes, when

$$r=\frac{R}{\sqrt{2}}=\cdot 707\ R, \qquad R-r=\cdot 293\ R.$$

The direct current is thus limited to an annulus of thickness $\cdot 293$ R, the return current occupying the whole interior, and having therefore a diameter of

$$2\times\cdot 707\ R=1\cdot 414\ R.$$