IV. *On the Discrimination of Maxima and Minima Solutions in the Calculus of Variations.*

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THE criteria for distinguishing between the maximum and minimum values of integrals have been investigated by many eminent mathematicians.* In 1786 LEGENDRE gave an imperfect discussion for the case where the function to be made a maximum is $\int f(x, y, dy/dx)\, dx$. Nothing further seems to have been done till 1797, when LAGRANGE pointed out, in his 'Théorie des Fonctions Analytiques,' published in 1797, that LEGENDRE had supplied no means of showing that the operations required for his process were not invalid through some of the multipliers becoming zero or infinite, and he gives an example to show that LEGENDRE's criterion, though necessary, was not sufficient. In 1806 BRUNACCI,† an Italian mathematician, gave an investigation which has the important advantage of being short, easily comprehensible, and perfectly general in character, but which is open to the same objection as that brought against LEGENDRE's method.† The next advance was made in 1836

* This sketch is founded on TODHUNTER's valuable 'History of the Calculus of Variations.'

† BRUNACCI's method may be explained as follows: Let

$$U = \iint \left(x,\ y,\ z,\ \frac{dz}{dx},\ \frac{dz}{dy}\right) dx\, dy$$

be the function to be made a maximum, and let us denote dz/dx and dz/dy by p and q respectively. Thus, the limits of z being supposed fixed,

$$\delta U = \iint \left(\frac{df}{dz}\delta z + \frac{df}{dp}\delta p + \frac{df}{dq}\delta q\right) dx\, dy$$
$$+ \tfrac{1}{2}\iint \left(\frac{d^2f}{dz^2}\delta z^2 + 2\frac{d^2f}{dz\, dp}\delta z\, \delta p + 2\frac{d^2f}{dz\, \delta q} + \frac{d^2f}{dp^2}\delta p^2 + 2\frac{d^2f}{dp\, dq}\delta p\, \delta q + \frac{d^2f}{dq^2}\delta q^2\right) dx\, dy. \quad . \quad . \quad (1)$$

The first integral must be made to vanish by a relation between z, x, and y. Eliminate z for the coefficients in the second integral of (1) by means of this relation, and let it be written—

$$\iint (A\, \delta z^2 + 2B\, \delta z\, \delta p + 2C\, \delta z\, \delta q + P\, \delta p^2 + 2R\, \delta p\, \delta q + Q\, \delta q^2)\, dx\, dy. \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Now, remark that, the limits being fixed, $\int \delta z^2 (\alpha\, dx + \beta\, dy)$ vanishes, whatever α and β may be, for

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by the illustrious JACOBI, who treats only of functions containing one dependent and one independent variable. JACOBI says (TODHUNTER, Art. 219, p. 243): "I have succeeded in supplying a great deficiency in the Calculus of Variations. In problems on maxima and minima which depend on this calculus no general rule is known for deciding whether a solution really gives a maximum or a minimum, or neither. It has, indeed, been shown that the question amounts to determining whether the integrals of a certain system of differential equations remain finite throughout the limits of the integral which is to have a maximum or a minimum value. But the integrals of these differential equations were not known, nor had any other method been discovered for ascertaining whether they remain finite throughout the required interval. I have, however, discovered that these integrals can be immediately obtained when we have integrated the differential equations which must be satisfied in order that the first variation may vanish."

JACOBI then proceeds to state the result of his transformation for the cases where the function to be integrated contains x, y, dy/dx, and x, y, dy/dx, d^2y/dx^2, and in this solution the analysis appears free from all objection, though, where he proceeds to consider the general case, the investigation does not appear to be quite satisfactory in form, inasmuch as higher and higher differential coefficients of δy are successively introduced into the discussion (see Art. 5). JACOBI'S analysis is much more complicated than BRUNACCI'S, its advantage being that the coefficients used in the transformation could be easily determined; hence it supplied the means of ascertaining whether they became infinite or not.

$\delta z = 0$ all along this curve, and it may, therefore, be added to the integral (2) without altering its value. Now,

$$\int \delta z^2 (\alpha\, dx + \beta\, dy) = 0 = \iint \left(\frac{d}{dx}(\beta\, \delta z^2) + \frac{d}{dy}(\alpha \delta z^2)\right) dx\, dy$$
$$= \iint \left(\left(\frac{d\alpha}{dy} + \frac{d\beta}{dx}\right)\delta z^2 + 2\beta\, \delta z\, \delta p + 2\alpha\, \delta z\, \delta q\right) dx\, dy.$$

Adding this quantity to the integral, we get for the quantity under the integral sign in (2) the expression

$$\left\{\left(A + \frac{d\alpha}{dy} + \frac{d\beta}{dx}\right)\delta z^2 + 2(B + \beta)\,\delta z\,\delta p + 2(C + \alpha)\,\delta z\,\delta q + P\,\delta p^2 + 2R\,\delta p\,\delta q + Q\,\delta q^2\right\} dx\, dy.$$

This expression cannot change sign if $PQ - R^2$ is positive, and

$$(PQ - R^2)\left(P\left(A + \frac{d\alpha}{dy} + \frac{d\beta}{dx}\right) - (B + \beta)^2\right) - (P(C + \alpha) - (B + \beta)R)^2 > 0. \quad . \quad . \quad . \quad (3)$$

We can determine α and β so that (3) shall be true, and hence, if $PQ - R^2$ be positive, the second variation will be invariable in sign.

The objection to this method is that there is no means of ascertaining whether α and β remain finite or not. If the region of integration be small, they can always be determined so as to satisfy (3). But in general they become infinite when the integration extends over a large area. Thus, to complete the solution, it is necessary to have some means of finding within what range of integration the criteria are sufficient. It was because BRUNACCI'S method did not easily lend itself to the discussion of this problem that JACOBI devised his far more intricate method.

JACOBI did not himself give a detailed account of his process, but said that "the analysis just indicated requires a good knowledge of the Integral Calculus." Various demonstrations were subsequently given by different mathematicians. That of DELAUNAY has been adopted by JELLETT and other English writers.

In 1852 MAINARDI devised a method somewhat similar to that of BRUNACCI, but he endeavoured to remedy the omission in the latter by showing how to determine the coefficients used in the transformation, the equations for determining these coefficients being supplied by JACOBI'S reasoning. But in this he was not successful, even in some of the simple cases which he discussed, and in the more complicated cases the equations appear to be quite unmanageable.

In 1853 EISENLOHR extended JACOBI'S method to double integrals. In 1854 a memoir by SPITZER was published which seems to have been more complete than MAINARDI'S; but the most important advances were made a few years later when the Theory of Determinants was applied by HESSE and by CLEBSCH to simplify and extend JACOBI'S methods.

From the foregoing sketch it will be seen that as early as 1806 the criteria had been correctly given and simply proved, with the exception of one point, namely, that there was no means of ascertaining for what range of integration the criteria ceased to be sufficient. JACOBI endeavoured, by the help of a complicated analysis, to remedy this defect; and, although all the efforts of later mathematicians have been directed to the extending or simplifying of JACOBI'S method, the analysis is still very complicated, and requires an intimate acquaintance with other branches of mathematics.

All these methods, however, are open to the objection stated in Art. 5, and, furthermore, it appears to me, for the reasons briefly indicated in Art. 12, that, although the results arrived at by these mathematicians are undoubtedly correct, it would be impossible to give a *strict* proof of them by any method based on transformations. However this may be, I cannot find that anyone has yet given a proof of them. JACOBI merely states the limits within which the criteria hold.

The chief object of the present paper is to show that a rigorous discussion of the discriminating conditions can be given without introducing any analytical transformations whatever, the results being obtained by reasoning from the fundamental conceptions of the Calculus of Variations. In the *first* Part of the paper will, however, be found an analytical method leading to JACOBI'S transformation, but free from any serious difficulty. It is inserted chiefly on account of the historic interest of the problem. I had extended this method to obtain the criteria for the case of any integral whatever before I was aware that the results were not altogether new. It was after finding the limits up to which the criteria were sufficient that I was led to the general method given in Part II.

For convenience, a summary of contents is given below; the numbers refer to the

articles. Those who desire to read only the general method will find Part II. complete in itself.

It was originally intended that a tolerably complete account of the treatment of the problem, when the limits were not all fixed, should be inserted, but the length to which the paper has extended seems to render this inexpedient.

PART I.

Algebraic Transformations of the Second Variation.

PART II.

The General Method.

PART I.

Algebraic Transformations of the Second Variation.

1. In the following pages the word "fluxion" will be used instead of the long expression "differential coefficient." When there is but one independent variable, x, the successive fluxions of the dependent variables will be expressed in the known notation

$$\frac{dy}{dx} = \dot{y}, \quad \frac{d^2y}{dx^2} = \ddot{y}, \quad \&c., \quad \&c., \quad \frac{d^ny}{dx^n} = y^{(n)}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

and the partial fluxions of a function (f) of these quantities with respect to any of them, say $y^{(n)}$, will be denoted by $df/dy^{(n)}$. It will often be convenient to use a bracket $[\ ]_0^1$ to denote the result of subtracting the value of the quantity in the bracket, when taken at the lower limit of integration, from its value for the upper limit; thus by $[y\, \delta y]_0^1$ is meant $y_1\, \delta y_1 - y_0\, \delta y_0$. In other cases, where it is unnecessary to write out the limiting terms, the letter L will be used as an abbreviation for the *expression* "terms depending on the limits only," as in the following equations:—

$$\int y\, dx = \mathrm{L} - \int x\dot{y}\, dx = \mathrm{L} + \tfrac{1}{2} \int x^2 \ddot{y}\, dx, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

where, though the letter L is the same in both equations, it does not necessarily denote the same quantity.

2. To obtain a clear insight into the nature of the problem before us, let us examine it in the most simple and familiar case, that in which there is but one dependent and one independent variable. Writing

$$\mathrm{U} = \int f(x, y, \dot{y}, \ddot{y} \ldots y^{(n)})\, dx, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

let us call V.U the *total* variation of U due to a change of form in y, by which it becomes $y + \delta y$ (y being always a function of x):

$$\left.\begin{aligned} \mathrm{V.U} = & \int \left(\frac{df}{dy}\delta y + \frac{df}{d\dot{y}}\delta\dot{y} + \&c. + \frac{df}{dy^{(n)}}\delta y^{(n)}\right) dx \\ & + \tfrac{1}{1\cdot 2}\int\left(\frac{d^2f}{dy^2}\delta y^2 + 2\frac{d^2f}{dy\, d\dot{y}}\delta y\, \delta\dot{y} + \&c. + \frac{d^2f}{dy^{(n)2}}\delta y^{(n)2}\right) dx \\ & + \tfrac{1}{1\cdot 2\cdot 3}\int\left(\frac{d^3f}{dy^3}\delta y^3 + \&c.\right) dx + \&c. \end{aligned}\right\}, \quad . \quad . \quad (4)$$

which we may write

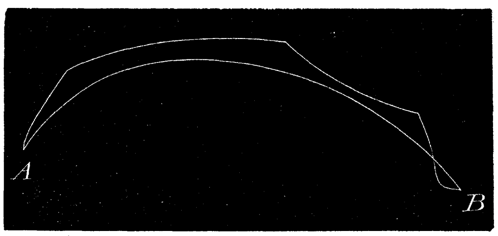
$$\mathrm{V.U} = \delta\mathrm{U} + \tfrac{1}{2}\delta^2\mathrm{U} + \tfrac{1}{1\cdot 2\cdot 3}\delta^3\mathrm{U} + \&c.,$$

where δU is the part of V. U depending on the first powers of δy and its fluxions, δ^2U that depending on their second powers, δ^3U on their third, and so on. As usual, these will be termed the first, second, and third variations of U, and so on.

3. The form of U having been determined so that the value of δU vanishes independently of the value of δy, the terms which we have to examine are those of δ^2U. This quantity, as obtained in the first instance, is explicitly a function of x, y, and δy. But, as the value of y obtained by making δU vanish is known in terms of x, δ^2U can be expressed in terms of x and δy. When so expressed, δ^2U may either remain of the same sign, whatever function δy may be of x, or else it may have one sign

when δy is a suitably chosen function of x, and the opposite sign when another expression is taken for δy. In the former case the integral must be either a maximum or a minimum; a maximum if the sign of $\delta^2 U$ is constantly negative, and a minimum if it is positive. In the latter case the integral is neither a maximum nor a minimum; its only characteristic is that the first variation vanishes. It will save much useless verbiage if we use a single word to express this latter class of integral, and the word "anticlastic," borrowed from geometry, seems a suitable one. It has also the advantage of suggesting another term, "synclastic," for those functions which give either a maximum or a minimum. With this explanation the problem before us is to ascertain whether the integral U is synclastic or anticlastic. If it be the former, a glance will enable us to determine which the result is, a maximum or a minimum.

4. We might reduce, by a real linear algebraic transformation, the part of $\delta^2 U$ under the integral sign to the sum of n squares, and say that, if their coefficients be positive all through the integration, then, whatever be the limiting values of the arbitrary variations, the second variation must be essentially positive (at least unless dx changes sign during the integration); but the converse of this is not true, for it does not follow that, if the coefficients of the squares have different signs, it is possible to make the second variation change sign. If δy, $\delta \dot{y}$ . . . . $\delta y^{(n)}$ were all independent, this converse would be true; hence one element in the problem before us is to introduce, in a suitable form, the interdependence of the quantities δy, $\delta \dot{y}$ . . . . $\delta y^{(n)}$. Again, if the limits are not all arbitrary, there will be further limitations to the range of values taken by δy and its fluxions; and the second element in the problem is to find how these limitations to δy and its fluxions affect the sign of the second variation.



5. JACOBI'S method appears to me open to the serious objection that it is necessary to its validity that the first $(2n-1)$ fluxion of δy should be continuous; so that the discussion only proves that a curve AB fulfilling the synclastic condition gives a better result than any infinitely near curve fulfilling the same limiting conditions, *and which is continuous to the* $2n^{th}$ *fluxion of* y. And it would not show that it was not possible to find other broken curves fulfilling the same limiting conditions, and giving, at our pleasure, a value to the integral either greater or less than that given by the curve AB. Thus, for instance, in the case of least action, it would not show that the action in the free trajectory was less than in *any* constrained path (which it is, in fact), but only that it was less than that in any path for which the tangent had but one position at every point. To show for the general case that JACOBI'S proof assumes

this continuity of δy would require a large amount of work, but it will be sufficient for our purpose to take the simple case where the function to be made synclastic contains only x, y, and $\dot{y}$.

Let us apply his *general* method to the integral *

$$\mathrm{U} = \int_{x_0}^{x_1} f(x,\ y,\ \dot{y})\, dx\,;$$

therefore

$$\delta\mathrm{U} = \int_{x_0}^{x_1} \left(\frac{df}{dy}\delta y + \frac{df}{d\dot{y}}\delta\dot{y}\right) dx\,;$$

integrating by parts,

$$\delta\mathrm{U} = \left[\frac{df}{d\dot{y}}\delta y\right]_0^1 + \int \left(\frac{df}{dy} - \frac{d}{dx}\frac{df}{d\dot{y}}\right)\delta y\, dx.$$

Hence

$$\delta^2\mathrm{U} = \left[\frac{d^2f}{dy\, d\dot{y}}\,\delta y^2 + \frac{d^2f}{d\dot{y}^2}\delta y\,\delta\dot{y}\right]_0^1 + \int \left(\frac{d^2f}{dy^2}\delta y + \frac{d^2f}{dy\, d\dot{y}}\,\delta\dot{y} - \frac{d}{dx}\left(\frac{d^2f}{dy\, d\dot{y}}\,\delta y + \frac{dy}{d\dot{y}^2}\,\delta\dot{y}\right)\delta y\, dx.\right.$$

It will save trouble if, in what follows, we use the abbreviations

$$\frac{df}{dy} = \mathrm{Y}_0,\quad \frac{df}{d\dot{y}} = \mathrm{Y}_1,\quad \text{and, in general, } \frac{df}{dy^{(r)}} = \mathrm{Y}_r,$$

$$\frac{d^2f}{dy^2} = \mathrm{Y}_{00},\quad \frac{dy}{dy\, d\dot{y}} = \mathrm{Y}_{01},\quad \text{and, in general, } \frac{d^2f}{dy^{(r)}\, dy^{(s)}} = \mathrm{Y}_{rs}.$$

Then we may write

$$\delta^2\mathrm{U} = \left[\mathrm{Y}_{01}\,\delta y^2 + \mathrm{Y}_{11}\,\delta y\,\delta\dot{y}\right]_0^1 + \int\left(\mathrm{Y}_{00}\,\delta y + \mathrm{Y}_{01}\,\delta\dot{y} - \frac{d}{dx}(\mathrm{Y}_{01}\,\delta y + \mathrm{Y}_{11}\,\delta\dot{y})\,\delta y\, dx\right),$$

of which the part under the integral sign is

$$\int\left(\mathrm{Y}_{00}\,\delta y + \mathrm{Y}_{01}\,\delta\dot{y} - \mathrm{Y}_{01}\,\delta\dot{y} - \frac{d\mathrm{Y}_{01}}{dx}\,\delta y - \frac{d\mathrm{Y}_{11}}{dx}\,\delta\dot{y} - \mathrm{Y}_{11}\,\delta\ddot{y}\right)\delta y\, dx$$

$$= \int\left\{\left(\mathrm{Y}_{00} - \frac{d\mathrm{Y}_{01}}{dx}\right)\delta y - \left(\frac{d\mathrm{Y}_{11}}{dx}\,\delta\dot{y} - \mathrm{Y}_{11}\,\delta\ddot{y}\right)\right\}\delta y\, dx. \quad . \quad . \quad . \quad . \qquad (5)$$

Now let z be a solution of the equation

$$\left(\mathrm{Y}_{00} - \frac{d\mathrm{Y}_{01}}{dx}\right)z - \frac{d\mathrm{Y}_1}{dx}\,\dot{z} - \mathrm{Y}_{11}\,\ddot{z} = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (6)$$

* JACOBI treats this case by a method not identical in form with the general method he gives.

Multiply (6) by $\delta y^2/z \,.\, dx$, and subtract it from the part under the integral sign in equation (5): then we have for the part under the integral sign

$$\int\left(-\frac{dY_{11}}{dx}\delta\dot{y}+\frac{dY_{11}}{dx}\frac{\dot{z}\,\delta y}{z}-Y_{11}\ddot{y}+Y_{11}\frac{\ddot{z}\,\delta y}{z}\right)\delta y\,dx=\int\frac{d}{dx}\left(Y_{11}(\dot{z}\,\delta y-z\,\delta\dot{y})\right)\frac{\delta y}{z}\,dx,$$

$$=\left[Y_{11}(\dot{z}\,\delta y-z\,\delta\dot{y})\frac{\delta y}{z}\right]_0^1-\int Y_{11}(\dot{z}\,\delta y-z\,\delta\dot{y})\frac{d}{dx}\left(\frac{\delta y}{z}\right)dx,$$

$$=\left[Y_{11}(\dot{z}\,\delta y-z\,\delta\dot{y})\frac{\delta y}{z}\right]_0^1+\int Y_{11}\left\{\frac{d}{dx}\left(\frac{\delta y}{z}\right)\right\}^2 z^2\,dx,$$

and, writing out the complete variation, we get

$$\delta^2 U=\left[Y_{01}\,\delta y^2+Y_{11}\,\delta y\,\delta\dot{y}+Y_{11}(\dot{z}\,\delta y-z\,\delta\dot{y})\frac{\delta y}{z}\right]_0^1+\int Y_{11}\left(\frac{d}{dx}\left(\frac{\delta y}{z}\right)\right)^2 z^2\,dx,$$

the final expression; from this it is evident that, if $Y_{11}\,dx$ changed sign in the course of integration, the function could not be synclastic.

6. If the variation δy were such that $\delta\dot{y}$ suddenly changed from one finite value to another, $\delta\ddot{y}$ must be infinite at that point, and the integration would not be permissible, or, at least, it would require a justification which, so far as I am aware, has never been considered necessary.

Again, it will be observed that the limiting values of $\delta\dot{y}$ are introduced in the process (though for this case it is easily seen that in the final result the terms in which they appear are identically zero).

What has been shown in this case is true in general: the first $2n$ fluxions of δy are brought in under the integral sign, and of these the last n fluxions $y^{(n+1)}\ldots y^{(2n)}$ are got rid of by integration. Again, the limiting values of the first $2n-1$ fluxions are introduced outside the integral, and, of these, the n fluxions $y^{(n)}\ldots y^{(2n-1)}$ disappear through their coefficients being identically zero; but the *direct* proof of this in the general case would, so far as I can see, require transformations of even greater length than those usually applied to the part under the integral sign.

7. It was with the object of ascertaining whether it was necessary, in the general case, to assume the continuity of the value of δy and its $2n-1$ fluxions that I first considered the problem; and I communicated to the British Association at Montreal an account of a method which was free from this objection, and had the additional advantage of a simplicity which enabled it to be easily extended to any number of dependent variables; and it is, in its main principles, applicable to the most general case.

8. The application of this method to the preceding case is as follows. Taking the second variation in the form on p. 99, equation (4), in expanding V.U, we have

$$\delta^2 U = \int (Y_{00}\,\delta y^2 + 2Y_{10}\,\delta y\,\delta\dot{y} + Y_{11}\,\delta\dot{y}^2)\,dx.$$

Writing in this $\theta\,\delta_1 y$ for δy, where θ is a function of x at present unknown in form, and expanding the result, we get

$$\begin{aligned}\delta^2 U &= \int (Y_{00}\theta^2\,\delta_1 y^2 + 2Y_{10}\theta\,\delta_1 y\,(\theta\,\delta_1\dot{y} + \dot{\theta}\,\delta_1 y) + Y_{11}\,(\dot{\theta}\,\delta_1 y + \theta\,\delta_1\dot{y})^2)\,dx \\ &= \int (\delta_1 y^2\,(Y_{00}\theta^2 + 2Y_{10}\theta\dot{\theta} + Y_{11}\dot{\theta}^2) + 2\delta_1 y\,\delta_1\dot{y}\,(Y_{10}\theta^2 + Y_{11}\theta\dot{\theta}) + Y_{11}\theta^2\,\delta_1\dot{y}^2)\,dx.\end{aligned}$$

Integrating by parts the term involving $\delta_1 y\,\delta_1\dot{y}$,

$$\delta^2 U = [(Y_{10}\theta^2 + Y_{11}\theta\dot{\theta})\,\delta_1 y^2]_0^1 + \int \left(Y_{00}\theta^2 + 2Y_{10}\theta\dot{\theta} + Y_{11}\dot{\theta}^2 - \frac{d}{dx}(Y_{10}\theta^2 + Y_{11}\theta\dot{\theta})\right)\delta_1 y^2\,\delta x$$
$$+ \int (Y_{11}\theta^2\,\delta_1\dot{y}^2)\,dx.$$

The multiplier θ is here quite arbitrary, and can therefore be determined so that the quantity multiplying δy^2 under the sign of integration shall vanish, and hence, when this value of θ is chosen, we have

$$\delta^2 U = [(Y_{10}\theta^2 + Y_{11}\theta\dot{\theta})\,\delta_1 y^2]_0^1 + \int Y_{11}\theta^2\,\delta_1\dot{y}^2\,dx.$$

This is the same form as that in the other reduction, and it is easy to prove that θ is a solution of the equation (6) used in finding z in JACOBI's method.

9. We have now given examples of the simplest cases. The method of treating $\delta^2 U$, where

$$U = \int f(x, y, \dot{y}, \ldots y^{(n)})\,dx,$$

will now be given in its shortest form.

Using the notation already adopted, we may write

$$\delta^2 \int f(x, y, \dot{y}, \ldots, y^{(n)})\,dx = \int \sum_0^n \sum_0^n Y_{rs}\,\delta y^{(r)}\,\delta y^{(s)}\,dx.$$

If we transform this by writing $z_1\,\delta_1 y$ for δy, and expanding the fluxions, we shall get an expression which may be written

$$\delta^2 U = \int \sum_{(0)}^{(n)} \sum_{(0)}^{(n)} A_{rs}\,\delta_1 y^{(r)}\,\delta_1 y^{(s)}\,dx, \quad \ldots \ldots \ldots \quad (7)$$

where $\delta_1\dot{y}$, for instance, means d/dx, $\delta_1 y$ and A_{rs} contains z_1 as well as x. This expression consists of terms in which $\delta_1 y$ itself enters, and other terms in which only its fluxions appear. Integrate by parts the terms containing $\delta_1 y$; thus, in the previous expression

$$\int Y_{0s}\,\delta y\,\delta y^{(s)}\,dy = [Y_{0s}\,\delta y\,\delta y^{(s-1)}]_0^1 - \int\left(\frac{dY_{0s}}{dx}\,\delta y\,\delta y^{(s-1)} + Y_{0s}\,\delta\dot{y}\,\delta y^{(s-1)}\right)dx$$

$$= [Y_{0s}\,\delta y\,\delta y^{(s-1)}]_0^1 - \left[\frac{dY_{0s}}{dx}\,\delta y\,\delta y^{(s-1)}\right]_0^1$$

$$+ \int\left(\frac{d^2Y_{0s}}{dx^2}\,\delta y\,\delta y^{(s-2)} - \frac{dY_{0s}}{dx}\,\delta\dot{y}\,\delta y^{(s-2)} - Y_{0s}\,\delta\dot{y}\,\delta y^{(s-1)}\right)dx\,;$$

and, finally,

$$\int Y_{0s}\,\delta y\,\delta y^s = \mathrm{L} \pm \int\left(\tfrac{1}{2}\,\frac{d^sY_{0s}}{dx^s}\,\delta y^2 - \delta\dot{y}\left(\frac{d^{s-1}Y_{0s}}{dx^{s-1}}\,\delta\dot{y} + \&c.\right)\right)dx,$$

where, with the exception of the term involving δy^2, the integral involves only fluxions of δy, and a similar reduction can, of course, be applied to any terms of the same form.

Reducing all the terms involving $\delta_1 y$ in (7), we get for $\delta^2\mathrm{U}$ an expression which we may write

$$\delta^2\mathrm{U} = \mathrm{L} + \int\left(\mathrm{A}_0\,\delta_1 y^2 + \sum_{(1)}^{(n)}\sum_{(1)}^{(n)}\mathrm{A}'_{rs}\,\delta_1 y^{(r)}\,\delta_1 y^{(s)}\right)dx,$$

where, in the terms under the double summation sign, $\delta_1 y$ only appears through its fluxions. If we determine z_1 by the equation $\mathrm{A}_0 = 0$, a differential equation of the $2n^{\text{th}}$ order, we have $\delta^2\mathrm{U}$ depending on the limits and on terms involving $\delta_1 y$ only through its fluxions. If this expression be transformed by writing $z_2\,\delta_2 y$ for $\delta_1\dot{y}$, then, after expanding, we get an expression which we may write

$$\delta^2\mathrm{U} = \mathrm{L} + \int\sum_{(0)}^{(n-1)}\sum_{(0)}^{(n-1)}\mathrm{B}_{rs}\,\delta_2 y^{(r)}\,\delta_2 y^{(s)}\,dx,$$

in which the highest fluxion of $\delta_2 y$ is of the $(n-1)^{\text{th}}$ order.

This expression can be reduced by the same method, and we get

$$\delta^2\mathrm{U} = \mathrm{L} + \int\left(\mathrm{B}_0\,\delta_2 y^2 + \sum_{(1)}^{(n-1)}\sum_{(1)}^{(n-1)}\mathrm{B}'_{rs}\,\delta_2 y^{(r)}\,\delta_2 y^{(s)}\right)dx.$$

Determine z_2 so that $\mathrm{B}_0 = 0$ for all values of x, then $\delta^2\mathrm{U}$ will depend on L and the fluxions of $\delta_2 y$: transform this by writing, in the part under the integral sign, $\delta_2\dot{y} = z_3\,\delta_3 y$, and reduce in the same way, and so on till we come to the last transformation but one, in which

$$\delta^2\mathrm{U} = \mathrm{L} + \int\left(\mathrm{M}_0\,(\delta_{n-1}y)^2 + \sum_{(1)}^{(2)}\sum_{(1)}^{(2)}\mathrm{M}_{rs}\,\delta_{n-1}y^{(r)}\,\delta_{n-1}y^{(s)}\right)dx,$$

which, treated similarly, gives

$$\delta^2\mathrm{U} = \mathrm{L} + \int(\mathrm{N}_0\,\delta_n y^2 + \mathrm{N}_{11}\,\delta_n\dot{y}^2)\,dx\,;$$

and, determining z_n so that $\mathrm{N}_0 = 0$, we get finally

$$\delta^2\mathrm{U} = \mathrm{L} + \int\mathrm{N}_{11}\,(\delta_n\dot{y})^2\,dx.$$

When we substitute for $\delta_n \dot{y}$ its value in terms of δy and its fluxions, we must clearly have an expression of the form

$$\delta^2 U = L + \int N_{11} \{ f_0(x) \, \delta y^{(n)} + f_1(x) \, \delta y^{(n-1)} + \&c. + f_n(x) \, \delta y \}^2 \, dx \, ;$$

and, as it is evident that the coefficient of $\delta y^{(n)2}$ is unaltered by the transformations, the expression for $\delta^2 U$ may be written in the form

$$L + \int Y_{nn} (\delta y^{(n)} + \alpha_1 \, \delta y^{(n-1)} + \&c. + \alpha_n \, \delta y)^2 \, dx,$$

where L contains δy and its fluxions up to, but not including, δy^n.

10. This is the same form as that in JACOBI's method, and it has been obtained on the supposition that integrations may be performed on $\delta y, \delta \dot{y}, \ldots \delta y^{(n)}$; this requires that $\delta y \ldots \delta y^{(n-1)}$ should be continuous in every sense, and that $\delta y^{(n)}$ should not become infinite, but it may suddenly change from one finite value to another. It is very important to note this in connection with the condition determining the point at which a curve ceases to fulfil the synclastic condition.

As yet, the equations determining the quantities $z_1, z_2, \ldots z_n$ used in the transformation have not been given, and it might be supposed that they would have to be found in order to determine the part of $\delta^2 U$ depending on the limits. It will, however, be seen that in this as well as in every other case under the Calculus of Variations it is only necessary to use these quantities to show that the reduction *can be made*. As a matter of interest, however, a short discussion is given below.*

* It may be interesting, though not required for our immediate purpose, to examine the relation which the coefficients z_1, &c., in the transformation bear to the value for y which gives the synclastic value to the integral. For this purpose it will be convenient to use $\Delta \, \delta U$ instead of $\delta^2 U$.

In the first place, it is evident that, if we have any expression for $\delta^2 U$ as a quadratic function of δy and its fluxions, we shall get $\delta \Delta U$ by taking the polar, with respect to the quadratic function, of the point whose coordinates are $\Delta y, \Delta \dot{y}, \Delta \ddot{y}$, and so on, using the word "point" in an extended sense. If a proof of this is desired, write $(\delta + K\Delta)$ for δ in

$$\delta^2 U = f(\delta y, \delta \dot{y}, \ldots \delta y^{(n)}),$$

and get

$$(\delta + K\Delta)^2 U = f\{(\delta + K\Delta) \, y, \ (\delta + K\Delta) \, \dot{y} \ldots (\delta + K\Delta) \, y^n\},$$

and, after expanding, equate coefficients of K, which is quite arbitrary. Hence, from equation (7) we obtain

$$\Delta \, \delta U = L + \int \sum_1^n \sum_1^n A_{rs} \, \Delta_1 y^r \, \delta_1 y^s \, dx \, ;$$

and, since the integral contains only fluxions of $\Delta_1 y_1$, $\Delta \, \delta U$ will depend only on the limits, whatever be the form of δy, provided $\Delta_1 y$, that is, $\Delta y / z_1$, is constant; and therefore z_1 is one of the values of Δy for which $\Delta \, \delta U$ is independent of the form of δU. Now,

$$\delta U = L + \int \left(Y_0 - \frac{dY_1}{dx} + \frac{d^2 Y_2}{dx^2} - \&c. \right) \delta y \, dx,$$

11. It is important to observe that this transformation is only valid, provided none of the quantities $z_1, z_2, \ldots z_n$ used in the transformation vanish for any value of x passed through in going from x_0 to x_1. For, suppose z_r vanishes, then, as $\delta_{r-1} y = z_r \delta_r y$, the corresponding value of $\delta_r y$ must be infinite. Now, since $z_1 \ldots z_n$

and therefore

$$\Delta \delta \mathrm{U} = \mathrm{L} + \int \Delta\left(\mathrm{Y}_0 - \frac{d\mathrm{Y}_1}{dx} + \ldots\right) \delta y \, dx;$$

and this can only be independent of the form of δy when

$$\Delta\left(\mathrm{Y}_0 - \frac{d\mathrm{Y}_1}{dx} + \ldots\right) = 0, \quad \ldots \quad (a)$$

that is to say, when $y + \Delta y$ satisfies the equation satisfied by y, *i.e.*,

$$\mathrm{Y}_0 - \frac{d\mathrm{Y}_1}{dx} + \&\text{c.} = 0. \quad \ldots \quad (b)$$

If, then, the solution of (b) be $y = f(x, c_1, c_2, \ldots c_{2n})$, that of (a) is obtained in the well-known form (see TODHUNTER, 'History of the Calculus of Variations,' p. 271, or JELLETT, 'Calculus of Variations,' p. 84)—

$$\Delta y = \frac{df}{dc_1} \Delta c_1 + \frac{df}{dc_2} \Delta c_2 + \ldots + \frac{df}{dc_{2n}} \Delta c_{2n}. \quad \ldots \quad (c)$$

It follows that z_1 must be $\mathrm{C}_1 \, df/dc_1 + \mathrm{C}_2 \, df/dc_2 + \&\text{c.}$, or, shortly, $z_1 = y_1$; $y_1, y_2, \ldots y_{2n}$ being independent solutions of (c).

In finding z_2, we employ a similar process. Since the part of $\delta^2 \mathrm{U}$ under the integral sign has been expressed in terms of the *fluxions* of $\delta_2 y$, that of $\Delta \, \delta \mathrm{U}$ can be expressed in terms of the fluxions of $\delta_2 y$ and $\Delta_2 y$. Hence, if we choose Δy, so that $\Delta_2 y =$ constant, the integral vanishes identically, and the whole variation $\Delta \, \delta \mathrm{U}$ depends only on limiting values of δy, and not on the general value. Hence, as before, Δy must be a solution of (a), and evidently it must not be the y_1 solution. Let us denote it by y_2 (observing that y_2, however, cannot be quite arbitrarily chosen from the remaining $2n - 1$ solutions because the equation for Δy is only of the $(2n-2)^{\text{th}}$ order, and can only have $2n - 2$ solutions). Then, since $z_2 = \Delta_1 \dot{y} / \Delta_2 y = 1/\Delta_2 y \, . \, d/dx \, (\Delta y / z_1)$, it easily follows that $z_2 = c \, d/dx \, (y_2/y_1)$. Similarly, when we come to the third transformation, we have $z_3 \Delta_3 y = \Delta_2 \dot{y}$, whence

$$z_3 = \frac{\Delta_2 \dot{y}}{\Delta_3 y} = \frac{1}{\Delta_3 y} \frac{d}{dx} \left[\frac{\frac{d}{dx}\left(\frac{\Delta y}{y_1}\right)}{\frac{d}{dx}\left(\frac{y_2}{y_1}\right)} \right]$$

and, when $\Delta_3 y =$ constant, Δy is a solution y_3 of (a). Hence,

$$z_3 = \frac{d}{dx} \left[\frac{\frac{d}{dx}\left(\frac{y_3}{y_1}\right)}{\frac{d}{dx}\left(\frac{y_2}{y_1}\right)} \right],$$

and similarly for the remainder. As nothing hangs on the discussion of these quantities, it is not worth writing out any more. In fact, the only application of the investigation to the present case would be to show that the transformations in Art. 9 are always possible, provided a sufficiently short length of the curve be taken; that is, to show that it is always possible to choose z_1, z_2, z_3, &c., so that none of them vanish for any value of x between the limits of integration. But it is easier to see that this is, in general, the case from the equations obtained for z_1, z_2, z_3, &c., in the course of the transformations themselves. For, z_r being a function of x and arbitrary constants, it follows that, if we put $z_r = 0$, we can solve for the value of x in terms of those arbitrary constants, and hence, by taking suitable values for the constants, we can, in general, ensure that z_r does not vanish for a value $x = x_0$.

are definite functions of x and the arbitrary constants introduced in the solution, it is evident that, x_0 being the initial value of (x), there will in general be some value x_1 at which it becomes impossible to determine the arbitrary constants, so that some one at least of the solutions $z_1 \ldots z_n$ shall not have changed sign. Up to this point the transformation must hold, and the conditions for synclasticism derived from it must be sufficient and necessary.

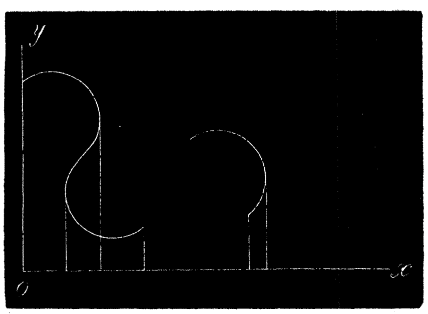
12. Confining our attention to integrals for which the transformation does hold (that is, integrals whose limits are not too widely separated), it is easy to see by the usual method that, unless $Y_{nn}dx$ retains the same sign throughout the integration, the integral cannot be synclastic.* For the integral

$$\delta^2 U = \int Y_{nn} (\delta y^{(n)} + a\, \delta y^{(n-1)} + \&c.)^2\, dx$$

may then be divided into two parts, one negative and the other positive, and, as the form of δy is arbitrary, we could make the numerical value of either of these parts exceed that of the other, and therefore $\delta^2 U$ would be capable of either sign.

But the condition that $Y_{nn}dx$ remains of the same sign throughout the integration is not sufficient to ensure that this integral shall be synclastic. This would be the proper place to examine the further condition if it could be derived from the preceding transformation, but it does not appear to me that we can avail ourselves of the analysis, for the following reasons :—Some of the quantities z_r used in the tranformation may vanish for some value of x included in the integration, and the investigation would not apply. Hence we should have to give an independent discussion to discover the limits of the integration within which the transformation does apply. But even then we should only have proved that the function was synclastic up to those limits at least, and we should have still to discover whether it might not be

* It is usually stated that, unless Y_{nn} preserves its sign, the integral could not be synclastic; but this is a mistake arising from the supposition that, because dx increases from the lower to the higher limit, it must have the same sign throughout the integration. But it is evident from the figure



that dx may change sign for a value of x *between* the limits, in which case there must be an even number of changes, or it may change sign an uneven number of times for values not numerically between the lower and higher limit, but yet passed through in going from the one to the other *viâ* the curve. It would be easy, by transforming the axes, to multiply examples of the latter, and in these Y_{nn} and dx will change sign together.

synclastic for wider limits. For it does not seem by any means evident that the function cannot be synclastic unless the transformation is valid. It might still be possible to reduce it to a sum of *several* squares, for instance, after it had ceased to be possible to reduce it to a single square term. At first sight this would appear not improbable, for it would mean that it was still possible to determine the first few coefficients, $z_1, z_2, \ldots . z_r$, &c., so that none of them vanished, although it was impossible to determine z_{r+1} so that it did not vanish; then the integral would be reduced to the sum of $(n-r)$ squares, and the conditions would be that all their coefficients were positive.

These difficulties seemed so great that it appeared better to attempt the problem by an altogether different method, namely, that of supposing that the synclastic property does hold for a given length of the curve and then ascertaining where the property ceases to hold.

It is evident that if this could be done it would be sufficient to find the synclastic condition for an infinitely small range of integration, and this suggested the method now to be given. As already stated, the discussion of the further condition for synclasticism will be postponed to Art. 21.

PART II.

The General Method.

13. A full account will now be given of the general investigation as applied to the case of two variables, and a somewhat shorter discussion of the general case will be found in Arts. 20, 21.

Consider the conditions under which the equation

$$\begin{aligned}\int_{x_0}^{x_1} f(x, y + \delta y, \dot{y} + \delta\dot{y}, \ldots y^{(n)} + \delta y^{(n)})\, dx &= \int_{x_0}^{x_1} f(x, y, \dot{y}, \ldots y^{(n)})\, dx \\ &+ \int_{x_0}^{x_1} \left(\frac{df}{dy}\,\delta y + \frac{df}{d\dot{y}}\,\delta\dot{y} + \ldots + \frac{df}{dy^{(n)}}\,\delta y^{(n)}\right) dx \\ &+ \tfrac{1}{2}\int_{x_0}^{x_1} \left(\frac{d^2f}{dy^2}\,\delta y^2 + 2\,\frac{d^2f}{dy\,d\dot{y}}\,\delta y\,\delta\dot{y} + \frac{d^2f}{d\dot{y}^2}\,\delta\dot{y}^2 + \ldots + \frac{df}{dy^{(n)2}}\,\delta y^{(n)2}\right) dx \\ &+ \tfrac{1}{2.3}\int_{x_0}^{x_1} \left(\frac{d^3f}{dy^3}\,\delta y^3 + 3\,\frac{d^3f}{dy^2\,d\dot{y}}\,\delta y^2\,\delta\dot{y} + \ldots + \frac{d^3f}{dy^{(n)3}}\,\delta y^{(n)3}\right) dx + \&c.\end{aligned}$$

is valid. It is obtained by writing $y + \delta y, \dot{y} + \delta\dot{y} \ldots$ for $y, \dot{y} \ldots$ in $f(x, y, \dot{y} \ldots y^{(n)})$, expanding by TAYLOR's Theorem and then integrating. TAYLOR's Theorem requires that numerical values of the quantities δy, $\delta\dot{y}$, &c., shall not exceed certain limits, and that the values of $x, y, \dot{y}$, &c., shall not be such as to make the coefficients in the expansion infinite. Hence, if $f(x, y, \dot{y} \ldots y^{(n)})$ satisfy the latter condition for every value of x included in the range of integration, and if we take δy, $\delta\dot{y}$, &c., small

enough, we can always ensure that the above expansion holds. So far, nothing has been said as to the continuity of δy, $\delta \dot{y}$, &c., but, $\delta \dot{y}$, &c., being the successive fluxions of a single quantity δy, they must all be continuous functions of x except $\delta y^{(n)}$, the highest fluxion, whose magnitude is not so restricted. For, if $\delta y^{(r)}$ changes suddenly from one finite value to another, for change of x from x' to $x' + dx$, *its* differential coefficient, $\delta y^{(r+1)}$, would become infinite for that value of x. But, as $\delta y^{(n+1)}$ does not occur in $f(x, y, \dot{y} \ldots y^{(n)})$, the validity of the expansion will not be affected by its becoming infinite, and therefore $\delta y^{(n)}$ may change from one finite value to another.

14. Hence, if we discuss the problem of maxima and minima by the usual method, the variation we give is of necessity restricted as follows: $\delta y, \delta \dot{y}, \ldots \delta y^{(n-1)}$ must all be continuous functions of x. $\delta y^{(n)}$ need not be continuous, but the magnitude of each fluxion must be restricted with certain limits, which will vary with the nature of the problem under discussion, but it will in all cases be sufficient to make them infinitely small. It will be convenient to consider δy as $\alpha\phi x$, where α is a small numerical coefficient, and ϕx is a function of x, such that it and any number of its fluxions may become zero, though in general they will be finite, while neither the function itself nor any of its fluxions up to and including the n^{th} can become infinite for any value of x occurring in the integration.

The coefficient α must be taken sufficiently small to ensure that, when considering only the sign and not the value of an expression involving it, we may neglect terms depending on α^2 or higher powers in comparison with those depending on α. Denoting, as usual, by δU, $\delta^2 U$, &c., the part of the expansion depending on the first, second, &c., powers of δy and its fluxions, we may say that δU is of the order α, $\delta^2 U$ of the order α^2, and so on. Hence, in the absence of special determinations of the form of δy, $\delta^2 U$, the part depending on α^2 will exceed all terms depending on α^3 and higher powers of α, and then the sign of the whole variation will be the same as that of $\delta^2 U$ (δU being zero when y has its synclastic value).

It is convenient to have a geometric representation, and the function y will be taken as the ordinate of a curve of which x is the abscissa, and the curve corresponding to the synclastic form for y will be called the synclastic curve.

15. We may now easily prove the following proposition:—

Let
$$U = \int_{x_0}^{x_1} f(x, y, \dot{y} \ldots y^{(n)})\, dx,$$
y being *any* function of x. Let the second variation $\delta^2 U$ be taken, subject to the condition that $\delta y, \delta \dot{y}, \ldots \delta y^{(n-1)}$ are zero at each limit. Then the sign of $\delta^2 U$ is the same as that of the term involving $\delta y^{(n)2}$ in the integral, provided the range of the integration be sufficiently small.

The second variation being written
$$\delta^2 U = \int_{x_0}^{x_1} (Y_{00}\, \delta y^2 + 2Y_{01}\, \delta y\, \delta \dot{y} + Y_{11}\, \delta \dot{y}^2 + \text{\&c.} + 2Y_{n-1,n}\, \delta y^{(n-1)}\, \delta y^{(n)} + Y_{nn}\, \delta y^{(n)2})\, dx,$$

the proposition will be proved when it is shown that throughout the integration $\delta y^{(n-1)}/\delta y^{(n)}$, $\delta y^{(n-2)}/\delta y^{(n-1)}$, &c., are all negligible, for then the term $Y_{nn}\,\delta y^{(n)2}$ is obviously the most important.

Now

$$\delta y_x^{(n-1)} = \int_{x_0}^{x} \delta y^{(n)}\,dx,$$

no constant being added, as $\delta y^{(n-1)}$ vanishes when $x = x_0$. Let the numerically greatest value of $x - x_0$ in the integration be β, and that of $\delta y^{(n)}$ be γ; then, numerically, $\delta y^{(n-1)} < \beta\gamma$.

A fortiori, $\delta y^{(n-1)} < \beta^2\gamma$, for, if β^2 be the greatest value of $\delta y^{(n-1)}$, $\delta y^{(n-2)} < \beta\,\beta'$. But $\beta' < \beta\gamma$; similarly $\delta y^{(n-3)} < \beta^3\gamma$, and so on. Hence $\delta y^{(n-1)}/\delta y^{(n)}$ is of the order β, and similarly for each of the fractions.*

It follows from this that the only term of the order α^2 in the expression for $\delta^2 U$ is

$$\int_{x_0}^{x_1} Y_{nn}\,\delta y^{(n)2}\,dx.$$

It follows that, if $Y_{nn}\,dx$ does not change sign in passing from x_0 to x_1, neither can $\delta^2 U$ change sign, whatever be the form of δy, and it is clear that $\delta^2 U/U$ is of the order α^2, exactly as if the integral were taken over a finite portion of the curve; and it should be remarked that the value of $\delta^2 U$ is of exactly the same order as if the limiting variations $\delta y \ldots \delta y^{(n-1)}$ had not been zero—that is, the order of $\delta^2 U$ is the same as if the most general variation possible had been given to y, although the actual variation is zero, as also are all its fluxions up to, but not including, the n^{th}.

It might possibly be objected that, as $\delta y^{(n-1)}$ is zero at each limit, therefore $\int_{x_0}^{x_1}\delta y^{(n)}\,dx$ is also zero. Hence, as dx may be taken to *increase* uniformly, $\delta y^{(n)}$ must change sign between $x = x_0$ and $x = x_1$; and, as these values are very close, $\delta y^{(n)}$ is everywhere very near its vanishing point, and is therefore everywhere very small. But this proceeds on the idea that $\delta y^{(n)}$ must be continuous. There is no difficulty when it is remembered that $\delta y^{(n)}$ may change suddenly from a positive value to a negative value.

16. Thus, without any analytical transformations, it has been shown that if Y_{nn} be positive the integral is a true minimum, and if Y_{nn} be negative it is a true maximum when the integration is extended over a very small range. We have still to consider how far the integration may be extended without annulling this property. Withdrawing the restriction that $x_1 - x_0$ is small, let us consider the continuity of the second variation

$$\delta^2 U = \int_{x_0}^{x_1} (Y_{00}\,\delta y^2 + 2Y_{01}\,\delta y\,\delta\dot{y} + \text{\&c.} + Y_{nn}\,\delta y^{(n)2})\,dx.$$

* It follows similarly that, provided β, the range of the integration, be not too great, the sign of $\delta^3 U$ is the same as that of $\int_{x_0}^{x_1}(d^3 f/\delta y^{(n)3}\,\delta y^{(n)3})\,dx$, that of $\delta^4 U$ the same as that of $\int_{x_0}^{x_1} d^4 f/dy^{(n)4}\,(\delta y^{(n)})^4\,dx$, and so on; and this does not in any way depend on y having its synclastic value.

It is, in general, a quantity of the order α^2 (Art. 14), and it is continuous if we can alter it by amounts infinitely small compared to α^2. This is always possible. First let the limiting values of x be unchanged; then, writing, for δy, $\delta y + \delta^2 y$, where $\delta^2 y$ is infinitely small compared to δy or α, it is evident that the change in the integral is infinitely small compared to α. If, still keeping the same variation $\delta y + \delta^2 y$, we change the limits by writing $x_1 + dx_1$, $x_0 + dx_0$, for x_1 and x_0, the most important additional terms due to this change,

$$\Big[(\mathrm{Y}_{00}\,\delta y^2 + 2\mathrm{Y}_{01}\,\delta y\,\delta\dot{y} + \ldots + \mathrm{Y}_{nn}\,\delta y^{(n)2})\,dx\Big]_0^1,$$

are also infinitely small compared to the original integral.

Now, as in the original expression for $\delta^2\mathrm{U}$ we may suppose $\delta y, \delta\dot{y}, \ldots \delta y^{(n-1)}$ zero at each limit x_1 and x_0, so we may suppose $\delta^2 y$ so determined that $\delta y + \delta^2 y$, $\delta\dot{y} + \delta^2\dot{y}, \ldots \delta y^{(n-1)} + \delta^2 y^{(n-1)}$, shall be zero for the values $x_0 + dx_0$, $x_1 + dx_1$.

Hence, representing by $\delta^2\mathrm{U}_{x_0}^{x_1}$ the value of the second variation when the value of δy is such that it and all its fluxions, up to the n^{th}, vanish for both limits of integration x_0 and x_1, we see that we can always alter the value of δy so that

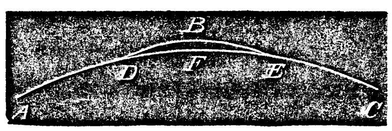
$$\frac{\delta^2\mathrm{U}_{x_0+dx_0}^{x_1+dx_1} - \delta^2\mathrm{U}_{x_0}^{x_1}}{\delta^2\mathrm{U}_{x_0}^{x_1}}$$

is infinitely small. (Of course we might also alter it so that it should be finite.)

It will simplify the further explanation if we represent the values of y by ordinates of a curve of which x is the abscissa,* and we will suppose that the value of Y_{nn} at the lower limit is negative, so that the curve obtained by making $\delta\mathrm{U}$ vanish is a maximum when the upper limit is very near the lower one.

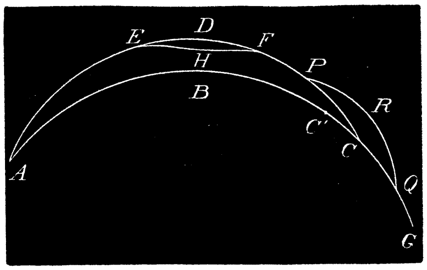
Considering the lower limit of integration x_0 as a fixed point A, and the higher one x_1 as an arbitrary point M, on the curve ABC, we know from § 15 that when M is sufficiently close to x_1 the integral is a maximum, *i.e.*, $\delta^2\mathrm{U}_{x_0}^{x_1}/(x_1 - x_0)$ is a negative quantity of the order α^2. Suppose the curve first ceases to give a maximum when M coincides with C. Then we may easily see that C is the point to which it first

* It is evident that what has been said about the admissibility of a variation may be expressed thus: δy is the difference between the ordinates of the new curve and old curve, and any curve is admissible, provided δy^n is nowhere infinite. Thus the broken curve ABC is admissible provided that at D and E,



the points of junction with AFC, it has contact of the $(n-2)^{\text{th}}$ order. Furthermore, the difference of the integrals taken along ABC and AFC is the same as the difference of the integrals along DBE and DFE, as is seen at once by regarding the sign of integration as one of summation.

becomes possible to draw a second curve such that $\delta^2 U_A^C = 0$. For, if $\delta^2 U_A^C$ could be a *positive quantity of the order* α^2, we could alter δy so that $\delta^2 U_A^{C'}$ should be positive and of the same order, C′ being between A and C, infinitely near C. Hence the



maximum property would have ceased at C′, but by supposition it does not cease till C, and therefore we have to find C as being the first point for which $\delta^2 U_A^C$ or $\delta^2 U_{x_0}^{x_1} = 0$.

Now $\delta^2 U_{x_0}^{x_1} = 0$, 1st, because it vanishes independently of the form of δy, or 2nd, because a particular form is assigned to δy, causing it to vanish, or 3rd, because $x_1 - x_0$ can be divided into separate parts, some of which vanish from the first cause and the rest from the others.

The first case can only occur if $\delta^2 U$ vanishes identically, *i.e.*, if its coefficients vanish, and will not be further discussed. The second case will occur when Y_{nn} does not vanish between A and the point for which $\delta^2 U_A^C = 0$ (this will be evident presently). The third case must be investigated on the supposition that the parts which vanish independently of the form of δy are infinitely small (for otherwise $\delta^2 U$ would vanish identically), and for these parts Y_{nn} must vanish, for when the limits of integration are infinitely close, and the limiting values of $\delta y, \delta \dot{y}, \ldots \delta y^{(n-1)}$ zero, $\delta^2 U = \int Y_{nn} \delta y^{(n)2} dx$, and this vanishes independently of the form of δy when $Y_{nn} = 0$.

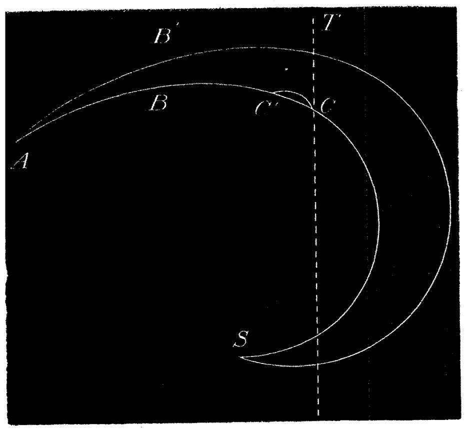
We have then only two cases to discuss—(a) Y_{nn} does not vanish throughout the integration, so that $\delta^2 U_A^C$ vanishes in consequence of a particular form being assigned to δy throughout the integration, and (b) that in which Y_{nn} does vanish.

First suppose that Y_{nn} does not vanish. Then C is evidently determined at the first point to which a second synclastic curve can be drawn, having at each limit contact of the $(n-2)^{\text{th}}$ order with ABC.*

* For it is evident that if $\delta^2 U$ vanishes in passing from ABC to ADC, and if any portion as EF of ADC were not itself synclastic, we could obtain an integral greater than that along ADC, and therefore greater than that along ABC, by joining EF by a synclastic curve EHF, having at E and F contact of the $(n-2)^{\text{th}}$ order with ADC; and therefore the preceding reasoning shows that C would not be the *first* point at which a curve can be drawn so that $\delta^2 U_A^C = 0$. Hence we must use the synclastic curve to get this limit.

Next suppose that, in fig. 4, the point S represents the first intersection of a consecutive synclastic curve AB′S with ABS, and that the dotted line, through T, is a line whose equation is $x = x'$, x' being the value of x for which $Y_{nn} = 0$ (if there is more than one value, take the first one you pass through in going from A *viâ* the curve). Then, in that figure, $\delta^2 U_A^C$ first becomes capable of the value zero at C, where the curve meets the ordinate through T. For the second variation vanishes in passing from ABC to a curve ABC′C coincident with ABC up to C′, a point infinitely close to C, and having at C and C′ contact of the $(n-2)^{\text{th}}$ order with ABC, since from A to C′ δy is zero, and from C to C′ Y_{nn} is zero.

Hence $\delta^2 U_A^C$ first becomes capable of a zero value when C coincides with the intersection* (S) of a consecutive synclastic curve having contact of the $(n-1)^{\text{th}}$ order with ABC, or with the point (T), given by $Y_{nn} = 0$, *whichever is first reached in passing from A along the curve.*



It follows, rigorously, that the synclastic property cannot cease *until* the first of these points is reached. It does not follow that, if the integration is extended beyond this limit, the integral is anticlastic.

Three cases arise:—1st, S is nearer A than T is; or, 2nd, T is nearer to A than S; or, 3rd, T and S coincide.

Case 1.—In this case we can easily show (see fig. 3) that, C being the intersection of a consecutive synclastic curve ADC with ABCG, the synclastic property does cease at C, that is, we can join A and G by curves the integral along which is either greater or less than that along ABCG. Let ADC be a consecutive synclastic curve for which $\delta^2 U_A^C = 0$, then evidently, representing by I (ABC . . .) the integral along the curve ABC . . . , we have

$$\mathrm{I}\,(\mathrm{ABC}) = \mathrm{I}\,(\mathrm{ADC});$$

add

$$\mathrm{I}\,(\mathrm{CG}) \equiv \mathrm{I}\,(\mathrm{CG}),$$

and get

$$\mathrm{I}\,(\mathrm{ABCG}) = \mathrm{I}\,(\mathrm{ADCG})$$

(which is legitimate, as δy^n is nowhere infinite). Now take two points P and Q on ADC and CG respectively, and join them by a synclastic curve PRQ having contact of the $(n-2)^{\text{th}}$ order with ADC and CG, P and Q being sufficiently close for the synclastic property to hold between them. Then, since I (PRQ) is a maximum,

$$\mathrm{I}\,(\mathrm{PRQ}) > \mathrm{I}\,(\mathrm{PCQ})\,;$$

add

$$\mathrm{I}\,(\mathrm{ADP}) + \mathrm{I}\,(\mathrm{QG}) \equiv \mathrm{I}\,(\mathrm{ADP}) + \mathrm{I}\,(\mathrm{QG}),$$

and get

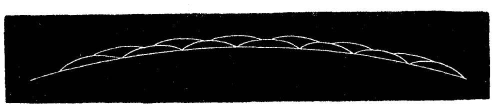
$$\mathrm{I}\,(\mathrm{APRQG}) > \mathrm{I}\,(\mathrm{ADCG}), \text{ and therefore } > \mathrm{I}\,(\mathrm{ABCG}),$$

showing that the integral taken along ABCG is not a maximum, and evidently it is not a minimum, for, if it were, every part of it would have to be a minimum, and the part from A to B, for instance, is not a minimum, but a maximum.*

Case 2.—In this case, if Y_{nn} changes sign as well as vanishes, the integral becomes anticlastic when the limit is beyond T (the point where $Y_{nn} = 0$). This is obvious, as the part immediately beyond T then gives a minimum value to the integral whose lower limit is T, while from A to T it gives a maximum value. If Y_{nn} does not change sign, it is easy to see that the maximum property does hold beyond C, with this nominal exception. A curve ABC′C″C‴G can be found the integral along which is *equal* to that along ABCG, C′ and C‴ being infinitely near C, and C′C″C‴ being any curve having contact of the $(n-1)^{\text{th}}$ order with ABCG at C′ and C‴. It is not difficult to see this by reasoning similar to the above, but it is shorter to observe that if we alter very slightly the value of Y_{nn}, so as just to make it preserve its sign without vanishing, we alter the value of the integral *very* slightly, and, therefore, &c.

Case 3.—Very slight consideration shows that the synclastic property ceases at T.

17. Before passing to the general case, it will not be amiss to add a few explanations.



The argument is not that it is possible to pass from the original curve to any other infinitely near curve by repeating again and again for each part of the curve a

* It is sometimes considered sufficient to say that, as $\delta \mathrm{U}_{\mathrm{A}}^{\mathrm{C}} = 0$, and $\delta^2 \mathrm{U}_{\mathrm{A}}^{\mathrm{C}} = 0$, while $\delta^3 \mathrm{U}_{\mathrm{A}}^{\mathrm{C}}$ changes sign with δy, the maximum property must cease at A. In TODHUNTER'S "ADAMS' Prize Essay on the Calculus of Variations" this reasoning is employed (Art. 24, p. 25). But it is invalid for two reasons. 1st, $\delta^3\mathrm{U}$ and all higher variations may vanish, as in the case of great circles on a sphere; and 2nd, it shows only that a curve can be got giving an integral greater than that corresponding to ABCG by terms of the *third*, not of the *second*, order. Now, as the second and third variations are quite independent of each other, there is nothing whatever to show that when a value is given to δy other than that which makes $\delta^2\mathrm{U} = 0$ this variation $\delta^2\mathrm{U}$ can be made to change sign. It is absolutely necessary to show this, for otherwise the integral from A to G would really be a true maximum, though of a very curious nature. —See the remarks at the end of Art. 17.

variation such as we have just given. No doubt, if we were to do this, we should at last get a curve differing from the original one, and, since Y_{nn} is the same if x be the same, it seems that we should get the variations all of the same sign, and that the result could thus at once be extended to a finite length of the curve; but, although it would be true that the sign of the *second* variation in passing from any one of the curves thus found to the consecutive one would have the same sign as $Y_{nn}\,dx$, yet the *first* variation would not vanish.

It is not difficult to see that these two alternative conditions, treated of in Cases 1 and 2 respectively, are independent. For, if the points obtained by the two criteria coincided, the least root of the determinant equation (a) in the note* must coincide with the least value of x for which $Y_{nn} = 0$, and it seems evident that there can be no such connection. It may, however, be as well to give an example, to show that the condition in Case 2 may cease to be fulfilled, while that in Case 1 still holds. Taking for U the expression $\int \dot{y}^4 x\,dx$, we have

$$\delta \mathrm{U} = 4\int \delta\dot{y}\,\dot{y}^3 x\,dx = \left[4\dot{y}^3 x\,\delta y\right]_0^1 - 4\int \frac{d}{dx}\,(\dot{y}^3 x)\,\delta y\,dx,$$

and the equation given by the calculus is

$$\frac{d}{dx}\,\dot{y}^3 x = 0\,;$$

integrating, we get

$$y = cx^{2/3} + c',$$

whence it is easy to see that $Y_{nn}\,dx$ or $4.3.\dot{y}^2 x\,dx$ changes sign as x passes through the value zero. (It might be thought sufficient to say that, as $\dot{y}^2$ cannot change sign,

* If $y = f(x,\ c_1,\ c_2, \ldots c_{2n})$ be the general solution obtained by making

$$Y_0 - \frac{dY_1}{dx} + \frac{d^2Y_2}{dx^2} + \text{\&c.} \pm \frac{d^nY_n}{dx^n} = 0,$$

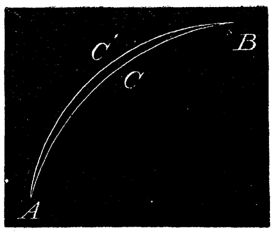
then, denoting df/dc_1, df/dc_2, &c., by y_1, y_2, &c., it is well known that we have for the value of x_1, to which a second curve of the species can be drawn from x_0, so as to have, at x_0 and the point we seek, contact of the $(n-2)^{\text{th}}$ order with the curve $y = f(x,\ c_1,\ c_2, \ldots c_{2n})$, the equation

$$\begin{vmatrix} y'_1, & y'_2, & \cdot\ \cdot\ \cdot\ \cdot\ \cdot & y'_{2n} \\ \dot{y}_1, & \dot{y}_2, & \cdot\ \cdot\ \cdot\ \cdot\ \cdot & \dot{y}_{2n} \\ \cdot\,, & \cdot\,, & \cdot\ \cdot\ \cdot\ \cdot\ \cdot & \cdot \\ {y'_1}^{(n-1)}, & {y'_2}^{(n-1)}, & \cdot\ \cdot\ \cdot\ \cdot & {y'_{2n}}^{(n-1)} \\ y_1, & y_2, & \cdot\ \cdot\ \cdot\ \cdot\ \cdot & y_{2n} \\ \dot{y}_1, & \dot{y}_2, & \cdot\ \cdot\ \cdot\ \cdot\ \cdot & \dot{y}_{2n} \\ \cdot\,, & \cdot\,, & \cdot\ \cdot\ \cdot\ \cdot\ \cdot & \cdot \\ y_1^{(n-1)}, & y_2^{(n-1)}, & \cdot\ \cdot\ \cdot\ \cdot & y_{2n}^{(n-1)} \end{vmatrix} = 0, \quad \ldots \ldots \ldots \quad (a)$$

where y' means the value of y when x_0 is substituted for x. The least value of x satisfying this, or more properly the value first reached in going from x_0 *viâ* the curve, gives the point in question.

this quantity must change sign with x, as dx is a constant quantity; this, however, is a mistake: it is necessary to know the nature of the curve in order to be sure that dx does not change sign when x does.) Hence, according to the first rule, the integral will be anticlastic, should the value $x = 0$ be included in the limits of integration. But, if we seek the limits within which the condition in the second proposition is fulfilled, we find that, if we start from a point $x_0 = -\alpha$, a second curve of the same species does not intersect the original curve until $x = +\alpha$. It follows that, so far as regards the second condition, the integration might extend from $x_0 = -\alpha$ to $x_1 = +\alpha$.

It may be well to observe that the proof given in Case 1, that when the integration is extended beyond the limits stated, *i.e.*, those for which δ^2U can vanish, the synclastic property ceases to hold, does not in any way depend on the supposition that δ^3U does not vanish. It is shown absolutely, and without exception, that when the limit stated is passed, δ^2U can change its sign (§ 16), and the values of δ^3U, δ^4U, for those limits will only enable us to find whether the synclastic property holds *at* the limit *up to which* it is known to hold, namely, whether it holds up to *and including*



the limits found in Prop. 2. Consider the case of a curve, and let ACB and AC′B be two consecutive curves satisfying the limits and the differential equation

$$\mathrm{Y}_0 - \frac{d\mathrm{Y}_1}{dx} + \frac{d^2\mathrm{Y}_2}{dx^2} - \&c.\,;$$

then both the first and second variations, δU and δ^2U, vanish in passing from the curve ACB to AC′B. But the third variation, in passing from A to B, will not, in general, vanish, and may be expressed as a function of the coordinate of A, $= f(x)$, suppose; now, as A moves along the curve, $f(x)$ will, in general, vanish at one or more points. Hence it follows that, in general, the synclastic property only holds *between* A and B, and does not hold *for the limits* A *and* B *actually*, though there the difference is only of the third order; but there may be certain points for which δ^3U vanishes and δ^4U is of the same sign as $\mathrm{Y}_{nn}dx$, and for these the curve joining A and B gives a truly synclastic value to the integral.

If the synclastic property ceases because Y_{nn} changes sign, it will hold up to (and including) the limit, and only cease as you pass beyond the limit.

18. It is often convenient to borrow a few terms from geometry when treating of functions depending on a number of independent variables $x_1, x_2, \ldots . x_n$. In the following paragraphs the word "point" will be used to denote any single set of values of the independent variables, while "region" will mean a continuous collection of

points whose boundary is defined by those values of $x_1 \ldots . x_n$ which make a certain function or functions vanish; as, for instance, all those sets of values which satisfy the two sets of inequalities

$$x^2_1 + x^2_2 + \ldots + x^2_m - r^2 < 0$$

and

$$(x_1 - a_1)^2 + (x_2 - a_2)^2 + \ldots + (x_m - a_m)^2 - r'^2 < 0.$$

The dependent variables being $y_1 \ldots . y_n$, the word "surface" will be used as an abbreviation for the term "set of equations expressing the dependent variables in terms of the independent ones." Not only is there much saving in labour, both to the writer and to the reader, in the adoption of these terms, but there is the additional advantage that the same explanation is applicable alike to the most general case and to that in which geometrical conceptions enable the argument to be grasped with a clearness unattainable in reasoning of a purely analytic character. The reason for adopting the word "surface" instead of "curve" is that, as the explanation for the curve has been already given, it would be superfluous to repeat it, while it seems a real advantage to give the investigation for the case of one dependent and two independent variables.

It will be necessary, in the first place, to examine the conditions under which the solution supplied by the rules of the Calculus of Variations is applicable.

Let the function to be made a minimum be

$$U = \int dx_1 \ldots \int dx_m f(x_1, \ldots x_m, y_1, \ldots y_n,),$$

where $f(x_1, \ldots x_m, y_1, \ldots y_n)$ includes the fluxions of $y_1 \ldots y_n$ with regard to the independent variables.

To find the variation in this expression, let us increase $y_1, y_2, \ldots y_n$ to y_1+h_1, $y_2+h_2, \ldots y_n+h_n$ (h_1, &c., being functions of $x_1, x_2, \ldots x_m$); then the fluxions of y_1, &c., will be increased by the corresponding fluxions of h_1, &c.

For the purpose of ascertaining the limits within which the quantities h_1, h_2, &c., must be confined in order that our reasoning may be valid, let us write, in place of h_1, h_2, &c., $\alpha\Delta y_1$, $\alpha\Delta y_2$, &c., where α is a constant, the same for all the variables, and Δy_1, Δy_2, &c., are finite functions of $x_1, x_2, \ldots x_m$, or, more accurately, are functions which, though they (and their fluxions contained in the integrals) may vanish, neither become infinite, nor give infinite values to these fluxions, for values of the independent variables included in the region of integration; they are *not-infinite* functions. If we represent by z, z', &c., any of the dependent variables *or* their fluxions, we may represent the corresponding change by $\alpha\Delta z$, $\alpha\Delta z'$. By TAYLOR's theorem we may write the new value of U corresponding to the new values for the dependent variables in the form

$$U_{y+h} = U_y + \int \ldots \int \left\{ \alpha\Sigma\left(\frac{df}{dz}\Delta z\right) + \tfrac{1}{2}\alpha^2\Sigma\left(\frac{d^2f}{dzdz'}\Delta z\Delta z'\right) + \frac{\alpha^3}{\underline{|3}}(\&c.)\right\} dx_1 \ldots dx_m. \quad (8)$$

For our purposes it will be necessary to take α of such a magnitude that the part depending on α^2 is greater than all the subsequent terms. (It seems well to observe that this does not, in general, imply that α is *infinitely* small.) But there are restrictions to the values of the quantities Δz, $\Delta z'$, &c., and in order to find them it will be necessary to give an outline of the method which is usually employed in the calculus. The part of (8) depending on the first power of α is reduced by successive integration by parts, so that the part of the integral not solely depending on the limiting values contains only the variations Δy_1, Δy_2, &c., and does not contain their fluxions. Thus, if D represent the operation by which z is got from y,* we shall get from the term

$$\alpha \int \ldots \int \frac{df}{dz} \Delta z \, dx_1 \ldots dx_m$$

such terms as

$$\alpha \int \ldots \int (\mathrm{B}_1 \, dx_2 \, dx_3 \ldots dx_m + \mathrm{B}_2 \, dx_1 \, dx_3 \ldots dx_m + \&\text{c.}) \\ \pm \alpha \int \ldots \int \left[\mathrm{D}.\left(\frac{df}{dz}\right)\right] dx_1 \ldots dx_m,$$

where the first part of the right-hand side depends only on limiting values. Applying similar reductions to all terms containing *fluxions* of variations, we get an expression of the form

$$\left.\begin{aligned} \delta \mathrm{U} = \alpha \int \ldots \int \{\mathrm{L}_1 \, dx_1 \, dx_2 \ldots dx_m + \mathrm{L}_2 \, dx_1 \, dx_2 \ldots dx_m + \&\text{c.}\} \\ + \alpha \int \ldots \int \{\mathrm{A}_1 \Delta y_1 + \mathrm{A}_2 \Delta y_2 + \&\text{c.}\} \, dx_1 \, dx_2 \ldots dx_m \end{aligned}\right\} \quad . \quad . \quad (9)$$

This integration by parts depends for its validity on the supposition that no one of the quantities Δz becomes infinite for any values of $x_1 \ldots x_m$ within the limits of integration. But it is very important to remark that the integration is legitimate, whether the variations of the *highest* fluxions are or are not discontinuous in the sense of suddenly changing from one *finite* (properly, *not-infinite*) value to another. It is to be observed that, in discussing the sign of $\delta^2 \mathrm{U}$, we introduce no limitations except those already implied in the usual treatment of $\delta \mathrm{U}$.

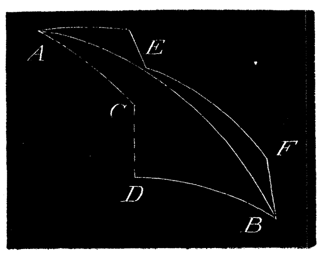
Again, it is to be observed that the terms in the limiting integrals in (9) will contain the limiting values of all but the highest fluxions of the variations. Hence, when we say that the limits are given, we mean that the values of the dependent variables and of all but their highest fluxions are given for all points on the boundary of the region of integration. Hence, if we determine the forms of the functions $y_1 \ldots y_n$ so as to satisfy the equations—

* Thus, when z represents $\dfrac{d^8 y_1}{dx_1^2 \, dx_2^3 \, dx_4^3}$, D will represent $\dfrac{d^8}{dx_1^2 \, dx_2^3 \, dx_4^3}$.

$$\left.\begin{aligned}
A_1 &\equiv \frac{df}{dy_1} - \frac{d}{dx_1}\frac{df}{d\left(\frac{dy_1}{dx_1}\right)} - \frac{d}{dx_2}\frac{df}{d\left(\frac{dy_1}{dx_2}\right)} - \&\text{c.} + \&\text{c.} = 0 \\
A_2 &\equiv \frac{df}{dy_2} - \frac{d}{dx_1}\frac{df}{d\left(\frac{dy_2}{dx_1}\right)} - \ldots\ldots\ldots = 0 \\
\cdot &\equiv \ldots\ldots\ldots\ldots\ldots\ldots = 0 \\
A_n &\equiv \frac{df}{dy_n} - \frac{d}{dx_1}\frac{df}{d\left(\frac{dy_n}{dx_1}\right)} - \&\text{c.}\ldots\ldots\ldots = 0
\end{aligned}\right\}, \quad . \quad . \quad . \quad (10)$$

and at the same time fulfil the conditions supplied by the given limiting values of the dependent variables and of all but their highest fluxions, the difference between the integrals for the surface so determined and that for any other surface will vanish so far as regards terms involving only the first power of α, provided only that the second surface can be obtained from the first by a change such that none of the Δ variations for A become infinite when $x_1, \ldots x_m$ have the values corresponding to any point within the region of integration. This, indeed, is true whatever be the order of α, but, in order that the sign of the difference between the integrals shall be the same as that of the second variation (the part depending on α^2), it is, in general, necessary that the quantities $\alpha\,\Delta z$, $\alpha\,\Delta z'$, &c., be small.

19. To determine whether the integral is a true maximum or minimum, we have now only to find whether the terms of this order α^2 in (8) will be always of the same sign when the variations are given in any values consistent with the conditions given in Art. 18. For it is evident that, if we restrict ourselves to a less general variation in examining the sign of the second variation, we could neither be sure that the conditions obtained were sufficient to ensure that the integral was synclastic, though they would be necessary; nor that the conditions that it should be anticlastic were necessary, though they would be sufficient. If, on the other hand, we were to admit a more general variation, the conditions for synclasticism would be sufficient, but not necessary, and those for anticlasticism would be neither sufficient nor necessary. In fact, it will be found that the conditions under which we are discussing the problem are really those necessary in order that it shall have a



meaning. For instance, in the case of least action, when we say that the action in the free path is less than in any other, we imply that there is to be no sudden change

in the value of y, such as would occur in a path AC . . . DB, but there may be sudden changes in $\dot{y}$, the inclination of the tangent (as in AEFB).

20. To facilitate the discussion, a fluxion will be said to be one of the "highest fluxions" when no fluxions of that fluxion appear in the function whose integral is to be made synclastic. Thus, $d^2y_1/dx_1\,dx_2$ may be one of the "highest" fluxions, although d^7y/dx^7 is not; for there may be no terms which can be written $\mathrm{D}\,.\,d^2y/dx_1\,dx_2$, where D represents any combination of d/dx, d/dx_2, &c., while there might be a term d/dx_2, $d^7y/dx_1{}^7$.

The limits being supposed fixed, the following proposition can be easily proved:—If the highest fluxions of the variable y_1 which occur in U be all of the same order n_1, those of y_2 of the same order n_2, and so on, then the conditions that U shall be synclastic when the integral is only extended over a *small* region R are the same as those that the quantity

$$\left(\frac{d^2f}{da^2}\mathrm{X}^2 + \frac{d^2f}{db^2}\mathrm{Y}^2 + \&\text{c.} + 2\frac{d^2f}{da\,db}\mathrm{XY} + \&\text{c.}\right) dx_1\,dx_2 \ldots dx_m$$

shall be incapable of a change of sign, a, b, &c., representing the highest fluxions, and X, Y, &c., being any arbitrary quantities.

The conditions for the case where the highest fluxions of any dependent variable are not all of the same order will be discussed afterwards. When it is said that the region R of integration is small it is meant that the greatest *ranges of value* of the coordinates $x_1 \ldots x_m$ are small. Thus, if the region is given by

$$x_1{}^2 + x_2{}^2 + \ldots + x_m{}^2 - r^2 < 0,$$

then r must be small; of the order β, suppose. This being so, it is easy to show that we may neglect the variations of all but the highest fluxions of the dependent variables when finding the sign of $\delta^2\mathrm{U}$. For the change in Δz in passing from a point P_0 on the boundary of R to any point P within it is

$$(\Delta z)_{\mathrm{P}} - (\Delta z)_{\mathrm{P}_0} = \int_{\mathrm{P}_0}^{\mathrm{P}} d\,\Delta z = \int_{\mathrm{P}_0}^{\mathrm{P}}\left(\frac{d\Delta z}{dx_1}dx_1 + \frac{d\Delta z}{dx_2}dx_2 + \&\text{c.}\right);$$

and, since the total range of $dx_1, \ldots dx_m$, in the integration is of order β, the order of the integral will be that of the quantities $\beta\, d\Delta z/dx_1$, $\beta\, d\Delta z/dx_2$, &c., or at least it cannot be greater. Again, the limits being fixed, $(\Delta z)_{\mathrm{P}_0}$, the limiting value of Δz, must be zero unless z is one of the "highest fluxions." Hence it follows that all the other fluxions are small (of the order β at least) compared to the highest; and that they can be omitted from the integral when all we wish to determine is its sign. It is, however, necessary to show that the entire integral does not vanish, for, as the values of all but the highest fluxions, and therefore of the highest but one, are given for all points of the boundary, it is evident that

$$\int_{P_0}^{P'_0} \frac{d\Delta z}{dx_1} dx = 0,$$

where $d\Delta z/dx_1$ is one of the highest fluxions, and P_0 and P'_0 are two points on the boundary for which the value of x_1 alone is different, those of $x_2, \ldots x_m$ being the same. It follows from this that $d\Delta z/dx_1$ must change sign once at least in passing from P_0 to P_0', and therefore, if it varied continuously, must everywhere be infinitely small. Since, however, there is nothing to prevent d/dx, $d\Delta z/dx_1$, being infinite (§ 18), $d\Delta z/dx_1$ may have a finite value within the region.

Remembering that the region is small, and that the order of magnitude of Δa, Δb, &c., a, b, &c., being "highest fluxions," does not depend in any way on that of the region of integration, we see that the sign of $\delta^2 U$ will be the same as that of

$$\int \ldots \int \left(\frac{d^2 f}{da^2}(\Delta a)^2 + 2\frac{d^2 f}{da\, db}(\Delta a\, \Delta b) + \text{\&c.}\right) dx_1, dx_2, \ldots dx_m, \quad . \quad . \quad . \quad (11)$$

Δa and Δb representing highest variations. As the region is small, d^2f/da^2, $d^2f/da\, db$, &c., may be considered as constants throughout the integration, a supposition which again involves neglect of small terms. (Δa, Δb, &c., cannot be regarded as constants, however small the region may be, for their differential coefficients may be infinite.) Now, the part inside the bracket can be resolved into a sum of squares, and, if the coefficient of any one (or more) of these squares is negative, the expression can be given either sign. For, although the quantities Δa, Δb, &c., are not independent of each other to the extent that all the other square terms could be made to vanish, yet they are so to the extent that any one term may be made to exceed all the others; for instance, if the region of integration be that for which

$$x_1^2 + x_2^2 + \text{\&c.} \ldots - r'^2 = < 0 \quad \text{or} \quad \phi' = < 0,$$

and

$$x_1^2 + x_2^2 + \text{\&c.} \ldots - r''^2 = < 0 \quad \text{or} \quad \phi'' = < 0,$$

r' and r'' being small quantities of the order β, we may assume

$$\Delta y_1 = \frac{1}{\beta^{2n_1}}(\phi')^{n_1}(\phi'')^{n_1} f_1\left(\frac{x_1}{\beta}, \frac{x_2}{\beta}, \ldots \frac{x_m}{\beta}\right),$$

$$\Delta y_2 = \frac{1}{\beta^{2n_2}}(\phi')^{n_2}(\phi'')^{n_2} f_2\left(\frac{x_1}{\beta}, \frac{x_2}{\beta}, \ldots \frac{x_m}{\beta}\right),$$

and so on for the others. For these assumptions will satisfy the limiting conditions, whatever be the forms of f_1, f_2, &c., which may be regarded as quite arbitrary, and they will give finite values to the "highest fluxions" of the dependent variables. If we resolve the quadratic expression in (11) into a sum of squares, and substitute for Δa, Δb, &c., their values in terms of f_1, f_2, &c., we shall, by solving a differential

equation, be able to make any one of these terms larger than the sum of the others. It is necessary, however, to show that such an equation has at least one real solution, and this can be done by the method of expansion in series just as it is usually done for the case of two variables only.

Thus the criterion for a maximum or minimum value of the function has been found when the region of integration is small. It is the same as that obtained by the methods of transformation; but the proof now given is free from the uncertainty which is connected with analytical proofs.

21. The criteria for the case where the region of integration has any finite magnitude can be derived from the preceding by considerations depending on the *continuity of the integrals*. Remembering that we are still treating of the case where the limits are fixed, we may prove the following proposition:—

If it is possible to take, around every point P in a region R of finite magnitude, a minor region (p), no matter how small, such that the integral U for that region is synclastic, then the integral for the entire region R will be synclastic, provided the further condition be fulfilled that it is impossible to take within the region R a second synclastic surface V′, having at all points of its limiting intersection with the first the same values for the dependent variables and for all their fluxions, *with the exception of the highest.* If it be possible to find such a surface, the integral U will be anticlastic for the region R.

Let us consider how it could happen that $\delta^2 U$ became capable of either sign at pleasure when the region of integration is extended. Let S be the region for which this change of sign *first* becomes possible. Hence (restricting ourselves to the case where the function U has a *minimum* value), when the integration extends over *any* region wholly contained in S, $\delta^2 U$ is positive, while, if it be extended over a region including S, $\delta^2 U$ can change sign. It is clear that this can only happen if the least value of $\delta^2 U$ is zero when the integration is extended over S. This may be shown thus. The second variation being written

$$\delta^2 U = \alpha^2 \int \ldots \int \Sigma \frac{d^2 f}{dz\, dz'} \Delta z\, \Delta z'\, dx_1 \ldots dx_m,$$

where Δz, $\Delta z'$, &c., are finite quantities, it is evidently capable of being changed by an amount infinitely small compared to α^2, by an infinitely small change in Δz, $\Delta z'$, &c., and the *new* values of Δz can be made to satisfy limiting conditions obtained from the previous ones by infinitely small changes. It follows hence that, if it were possible to take such values for Δz, $\Delta z'$, &c., that $\delta^2 U$ should be a negative quantity of the order α^2 when the integration extends over a region $S + dS$, greater than, but differing infinitely little from, S, we could, by an infinitely small change in the variations, obtain values making $\delta^2 U$ negative and of the order α^2 within the region $S - dS$, which by the supposition is impossible, as within S the integration of $\delta^2 U$ gives a result which is always positive. It follows that the greatest negative value of $\delta^2 U$

for the region $S + dS$ can only differ from zero by a quantity infinitely small compared to α^2, and therefore the least positive value of δ^2U must vanish for the region S.

It has now to be proved that δ^2U must be capable of a negative value of order α^2 when the region of integration is extended beyond the region S. It sometimes seems to be considered that this may be inferred from the fact of δ^3U changing sign when $\delta^2U = 0$. But, as δ^3U does not represent the increment of δ^2U *due to extending the region of integration beyond S*, this is not admissible. If we draw an imperfect analogy from algebra, we may say that what we have to prove is that in no case does δ^2U behave as if it had a square factor the value of which, after vanishing, remains of the same sign, but that, if S be a region of integration of the character supposed in Proposition 2, for which it is possible to make δ^2U zero, then for any region including S it is possible to make δ^2U take either sign, the limits being in each case supposed fixed.

To prove this, let us suppose that $y_1, y_2, \ldots y_n$ represent values of the dependent variables which make the first variation vanish. Let $\alpha\Delta_1 y_1$, $\alpha\Delta_1 y_2$, &c., represent the variations for which $\delta^2U = 0$ (that is, $\Delta_1{}^2U = 0$) when the integration is extended over S. The limiting values of all variations, except those of the highest fluxions, are zero. Let S′ be a region including S, and let $\alpha\Delta_2 y_1$, $\alpha\Delta_2 y_2$, &c., be variations having at all points of S the values $\alpha\Delta_1 y_1$, $\alpha\Delta_1 y_2$, &c., for all the variations, and having at all points of S′ not common to it and S the values zero for all the variations.

Now take a third region S″, wholly included in S′, and of which a portion Σ_1 is included in S, and the remainder Σ_2 excluded from S; and let $\alpha\Delta_3 y_1$, $\alpha\Delta_3 y_2$, &c., be a variation having at all points of S′ — S″ (representing in that way the points contained in S′ and excluded from S″) the values $\alpha\Delta_2 y_1$, $\alpha\Delta_2 y_2$, &c., while, over the region S″, $\alpha\Delta_3 y_1$, $\alpha\Delta_3 y_2$, &c., are determined by the conditions that $y_1 + \alpha\Delta_3 y_1$, $y_2 + \alpha\Delta_3 y_2$, &c., are the values which make U a true minimum when the integration extends over the region S″, and the limiting conditions are that, for all except the highest fluxions of the variations, $\Delta_3 z = \Delta_2 z$ all over the boundary of S″, z representing, as before, any dependent variable or fluxion. (By taking the region S″ small enough, these conditions can always be satisfied.)

Then the variations represented by Δ_1, Δ_2, and Δ_3 are admissible ones (Art. 18). Then $\Delta_1{}^2U = 0$ when the integration extends over S, and $\Delta_2{}^2U$, when the integration extends over S′, is identically equal to it, and therefore vanishes (for the Δ_2 variations are zero except over S, where they equal the Δ_1 variations). Again the Δ_3 variations are the same as the Δ_2 ones, except over S″; and over S″ the functions $y_1 + \alpha\Delta_3 y_1$, &c., give a smaller value to the integral than do $y_1 + \alpha\Delta_2 y_1$ (because they make it an absolute minimum compared to all near values), and therefore $\Delta_3{}^2U$ is smaller than $\Delta_2{}^2U$, the integration being extended over S′. But $\Delta_2{}^2U = 0$, and therefore $\Delta_3{}^2U$ is negative. But clearly δ^2U may be positive, and we have now shown that it may be negative, for one value is $\alpha^2\Delta_3{}^2U$; hence it is capable of either sign.

It may be objected that exceptional cases might occur, in which $\Delta_3 y$ and $\Delta_2 y$ coincided over S″ as well as over S′—S″. But it is directed that, of S″, Σ_1 is in the part common to S and S′, and Σ_2 is outside S. The surface $y_1 + \alpha \Delta_2 y_1$, &c., is therefore one in which discontinuous values for the fluxions of $y_1 + \alpha \Delta_2 y_1$, $y_2 + \alpha \Delta_2 y_2$, &c., would appear in the equations (10), Art. 18 ; and therefore that equation cannot be satisfied by these values. Hence Δ_2 is not the same operation as Δ_3.

Observe that the whole point of the proof consists in the fact that the Δ_3 variations are sufficiently continuous, notwithstanding the discontinuity in the highest fluxions.

To complete the proof, it only remains to show that the surface V′, for which $\delta^2 U = 0$, satisfies the equations (10), Art. 18, for every point of the region S. Suppose that it did not do so for a portion S_1 of S. Take a compound surface made up of V_1 over S_1 and V′ over $S - S_1$, where V_1 is the synclastic surface, having at all points of the boundary of S_1 the same values of Δy_1, Δy_2, &c., and all but their highest fluxions, as those of the function V′. This compound surface gives us an admissible variation, and the integral over it is less than that over V′. But, by hypothesis, it is impossible within the region S to find a surface giving a smaller value to the integral than that given by V. Hence V′ must be a synclastic surface, and the proposition is proved.

Hence a function will be synclastic provided, first, that the condition given in Art. 20 is fulfilled for every point in the region of integration; and second, that it is impossible, within that region, to draw another synclastic surface with the same limits. It will be easily seen that these conditions are independent of each other, and that, if either or both fail, the function becomes anticlastic.

22. When the highest fluxions of any dependent variable are not all of the same order of differentiation, the conditions found in Art. 20, although sufficient, are not all necessary. For, as will be proved presently, the values of the highest fluxions of the Δ variations are not all of the same order of magnitude. To ascertain the comparative orders of different fluxions, let us consider the equations

$$\Delta z_P - \Delta z_{P_{01}} = \int_{x_1'}^{x_1} \frac{d\Delta z}{dx_1} dx_1,$$

$$\Delta z_P - \Delta z_{P_{02}} = \int_{x_2'}^{x_2} \frac{d\Delta z}{dx_2} dx_2, \text{ and so on,}$$

where Δz_P means the value at the point P, or $x_1, x_2, \ldots x_m$; $\Delta z_{P_{01}}$ that at the point on the boundary which has the same values for all the coordinates except x_1, which has the value x_1' given by $\phi(x_1' x_2 \ldots x_m) = 0$, where $\phi = 0$ is the equation of the boundary; and similarly for the other coordinates. As $x_1 - x_1'$, $x_2 - x_2'$, &c., are everywhere of the same order, β, it might appear that $d\Delta z/dx_1$, $d\Delta z/dx_2$, must each be of the order $\Delta z/\beta$. This, however, is only the case when $d\Delta z/dx_1$, $d\Delta z/dx_2$, &c., do not rapidly change sign. For, while it is clear that $d\Delta z/dx$ must exceed Δz in the ratio $1/\beta$ *at least*, it is easily seen that the ratio may be greater if the terms in the

integral change sign rapidly. In fact, if the order of $d\Delta z/dx_1$ is unity, and its fluctuations recur at intervals of order β^n, then the terms in the integral

$$\int_{x_1'}^{x_1} \frac{d\Delta z}{dx_1} dx_1$$

will cancel, provided $x_1 - x_1' = m\beta^n$, m being any integer; and so the integration up to any point can only contain terms of the order β^n. Moreover, since, if any function is periodic with respect to any variable, all its fluxions with respect to that variable are of the same period, it follows that, if *one* differentiation introduces a coefficient of the order β^{-n}, *two* will bring in β^{-2n}, and so on.

It thus appears that, if we assume $\Delta y_1 = \beta^p$, multiplied by a function whose fluctuation-periods with regard to x_1, x_2, &c., are p_1, p_2, &c., the order of

$$\frac{d\Delta y_1}{dx_1} \text{ will be that of } \beta^{(p-p_1)};$$

$$\frac{d\Delta y_2}{dx_2} \quad \ldots\ldots\ldots\ldots \quad \beta^{(p-p_2)};$$

$$\ldots\ldots\ldots\ldots\ldots\ldots,$$

and, in general, that of $d^{a_1+a_2+\&c.}\Delta y_1/dx_1^{a_1}\,dx_2^{a_2}\ldots$ will be $\beta^{p-a_1p_1-a_2p_2-\&c.}$. Now, by the conditions in Art. 18, none of the quantities Δz, &c., can be infinite; hence we must reject any values of Δy_1, Δy_2, &c., which do not fulfil this condition. Thus the term just considered must be rejected if it makes any of the expressions

$$p - a_1p_1 - a_2p_2 - \&c. - a_mp_m < 0,$$

where $a_1, a_2, \ldots a_m$ have the values corresponding to any fluxion occurring in U. And it may be rejected as useless unless it makes some one, at least, of these expressions zero. An example will render this more intelligible. Suppose we consider a function in which d^8y_1/dx_1^8, $d^4y_1/dx_1^3\,dx_2$, $d^4y_1/dx_1^2\,dx_2^2$, d^5y_1/dx_2^5, are the highest fluxions. Here the equations to be satisfied are

$$p - 8p_1 = > 0 \quad \ldots \quad (a); \qquad p - 3p_1 - p_2 = > 0 \quad \ldots \quad (b);$$
$$p - 2p_1 - 2p_2 = > 0 \quad \ldots \quad (c); \qquad p - 5p_2 = > 0 \quad \ldots \quad (d),$$

together with those obtained from the lower fluxions. But, as the lower fluxions must be smaller than the higher in the ratio $1/\beta$ *at least*, it is unnecessary to consider the conditions obtained from them. If we put $p_1 = 1$, we get, from (a), $p = 8$, and it is obvious that $p_2 = 1$ will make (b), (c), and (d) each > 0. If we put $(b) = 0$ and eliminate p from (1), (c), and (d), we get, as our conditions from (a), $p_2 - 5p_1 > 0$; therefore $p_2 > 5$, as $p_1 = 1$; from (3), $p_1 - p_2 > 0$, and therefore $p_2 < 1$. Hence no value for p_1, p_2, or p can satisfy $(b) = 0$ and the other inequalities. Hence we may leave $d^4\Delta y/dx_1^3\,dx_2$ out of the final condition, as it could only give terms

multiplied by positive powers of α. Similarly, if $(c) = 0$, $p = 2p_1 + 2p_2$, and from (a) $2p_2 - 6p_1 = > 0$, but from (d) $2p_1 - 3p_2 = > 0$, and, as these are incompatible for positive values of p_1 and p_2, $d^4\Delta y_1/dx_1^2\,dx_2^2$ cannot appear in the quadratic function, on whose sign the problem depends. For, if we make any assumption which gives to it a finite value, we give, to other quantities in the integral, values involving negative powers of β. If β be infinitely small, this would imply infinite values for the Δ variations. But the values $p = 8$, $p_1 = 1$, $p_2 = \frac{8}{5}$, will satisfy $(d) = 0$, and the rest either $=$ or > 0. Hence, in the case in question, the sign of the variation depends on that of

$$\left\{\frac{d^2f}{\left(d\left(\frac{d^8y_1}{dx_1^{\,8}}\right)\right)^2}\mathrm{X}^2 + 2\frac{d^2f}{\left(d\left(\frac{d^8y_1}{dx_1^{\,8}}\right)\right)\left(d\left(\frac{d^5y_1}{dx_2^{\,5}}\right)\right)}\mathrm{XY} + \frac{d^2f}{\left(d\left(\frac{d^5y_1}{dx_2^{\,5}}\right)\right)^2}\mathrm{Y}^2\right\}dx_1\,dx_2.$$

If the highest fluxions had been

$$\frac{d^8y_1}{dx_1^{\,8}},\quad \frac{d^8y_1}{dx_1^{\,7}\,dx_2},\quad \frac{d^7y_1}{dx_1^{\,5}\,dx_2^{\,2}},\quad \frac{d^3y_1}{dx_2^{\,3}},$$

the conditions would be, if we write $p_1 = 1$ all through, $p - 8 = > 0\,(a)$; $p - 7 - p_2 = > 0\,(b)$; $p - 5 - 2p_2 = > 0\,(c)$; $p - 3p_2 = > 0\,(d)$. Putting $p = 8$, $p_2 = 1$, (a) and $(b) = 0$, and (c) and $(d) > 0$: hence $d^8\,\Delta y_1/dx_1^{\,8}$ and $d^8\,\Delta y_1/dx_1^{\,7}\,dx_2$ will remain in the condition. Putting $p = 9$, $p_2 = 2$, (b) (and (c)) $= 0$, while (a) and $(d) > 0$: hence $d^7\,\Delta y_1/dx_1^{\,5}\,dx_2^{\,2}$ remains. Again, putting $p_2 = 5$ and $p = 15$, we get $(d) = 0$, while $(c) = 0$ and (a) and $(b) > 0$, and hence $d^3\,\Delta y_1/dx_2^{\,3}$ appears. Hence in this case all the fluxions remain.

To complete the proof, we must show that it is possible to assign values for the variations Δy_1, Δy_2, &c., which shall satisfy the limiting conditions as well as those given above. In order to show what kind of assumption must be made, it is necessary to remark that, if z be any quantity such that $z = 0$, and $dz/dx_1 = 0$ all over any surface $\phi(x_1, x_2, \ldots x_m) = 0$, then will every other fluxion, as dz/dx_r, $= 0$ over that surface. For, as $z = 0$,

$$0 = dz = \frac{dz}{dx_1}dx_1 + \frac{dz}{dx_2}dx_2 + \ldots \frac{dz}{dx_m}dx_m$$

whenever $dx_1, dx_2, \ldots dx_m$ satisfy the equation $\phi = 0$. The only relation among $dx_1, dx_2, \ldots dx_m$ imposed by this limitation is

$$\frac{d\phi}{dx_1}dx_1 + \frac{d\phi}{dx_2}dx_2 + \text{\&c.} = 0.$$

Hence, comparing coefficients,

$$\frac{dz/dx_1}{d\phi/dx_1} = \frac{dz/dx_2}{d\phi/dx_2} = \text{\&c.};$$

but, as dz/dx_1 vanishes, so do all the other fluxions. It follows that, if the boundary conditions are $\Delta y_1 = 0$, $d/dx_r \,.\, \Delta y_1 = 0$, $d/dx_s \,.\, d/dx_t \,.\, \Delta y_1 = 0$, and so on up to $d^{\alpha_1+\alpha_2\cdots}/dx_1^{\alpha_1}\, dx_2^{\alpha_2}\cdots \,.\, \Delta y_1$, all the fluxions whose order does not exceed $\alpha_1 + \alpha_2 \ldots + \alpha_m$ must vanish at the boundary. For, as Δy_1 and $d\Delta y_1/dx_r = 0$, so do all other first fluxions, and therefore $d\Delta y_1/dx_s = 0$: but this, with $d/dx_t \,.\, d\Delta y_1/dx_s$, shows that all second fluxions with dx_s in them vanish. Hence any fluxion $d^2\Delta y_1/dx_a\, dx_b = 0$ for $d\Delta y_1/dx_a = 0$ and $d^2\Delta y_1/dx_s\, dx_a = 0$, and hence all second fluxions of $d\Delta y_1/dx_a$ vanish, and therefore, &c.

It follows from the preceding that in the examples in question all fluxions up to and including those of the 7th order must vanish for points on the boundary, and if we assume for Δy_1 an algebraic form we must write

$$\Delta y_1 = (\phi\,(x_1, x_2, \ldots x_m)\,)^8 f(xy),$$

where $\phi = 0$ is the equation of the bounding surface. Suppose we adapt this to the last example, the origin being taken as the point P in Proposition 1, and the bounding surface as

$$x_1^2 + x_2^2 - r^2 = 0,$$

so that the integration extends over all values of x_1 and x_2 which make the left-hand negative. Hence r is to be a quantity of the order β, and, to adapt the expression to the preceding formula, we must write

$$\Delta y_1 = \left[\left(\frac{x_1}{r}\right)^2 + \left(\frac{x_2}{r}\right)^2 - 1\right]^8 \left[\beta^8 f\left(\frac{x_1}{r}, \frac{x_2}{r}\right) + \beta^9 \cos\left(\frac{x_2}{\beta^2}\right) + \beta^{15} \cos\left(\frac{x_2}{\beta^5}\right)\right],$$

where, however, $\cos(x_2/\beta^2)$ and $\cos(x_2/\beta^5)$ are to be considered as abbreviations for any fluctuating functions of periods β^2 and β^5 respectively.

Similar assumptions can easily be made when other fluxions appear. The convenience in choosing $p_1 = 1$ is now evident, though, as far as the equations were concerned, it made no difference, as only the ratios of p, p_1, and p_2 entered into them.

The limits within which the property holds are evidently given by the discussion of Art. 21.

23. If all the highest of all the fluxions of any of the dependent variables appear in U in the first degree only, the foregoing reasoning would not hold, as *all* the variations of that dependent variable appearing in δ^2U would then vanish at the limits, and therefore the variation δ^2U when taken over a small region would be zero compared to quantities of the order α^2. (When there is only one independent variable there *must* be some *one* highest fluxion, but in general there is a group of fluxions higher than any others, not usually identical with what have been called the "highest fluxions," Art. 20, but included in them; it is only when each member of this group vanishes that the exception occurs.) It is known, however, that in this case it is, in general, impossible to fulfil the limiting conditions by means of the arbitrary

functions arising from the solution of the partial differential equations. This exception is well known in the simpler cases, but I am not aware that it has been generally discussed.

A partial differential equation between independent variables $x_1, \dots x_m$ and dependent variables $y_1, \dots y_n$ will, in general, require for the complete determination of $y_1, \dots y_n$ several sets of limiting conditions; for instance, one set when x_m has its limiting values $x_m = f_0(x_1, \dots x_{m-1})$ and $f_1(x_1, \dots x_{m-1})$, and another set when $x_{m-1} = f_0\ (x_1, \dots x_{m-2})$ and $f_1(x_1, \dots x_{m-2})$, and so on until, finally, there is a set derived from $x_1 = c_0$ and $x_1 = c_1$. But in the particular case in which the conditions at the limits are the values of $y_1, \dots y_n, dy_1/dx_m, \dots dy_n/dx_m, d^2y_1/dx^2_m$, &c., for a single surface $f(x_1, x_2, \dots x_m) = 0$ the functions $y_1, \dots y_n$ will be completely determined without any further limiting conditions (provided the proper number of conditions be given), and when the limiting conditions are of this character there is no difficulty in finding the requisite number of conditions relative to each variable. The limiting equations furnished by the Calculus of Variations are not, however, so simple as this, being of the dual character above. But, as this does not affect the *number* of the conditions at the limiting values of x_m, it will serve our purpose to find the number of conditions necessary in the simpler case. Let there be n equations, represented by (1), (2), . . . (n), between the n dependent variables $y_1, y_2, \dots y_n$, and let the highest order of differentiation with respect to any independent variable in which y_r appears in (s) be $[r, s]$. Then, provided each dependent variable appears in each equation, it can easily be shown that the number of conditions necessary to determine the dependent variables is the greatest of the sets of numbers $\Sigma\,[r, s]$, so chosen that in each set there is one term corresponding to each variable and one to each equation. For from § 22 it is evident that the number of functions required is *the same as if there was but one independent variable*, for the values of the single set $y, dy/dx, d^2y/dx^2$, &c., at the boundary determine those of all the other fluxions of y. We may, therefore, discuss the question on the supposition that there is but one independent variable. Now it is evident that, if we could determine all the successive differential coefficients of each function for each point of the bounding surface, we could, by TAYLOR'S theorem, expand the function in a series; and we know the limiting values of the differential coefficients of the y functions with regard to all variables x_1, x_2, &c., when we know those for any variable (Art. 22). Hence the problem will be solved if we show how many of the limiting values we must assume in order to determine all the rest. But we can show that it is possible to find the first Υ_1 quantities of the series $y_1, dy_1/dx_1, \dots d^{\Upsilon_1-1}y_1/dx_1^{\Upsilon_1-1}$; the first Υ_2 quantities $y_2, dy_2/dx_1, \dots d^{\Upsilon_2-1}y_2/dx_1^{\Upsilon_2-1}$; and so on for the rest, provided we assume $\Sigma\,[r, s]$ of these functions; Υ_1 and Υ_2, &c., being, if necessary, indefinitely large. For differentiate the equation (1) a_1 times, the equation (2) a_2 times, and so on. Since we are not to introduce any differential coefficients of orders higher than $\Upsilon_1 - 1$, $\Upsilon_2 - 1$, these being the orders of the highest fluxions in

Y_1, Y_2, &c., we must have, considering only fluxions of y_1, $a_1 + [1, 1] \overline{\overline{<}} Y_1 - 1$, $a_2 + [1, 2] \overline{\overline{<}} Y_1 - 1$, and so on, one inequality from each equation. Again, from considering the order of differentiation of y_2, $a_1 + [2, 1] \overline{\overline{<}} Y_2 - 1$, $a_2 + [2, 2] \overline{\overline{<}} Y_2 - 1$, and so on. Since every differentiation gives us a new equation, among the quantities in question we get altogether $n + a_1 + a_2 +$ &c. $+ a_n$ equations; hence the difference between this and $Y_1 + Y_2 +$ &c. must be equal to the number of quantities to be assumed in order to solve these equations. It is easy to see that the most favourable way in which the differentiations a_1, a_2, &c., can be disposed consistently with the inequalities to be satisfied will give the number stated above as the least number of this difference. Now suppose the equations obtained in the Calculus of Variations from making the coefficients of δy_1, δy_2, . . . δy_n vanish are denoted by (1), (2), . . . (n), and let the [1] denote the highest differential coefficient of y_1 occurring in the second or in a higher degree in the function to be made synclastic, [2] that of y_2, and so on; then it is easy to see that y_r cannot enter into the equation (p) by fluxions of order higher than $[r] + [p]$, and that y_p will enter into it in the order $2p$. Hence in this case $[p] + [r] = [pr]$, $[r] + [p] = [rp]$, and we have for this case to find the set for which $\Sigma [p, r]$ is greatest. Now, since we are to take one index for each equation, in our $\Sigma \{(p) + (r)\}$ we are to take only one term from equation (r), and hence (r) on the right-hand side is to appear only once. Moreover, we are to take only one term from each variable, and therefore we are simply to take $\Sigma \{(p) + \Sigma (r)\}$, where each refers to the values (1), (2), (3), &c.; and hence, in all, double the sum of the order of the highest of all the fluxions of each variable in the expression to be made synclastic. But this is exactly the number of the conditions supplied by equating the limiting terms of δU to zero in the expression to be integrated, except in the case we are at present discussing (where some of the highest of all the fluxions do not appear in the second degree, but in the first only); and therefore in this case the limiting conditions cannot be satisfied, and the problem becomes, in general, incapable of solution.