V. *On Ellipsoidal Current-Sheets.*

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It is a problem of some interest in Electromagnetism to determine the natural modes of decay, and the corresponding persistencies, of free currents in a given conductor. When this has been solved it is an easy matter to find the currents induced by given varying electromotive forces.

The general theory for a system of *linear* circuits is of course well known. If the variables $\dot{y}_1, \dot{y}_2, \ldots \dot{y}_n$, which specify the currents, be so chosen that the electrokinetic energy T and the dissipation-function F are both expressed by sums of squares, say

$$2T = L_1\dot{y}_1^2 + L_2\dot{y}_2^2 + \ldots + L_n\dot{y}_n^2,$$

$$2F = R_1\dot{y}_1^2 + R_2\dot{y}_2^2 + \ldots + R_n\dot{y}_n^2,$$

then $y_1, y_2, \ldots y_n$ are for the present purpose the "normal coordinates" of the system; and the equations of motion of electricity are of the form

$$L\ddot{y} + R\dot{y} = E,$$

where E is the external electromotive force of the type in question. In the case of free currents, $E = 0$, and consequently

$$\dot{y} = Ae^{-\lambda t},$$

where

$$\lambda = R/L.$$

If we put

$$\tau = \lambda^{-1} = L/R,$$

then τ may be called the "modulus of decay," or the "persistency," of free currents of this type.

In considering the effect of varying electromotive forces, it is convenient to suppose these expressed, as regards the time, in a series of simple harmonic terms, each of which may be taken separately. Assuming, then, that $E \propto e^{ipt}$, we have, for the induced current,

$$\dot{y} = \frac{\mathrm{E}}{\mathrm{R} + ip\mathrm{L}} = \frac{\mathrm{E}}{\mathrm{R}(1 + ip\tau)} = \frac{\mathrm{E}\cos\theta}{\mathrm{R}} \cdot e^{-i\theta},$$

if

$$\tan\theta = p\tau.$$

Hence the phase of the currents lags behind that of the inducing electromotive force by an amount arc tan $p\tau$. This remark, obvious as it is, is of some importance in relation to the practically interesting question of the rotation of a conductor about an axis of symmetry in a constant magnetic field. The magnetic potential of any normal type will be proportional to cos $s\omega$ or sin $s\omega$, where ω is the azimuth about the axis of symmetry, and s is integral (or zero). If now, as in MAXWELL's 'Electricity,' § 600, we employ coordinate axes moving with the conductor, the electromotive forces relative to these will vary as e^{ispt}, where p is the angular velocity of rotation. On account of the symmetry about the axis, the retardation of phase above spoken of comes to this, that the system of currents of any normal type is, owing to its inertia, displaced relatively to the field through an angle $1/s$ arc tan $sp\tau$, where τ is the modulus of decay proper to the type. (See §§ 7, 16, below.)

For other than linear conductors the problem above stated was first solved by MAXWELL in the case of an infinite plane sheet of uniform conductivity. The cases of solid spherical and cylindric conductors, and of thin spherical and cylindric shells, have been treated by Prof. C. NIVEN,* Lord RAYLEIGH,† and the writer.‡ It is remarkable that, with a certain exception,§ no difference of electric potential, and consequently no surface distribution of electricity, is called into existence during the decay of free currents in conductors of the forms mentioned.

In § 675 of his 'Electricity and Magnetism,' MAXWELL has indicated a certain arrangement of currents over the surface of an ellipsoid, which produces a uniform magnetic field in the interior. I do not know that it has yet been noticed that this arrangement fulfils the conditions for a natural mode of decay of free currents in a thin ellipsoidal film whose conductivity (per unit area) varies as the perpendicular from the centre on the tangent plane; or, say, in a thin shell of uniform material bounded by similar and coaxial ellipsoids. This is proved in Part I. of the following paper; and we thence easily find the currents induced in such a shell when situate in a uniform magnetic field of varying intensity; or, again, the currents induced by rotation of the shell in a uniform and constant field.

I have attempted to generalise these results and to ascertain the remaining normal types of currents in a shell of the kind indicated. In Part II. is given the complete solution of this problem, including the determination of the corresponding persistencies,

* 'Phil. Trans.', 1882.

† 'Brit. Assoc. Rep.', 1882.

‡ 'Phil. Trans.', 1883; 'London Math. Soc. Proc.', vol. 15, pp. 139 and 270.

§ That of the currents of the "Second Type" in a spherical conductor. Such currents cannot, however, be excited by any electromagnetic operations outside the sphere.

for the case where two of the axes of the ellipsoid are equal, when the LAMÉ's functions which naturally present themselves in such an investigation reduce to spherical harmonics, and so can be handled with comparative facility. The solution of the problem of *induced* currents can then be obtained in a very simple manner.

Of the special forms which the conducting shell may assume, the most interesting is that in which the third axis (that of symmetry) is infinitesimal, so that we have practically a circular disk whose resistance varies as $\sqrt{(a^2 - r^2)}$, where r is the distance of any point from the centre, and a the radius. In view of the physical interest attaching to the question, it would be desirable to have a solution for the case of a *uniform* circular disk rotating in any magnetic field; but, in the absence of this, the solution for the more special kind of disk here considered may not be uninstructive.

It appears that, except in the case of currents symmetrical about the axis, when the ellipsoid is one of revolution, there is always a surface distribution of electricity in the problems considered in this paper.

I.

1. If u', v', w', be the components of electric current at any point of a thin conducting film; F, G, H, those of electric momentum at any point (x, y, z) of space; the following conditions must be satisfied. At all points external to the film we must have

$$\nabla^2 \mathrm{F} = 0, \qquad \nabla^2 \mathrm{G} = 0, \qquad \nabla^2 \mathrm{H} = 0, \quad \ldots \ldots \quad (1)$$

where $\nabla^2 = d^2/dx^2 + d^2/dy^2 + d^2/dz^2$. The functions F, G, H, are everywhere continuous, but their derivatives are discontinuous at the film; viz., we have

$$\frac{d\mathrm{F}}{d\nu_0} + \frac{d\mathrm{F}}{d\nu_1} = -4\pi u', \qquad \frac{d\mathrm{G}}{d\nu_0} + \frac{d\mathrm{G}}{d\nu_1} = -4\pi v', \qquad \frac{d\mathrm{H}}{d\nu_0} + \frac{d\mathrm{H}}{d\nu_1} = -4\pi w', \quad . \quad (2)$$

where $d\nu_0$, $d\nu_1$, are elements of the normal drawn from the film on the two sides. If u', v', w', satisfy the solenoidal condition over the film, these conditions ensure that

$$\frac{d\mathrm{F}}{dx} + \frac{d\mathrm{G}}{dy} + \frac{d\mathrm{H}}{dz} = 0 \quad \ldots \ldots \ldots \ldots \quad (3)$$

everywhere. The electric potential ψ satisfies the equation

$$\nabla^2 \psi = 0$$

at all points external to the film; it is everywhere continuous, but its normal derivatives may be discontinuous at either or both of the surfaces of the film.

If ρ' be the resistance of the film, per unit area, the equations of electromotive force are

$$\rho' u' = -\frac{dF}{dt} - \frac{d\psi}{dx}, \qquad \rho' v' = -\frac{dG}{dt} - \frac{d\psi}{dy}, \qquad \rho' w' = -\frac{dH}{dt} - \frac{d\psi}{dz}. \quad . \quad (4)$$

In these equations ψ is supposed to have the value appropriate to the space included between the two surfaces of the film, which may differ in form from the values which it has in the external space on either side.

In any natural mode of decay the time occurs through a factor of the form $e^{-\lambda t}$, where λ is real and positive. The preceding equations then become

$$\rho' u' = \lambda F - \frac{d\psi}{dx}, \qquad \rho' v' = \lambda G - \frac{d\psi}{dy}, \qquad \rho' w' = \lambda H - \frac{d\psi}{dz}. \quad . \quad . \quad . \quad (5)$$

2. Let us apply this to the case of an ellipsoidal shell whose thickness varies as the perpendicular ϖ from the centre on the tangent plane; say it equals $\epsilon\varpi$, where ϵ is a small numerical constant. If ρ be the specific resistance of the material, we then have

$$\rho' = \rho/\epsilon\varpi.$$

Let the semi-axes of the shell be a, b, c, and let the axes of coordinates be taken along these. In the most important type of free currents the lines of flow are in planes perpendicular to a principal axis. If this axis be that of z, the current-function over the surface of the ellipsoid is of the form

$$\phi = Cz.$$

The corresponding values of u', v', w', are

$$u' = -\frac{\varpi y}{b^2} C, \qquad v' = \frac{\varpi x}{a^2} C, \qquad w' = 0.$$

The values of F, G, H, in the internal space are

$$\left.\begin{aligned} F &= -2\pi abc\, C \,.\, y \int_0^\infty \frac{dq}{(b^2+q)\,Q} \\ G &= 2\pi abc\, C \,.\, x \int_0^\infty \frac{dq}{(a^2+q)\,Q} \\ H &= 0 \end{aligned}\right\}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

where

$$Q = \{(a^2+q)(b^2+q)(c^2+q)\}^{\frac{1}{2}}.$$

The corresponding values for the external space are obtained by replacing the lower limit of the integrals by the positive root of

$$\frac{x^2}{a^2+q} + \frac{y^2}{b^2+q} + \frac{z^2}{c^2+q} = 1. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

These values of F, G, H, satisfy (1); they are continuous at the film, and their normal derivatives satisfy (2).*

We now find that the equations (5) are satisfied, provided we assume for the value of the electric potential within the substance of the film

$$\psi = Axy, \quad \ldots \ldots \ldots \quad (8)$$

and properly determine A. If we write, for shortness,

$$L = 2\pi\, abc \int_0^\infty \frac{dq}{(a^2+q)\,Q}, \qquad M = 2\pi\, abc \int_0^\infty \frac{dq}{(b^2+q)\,Q}, \qquad N = 2\pi\, abc \int_0^\infty \frac{dq}{(c^2+q)\,Q},$$

the equations in question reduce to

$$-\rho C/\epsilon b^2 = -\lambda MC - A,$$

$$\rho C/\epsilon a^2 = \lambda LC - A,$$

whence for the "persistency" we have

$$\tau = \lambda^{-1} = \frac{\epsilon}{\rho}\,\frac{L+M}{1/a^2 + 1/b^2} = \frac{\epsilon}{\rho}\,(4\pi - N)\frac{a^2b^2}{a^2+b^2}. \quad \ldots \ldots \quad (9)$$

Also

$$A = \frac{\rho}{\epsilon}\cdot\frac{La^2 - Mb^2}{(L+M)\,a^2b^2}\,C. \quad \ldots \ldots \ldots \quad (10)$$

The value (8) of ψ will obtain throughout the internal cavity of the ellipsoid, but in the external space we shall have

$$\psi = A_1 xy \int_q^\infty \frac{dq}{(a^2+q)(b^2+q)\,Q},$$

the lower limit being defined by (7). The continuity of ψ at the outer surface of the film requires

$$A_1 = A \div \int_0^\infty \frac{dq}{(a^2+q)(b^2+q)\,Q} = 2\pi\, abc\,.\,\frac{a^2-b^2}{M-L}\cdot A.$$

Unless $a = b$, there will be a distribution of electricity over the *outer* surface, the density σ being determined by

$$-\frac{4\pi\sigma}{K} = \frac{d\psi}{d\nu} + \frac{d\psi}{d\nu_1}, \quad \ldots \ldots \ldots \quad (11)$$

where K is the specific inductive capacity of the surrounding medium, and $d\nu$, $d\nu_1$,

* See, for instance, FERRERS' 'Spherical Harmonics,' chap. vi.

denote elements of the normal, drawn from the surface on the two sides. Since $dq/d\nu_1 = 2\varpi$, we find without difficulty

$$\frac{\sigma}{\mathrm{K}} = \frac{\rho}{\epsilon}\left(\frac{1}{a^2} - \frac{1}{b^2}\right)\frac{\mathrm{L}a^2 - \mathrm{M}b^2}{\mathrm{L}^2 - \mathrm{M}^2}\cdot\frac{\varpi xy}{a^2b^2}\cdot\mathrm{C}. \qquad (12)$$

3. Some particular cases of the formula (9) may be noticed. For a spherical shell we have $\mathrm{L} = \frac{4}{3}\pi$, and thence

$$\tau = \frac{4\pi}{3}\cdot\frac{\epsilon a}{\rho}\cdot a,$$

which is right. For an ellipsoid of revolution $(a = b)$

$$\tau = \frac{4\pi - \mathrm{N}}{2}\,\frac{\epsilon a}{\rho}\cdot a \qquad (13)$$

when the currents are symmetrical round the axis; whilst, in the case of currents in planes parallel to the axis (say $\phi = \mathrm{C}x$),

$$\tau = (4\pi - \mathrm{L})\frac{\epsilon a}{\rho}\cdot\frac{c^2 a}{c^2 + a^2}. \qquad (14)$$

For the prolate form we have

$$\left.\begin{aligned} \mathrm{L} = \mathrm{M} &= 2\pi\left(\frac{1}{e^2} - \frac{1-e^2}{2e^3}\log\frac{1+e}{1-e}\right) \\ \mathrm{N} &= 4\pi\left(\frac{1}{e^2} - 1\right)\left(\frac{1}{2e}\log\frac{1+e}{1-e} - 1\right) \end{aligned}\right\}, \qquad (15)$$

and for the oblate form

$$\left.\begin{aligned} \mathrm{L} = \mathrm{M} &= 2\pi\left(\frac{\sqrt{(1-e^2)}}{e^3}\,\text{arc}\sin e - \frac{1-e^2}{e^2}\right) \\ \mathrm{N} &= 4\pi\left(\frac{1}{e^2} - \frac{\sqrt{(1-e^2)}}{e^3}\,\text{arc}\sin e\right) \end{aligned}\right\}, \qquad (16)^*$$

e denoting in each case the excentricity of the meridian section.

Again, if we make $c = \infty$, we get an elliptic "homœoidal" cylinder. We then have

$$\mathrm{L} = \frac{4\pi b}{a+b}, \qquad \mathrm{M} = \frac{4\pi a}{a+b}, \qquad \mathrm{N} = 0; \qquad (17)$$

* Maxwell's 'Electricity,' § 438.

so that for the case of currents circulating round the cylinder

$$\tau = \frac{4\pi\epsilon}{\rho} \cdot \frac{a^2b^2}{a^2 + b^2} \cdot \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

The surface density of free electricity is then given by

$$\frac{4\pi\sigma}{\text{K}} = \frac{\rho}{\epsilon} \cdot \frac{a^2 - b^2}{a^3b^3} \varpi xy \,.\, \text{C}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

For a circular cylinder (18) gives

$$\tau = 2\pi \cdot \frac{\epsilon a}{\rho} \cdot a,$$

which is right.

For currents parallel to the axis of the cylinder (say $\phi = \text{C}x$),

$$\tau = \frac{4\pi\epsilon}{\rho} \cdot \frac{ab^2}{a + b} \cdot \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

If in (18) or (20) we make a infinite, we get the case of two uniform parallel plane sheets at a distance $2b$ apart. The persistency of uniform parallel straight currents flowing in opposite directions in the two planes is then

$$\tau = \frac{4\pi b}{\rho'} \cdot \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

4. Such special results as these may, of course, be obtained more easily by independent processes. Thus for a cylindrical shell of any form, if a current of strength C circulate round each unit length, the magnetic induction in the interior is parallel to the axis and equal to $4\pi\text{C}$. Hence, if R be the resistance of unit length to currents circulating round it,

$$\text{RC} = -\frac{d}{dt}(4\pi\text{CS}),$$

where S is the area of the cross-section. This gives

$$\tau = 4\pi\text{S}/\text{R}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (22)$$

For a "homœoidal" cylinder we have, if ds be an element of the elliptic contour, and ξ the "excentric angle,"

$$\rho'\, ds = \rho\, ds/\epsilon\varpi = \rho\, ab\, d\xi/\epsilon\varpi^2,$$

whence

$$R = \int \rho' ds = \frac{\pi\rho}{\epsilon} \cdot \frac{a^2 + b^2}{ab},$$

giving, of course, the same value (18) of τ as before.

If we wish to determine, not merely the persistency, but also the distribution of free electricity, we may proceed somewhat as follows. Taking the case of an elliptic cylinder, and resolving parallel to the principal axes, we have, as before,

$$u' = -\frac{\varpi y}{b^2} C, \quad v' = \frac{\varpi x}{a^2} C, \quad w' = 0,$$

and, at the surface,

$$F = -My \,.\, C, \quad G = Lx \,.\, C, \quad H = 0,$$

where L, M, have the values (17). Resolving in the direction of the current,

$$\begin{aligned} \rho' C &= \lambda \left(F \frac{dx}{ds} + G \frac{dy}{ds} \right) - \frac{d\psi}{ds} \\ &= \frac{4\pi\, ab}{a+b} \lambda\varpi \left(\frac{x^2}{a^3} + \frac{y^2}{b^3} \right) C - \frac{d\psi}{ds}. \end{aligned}$$

The resistance ρ' is here supposed independent of z, but is otherwise unrestricted. Introducing the excentric angle ξ, we have, since $\varpi\, ds = ab\, d\xi$,

$$\frac{d\psi}{d\xi} = -\rho' \frac{ds}{d\xi} C + \frac{4\pi ab}{a+b} \lambda \left(a \sin^2 \xi + b \cos^2 \xi \right) C. \quad . \quad . \quad . \quad . \quad . \quad (23)$$

Integrating from $\xi = 0$ to $\xi = 2\pi$,

$$4\pi^2 ab\lambda = \int \rho' ds,$$

which agrees with (22). The value of ψ over the film is found by integration of (23), ρ' being supposed a known function of ξ. We can then find two functions which satisfy $\nabla^2 \psi = 0$ and are finite, &c., throughout the interior and exterior spaces respectively, and coincide at the film with value just indicated. Again, within the substance of the film itself, the electromotive force in the direction of the normal must be zero. This gives

$$0 = \frac{\varpi x}{a^2} \lambda F + \frac{\varpi y}{b^2} \lambda G - \frac{d\psi}{d\delta},$$

or

$$\frac{d\psi}{d\delta} = 4\pi \frac{a-b}{a+b} \lambda C \frac{\varpi xy}{ab}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (24)$$

where $d\delta$ is an element of the thickness of the film, directed towards the outer surface. If σ_0, σ_1, be the densities of free electricity at the inner and outer surfaces respectively, we then have

$$\left.\begin{aligned} -\frac{4\pi\sigma_0}{\mathrm{K}} &= \frac{d\psi}{d\nu_0} + \frac{d\psi}{d\delta} \\ -\frac{4\pi\sigma_1}{\mathrm{K}} &= \frac{d\psi}{d\nu_1} - \frac{d\psi}{d\delta} \end{aligned}\right\}, \qquad (25)$$

$d\nu_0$, $d\nu_1$, denoting, as before, elements of the normal, drawn from the film, on the inside and outside respectively.

For the case of a homœoidal cylinder this process leads to the result already obtained. For other laws of thickness there will, in general, be a distribution of electricity on *both* surfaces of the film.*

5. Another case of interest is obtained by supposing c infinitesimal, so that the conductor may be taken to be an elliptic *disk* whose resistance per unit area varies according to the law

$$\rho' = \rho'_0 \surd\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right).$$

In the case of a circular disk the formula (13) is replaced by

$$\tau = (4\pi - \mathrm{N}) \frac{a^2}{4\rho'\varpi},\dagger$$

where we may put

$$\frac{\mathrm{N}}{4\pi} = 1 - \frac{\pi}{2}\frac{c}{a},$$

$$\varpi = \frac{ac}{\surd(a^2 - r^2)},$$

$$\rho' = \rho'_0 \surd\left(1 - \frac{r^2}{a^2}\right).$$

Hence

$$\tau = \frac{\pi^2 a}{2\rho_0'}. \qquad (26)\ddagger$$

* I find, however, that in the case of a film bounded by *confocal* cylinders the *inner* surface alone becomes electrified.

† The symbol ρ' here refers to the *disk*. Since this is the limit of a double film, its resistance at any point is half that of the corresponding portion of the film on either side.

‡ For an elliptic disk

$$\mathrm{N}/4\pi = 1 - \frac{\mathrm{E}_1(e)}{\surd(1-e^2)}\frac{c}{a},$$

where a is the semi-major axis, e the excentricity, and E_1 the complete elliptic integral of the second kind. This gives

$$\tau = \frac{2\pi ab}{a^2+b^2}\mathrm{E}_1(e).\frac{a}{\rho_0'}.$$

The current at any point is proportional to

$$\frac{r}{\sqrt{(a^2 - r^2)}}.$$

It would be interesting for many reasons to have a solution for the case of a uniform disk, but at all events the above result shows that the time-constant of a disk of radius a and uniform resistance ρ' must be considerably less than $4 \cdot 93\ a/\rho'$. I find, by methods similar to those employed in Lord RAYLEIGH'S 'Sound' (§§ 89, 305, &c.), that the true value lies between $\pi a/\rho'$ and $2 \cdot 26\ a/\rho'$, the latter value being probably not far removed from the truth.* For a disk of copper ($\rho = 1600$ C.G.S.) whose radius is a decimetre and thickness a millimetre this lower limit gives ·0014 sec. For disks of different dimensions the result will vary as the radius and the thickness conjointly.

6. Let us next calculate the currents induced in a homœoidal shell when situate in a uniform magnetic field $(\bar{\alpha}, \bar{\beta}, \bar{\gamma})$ of varying intensity. It is sufficient to consider the case where the lines of force are parallel to a principal axis. Also the expression for the magnetic force may be supposed resolved, as regards the time, into a series of simple harmonic terms, each of which may be taken separately. Putting, then,

$$\bar{\alpha} = 0, \qquad \bar{\beta} = 0, \qquad \bar{\gamma} = \mathrm{I}\, e^{ipt},$$

and denoting by $\bar{\mathrm{F}}, \bar{\mathrm{G}}, \bar{\mathrm{H}}$, the components of vector potential due to the field, we may write

$$\bar{\mathrm{F}} = -\tfrac{1}{2}\mathrm{I}\, y\, e^{ipt}, \qquad \bar{\mathrm{G}} = \tfrac{1}{2}\mathrm{I}\, x\, e^{ipt}, \qquad \bar{\mathrm{H}} = 0.$$

The induced currents will be of the type

$$u' = -\frac{\varpi y}{b^2}\mathrm{C}\, e^{ipt}, \qquad v' = \frac{\varpi x}{a^2}\mathrm{C}\, e^{ipt}, \qquad w' = 0,$$

and the corresponding components of electric momentum at the film will be

$$\mathrm{F} = -\mathrm{MC}\, y\, e^{ipt}, \qquad \mathrm{G} = \mathrm{LC}\, x\, e^{ipt}, \qquad \mathrm{H} = 0.$$

Assuming

$$\psi = \mathrm{A}xy\, e^{ipt},$$

and substituting in the equations

$$\rho' u' = -\frac{d\mathrm{F}}{dt} - \frac{d\bar{\mathrm{F}}}{dt} - \frac{d\psi}{dx}, \text{ \&c., \&c.,}$$

we find

$$-\rho\mathrm{C}/\epsilon b^2 = \quad ip\,\mathrm{MC} + \tfrac{1}{2}\, ip\,\mathrm{I} - \mathrm{A},$$
$$\rho\mathrm{C}/\epsilon a^2 = -ip\,\mathrm{LC} - \tfrac{1}{2}\, ip\,\mathrm{I} - \mathrm{A},$$

whence

$$\left\{\frac{\rho}{\epsilon}\left(\frac{1}{a^2} + \frac{1}{b^2}\right) + (\mathrm{L} + \mathrm{M})\, ip\right\}\mathrm{C} = -ip\,\mathrm{I},$$

* [See 'Roy. Soc. Proc.,' vol. 42, 1887, p. 294.]

or, by (9),

$$C = \frac{-ip\tau}{1+ip\tau} \cdot \frac{I}{L+M}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (27)$$

The retardation of phase of the induced currents relatively to the electromotive forces of the field is arc tan $p\tau$, as usual. When p is very great in comparison with τ^{-1} this is equal to $\pi/2$. Since the magnetic force in the interior of the shell, due to the currents alone, is given by

$$\frac{dG}{dx} - \frac{dF}{dy} = (L + M)\, C\, e^{ipt},$$

we see that in this case the currents just neutralise, in the interior, the magnetic action of the field, in accordance with a well-known principle.

7. Take next the case where the shell rotates with constant angular velocity p about a principal axis (z) in a uniform and constant magnetic field. It is shown in MAXWELL'S 'Electricity,' § 600, that the problem is the same if we suppose the shell to be fixed, and the field to rotate in the opposite direction, provided we add to the electric potential the function

$$\psi' = p\,(yF - xG). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (28)$$

First let us suppose the lines of force to be perpendicular to the axis of rotation, so that we may write for the components of the field

$$\bar{\alpha} = I \cos pt, \qquad \bar{\beta} = -I \sin pt, \qquad \bar{\gamma} = 0\,;$$

whence

$$\bar{F} = 0, \qquad \bar{G} = 0, \qquad \bar{H} = I\,(x \sin pt + y \cos pt) = I\,(y - ix)\, e^{ipt}, \quad . \quad (29)$$

if, as usual, we retain in the end only the real parts. Hence the solution of our problem follows by superposition from the results of the preceding section. Omitting the time-factor e^{ipt}, we assume for the current-function

$$\phi = Cx + Dy,$$

which gives

$$\left.\begin{aligned} u' &= \frac{\varpi z}{c^2} D \\ v' &= -\frac{\varpi z}{c^2} C \\ w' &= \frac{\varpi y}{b^2} C - \frac{\varpi x}{a^2} D \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (30)$$

The corresponding values of F, G, H, at the film are

$$\left.\begin{aligned} \mathrm{F} &= \mathrm{DN}z \\ \mathrm{G} &= -\mathrm{CN}z \\ \mathrm{H} &= \mathrm{CM}y - \mathrm{DL}x \end{aligned}\right\} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (31)$$

If we further assume

$$\psi + \psi' = (\mathrm{A}x + \mathrm{B}y)z, \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (32)$$

and substitute in the equations

$$\rho' u' = -\frac{d\mathrm{F}}{dt} - \frac{d\overline{\mathrm{F}}}{dt} - \frac{d}{dx}(\psi + \psi'), \text{ \&c., \&c.};$$

then, equating coefficients, we obtain the following four equations to determine A, B, C, D :—

$$\left.\begin{aligned} \rho/\epsilon c^2 \,.\, \mathrm{D} &= -ip\,\mathrm{N}\,.\,\mathrm{D} - \mathrm{A} \\ -\rho/\epsilon c^2 \,.\, \mathrm{C} &= ip\,\mathrm{N}\,.\,\mathrm{C} - \mathrm{B} \\ -\rho/\epsilon a^2 \,.\, \mathrm{D} &= ip\,\mathrm{L}\,.\,\mathrm{D} - p\mathrm{I} - \mathrm{A} \\ \rho/\epsilon b^2 \,.\, \mathrm{C} &= -ip\,\mathrm{M}\,.\,\mathrm{C} - ip\mathrm{I} - \mathrm{B} \end{aligned}\right\} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (33)$$

Hence

$$\left\{\frac{\rho}{\epsilon}\left(\frac{1}{b^2} + \frac{1}{c^2}\right) + ip\,(\mathrm{M} + \mathrm{N})\right\}\mathrm{C} = -ip\,\mathrm{I},$$

$$\left\{\frac{\rho}{\epsilon}\left(\frac{1}{a^2} + \frac{1}{c^2}\right) + ip\,(\mathrm{L} + \mathrm{N})\right\}\mathrm{D} = p\mathrm{I}.$$

If τ_1, τ_2, denote the persistencies of free currents of the types $\phi = \mathrm{C}x$, $\phi = \mathrm{D}y$, respectively, these equations may be written

$$\left.\begin{aligned} \mathrm{C} &= \frac{-ip\tau_1}{1 + ip\,\tau_1} \cdot \frac{\mathrm{I}}{\mathrm{M} + \mathrm{N}} \\ \mathrm{D} &= \frac{p\tau_2}{1 + ip\,\tau_2} \cdot \frac{\mathrm{I}}{\mathrm{L} + \mathrm{N}} \end{aligned}\right\} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (34)$$

The components of magnetic force within the shell, due to the currents alone, are found to be

$$\alpha = (\mathrm{M} + \mathrm{N})\,\mathrm{C}, \qquad \beta = (\mathrm{L} + \mathrm{N})\,\mathrm{D}, \qquad \gamma = 0,$$

so that we have for the total magnetic field inside

$$\alpha + \overline{\alpha} = \frac{\mathrm{I}}{1 + ip\,\tau_1}, \qquad \beta + \overline{\beta} = \frac{i\mathrm{I}}{1 + ip\,\tau_2}, \qquad \gamma + \overline{\gamma} = 0. \quad . \; . \; . \quad (35)$$

These diminish indefinitely as p increases.

If we write

$$p\tau_1 = \tan\omega_1, \qquad p\tau_2 = \tan\omega_2,$$

and restore the time-factor, the expression for the current-function becomes, on discarding the imaginary part,

$$\phi = \frac{\mathrm{I}\sin\omega_1}{\mathrm{M}+\mathrm{N}} x \sin(pt - \omega_1) + \frac{\mathrm{I}\sin\omega_2}{\mathrm{L}+\mathrm{N}} y \cos(pt - \omega_2). \quad . \quad . \quad . \quad . \quad (36)$$

The currents flow at any instant in a system of ellipses whose planes are parallel to one another and to the axis of rotation. When the ellipsoid is one of revolution about z we have $\mathrm{L} = \mathrm{M}$, $\omega_1 = \omega_2$. The planes of the currents are dragged round, as it were, in the direction of the rotation of the shell, through a constant angle ω_1 from the direction of the magnetic force in the inducing field, in accordance with a general principle pointed out at the beginning of this paper.*

8. When the lines of force are parallel to the axis of rotation there are no induced currents, but only a superficial distribution of electricity. The calculation of this distribution involves assumptions which vary with the particular theory of electro-magnetism adopted; and even MAXWELL'S theory has been differently interpreted in this respect by different writers. It may be well, therefore, to state with some care the view here taken.

Considering, for the sake of simplicity, the case of a solid conductor rotating in a field of uniform intensity γ about an axis (z) parallel to the lines of force, and supposing the axes of x, y, to move with the solid, then, on the hypothesis that there are no currents, we have, throughout the interior,

$$\left.\begin{aligned} 0 &= \gamma \, . \, px - \frac{d\psi}{dx} \\ 0 &= \gamma \, . \, py - \frac{d\psi}{dy} \\ 0 &= \qquad\quad - \frac{d\psi}{dz} \end{aligned}\right\}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (37)$$

whilst in the surrounding dielectric (taken to be sensibly at rest)—

* The case of a *spherical* shell has been discussed by C. NIVEN (*loc. cit.*) and J. LARMOR, 'Phil. Mag.,' Jan., 1884.

MAXWELL has considered the currents induced by rotation of a *solid* ellipsoid (see STEWART and TAIT, 'Roy. Soc. Proc.,' vol. 15, 1867, p. 291), leaving out of account, however, the mutual action of the currents themselves. This is equivalent to supposing the period of rotation to be long in comparison with the modulus of decay of free currents.

$$\left.\begin{aligned}\frac{4\pi f}{\mathrm{K}} &= -\frac{d\psi}{dx}\\ \frac{4\pi g}{\mathrm{K}} &= -\frac{d\psi}{dy}\\ \frac{4\pi h}{\mathrm{K}} &= -\frac{d\psi}{dz}\end{aligned}\right\}, \quad \ldots \ldots \quad (38)$$

where f, g, h, are the components of dielectric polarisation, and K the specific inductive capacity. We have also the solenoidal condition

$$\frac{df}{dx} + \frac{dg}{dy} + \frac{dh}{dz} = 0. \quad \ldots \ldots \quad (39)$$

It is to be carefully borne in mind that nothing is known of the function ψ beyond what is contained in these equations, except that it is everywhere continuous. The familiar *electrostatic* relations of ψ may be deduced from these equations by putting $p = 0$. In the present problem we have

$$\psi = \tfrac{1}{2} p \gamma (x^2 + y^2) + \text{const.} \quad \ldots \ldots \quad (40)$$

throughout the conductor, whilst in the external space ψ satisfies the equation

$$\nabla^2 \psi = 0,$$

with the conditions that its value at the surface shall agree with (40) and its first derivatives vanish at infinity. The surface density σ is then given by

$$\sigma = lf + mg + nh = -\frac{\mathrm{K}}{4\pi} \frac{d\psi}{d\nu_1}, \quad \ldots \ldots \quad (41)$$

where l, m, n, are the direction-cosines of the element $d\nu_1$ of the normal drawn outwards.

The solution of this problem for an ellipsoidal conductor is obtained by an adaptation of the results given by FERRERS ('Spherical Harmonics,' chapter 6, §§ 29, 30). I do not think it worth while to transcribe these, as the result for the more specially interesting case of an ellipsoid of revolution, and in particular for a circular *disk*, is given below in § 15.

II.

9. The result of § 2 can be generalised, and it can be shown that the different normal types of free currents in a homœoidal shell are obtained by equating the current-function ϕ to the LAMÉ's functions of various orders. But it may be sufficient here to consider the case where two of the axes of the shell are equal, when the functions in question reduce to spherical harmonics.

Taking first the case where the ellipsoid is of the *prolate* form, we transform to elliptic coordinates (ζ, μ, ω) or (η, θ, ω) by writing

$$\left.\begin{aligned} x &= k\sqrt{(1-\mu^2)}\sqrt{(\zeta^2-1)}\cos\omega &&= k\sin\theta\sinh\eta\cos\omega \\ y &= k\sqrt{(1-\mu^2)}\sqrt{(\zeta^2-1)}\sin\omega &&= k\sin\theta\sinh\eta\sin\omega \\ z &= k\zeta\mu &&= k\cos\theta\cosh\eta \end{aligned}\right\}, \quad . \quad . \quad . \quad . \quad (42)$$

the axis of z being that of symmetry. The value of μ may range from -1 to $+1$, that of ζ from 1 to ∞. The surfaces $\zeta = \text{const.}$ are confocal ellipsoids of revolution, whose semi-axes are

$$\begin{aligned} a = b &= k\sqrt{(\zeta^2-1)} = k\sinh\eta, \\ c &= k\zeta \qquad\qquad\;\; = k\cosh\eta, \end{aligned}$$

the distance between the common foci being $2k$. The value of ζ for the surface of the shell will be distinguished, where necessary, by ζ_0. The perpendicular on the tangent plane at any point of the shell is

$$\varpi = k\frac{\zeta_0\sqrt{(\zeta_0^2-1)}}{\sqrt{(\zeta_0^2-\mu^2)}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (43)$$

LAPLACE's equation $\nabla^2 V = 0$ transforms into

$$\frac{d}{d\mu}\left\{(1-\mu^2)\frac{dV}{d\mu}\right\} + \frac{1}{1-\mu^2}\frac{d^2V}{d\omega^2} = \frac{d}{d\zeta}\left\{(1-\zeta^2)\frac{dV}{d\zeta}\right\} + \frac{1}{1-\zeta^2}\frac{d^2V}{d\omega^2} \quad . \quad . \quad . \quad (44)$$

Considered as a function of μ, ω, V may be expanded in a series of spherical harmonics whose coefficients are functions of ζ, and it is easily seen that each term of the expansion must separately satisfy (44). Taking first the case of the zonal harmonic, if we put

$$V = P_n(\mu) . Z,$$

where

$$P_n(\mu) = \frac{1.3.5\ldots(2n-1)}{\lfloor n}\left\{\mu^n - \frac{n(n-1)}{2(2n-1)}\mu^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2.4(2n-1)(2n-3)}\mu^{n-4} - \ldots\right\}, \quad (45)$$

we find

$$\frac{d}{d\zeta}\left\{(1-\zeta^2)\frac{dZ}{d\zeta}\right\} + n(n+1)Z = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (46)$$

showing that Z must be a zonal harmonic of order n, of the first or second kind. We thus obtain the solutions

$$\left.\begin{aligned} V &= P_n(\mu) \,.\, P_n(\zeta)^{*} \\ V &= P_n(\mu) \,.\, Q_n(\zeta) \end{aligned}\right\}, \qquad (47)$$

where Q_n denotes the zonal harmonic of the second kind, viz. :—

$$\begin{aligned} Q_n(\zeta) &= P_n(\zeta)\int_\zeta^\infty \frac{d\zeta}{\{P_n(\zeta)\}^2(\zeta^2-1)}, \\ &= \tfrac{1}{2}P_n(\zeta)\log\frac{\zeta+1}{\zeta-1} - \frac{2n-1}{1\,.\,n}P_{n-1}(\zeta) - \frac{2n-5}{3(n-1)}P_{n-3}(\zeta) - \ldots, \\ &= \frac{\lfloor n}{1.3..(2n+1)}\left\{\zeta^{-n-1} + \frac{(n+1)(n+2)}{2(2n+3)}\zeta^{-n-3}\right. \\ &\qquad \left. + \frac{(n+1)(n+2)(n+3)(n+4)}{2\,.\,4(2n+3)(2n+5)}\zeta^{-n-5} + \ldots\right\} \end{aligned} \qquad (48)$$

The former of the solutions (47) is appropriate to the space inside the shell, the latter to the external space.

In like manner, when V involves ω, we have the solutions

$$\left.\begin{aligned} V &= (1-\mu^2)^{s/2}\frac{d^sP_n(\mu)}{d\mu^s}(\zeta^2-1)^{s/2}\frac{d^sP_n(\zeta)}{d\zeta^s}\left.\begin{matrix}\cos\\ \sin\end{matrix}\right\}s\omega \\ V &= (1-\mu^2)^{s/2}\frac{d^sP_n(\mu)}{d\mu^s}(\zeta^2-1)^{s/2}\frac{d^sQ_n(\zeta)}{d\zeta^s}\left.\begin{matrix}\cos\\ \sin\end{matrix}\right\}s\omega \end{aligned}\right\} \qquad (49)^{\dagger}$$

10. Proceeding now to the problem of free currents, we shall show that the conditions for a normal type are satisfied whenever ϕ, considered as a function of μ and ω,

* The following are the values of the first four solid harmonics of this type, expressed in terms of x, y, z :—

$$\begin{aligned} &kP_1(\mu)\,P_1(\zeta) = z, \\ &k^2P_2(\mu)\,P_2(\zeta) = \tfrac{1}{4}\{6z^2 - 3(x^2+y^2) - 2k^2\}, \\ &k^3P_3(\mu)\,P_3(\zeta) = \tfrac{1}{4}z\{10z^2 - 15(x^2+y^2) - 6k^2\}, \\ &k^4P_4(\mu)\,P_4(\zeta) = \tfrac{1}{64}\{280z^4 - 840z^2(x^2+y^2) + 105(x^2+y^2)^2 - 240k^2z^2 + 120k^2(x^2+y^2) + 24k^4\}. \end{aligned}$$

† I have here only recapitulated, for purposes of reference, the principal steps in the deduction of the solutions (47), (49). For details see HEINE, 'Kugelfunctionen,' vol. 2, part ii., chap. ii.; or FERRERS, 'Spherical Harmonics,' chap. vi.

The following are the values of a few of the more important solid harmonics of the form given by the first of equations (49) :—

$n=1,$	$s=1:$	$x,$	$y.$
$n=2,$	$s=1:$	$9xz,$	$9yz.$
$n=2,$	$s=2:$	$9(x^2-y^2),$	$18xy.$
$n=3,$	$s=1:$	$\frac{9}{4}x\{20z^2-5(x^2+y^2)-4k^2\},$	$\frac{9}{4}y\{20z^2-5(x^2+y^2)-4k^2\}.$
$n=3,$	$s=2:$	$225z(x^2-y^2),$	$450xyz.$
$n=3,$	$s=3:$	$225(x^3-3xy^2),$	$225(3x^2y-y^3).$
$n=4,$	$s=1:$	$\frac{25}{4}zx\{28z^2-21(x^2+y^2)-12k^2\},$	$\frac{25}{4}zy\{28z^2-21(x^2+y^2)-12k^2\}.$

is a zonal, tesseral, or sectorial harmonic of integral order. Since any arbitrary value of ϕ can be expanded in a series of such harmonics, the results thus obtained will enable us to represent the decay of any initial distribution of current whatever.

If ds_μ, ds_ω, be linear elements drawn on the surface along a meridian and a parallel of latitude respectively, viz.,

$$\left.\begin{aligned} ds_\mu &= k\frac{\sqrt{(\zeta_0^2-\mu^2)}}{\sqrt{(1-\mu^2)}}\,d\mu \\ ds_\omega &= k\sqrt{(\zeta_0^2-1)}\sqrt{(1-\mu^2)}\,d\omega \end{aligned}\right\}, \qquad (50)$$

the current may be resolved into the components

$-\dfrac{d\phi}{ds_\omega}$ along the meridian, towards the positive pole; and

$\dfrac{d\phi}{ds_\mu}$ along the parallel, in the direction of ω increasing.

Take, first, the case of the zonal harmonic

$$\phi = \mathrm{C}\,.\,\mathrm{P}_n(\mu). \qquad (51)$$

The currents then flow in circles round the axis of z, the strength of the current at any point being

$$\mathrm{C}\,\frac{\varpi}{k^2\zeta_0\sqrt{(\zeta_0^2-1)}}\sqrt{(1-\mu^2)}\,\frac{d\mathrm{P}_n(\mu)}{d\mu}. \qquad (52)$$

If Ω be the magnetic potential due to these currents, we have

$$\nabla^2\Omega = 0,$$

with the conditions that at the surface ζ_0

$$\Omega_1 = \Omega_0 + 4\pi\phi,$$

(the suffixes denoting the values on the two sides), whilst the normal derivative is continuous. Assuming

$$\left.\begin{aligned} \Omega_0 &= \mathrm{A}\,\mathrm{P}_n(\mu)\,\mathrm{P}_n(\zeta) \\ \Omega_1 &= \mathrm{B}\,\mathrm{P}_n(\mu)\,\mathrm{Q}_n(\zeta) \end{aligned}\right\}, \qquad (53)$$

the surface conditions give

$$\mathrm{B}\,\mathrm{Q}_n(\zeta_0) = \mathrm{A}\,\mathrm{P}_n(\zeta_0) + 4\pi\mathrm{C},$$
$$\mathrm{B}\,\mathrm{Q}_n'(\zeta_0) = \mathrm{A}\,\mathrm{P}_n'(\zeta_0);$$

whence, in virtue of the relation

$$\mathrm{P}_n'(\zeta)\,\mathrm{Q}_n(\zeta) - \mathrm{P}_n(\zeta)\,\mathrm{Q}_n'(\zeta) = \frac{1}{\zeta^2-1}, \qquad (54)$$

we find

$$\left.\begin{aligned} \mathrm{A} &= 4\pi\,(\zeta_0^2 - 1)\ \mathrm{Q}_n'(\zeta_0)\,.\,\mathrm{C} \\ \mathrm{B} &= 4\pi\,(\zeta_0^2 - 1)\ \mathrm{P}_n'(\zeta_0)\,.\,\mathrm{C} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (55)^*$$

Owing to the symmetry about the axis, there will be no difference of electric potential, and the electromotive force of induction will be everywhere in the direction of the current. The magnetic induction across any element of the surface will be

$$\begin{aligned} -\frac{d\Omega}{d\nu_1}\,ds_\mu\,ds_\omega &= -\frac{\varpi}{k^2\zeta_0}\frac{d\Omega}{d\zeta}\,ds_\mu\,ds_\omega \\ &= -k\,(\zeta_0^2 - 1)\frac{d\Omega}{d\zeta}\,d\mu\,d\omega. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (56) \end{aligned}$$

Substituting from (53), and integrating over the portion of the surface bounded by a parallel of latitude and including the positive pole, we find for the total induction through the parallel

$$-\,8\pi^2 k\,(\zeta_0^2 - 1)^2\,\mathrm{P}_n'(\zeta_0)\,\mathrm{Q}_n'(\zeta_0)\int_\mu^1 \mathrm{P}_n(\mu)\,d\mu\,.\,\mathrm{C}.$$

If the system of currents defined by (51) remain always similar to itself, the electromotive force round a parallel is equal to $-\,d/dt$ of this. The electromotive force at a point is derived by dividing by $2\pi k\sqrt{(\zeta_0^2 - 1)}\sqrt{(1 - \mu^2)}$. Since

$$\int_\mu^1 \mathrm{P}_n(\mu)\,d\mu = \frac{1-\mu^2}{n\,(n+1)}\,\frac{d\mathrm{P}_n(\mu)}{d\mu},$$

we obtain in this way

$$\frac{4\pi}{n\,(n+1)}\,(\zeta_0^2 - 1)^{\frac{3}{2}}\,\mathrm{P}_n'(\zeta_0)\,\mathrm{Q}_n'(\zeta_0)\sqrt{(1-\mu^2)}\,\frac{d\mathrm{P}_n(\mu)}{d\mu}\cdot\frac{d\mathrm{C}}{dt}. \quad . \quad . \quad . \quad (57)$$

Equating this to

$$\rho'\frac{d\phi}{ds_\mu},$$

where $d\phi/ds_\mu$ has the value (52), and $\rho' = \rho/\epsilon\varpi$, we find that all the conditions of the problem are satisfied, provided

$$\frac{d\mathrm{C}}{dt} + \frac{\mathrm{C}}{\tau} = 0,$$

where

$$\tau = -\,\frac{4\pi k^2}{n(n+1)}\,\frac{\epsilon}{\rho}\,\zeta_0\,(\zeta_0^2 - 1)^2\,\mathrm{P}_n'(\zeta_0)\,\mathrm{Q}_n'(\zeta_0). \quad . \quad . \quad . \quad . \quad . \quad (58)$$

* We may prove in a similar manner that in the case of an ellipsoidal shell with unequal axes magnetised everywhere in the direction of the normal so that the "strength" ϕ is proportional to a LAMÉ's function, the potential on either side is everywhere proportional to ϕ.

Since

$$P_1(\zeta) = \zeta,$$
$$Q_1(\zeta) = \frac{\zeta}{2}\log\frac{\zeta+1}{\zeta-1} - 1,$$

this gives, for $n = 1$,

$$\tau = \frac{2\pi k^2 \epsilon}{\rho}\zeta_0(\zeta_0^2 - 1)^2\left\{\frac{\zeta_0}{\zeta_0^2 - 1} - \tfrac{1}{2}\log\frac{\zeta_0 + 1}{\zeta_0 - 1}\right\};$$

or, putting

$$a = k\sqrt{(\zeta_0^2 - 1)}, \qquad \zeta_0 = 1/e,$$
$$\tau = \frac{2\pi a^2 \epsilon}{\rho}\left\{\frac{1}{e^2} - \frac{1 - e^2}{2e^3}\log\frac{1+e}{1-e}\right\}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (59)$$

which agrees with (13).

For $n = 2$, I find

$$\tau = \frac{2\pi a^2 \epsilon}{\rho}\left\{\frac{3 - 2e^2}{e^4} - \frac{3}{2}\,\frac{1 - e^2}{e^5}\log\frac{1+e}{1-e}\right\}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (60)$$

The case of a *spherical* shell may be deduced from (58) by making $k = 0$, $\zeta = \infty$, $k\zeta = a$. For large values of ζ we need only retain the first terms of the series in (45), (48). In this way we reproduce the known result

$$\tau = \frac{4\pi a^2 \epsilon}{(2n+1)\rho}.$$

11. The simplest plan of dealing with the case where the current-function is a tessaral or sectorial harmonic is to consider the current round any infinitely small circuit bounded by meridians and parallels. If R, S, be the components of electric momentum along a meridian and a parallel respectively, we have

$$\left.\begin{aligned} -\rho'\frac{d\phi}{ds_\omega} &= -\frac{dR}{dt} - \frac{d\psi}{ds_\mu} \\ \rho'\frac{d\phi}{ds_\mu} &= -\frac{dS}{dt} - \frac{d\psi}{ds_\omega} \end{aligned}\right\}; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (61)$$

or, putting

$$\rho' = \frac{\rho}{\epsilon\varpi} = \frac{\rho}{k\epsilon}\cdot\frac{\sqrt{(\zeta_0^2 - \mu^2)}}{\zeta_0\sqrt{(\zeta_0^2 - 1)}},$$

$$\left.\begin{aligned} -\frac{\rho}{k\epsilon}\cdot\frac{\zeta_0^2 - \mu^2}{\zeta_0(\zeta_0^2 - 1)(1 - \mu^2)}\frac{d\phi}{d\omega} &= -\frac{dR}{dt}\frac{ds_\mu}{d\mu} - \frac{d\psi}{d\mu} \\ \frac{\rho}{k\epsilon\zeta_0}\cdot(1 - \mu^2)\frac{d\phi}{d\mu} &= -\frac{dS}{dt}\frac{ds_\omega}{d\omega} - \frac{d\psi}{d\omega} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (62)$$

Eliminating ψ,

$$\frac{\rho}{k\epsilon\zeta_0}\left[\frac{d}{d\mu}\left\{(1 - \mu^2)\frac{d\phi}{d\mu}\right\} + \frac{\zeta_0^2 - \mu^2}{(\zeta_0^2 - 1)(1 - \mu^2)}\frac{d^2\phi}{d\omega^2}\right]$$
$$= \frac{d}{dt}\left\{\frac{d}{d\omega}\left(R\frac{ds_\mu}{d\mu}\right) - \frac{d}{d\mu}\left(S\frac{ds_\omega}{d\omega}\right)\right\}. \quad . \quad . \quad . \quad (63)$$

If we now assume

$$\phi = \mathrm{C}(1-\mu^2)^{s/2}\frac{d^s\mathrm{P}_n(\mu)}{d\mu^s}\sin s\omega, \quad \ldots \ldots \quad (64)$$

the first member of (63) reduces to

$$-\frac{\rho}{k\epsilon\zeta_0}\left\{n(n+1)+\frac{s^2}{\zeta_0^2-1}\right\}\phi, \quad \ldots \ldots \quad (65)$$

by the differential equation of tessaral harmonics. Again

$$\frac{d}{d\omega}\left(\mathrm{R}\frac{ds_\mu}{d\mu}\right)-\frac{d}{d\mu}\left(\mathrm{S}\frac{ds_\omega}{d\omega}\right) = -k(\zeta_0^2-1)\frac{d\Omega}{d\zeta}, \quad \ldots \ldots \quad (66)$$

where Ω is the magnetic potential due to the currents (64); for, if we multiply each member by $d\mu\, d\omega$, the left-hand side is equal to the line-integral of the electric momentum round an infinitesimal circuit, and the right-hand side gives the magnetic induction through the circuit. To find Ω, we assume

$$\left.\begin{aligned}\Omega_0 &= \mathrm{A}(1-\mu^2)^{s/2}\frac{d^s\mathrm{P}_n(\mu)}{d\mu^s}(\zeta^2-1)^{s/2}\frac{d^s\mathrm{P}_n(\zeta)}{d\zeta^s}\sin s\omega\\ \Omega_1 &= \mathrm{B}(1-\mu^2)^{s/2}\frac{d^s\mathrm{P}_n(\mu)}{d\mu^s}(\zeta^2-1)^{s/2}\frac{d^s\mathrm{Q}_n(\zeta)}{d\zeta^s}\sin s\omega\end{aligned}\right\} \quad \ldots \ldots \quad (67)$$

for the spaces inside and outside the shell respectively. If we write, for shortness,

$$\left.\begin{aligned}\mathrm{T}_n^s(\zeta) &= (\zeta^2-1)^{s/2}\frac{d^s\mathrm{P}_n(\zeta)}{d\zeta^s}\\ \mathrm{U}_n^s(\zeta) &= (\zeta^2-1)^{s/2}\frac{d^s\mathrm{Q}_n(\zeta)}{d\zeta^s}\end{aligned}\right\}, \quad \ldots \ldots \quad (68)$$

the conditions to be satisfied at the surface of the shell give

$$\begin{aligned}\mathrm{BU}_n^s(\zeta_0) &= \mathrm{AT}_n^s(\zeta_0)+4\pi\mathrm{C},\\ \mathrm{B}\frac{d\mathrm{U}_n^s(\zeta_0)}{d\zeta_0} &= \mathrm{A}\frac{d\mathrm{T}_n^s(\zeta_0)}{d\zeta_0};\end{aligned}$$

whence

$$\left.\begin{aligned}\mathrm{A} &= (-)^s 4\pi\frac{\lfloor n-s}{\lfloor n+s}(\zeta_0^2-1)\frac{d\mathrm{U}_n^s}{d\zeta_0}\mathrm{C}\\ \mathrm{B} &= (-)^s 4\pi\frac{\lfloor n-s}{\lfloor n+s}(\zeta_0^2-1)\frac{d\mathrm{T}_n^s}{d\zeta_0}\mathrm{C}\end{aligned}\right\}, \quad \ldots \ldots \quad (69)$$

the reduction being effected by the formula

$$U_n^s \frac{dT_n^s}{d\zeta} - T_n^s \frac{dU_n^s}{d\zeta} = (-)^s \frac{\lfloor n+s}{\lfloor n-s} \frac{1}{\zeta^2-1} \quad . \quad . \quad . \quad . \quad . \quad . \quad (70)^*$$

Hence the right-hand side of (63)

$$= -k(\zeta_0^2-1)\frac{d}{dt}\frac{d\Omega}{d\zeta},$$

$$= -k(\zeta_0^2-1)\frac{dA}{dt}\frac{dT_n^s}{d\zeta_0}(1-\mu^2)^{s/2}\frac{d^sP_n(\mu)}{d\mu^s}\sin s\omega,$$

$$= (-)^{s-1}4\pi k\frac{\lfloor n-s}{\lfloor n+s}(\zeta_0^2-1)^2\frac{dT_n^s}{d\zeta_0}\cdot\frac{dU_n^s}{d\zeta_0}\cdot\frac{d\phi}{dt} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (71)$$

by (64) and (69). Equating (65) and (71), we see that the assumption (64) satisfies the conditions of a normal type, and that the corresponding modulus of decay is

$$\tau = (-)^{s-1}\frac{4\pi k^2\epsilon}{\rho}\frac{\lfloor n-s}{\lfloor n+s}\frac{\zeta_0(\zeta_0^2-1)^3}{n(n+1)(\zeta_0^2-1)+s^2}\frac{dT_n^s(\zeta_0)}{d\zeta_0}\frac{dU_n^s(\zeta_0)}{d\zeta_0}. \quad . \quad . \quad . \quad (72)$$

The accuracy of this result may be tested by putting $k=0$, $\zeta_0=\infty$, $k\zeta_0=a$, when we obtain the correct result

$$\frac{4\pi a^2\epsilon}{(2n+1)\rho}$$

for the case of a spherical shell.

The case of the *sectorial* harmonic is obtained by putting $s=n$ in (72). When $n=1$, in which case $\phi \propto y$, we obtain

$$\tau = \frac{2\pi k^2\epsilon}{\rho}\frac{\zeta_0^2(\zeta_0^2-1)^2}{2\zeta^2-1}\left\{\frac{\zeta_0}{2}\log\frac{\zeta_0+1}{\zeta_0-1}-\frac{\zeta_0^2-2}{\zeta_0^2-1}\right\};$$

or, writing

$$\zeta_0 = 1/e, \qquad k^2(\zeta_0^2-1)=a^2,$$

$$\tau = \frac{2\pi a^2\epsilon}{\rho}\frac{1}{2-e^2}\left\{\frac{1-e^2}{2e^3}\log\frac{1+e}{1-e}-\frac{1-2e^2}{e^2}\right\},$$

which will be found to agree with (14).

The results (58) and (72) were originally obtained by a method more analogous to that of § 2, the currents and the electric momentum being resolved parallel to x, y, z. This method is much longer than that here given, and involves the determination of

* TODHUNTER, 'Functions of LAPLACE, &c.,' § 109.

the electric potential ψ. It may be worth while, however, to record the value of ψ thus obtained for the space included between the two surfaces of the film, viz.:—

$$\psi = \mathrm{D}\,(\mathfrak{T}_{n+1}^{s} - \mathfrak{T}_{n-1}^{s}) \cos s\omega + \mathrm{E}\,(\mathfrak{U}_{n+1}^{s} - \mathfrak{U}_{n-1}^{s}) \cos s\omega, \quad . \quad . \quad . \quad (73)$$

where

$$\mathfrak{T}_{n}^{s} = \frac{\underline{|n-s}}{\underline{|n+s}} (1-\mu^2)^{s/2} \frac{d^s \mathrm{P}_n(\mu)}{d\mu^s} (\zeta^2 - 1)^{s/2} \frac{d^s \mathrm{P}_n(\zeta)}{d\zeta^s},$$

$$\mathfrak{U}_{n}^{s} = \frac{\underline{|n-s}}{\underline{|n+s}} (1-\mu^2)^{s/2} \frac{d^s \mathrm{P}_n(\mu)}{d\mu^s} (\zeta^2 - 1)^{s/2} \frac{d^s \mathrm{Q}_n(\zeta)}{d\zeta^s},$$

and D, E, are certain constants. The first part of (73) corresponds to a distribution of electricity on the *outer* surface of the film, the second to a distribution on the *inner* surface. In the case of the sectorial harmonic, $\mathrm{E} = 0$.

12. When the shell is of the *oblate* form, the elliptic coordinates to be employed are as follows. We write

$$\left.\begin{aligned} x &= k\sqrt{(1-\mu^2)}\sqrt{(\zeta^2+1)} \cos\omega = k \sin\theta \cosh\eta \cos\omega \\ y &= k\sqrt{(1-\mu^2)}\sqrt{(\zeta^2+1)} \sin\omega = k \sin\theta \cosh\eta \sin\omega \\ z &= k\mu\zeta \qquad\qquad\qquad\qquad\qquad\quad = k \cos\theta \sinh\eta \end{aligned}\right\}, \quad . \quad . \quad . \quad (74)$$

where ζ may range from 0 to ∞. The surfaces $\zeta =$ const. are confocal ellipsoids of revolution, the extreme case $\zeta = 0$ being a circular disk of radius k in the plane xy. Comparing these equations with (42), we see that they may be obtained by writing $i\zeta$ for ζ and $-ik$ for k. The equation $\nabla^2 \mathrm{V} = 0$ therefore becomes

$$\frac{d}{d\mu}\left\{(1-\mu^2)\frac{d\mathrm{V}}{d\mu}\right\} + \frac{1}{1-\mu^2}\frac{d^2\mathrm{V}}{d\omega^2} = -\frac{d}{d\zeta}\left\{(1+\zeta^2)\frac{d\mathrm{V}}{d\zeta}\right\} + \frac{1}{1+\zeta^2}\frac{d^2\mathrm{V}}{d\omega^2}. \quad . \quad (75)$$

The type of solutions symmetrical about the axis is

$$\mathrm{V} = \mathrm{P}_n(\mu) \,.\, \mathrm{Z},$$

where

$$\frac{d}{d\zeta}\left\{(1+\zeta^2)\frac{d\mathrm{Z}}{d\zeta}\right\} - n(n+1)\mathrm{Z} = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (76)$$

One integral of this is

$$p_n(\zeta) = \frac{1.3.5\ldots(2n-1)}{\underline{|n}}\left\{\zeta^n + \frac{n(n-1)}{2(2n-1)}\zeta^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2.4(2n-1)(2n-3)}\zeta^{n-4} + \ldots\right\}; \quad (77)^*$$

* See FERRERS, 'Spherical Harmonics,' chap. vi.

this becomes infinite for $\zeta = \infty$. The second solution is

$$q_n(\zeta) = p_n(\zeta)\int_\zeta^\infty \frac{d\zeta}{\{p_n(\zeta)\}^2(\zeta^2+1)},$$

$$= (-)^n\left\{p_n(\zeta)\operatorname{arc\,cot}\zeta - \frac{2n-1}{1\,.\,n}\,p_{n-1}(\zeta) + \frac{2n-5}{3(n-1)}\,p_{n-3}(\zeta) - \ldots\right\},$$

$$= \frac{\underline{|n}}{1\,.\,3\,..\,(2n+1)}\left\{\zeta^{-n-1} - \frac{(n+1)(n+2)}{2(2n+3)}\zeta^{-n-3} + \frac{(n+1)(n+2)(n+3)(n+4)}{2\,.\,4(2n+3)(2n+5)}\zeta^{-n-5} - \ldots\right\}, \quad . \quad (78)$$

the latter expansion, however, being only convergent when $\zeta > 1$. This function $q_n(\zeta)$ vanishes at infinity.

Hence we have the following solutions of (75):

$$\left.\begin{aligned} V &= P_n(\mu)\,.\,p_n(\zeta) \\ V &= P_n(\mu)\,.\,q_n(\zeta) \end{aligned}\right\}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (79)$$

the former being appropriate to the space inside the ellipsoid, the latter to the external space.*

In like manner, when V involves ω we have the solutions

$$\left.\begin{aligned} V &= (1-\mu^2)^{s/2}\frac{d^s P_n(\mu)}{d\mu^s}(\zeta^2+1)^{s/2}\frac{d^s p_n(\zeta)}{d\zeta^s}\left.\begin{matrix}\cos\\ \sin\end{matrix}\right\}s\omega \\ V &= (1-\mu^2)^{s/2}\frac{d^s P_n(\mu)}{d\mu^s}(\zeta^2+1)^{s/2}\frac{d^s q_n(\zeta)}{d\zeta^s}\left.\begin{matrix}\cos\\ \sin\end{matrix}\right\}s\omega \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (80)$$

It seems unnecessary to go through the details of the investigations corresponding to those of §§ 10, 11. The results are, for the zonal harmonic,

$$\phi = C\,.\,P_n(\mu),$$

$$\tau = -\frac{4\pi k^2}{n(n+1)}\frac{\epsilon}{\rho}\zeta_0(\zeta_0^2+1)^2\frac{dp_n(\zeta_0)}{d\zeta_0}\,\frac{dq_n(\zeta_0)}{d\zeta_0}; \quad . \quad . \quad . \quad . \quad . \quad . \quad (81)$$

and, for the tessaral harmonic,

$$\phi = C\,.\,(1-\mu^2)^{s/2}\frac{d^s P_n(\mu)}{d\mu^s}\sin s\omega,$$

$$\tau = (-)^{s-1}\frac{4\pi k^2\epsilon}{\rho}\,\frac{\underline{|n-s}}{\underline{|n+s}}\,\frac{\zeta_0(\zeta_0^2+1)^3}{n(n+1)(\zeta_0^2+1)-s^2}\,\frac{dt_n^s(\zeta_0)}{d\zeta_0}\,.\,\frac{du_n^s(\zeta_0)}{d\zeta_0}, \quad . \quad . \quad . \quad (82)$$

* The second solution is finite even when $\zeta = 0$, but its space derivatives are infinite at the *focal circle* $x^2 + y^2 = k^2$, $z = 0$.

where

$$t_n^s(\zeta) = (\zeta^2 + 1)^{s/2} \frac{d^s p_n(\zeta)}{d\zeta^s},$$

$$u_n^s(\zeta) = (\zeta^2 + 1)^{s/2} \frac{d^s q_n(\zeta)}{d\zeta^s}.$$

When $n = 1$, (81) becomes

$$\tau = \frac{2\pi k^2 \epsilon}{\rho} \zeta_0 (\zeta_0^2 + 1)^2 \left\{ \text{arc cot } \zeta_0 - \frac{\zeta_0}{\zeta_0^2 + 1} \right\};$$

or, putting

$$k\surd(\zeta_0^2 + 1) = a, \qquad \zeta_0^2 + 1 = 1/e^2,$$

$$\tau = \frac{2\pi a^2 \epsilon}{\rho} \left\{ \frac{\surd(1 - e^2)}{e^3} \text{ arc sin } e - \frac{1 - e^2}{e^2} \right\}, \quad . \quad . \quad . \quad . \quad . \quad (83).$$

which agrees with (13).

13. If we make ζ_0 infinitesimal, we get results applicable to a circular disk, of the kind considered in § 5, provided $n - s$ be *odd*. Putting $k = a$, $k\zeta_0 = c$, and

$$\rho/\epsilon = 2\rho'\varpi = 2\rho_0' c,$$

where c is ultimately made to vanish, the formulæ for the symmetrical currents become

$$\phi = \text{C} . \text{P}_n \left\{ \left(1 - \frac{r^2}{a^2} \right)^{\frac{1}{2}} \right\}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (84)$$

$$\tau = - \frac{2\pi a}{n(n+1)\rho_0'} \left[p_n'(\zeta) \, q_n'(\zeta) \right]_{\zeta = 0}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (85)$$

where r denotes the distance of any point of the disk from the centre. For small values of ζ we have, n being odd,

$$p_n'(\zeta) = \frac{1 . 3 \ldots n}{2 . 4 \ldots (n - 1)}, \qquad q_n' = - \frac{\pi}{2} p_n',$$

whence

$$\tau = \frac{n+1}{n} \frac{\pi^2 a}{\rho_0'} \left\{ \frac{1 . 3 \ldots n}{2 . 4 \ldots (n+1)} \right\}^2 . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (86)$$

For $n = 1$, this gives

$$\tau = \pi^2 a / 2\rho_0',$$

as in § 5. The persistencies of the successive symmetrical types are as

$$1, \tfrac{3}{8}, \tfrac{15}{64}, \text{\&c.}$$

For the tessaral harmonic

$$\phi = \mathrm{C}\frac{r^s}{a^s}\frac{d^s\mathrm{P}_n(\mu)}{d\mu^s}\sin s\omega,^{*}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (87)$$

where

$$\mu = \sqrt{\left(1 - \frac{r^2}{a^2}\right)},$$

we have

$$\tau = (-)^{s-1}\frac{2\pi a}{\rho_0{}'}\frac{\underline{|n-s}}{\underline{|n+s}}\frac{1}{n(n+1)-s^2}\left[\frac{dt_n{}^s}{d\zeta}\cdot\frac{du_n{}^s}{d\zeta}\right]_{\zeta=0}.$$

From the series (77) we find, without difficulty,

$$\left[\frac{dt_n{}^s}{d\zeta}\right]_{\zeta=0} = \frac{1\,.\,3\,\ldots\,(n+s)}{2\,.\,4\,\ldots\,(n-s-1)},$$

$n - s$ being *odd*. Also from the second line of (78), under the same restriction,

$$\frac{du_n{}^s}{d\zeta} = (-)^n\frac{\pi}{2}\frac{dt_n{}^s}{d\zeta},$$

when $\zeta = 0$. Hence

$$\tau = \frac{\pi^2 a}{\{n(n+1)-s^2\}\rho_0{}'}\cdot\frac{\underline{|n-s}}{\underline{|n+s}}\cdot\left\{\frac{1.3\ldots(n+s)}{2.4\ldots(n-s-1)}\right\}^2. \quad . \quad . \quad . \quad . \quad . \quad (88)$$

The most important of the types (87) is that in which $n = 2$, $s = 1$, when

$$\tau = \tfrac{3}{10}\pi^2 a/\rho_0{}'.$$

14. The methods of §§ 10, 11, might be applied to determine the currents induced by simple harmonic variations of a magnetic field; but it is unnecessary to go through the calculations, as the result can be written down at once from the following considerations.

We must first suppose the magnetic potential ($\overline{\Omega}$, say) due to the field to be expanded, for the space near the conductor, in a series of terms of the forms given by the first lines of (47) and (49)†; or of (79) and (80), as the case may be. Each of these terms will act by itself, and produce a current-function ϕ of the type (51) or (64). Now, in § 11, the equation of free currents of any normal type was brought to the form

$$\mathrm{I}\rho\phi = \mathrm{J}\frac{d\phi}{dt}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (89)$$

* The current lines are the orthogonal projections on the plane xy of the contour lines of spherical harmonics drawn on a sphere of radius a.

† F. E. NEUMANN has shown ('Crelle,' vol. 37) how to expand the potential of a single magnetic pole in this way.

the left-hand side being obtained as the electromotive force necessary to balance the resistance, and the right-hand side as the electromotive force of induction due to the decay of the currents. If τ be the modulus of decay of the type in question,

$$\tau = -\mathrm{J}/\mathrm{I}\rho.$$

Now let $\bar{\phi}$ represent a fictitious distribution of current over the ellipsoid, which shall have the same magnetic effect in the interior as the actual inducing field. This distribution is found at once from (69). The equation of induced currents will then be

$$\mathrm{I}\rho\phi = \mathrm{J}\left(\frac{d\phi}{dt} + \frac{d\bar{\phi}}{dt}\right)$$

or

$$\phi = -\tau\left(\frac{d\phi}{dt} + \frac{d\bar{\phi}}{dt}\right). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (90)$$

When the free currents have died away all our functions will vary as e^{ipt}, where p measures the rapidity of the changes in the field. Substituting in (90), we find

$$\phi = \frac{-ip\tau}{1 + ip\tau}\,\bar{\phi}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (91)$$

When $p\tau$ is very great this becomes

$$\phi = -\bar{\phi}.$$

The result (91) may be verified in the problem of § 6.

15. Let us next consider the rotation of the shell in a constant field. There will be no currents due to those terms in the expansion of $\bar{\Omega}$ which are *zonal* solid harmonics; the only effect of these being a certain surface electrification. We may complete the investigation of § 8 by finding the density of this electrification in the case of an oblate ellipsoid of revolution rotating in a uniform field about an axis parallel to the lines of force. We have to find a function ψ which shall have the value

$$\psi = \tfrac{1}{2}p\gamma\,(x^2 + y^2) + \mathrm{C} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (92)$$

at the surface of the conductor ($\zeta = \zeta_0$), and shall satisfy $\nabla^2\psi = 0$ throughout the external space. Denoting by a the equatorial radius, (92) may be put in the form

$$\psi = \mathrm{C} + \tfrac{1}{3}p\gamma a^2 - \tfrac{1}{3}p\gamma a^2 \,.\, \mathrm{P}_2(\mu) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (93)$$

Hence in the external space

$$\psi = (\mathrm{C} + \tfrac{1}{3}p\gamma a^2)\,\frac{q_0(\zeta)}{q_0(\zeta_0)} - \tfrac{1}{3}p\gamma a^2\mathrm{P}_2(\mu)\,\frac{q_2(\zeta)}{q_2(\zeta_0)}.$$

But

$$q_0'(\zeta) = -\frac{1}{\zeta^2+1},$$

$$\frac{q_2'(\zeta)}{q_2(\zeta)} = -\frac{1}{p_2(\zeta)\,q_2(\zeta)\,(\zeta^2+1)} + \frac{p_2'(\zeta)}{p_2(\zeta)},$$

and $k^2\,d\zeta/d\nu_1 = \varpi/\zeta_0$. Hence, by (41),

$$\frac{4\pi\sigma}{\mathrm{K}} = -\frac{d\psi}{d\nu_1} = \frac{\varpi}{k^2\zeta_0\,(\zeta_0^2+1)\operatorname{arc\,cot}\zeta_0}\,(\mathrm{C}+\tfrac{1}{3}p\gamma a^2)$$
$$-\tfrac{1}{3}p\gamma a^2\mathrm{P}_2(\mu)\left\{\frac{1}{p_2(\zeta_0)\,q_2(\zeta_0)\,\zeta_0\,(\zeta_0^2+1)} - \frac{p_2'(\zeta_0)}{\zeta_0 p_2(\zeta_0)}\right\}\frac{\varpi}{k^2}.$$

For the case of a *disk* we have, as in § 5,

$$\varpi = \frac{ac}{\sqrt{(a^2-r^2)}}, \qquad \mu = \sqrt{\left(1-\frac{r^2}{a^2}\right)},$$

and

$$a\zeta_0 = c,$$

where c is ultimately made $= 0$. Also

$$p_2(0) = \tfrac{1}{2}, \qquad q_2(0) = \frac{\pi}{4}, \qquad p_2'(0) = 0,$$

whence

$$\frac{4\pi\sigma}{\mathrm{K}} = \frac{2}{\pi\sqrt{(a^2-r^2)}}\,(\mathrm{C}+\tfrac{1}{3}p\gamma a^2) - \tfrac{4}{3}\,\frac{p\gamma}{\pi\sqrt{(a^2-r^2)}}\,(2a^2-3r^2)$$
$$= \frac{2}{\pi\sqrt{(a^2-r^2)}}\,\{\mathrm{C}+p\gamma\,(2r^2-a^2)\}. \qquad (94)$$

The total charge on both surfaces of the disk is

$$\frac{2a}{\pi}\,(\mathrm{C}+\tfrac{1}{3}p\gamma a^2)\,\mathrm{K}.$$

The constant C is of course to be determined by the other conditions of the problem. If the axis of the disk be uninsulated, we shall have $\mathrm{C} = 0$.

16. The only terms in the value of $\bar{\Omega}$ which give rise to sensible currents in a rotating *disk* are those tessaral solid harmonics for which $n - s$ is odd.

If the value of $\bar{\Omega}$, referred to fixed axes, be

$$\bar{\Omega} = \bar{\mathrm{A}}\,.\,(1-\mu^2)^{s/2}\frac{d^s\mathrm{P}_n(\mu)}{d\mu^s}\,(\zeta^2+1)^{s/2}\,\frac{d^s p_n(\zeta)}{d\zeta^s}\cos s\omega, \qquad (95)$$

the corresponding value of $\bar{\phi}$ will be

$$\bar{\phi} = 2\bar{C}\,(1-\mu^2)^{s/2}\frac{d^s\,P_n(\mu)}{d\mu^s}\cos s\omega,^* \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (96)$$

where, by the formulæ analogous to (69),

$$\begin{aligned} \bar{A} &= (-)^s\,4\pi\,\frac{\lfloor n-s}{\lfloor n+s}\left[\frac{du_\nu{}^s}{d\zeta_0}\right]_{\zeta=0}\cdot\,\bar{C}, \\ &= -\,2\pi^2\,\bar{C}\,.\,\frac{\lfloor n-s}{\lfloor n+s}\,.\,\frac{1\,.\,3\,.\,.\,(n+s)}{2\,.\,4\,.\,.\,(n-s-1)}, \\ &= -\,2\pi^2\,\bar{C}\,\cdot\,\frac{1\,.\,3\,.\,.\,.\,(n-s)}{2\,.\,4\,.\,.\,.\,(n+s-1)}\,. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (97) \end{aligned}$$

If now, to use MAXWELL'S artifice, we pass to axes of x, y, moving with the disk, we must write

$$\bar{\phi} = \bar{C}\,(1-\mu^2)^{s/2}\frac{d^s\,P_n(\mu)}{d\mu^s}\cos s(\omega + pt),$$

where p is the angular velocity of the rotation. For the trigonometrical term at the end we may write $e^{is(\omega+pt)}$ if we retain in the end only the real part. Hence for the induced currents we have, by (91),

$$\phi = \frac{-\,isp\tau}{1+isp\tau}\,\bar{\phi},$$

where τ is the persistency of free currents of the type (n, s).

Putting

$$\eta = \text{arc tan}\,sp\tau,$$

we find, finally, on returning to *fixed* axes of x, y,

$$\phi = \bar{C}\sin\eta\,(1-\mu^2)^{s/2}\frac{d^s\,P_n(\mu)}{d\mu^s}\sin\,(s\omega-\eta). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (98)$$

The system of currents is stationary in space, but is displaced relatively to the field by a greater or less angle

$$\frac{1}{s}\,\text{arc tan}\,sp\tau,$$

according to the speed of rotation. The maximum value of this is $\pi/2s$ for a sufficiently rapid rotation.

* This represents a fictitious distribution of currents which would give at all points of the disk the same *normal force* as the actual field.

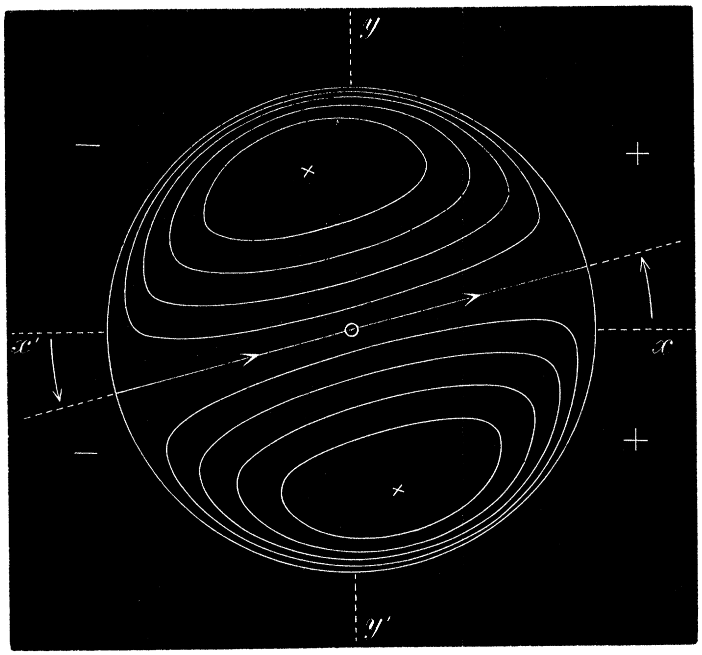
The most important type of induced currents is got by putting $n = 2$, $s = 1$, in (95). In this case

$$\overline{\Omega} \propto xz,$$

so that the lines of force at the disk are normal to it, but the direction of the force is reversed as we cross the axis of y. The current-function relatively to axes displaced through the proper angle η varies as

$$y\sqrt{\left(1 - \frac{r^2}{a^2}\right)}.*$$

The current-lines for this case are shown in the figure. The signs + and − indicate where the normal force due to the field is towards or from the spectator.



In the next type we have $n = 3$, $s = 2$, so that

$$\Omega \propto z(x^2 - y^2),$$

and the current-function (relatively to displaced axes as before) varies as

$$xy\sqrt{\left(1 - \frac{r^2}{a^2}\right)}.$$

* [It may be shown that, referred to the same axes, the potential ψ varies as

$$-\rho_0 x(1 - 2r^2/a^2).$$

The equipotential lines are not orthogonal to the current-lines, except in the case of the circle $r = a/\sqrt{2}$. —Note added June 30, 1887.]