X. *On the Induction of Electric Currents in Conducting Shells of Small Thickness.*

By S. H. Burbury, *M.A., formerly Fellow of St. John's College, Cambridge.*

Communicated by H. W. Watson, *D.Sc., F.R.S.*

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If a conductor be placed in a magnetic field, and the field be made to vary, closed electric currents will be induced on the surface or within the substance of the conductor. If the conductor be a wire forming a closed curve in which the electric current is regarded as having only one degree of freedom, the laws of induction in it are well known.

The problem of induction of electric currents in solids or hollow shells has been treated by several writers, generally with reference only to conductors of particular shape, as spheres, infinite planes, cylinders, or ellipsoids, and with reference to special variations of the external magnetic field.*

The object of this paper is to investigate a general theory applicable to surfaces of any shape in presence of an external magnetic field varying in any manner.

It will be necessary to make use of one conclusion at which Professor Lamb arrives in the second of the memoirs above referred to, namely, that the displacement currents, which according to Maxwell's theory exist in the dielectric, have in all cases subject to experiment no sensible influence in modifying the currents of conduction which may be induced in metallic conductors. We may, therefore, treat the currents of conduction as the only possible currents.

The Condition of Continuity.

Let u, v, w denote the components of electric current referred to unit of area. Then, if there be no time variation of free electricity, the condition of continuity requires that

$$\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz} = 0$$

* Professor Niven, 'Phil. Trans.,' 1881, p. 307; Professor H. Lamb, 'Phil. Trans.,' 1883, p. 519, 1887, p. 131; Professor Larmor, 'Phil. Mag.,' January, 1884; Mr. O. Heaviside, 'Phil. Mag.,' August, September, 1886.

at every point. We will suppose that a closed surface can be described completely enclosing all the currents of the system. Then the stream lines must, in order that the above condition may be everywhere fulfilled, be in closed curves, and the currents are then said to be *closed currents.*

Of Current Sheets and Shells and Superficial Currents.

2. Any surface to which the resultant currents or stream lines are everywhere tangential is defined to be *a current sheet.* The space between any two current sheets infinitely near each other may be defined to be *a current shell.* If a line be drawn on a current sheet perpendicular to the stream line at any point P, and dc be any element of that line at P, and h be the distance at P between the two current sheets forming the shell, then the ratio of the quantity of electricity which flows through the area hdc in unit of time to dc is the *superficial current,** or current referred to unit of length, at P.

If the current sheet be a closed surface, $S=0$, the stream lines form closed curves upon it, and no two of them intersect each other. Let there be given on S any system of superficial currents. We can suppose the stream lines traced out on the surface. Then the superficial current flowing along the belt between any two of them is inversely proportional at any point in its course to the distance between the two stream lines at the point.

Of the Current Function.

3. For any system of superficial currents which can be formed on a closed surface $S=0$ there exists a function ϕ, called *the current function*, such that the component superficial currents at any point are

$$U = n\frac{d\phi}{dy} - m\frac{d\phi}{dz}$$

$$V = l\frac{d\phi}{dz} - n\frac{d\phi}{dx}$$

$$W = m\frac{d\phi}{dx} - l\frac{d\phi}{dy}$$

where l, m, n are the direction cosines of the normal to the surface. We will use U, V, W for the superficial currents; u, v, w for the currents referred to unit of area.

For there exists a function, ϕ, of x, y, and z, which has any arbitrarily assigned

* It is usual to employ the term *superficial current* only in cases where the quantity of electricity in question is infinitely great compared with h by analogy to the definition of a superficial distribution of electricity in electrostatics. But the definition above given is unambiguous, and includes as a particular case the case of a finite current in an infinitely thin shell.

constant value along each stream line. Therefore there exists a function, ϕ, which is constant along each stream line, and such that $d\phi/dc$ is equal to the given superficial current at every point, dc being an element of a line drawn on the surface at right angles to the stream line. Then ϕ is the required function.

For U, V, and W so defined are proportional to the direction cosines of the common section of the surface S, and the surface $\phi =$ constant, that is, to the direction cosines of the stream line.

Also

$$\begin{aligned}\mathrm{U}^2 + \mathrm{V}^2 + \mathrm{W}^2 &= (m^2 + n^2)\left(\frac{d\phi}{dx}\right)^2 + (l^2 + n^2)\left(\frac{d\phi}{dy}\right)^2 + (l^2 + m^2)\left(\frac{d\phi}{dz}\right)^2 - 2mn\frac{d\phi}{dy}\frac{d\phi}{dz} \\ &\quad - 2ln\frac{d\phi}{dx}\frac{d\phi}{dz} - 2lm\frac{d\phi}{dx}\frac{d\phi}{dy} \\ &= \left(\frac{d\phi}{dx}\right)^2 + \left(\frac{d\phi}{dy}\right)^2 + \left(\frac{d\phi}{dz}\right)^2 - \left(l\frac{d\phi}{dx} + m\frac{d\phi}{dy} + n\frac{d\phi}{dz}\right)^2 \\ &= \left(\frac{d\phi}{d\nu'}\right)^2 - \left(\frac{d\phi}{d\nu}\right)^2\end{aligned}$$

(if $d\phi/d\nu'$ be the rate of increase of ϕ per unit of length of the normal to the surface $\phi =$ constant, and $d\phi/d\nu$ be the rate of increase of ϕ per unit of length of the normal to the sheet)

$$= \left(\frac{d\phi}{dc}\right)^2.$$

There may be an infinite variety of functions which satisfy the conditions for ϕ, but all of them give the same value for U, V, and W. If ϕ be given, it completely determines U, V, and W. Conversely, if U, V, and W be given at every point, they completely determine the values of ϕ on S subject to the addition of an arbitrary constant.

Of the Currents per Unit of Area.

4. Let there be any finite space, and two functions S and ϕ such that within the space

$$u = \frac{d\mathrm{S}}{dz}\frac{d\phi}{dy} - \frac{d\mathrm{S}}{dy}\frac{d\phi}{dz}$$

$$v = \frac{d\mathrm{S}}{dx}\frac{d\phi}{dz} - \frac{d\mathrm{S}}{dz}\frac{d\phi}{dx}$$

$$w = \frac{d\mathrm{S}}{dy}\frac{d\phi}{dx} - \frac{d\mathrm{S}}{dx}\frac{d\phi}{dy}$$

and $lu + mv + nw = 0$ at all points on the bounding surface.

Then u, v, and w, satisfying these conditions, may be the components of a system of finite currents per unit of area within the space. For they satisfy the condition

$$\frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz}=0.$$

They also satisfy the conditions

$$u\frac{dS}{dx}+v\frac{dS}{dy}+w\frac{dS}{dz}=0,$$

$$u\frac{d\phi}{dx}+v\frac{d\phi}{dy}+w\frac{d\phi}{dz}=0\,;$$

and, therefore, in such a system any surface, $S=$ constant, or $\phi=$ constant, is a *current sheet.*

If the surface $S=$ constant be a closed surface within the space the currents upon it are closed currents.

If we form a shell between two neighbouring surfaces, $S=c$ and $S=c+dc$, the superficial currents in that shell on $S=c$ are determined by the current function $\phi\,dc$, so that

$$U=dc\left(n\frac{d\phi}{dy}-m\frac{d\phi}{dz}\right),\ \&c.$$

The currents per unit of area being assumed finite, the superficial currents are of course infinitely small in an infinitely thin shell. If h be the thickness of the shell,

$$\frac{dS}{dz}=\frac{ndc}{h},\qquad \frac{dS}{dy}=\frac{mdc}{h},\qquad \frac{dS}{dx}=\frac{ldc}{dh}.$$

And, therefore,

$$U=hu,\qquad V=hv,\qquad W=hw.$$

Of the Vector Potential of a System of Superficial Currents.

5. If S be any current sheet, ϕ the current function, we have for the components vector potential

$$F=\iint\frac{hu}{r}\,ds=\iint\frac{U}{r}\,dS$$

$$=\iint\left(n\frac{d\phi}{dy}-m\frac{d\phi}{dz}\right)\frac{1}{r}\,dS$$

$$G=\iint\left(l\frac{d\phi}{dz}-n\frac{d\phi}{dx}\right)\frac{1}{r}\,dS$$

$$H=\iint\left(m\frac{d\phi}{dx}-l\frac{d\phi}{dy}\right)\frac{1}{r}\,dS$$

where r is the distance from a point in the shell to the point at which F, G, H are required. If the current sheet be a closed surface, or if it be a bounded surface, and ϕ be zero at the boundary, these expressions can be put in another form, as follows:—

Applying STOKES's theorem to any bounded surface S, and the function ϕ/r, we have

$$\int \frac{\phi}{r} \frac{dx}{ds} ds = \iint \left(m \frac{d}{dz} - n \frac{d}{dy}\right) \frac{\phi}{r} dS,$$

in which the integral on the left hand side is round the bounding curve. Therefore for any closed surface, or any unclosed surface, provided that ϕ is continuous and vanishes at the boundary,

$$\iint \left(m \frac{d}{dz} - n \frac{d}{dy}\right) \frac{\phi}{r} dS = 0\ ;$$

and, therefore,

$$\begin{aligned} F &= \iint \frac{1}{r} \left(n \frac{d\phi}{dy} - m \frac{d\phi}{dz}\right) dS \\ &= \iint \phi \left(m \frac{d}{dz} - n \frac{d}{dy}\right) \frac{1}{r} dS. \end{aligned}$$

F is therefore a linear function of all the ϕ's with coefficients functions of the coordinates; G and H have corresponding values. Given S and ϕ, F, G, and H are completely determinant, and are independent of h.

Corollary.—The vector potential due to any spherical current sheet is tangential to every spherical surface concentric with the sheet, as shown by MAXWELL, § 671.

Of the Energy of a System of Current Sheets.

6. The electrokinetic energy of a system of currents over the surface or system of surfaces S is

$$\begin{aligned} T &= \tfrac{1}{2} \iint (Fhu + Ghv + Hhw)\, dS \\ &= \tfrac{1}{2} \iint \left\{ F \left(n \frac{d\phi}{dy} - m \frac{d\phi}{dz}\right) + G \left(l \frac{d\phi}{dz} + n \frac{d\phi}{dx}\right) + H \left(m \frac{d\phi}{dx} - l \frac{d\phi}{dy}\right) \right\} dS. \end{aligned}$$

We can transform this by STOKES's theorem in the same manner as we transformed the integral

$$F = \iint \frac{1}{r} \left(n \frac{d\phi}{dy} - m \frac{d\phi}{dz}\right) dS.$$

For, if the surfaces be closed, or if ϕ be continuous and vanish at the boundary,

$$\iint \left(n \frac{d}{dy} - m \frac{d}{dz}\right)(\mathrm{F}\phi)\, d\mathrm{S} = 0,$$

or

$$\iint \mathrm{F}\left(n \frac{d\phi}{dy} - m \frac{d\phi}{dz}\right) d\mathrm{S} = \iint \phi \left(m \frac{d\mathrm{F}}{dz} - n \frac{d\mathrm{F}}{dy}\right) d\mathrm{S}.$$

Treating $\mathrm{G}\phi$ and $\mathrm{H}\phi$ in the same way, and arranging, we obtain

$$\mathrm{T} = \tfrac{1}{2} \iint \phi \left\{ l \left(\frac{d\mathrm{H}}{dy} - \frac{d\mathrm{G}}{dz}\right) + m \left(\frac{d\mathrm{F}}{dz} - \frac{d\mathrm{H}}{dx}\right) + n \left(\frac{d\mathrm{G}}{dx} - \frac{d\mathrm{F}}{dy}\right) \right\} d\mathrm{S}.$$

But

$$\frac{d\mathrm{H}}{dy} - \frac{d\mathrm{G}}{dz}, \quad \frac{d\mathrm{F}}{dz} - \frac{d\mathrm{H}}{dx}, \quad \text{and} \quad \frac{d\mathrm{G}}{dx} - \frac{d\mathrm{F}}{dy}$$

are respectively the x, y, and z components of magnetic force, or which is here the same thing, magnetic induction. That is, if Ω be the magnetic potential of the system,

$$\frac{d\mathrm{H}}{dy} - \frac{d\mathrm{G}}{dz} = - \frac{d\Omega}{dx}, \text{ \&c.}$$

Hence

$$\mathrm{T} = - \tfrac{1}{2} \iint \phi \left(l \frac{d\Omega}{dx} + m \frac{d\Omega}{dy} + n \frac{d\Omega}{dz}\right) d\mathrm{S}$$

$$= - \tfrac{1}{2} \iint \phi \frac{d\Omega}{d\nu}\, d\mathrm{S}.$$

This value of T is unambiguous; because, as is well known, $d\Omega/d\nu$ is not discontinuous at the sheet, even when the superficial currents are finite. To show this, it is sufficient to take the tangent plane at any point for the plane of x, y. Then

$$\frac{d\Omega}{d\nu} = \frac{d\Omega}{dz} = \frac{d\mathrm{F}}{dy} - \frac{d\mathrm{G}}{dx}$$

Now, F and G, or $\iint (hu/r)\, d\mathrm{S}$ and $\iint (hv/r)\, d\mathrm{S}$, are the potentials of imaginary matter distributed over S of surface densities hu and hv respectively. Therefore, $d\mathrm{F}/dy$ and $d\mathrm{G}/dx$, corresponding to tangential components of force, are continuous, although, if hu be finite, $d\mathrm{F}/dz$ may be discontinuous, at the sheet.

Since

$$- \frac{d\Omega}{d\nu} = l \left(\frac{d\mathrm{H}}{dy} - \frac{d\mathrm{G}}{dz}\right) + \text{\&c.},$$

and

$$\mathrm{F} = \iint \phi \left(m \frac{d}{dz} - n \frac{d}{dy}\right) \frac{1}{r}\, d\mathrm{S},$$

we see that T can be expressed as a quadratic function of all the ϕ's with coefficients functions of the coordinates.

Evidently, if we have two systems of currents on different sheets or on the same sheet, their energy of mutual action is $\frac{1}{2}\iint \phi' \, d\Omega/d\nu \, dS = \frac{1}{2}\iint \phi \, d\Omega'/d\nu \, dS$, where ϕ and Ω relate to one system, and ϕ' and Ω' to the other.

Comparison with Magnetic Shells.

7. If a system of magnetic shells be formed over the surface S, and ϕ be the strength at any point, regarded as positive when the positive face is outwards, the components of vector potential of magnetic induction due to the system at any point not within the substance of the shells are (MAXWELL, § 416)—

$$F = \iint \phi \left(m \frac{d}{dz} - n \frac{d}{dy} \right) \frac{1}{r} \, dS,$$

$$G = \iint \phi \left(n \frac{d}{dx} - l \frac{d}{dz} \right) \frac{1}{r} \, dS,$$

$$H = \iint \phi \left(l \frac{d}{dy} - m \frac{d}{dx} \right) \frac{1}{r} \, dS.$$

They are, therefore, the same as the components of vector potential of the system of electric currents over S, determined by ϕ as current function.

It follows that the components of magnetic force or magnetic induction, namely, $dH/dy - dG/dz$, &c., are at any point not within the substance of the shells the same for the system of shells whose strength is ϕ as for the system of currents whose current function is ϕ over the same surface; or, as we may otherwise express it, the magnetic potential due to the system of shells differs from that due to the system of currents by some constant at all points external to the sheet, and by some, but not necessarily the same, constant at all points within the sheet, the particular constants depending on the definition we choose to adopt of the magnetic potential due to a current shell.

8. *Proposition.*—There exists a determinate system of magnetic shells over any closed surface, S, which has magnetic potential at each point on or within S equal to that of any arbitrarily assigned external magnetic system.

For let P_0 be the potential of the external system.

Let q be the density of a distribution of matter over S, the potential of which is equal to P_0 at all points on S, and, therefore, also at all points within S.

Then q is possible and determinate by known theorems.

Let ϕ be that function of x, y, and z of negative degree which satisfies the conditions $\nabla^2\phi = 0$ at all points outside of S, and $d\phi/d\nu = q$ at all points on S,

where $d\nu$ is an element of the normal to S, measured outwards. Then ϕ is possible and determinate by known theorems; and ϕ is the strength of the required magnetic shell.

For let r be the distance of any point from an internal point O. Then at O the potential of the system of magnetic shells whose strength is ϕ is

$$P = \iint \phi \frac{d}{d\nu} \frac{1}{r} \, dS.$$

But, by GREEN's theorem, applied to the functions ϕ and $1/r$ and the infinite space outside of S,

$$\iint \phi \frac{d}{d\nu} \frac{1}{r} \, dS - \iiint \phi \, \nabla^2 \frac{1}{r} \, dx \, dy \, dz = \iint \frac{1}{r} \frac{d\phi}{d\nu} \, dS - \iiint \frac{1}{r} \nabla^2 \phi \, dx \, dy \, dz.$$

But $\nabla^2 1/r = 0$, because O is within S, and $\nabla^2 \phi = 0$ by definition at all external points.

Therefore,

$$\begin{aligned} P &= \iint \phi \frac{d}{d\nu} \frac{1}{r} \, dS \\ &= \iint \frac{1}{r} \frac{d\phi}{d\nu} \, dS \\ &= \iint \frac{q}{r} \, dS \\ &= P_0. \end{aligned}$$

Corollary.—There exists one determinate system of closed electric currents over any closed surface, S, whose magnetic potential, together with that of any arbitrarily given external magnetic system, is constant at all points on or within S; namely, the system of currents whose current function is $-\phi$, where ϕ is determined as in the principal proposition.

9. The magnetic induction due to the combined systems is, therefore, zero at every point within S. We will define the system of currents on S which has this property to be the *magnetic screen* on S to the external system.

Evidently the proposition and its corollary will apply equally to a system of magnetic shells or electric currents on S having at all points in external space the same magnetic effect as that of a magnetic system wholly within S.

Example of Magnetic Screen.

10. Let S be a sphere of radius a. Then P, the magnetic potential of the external system, may, as regards its value on S, be expanded in a series of spherical surface harmonics, including generally a constant term, namely,

$$P = A_0 + \Sigma A_n Y_n.$$

Hence,

$$q = \frac{A_0}{4\pi a} + \Sigma \frac{2n+1}{4\pi a} A_n Y_n.$$

Then ϕ is the solid harmonic

$$\phi = -\frac{A_0 a}{4\pi r} - \frac{1}{4\pi} \Sigma \frac{2n+1}{n+1} \left(\frac{a}{r}\right)^{n+1} A_n Y_n,$$

which satisfies the condition

$$\frac{d\phi}{d\nu} = \frac{d\phi}{dr} = q \text{ at all points on S};$$

and the value of ϕ on S is

$$\phi = -\frac{A_0}{4\pi} - \frac{1}{4\pi} \Sigma \frac{2n+1}{n+1} A_n Y_n.$$

The constant term $-A_0/4\pi$ corresponds to a magnetic shell of uniform strength over S, which gives constant potential A_0 at all internal points, and zero at all external points. It corresponds to no system of electric currents.

Of a certain Function called the Associated Function.

11. If S be any closed surface, and if P, Q, R be the components of a vector, such that $dP/dx + dQ/dy + dR/dz = 0$ at all points within S, it follows that

$$\iint (lP + mQ + nR)\, dS = 0.$$

Therefore there exists a determinate function, ψ, of x, y, and z, which satisfies the conditions $d\psi/d\nu = lP + mQ + nR$ at all points on S, and $\nabla^2\psi = 0$ at all points within S.

I shall call this the *associated function* to P, Q, R for the surface S.

Evidently the vector whose components are $P - d\psi/dx$, $Q - d\psi/dy$, $R - d\psi/dz$ is tangential to S at every point, and forms closed curves within S. If the conditions $dP/dy = dQ/dx$, &c., are satisfied at all points within S, then $P - d\psi/dx$, $Q - d\psi/dy$,

and $R - d\psi/dz$ are severally zero at all points within S. For, since these conditions are satisfied, there must exist a function χ of x, y, and z such that $P = d\chi/dx$, $Q = d\chi/dy$, $R = d\chi/dz$ at all points within S. And therefore $P - d\psi/dx = d(\chi - \psi)/dx$, &c.; and therefore $d(\chi - \psi)/d\nu = 0$ at all points on S, and $\nabla^2(\chi - \psi) = 0$ at all points within S; and therefore $\chi - \psi =$ constant, and $P - d\psi/dx = 0$, &c., at all points within S.

12. If P, Q, R be the components of an electromotive force, and if S be a conductor, then, whether the condition $P - d\psi/dx = 0$, &c., be satisfied or not, they will produce on S a distribution of electricity having potential ψ. For let

$$P = P_1 + P',$$

and

$$P_1 = \frac{d\psi}{dx}.$$

Similarly,

$$Q = Q_1 + Q' = \frac{d\psi}{dy} + Q',$$

$$R = R_1 + R' = \frac{d\psi}{dz} + R';$$

then the vector or electromotive force whose components are P_1, Q_1, R_1 is derived from the potential ψ, and produces on S a distribution having potential ψ. The vector whose components are P', Q', R', or $P - d\psi/dx$, $Q - d\psi/dy$, $R - d\psi/dz$, forms closed curves within or upon S, and cannot affect the potential.

13. Now let F_0, G_0, H_0 be the components of vector potential due to a magnetic system external to S. Let $-\psi_0$ be their associated function. Let the magnetic screen to the external system be formed on S. Let F, G, H be the components of vector potential due to it, and let $-\psi$ be their associated function. Then for the two systems together we have a vector whose components are $F_0 + F$, &c., with $-(\psi_0 + \psi)$ for associated function; and, since $d(F_0 + F)/dy = d(G_0 + G)/dx$, &c., within S because of the screen, it follows that

$$\left.\begin{aligned} F_0 + F + \frac{d}{dx}(\psi_0 + \psi) &= 0 \\ G_0 + G + \frac{d}{dy}(\psi_0 + \psi) &= 0 \\ H_0 + H + \frac{d}{dz}(\psi_0 + \psi) &= 0 \end{aligned}\right\} \text{at all points within S.}$$

14. We are now in a position to consider the general problem of induction, when electric conductors of any shape are in a magnetic field and the field is made to vary. Electric currents are generated by induction in or on the surface of the conductors.

These induced currents will, in any observed case, rapidly decay by resistance, and of course any calculations based on the hypothesis of there being no resistance cannot express any actually observed phenomena. But as the currents vary from two causes, (1) by induction, (2) by resistance, it is legitimate, for mathematical purposes, to calculate the effect of each cause separately. With this object, we may, in determining the law of formation of the induced currents, assume the resistance to be zero.

Let us take the case in which the conductors on which induced currents are to be found are hollow conducting shells of any shape. Let their surfaces be denoted by S.

15. In order that the application of LAGRANGE'S equations may be legitimate, without introducing equations of condition, we must express the energy in terms of as many variables, and no more, as there are degrees of freedom. Now the expression $2T = \iiint (Fu + Gv + Hw) dx\, dy\, dz$ contains u, v, w as the variables, and F, G, H linear functions of them. But u, v, w have to satisfy at each point two conditions, namely, (1) the condition of continuity, (2) the condition $lu + mv + nw = 0$ at the surface of the conductors. The number of variables u, v, w is greater than the number of degrees of freedom.

Let us then take ϕ, the current function, for independent variable, as it is subject to no condition on any surface. Further, the given magnetic field either consists of, or may be represented by, a system of current sheets, denoted by S_0, on which the current function is ϕ_0, and the magnetic potential due to it is Ω_0.

16. We may, therefore, without loss of generality, assume the given external magnetic field to be of that character. Then the electrokinetic energy at any instant due to the system of currents, as well original as induced, is

$$\begin{aligned} 2T = &- \iint \phi_0 \left(\frac{d\Omega}{d\nu} + \frac{d\Omega}{d\nu}\right) dS_0 \\ &- \iint \phi \left(\frac{d\Omega_0}{d\nu} + \frac{d\Omega}{d\nu}\right) dS \end{aligned}$$

in which Ω is the magnetic potential of the induced currents, and the first integral is over all the surfaces whereon ϕ_0 the current function is given, and the second over all the conductors on which ϕ is to be determined by induction.

If the system have any other form of energy, as for instance, that of any statical distribution, the expression for that energy cannot contain ϕ.

If, therefore, the given external system vary continuously with the time, the corresponding variation of the induced system is found by making

$$\frac{d}{dt}\frac{dT}{d\phi} = 0$$

or

$$\frac{d}{dt}\left(\frac{d\Omega_0}{d\nu} + \frac{d\Omega}{d\nu}\right) = 0$$

at every point on each of the surfaces S; that is

$$\frac{d}{d\nu}\left(\frac{d\Omega_0}{dt} + \frac{d\Omega}{dt}\right) = 0$$

at every point on each of the surfaces S.

But also

$$\nabla^2\left(\frac{d\Omega_0}{dt} + \frac{d\Omega}{dt}\right) = 0$$

at each point within S.

It follows that

$$\frac{d\Omega_0}{dt} + \frac{d\Omega}{dt} = 0$$

or constant, at all points on or within S.

But $d\Omega_0/dt$ being given, there is one, and only one, determinate system of closed electric currents on S which has this effect, namely, the particular system determined by the method of (8) and which we called *the magnetic screen.* This, then, is the system of currents which will be formed from instant to instant on the surfaces S in response to the continuous variation of the external magnetic system. This result is stated by MAXWELL, §§ 654, 655.

Case of a Solid Conductor.

17. I have assumed the conductors S to be hollow conducting shells. But if there be within any of them any solid conductor, the proof shows that no closed electric currents will, as the immediate effect of induction, be formed upon or within it, because, as the immediate effect of induction, the magnetic force undergoes no change within S. The outer shell S, with the induced currents upon it, acting for the instant as a complete *magnetic screen,* completely shelters the enclosed solid from the direct magnetic influence of the external system. As the superficial currents decay by resistance, they cease to be a complete screen, and the interior solid becomes exposed, in general very rapidly, to the influence of the external system. This effect we shall have to consider later. But the immediate effect of induction is to produce only superficial currents in the outer shell. And as this is true whatever be the form of the enclosed solid, it is true if S consists of a solid conductor, instead of a hollow shell.

It may be said that we cannot conceive an electric current otherwise than as existing in a conducting stratum of some finite thickness, nor as independent of resistance, which our expressions hitherto obtained are. And questions may be raised concerning the thickness of the solid actually occupied by the currents at any time during the induction, that is the rate at which the currents penetrate the solid. We shall see reasons later, see (31) *post,* for determining the rate of penetration in certain cases as a function of the resistance of the material. In the meantime we may treat of the

currents as produced from instant to instant in a thin shell, whatever its thickness may be.

The Potential Induced on a Conductor.

18. Let now dF_0/dt, dG_0/dt, dH_0/dt denote the time variation of the components of vector potential for the external system. Then $-dF_0/dt$, $-dG_0/dt$, $-dH_0/dt$, have an *associated function*, defined as in (11), which we will call ψ_0.

Similarly, if $-dF/dt$, $-dG/dt$, $-dH/dt$ relate to the induced currents, they have an associated function ψ.

The functions $-dF_0/dt - dG_0/dt - dH_0/dt$ are the components of an electromotive force, and, therefore, by (12), produce on the conductor a distribution having potential ψ_0.

Similarly the functions $-dF/dt - dG/dt - dH/dt$ produce on the conductor a distribution having potential ψ. Initially on the formation of the induced currents they satisfy the conditions

$$-\frac{dF_0}{dt} - \frac{dF}{dt} - \frac{d}{dx}(\psi_0 + \psi) = 0, \text{ \&c.}$$

19. Hence we arrive at the conclusion that any variation of the magnetic field outside of a conductor causes on the surface of the conductor—

(1.) A system of closed currents whose magnetic potential at the instant of their creation is equal and opposite to the time variation of the magnetic potential of the given system at all points on or within the conductor; that is, a complete *magnetic screen.*

(2.) It creates and maintains a difference of potential at different points on the conductor, and this may be used to produce an electric current in a system connected with the conductor.

20. The electrostatic distribution has energy, but such energy exists side by side with the electrokinetic or magnetic energy of the closed currents, without (so to speak) mixing. That is, there is no term involving products of U, V, W with $d\psi/dx$, &c.

For, if we have any system whatever of closed currents within any closed surface S, and in the field of a potential ψ,

$$\iiint \left\{u\frac{d\psi}{dx} + v\frac{d\psi}{dy} + w\frac{d\psi}{dz}\right\} dx\,dy\,dz = \iint \psi\,(lu + mv + nw)\,dS$$
$$- \iiint \psi\left(\frac{du}{dx} - \frac{dv}{dy} + \frac{dw}{dz}\right) dx\,dy\,dz = 0,$$

because we may take S so distant that $lu + mv + nw$ shall be zero everywhere upon it.

The Effect of Resistance.

21. If σ be the specific resistance per unit of area of the material of which a shell is composed, the components of electromotive force must be, by OHM's law, σu, σv, σw, where u, v, w are the component currents per unit of area. That is, (σ/h) U, (σ/h) V, and (σ/h) W, where h is the thickness of the shell, and U, V, W the components of superficial current. We see then that σ/h stands to U, V, W in the same relation as σ to u, v, w. And the heat generated by resistance per unit of time and unit area of the sheet is $h\sigma(u^2 + v^2 + w^2)$ or $\frac{\sigma}{h}(\mathrm{U}^2 + \mathrm{V}^2 + \mathrm{W}^2)$.

22. Let $d\mathrm{F}/dt$, $d\mathrm{G}/dt$, and $d\mathrm{H}/dt$ denote the time variations of vector potential.

Let Ψ be the electrostatic potential. Then the law of decay of the system, if left to itself to decay uninfluenced by induction, is expressed by the equations

$$\begin{aligned} \sigma u &= \frac{\sigma}{h}\mathrm{U} = -\frac{d\mathrm{F}}{dt} - \frac{d\Psi}{dx} \\ \sigma v &= \frac{\sigma}{h}\mathrm{V} = -\frac{d\mathrm{G}}{dt} - \frac{d\Psi}{dy} \\ \sigma w &= \frac{\sigma}{h}\mathrm{W} = -\frac{d\mathrm{H}}{dt} - \frac{d\Psi}{dz} \,. \end{aligned} \qquad \text{(A)}$$

These conditions must be fulfilled with some value of Ψ or other at every point within the substance of the shell. If σ, the specific resistance, be constant, we obtain by differentiation

$$\frac{d}{dt}\left(\frac{d\mathrm{F}}{dx} + \frac{d\mathrm{G}}{dy} + \frac{d\mathrm{H}}{dz}\right) + \nabla^2\Psi + \sigma\left(\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}\right) = 0$$

at every point within the substance of the shell.

23. If the currents decay as closed currents, their variations being given by that of a current function, this requires $\nabla^2\Psi = 0$ at every point within the substance of the shell.

But we have also

$$l\frac{d\mathrm{F}}{dt} + m\frac{d\mathrm{G}}{dt} + n\frac{d\mathrm{H}}{dt} + \frac{d\Psi}{d\nu} = -\sigma(lu + mv + nw) = 0$$

at every point on the surface.

Therefore, if the currents decay as closed currents, Ψ is *the associated function* above defined to $d\mathrm{F}/dt$, $d\mathrm{G}/dt$, $d\mathrm{H}/dt$, which we denote by ψ. The shell being of very small thickness, it matters not whether the distribution whose potential is ψ be on the inner or the outer surface. In either case $d\mathrm{F}/dt + d\psi/dx$, &c., are the tangential components of the time variation of vector potential.

Generally an arbitrarily given system of closed currents on a shell of given form

will not decay in this manner, unless the thickness of the shell be properly assigned at every point, or unless σ be properly assigned. For if $d\phi/dt$, the time variation of the current function, be given, $dF/dt + d\psi/dx$, and $dG/dt + d\psi/dy$, and $dH/dt + d\psi/dz$ are determinate. Now the equations A constitute two independent conditions to be fulfilled at every point. If, therefore, σ/h be given, it is not generally possible to satisfy the conditions A by any value of $d\phi/dt$.

The complete solution of any problem of this kind is the determination of $d\phi/dt$ at every point as a function of the time. That can be effected in special cases only.

Of Self-inductive Systems.

24. The class of cases most amenable to mathematical treatment is that in which $d\phi/dt = -\kappa\phi$, and, therefore,

$$\frac{dF}{dt} = -\kappa F, \quad \frac{dG}{dt} = -\kappa G, \quad \frac{dH}{dt} = -\kappa H,$$

where κ is a constant, independent both of the time and of position on the surface.

In any such case, if F_1, G_1, H_1 denote the initial values of F, G, H, then $F = F_1\epsilon^{-\kappa t}$, $G = G_1\epsilon^{-\kappa t}$, $H = H_1\epsilon^{-\kappa t}$, when the system decays in its own field.

The same must be the case with U, V, and W, and all linear functions of them, so that $U = U_1\epsilon^{-\kappa t}$, &c., and, Ω being a linear function of U, V, and W, $\Omega = \Omega_1\epsilon^{-\kappa t}$.

Also, since T is a quadratic function of U, V, W, we have $dT/dt = -2\kappa T_1$ and $T = T_1\epsilon^{-2\kappa t}$, giving the rate at which heat is generated in the decaying system.

A system of currents in a shell which has this property shall be here defined to be a *self-inductive system*. Professor LAMB, in his paper ('Phil. Trans.,' A., 1887, p. 131), calls this mode of decay "the natural decay." If the system be left to itself to decay in its own field, all the currents diminish proportionally, and the system varies in intensity but not in form.

25. Let now χ be the associated function to F, G, H, that is the function for which

$$\frac{d\chi}{d\nu} = lF + mG + nH \text{ on S},$$

and $\nabla^2\chi = 0$ within S.

Then evidently

$$\frac{d\psi}{dx} = \kappa\frac{d\chi}{dx}, \quad \frac{d\psi}{dy} = \kappa\frac{d\chi}{dy}, \quad \frac{d\psi}{dx} = \kappa\frac{d\chi}{dz},$$

and our equations (A) become

$$\frac{\sigma}{h} = \kappa\frac{F - d\chi/dx}{U} = \kappa\frac{G - d\chi/dy}{V} = \kappa\frac{H - d\chi/dz}{W}. \quad . \quad . \quad . \quad . \quad (B)$$

Now if the current function ϕ be given for a surface S, U, V, W, F, G, H, and χ are determinate.

Also the resultant of $F - d\chi/dx$, $G - d\chi/dy$, $H - d\chi/dz$ is necessarily in the tangent plane. But it is not necessarily in the same direction in that plane with the resultant current.

But the equations (B), or

$$\frac{F - d\chi/dx}{U} = \frac{G - d\chi/dy}{V} = \frac{H - d\chi/dz}{W},$$

cannot be satisfied unless the resultant of $F - d\chi/dx$, &c., *is* in the same direction in the tangent plane with the resultant current. They, therefore, express a necessary condition which the current function, ϕ, must satisfy, in order that the system may be capable of being made *self-inductive*. Evidently they express only one condition at every point, that a line known to be in the tangent plane shall have a particular direction in that plane. It may be put in the form of a partial differential equation to be satisfied at every point on S, namely,

$$\left(F - \frac{d\chi}{dx}\right)\frac{d\phi}{dx} + \left(G - \frac{d\chi}{dy}\right)\frac{d\phi}{dy} + \left(H - \frac{d\chi}{dz}\right)\frac{d\phi}{dz} = 0.$$

Since, if ϕ be given, χ is determinate, this is a partial differential equation in ϕ only. We may assume that, inasmuch as there are as many disposable quantities, namely, the values of ϕ at every point, as there are conditions to be fulfilled, there must for every surface S be one or more solutions. As we shall see, if S be a sphere, and in certain other special cases, there are many.

26. If this necessary condition for ϕ be fulfilled, it makes the three quantities

$$\frac{F - d\chi/dx}{U}, \quad \frac{G - d\chi/dy}{V}, \quad \frac{H - d\chi/dz}{W}$$

equal to each other at every point. But they will generally differ in values from point to point on the surface.

Let each of them be denoted by Q. Then Q is a function depending on the form of S and ϕ and the position on the surface.

Then in order that the system, with ϕ so chosen, may actually be self-inductive, we must so regulate the thickness, h, of the shell, as that

$$\frac{\sigma}{h} = \kappa Q = \kappa\,\frac{F - (d\chi/dx)}{U}, \text{ \&c.}$$

at every point, κ being a constant.

Of course we may make σ vary instead of, or as well as, h. But it is most convenient to keep σ constant, or the shell of uniform material.

If h be so chosen with σ constant, it harmonises the equations (A)

$$\left.\begin{aligned} \frac{\sigma}{h}\mathrm{U} &= -\frac{d\mathrm{F}}{dt} - \frac{d\psi}{dx} = \kappa\left(\mathrm{F} - \frac{d\chi}{dx}\right) \\ \frac{\sigma}{h}\mathrm{V} &= -\frac{d\mathrm{G}}{dt} - \frac{d\psi}{dy} \\ \frac{\sigma}{h}\mathrm{W} &= -\frac{d\mathrm{H}}{dt} - \frac{d\psi}{dz} \end{aligned}\right\} \text{(A)};$$

and the shell so formed is, with the given current function ϕ, *self-inductive.*

It follows from the above that if a shell be self-inductive to a system of currents denoted by U, V, W, then if the components of electromotive force due to the external field be at every point on the surface proportional to the components of those currents, they will induce in the shell a system of that type.

We see further that, if ϕ do *not* satisfy the required condition, the shell cannot be made self-inductive, however we may choose σ/h.

27. The constant κ determines the rate of decay of the currents with the time. Since $\kappa = \sigma/\mathrm{Q}h$, we see that κ varies directly as σ, and inversely as h; that is, if the thickness of the shell be increased at every point in the same ratio, κ is diminished in that same ratio. Also κ varies inversely as Q, and Q depends on the forms of S and ϕ. As an example, if S be a sphere of radius a, and ϕ a spherical surface harmonic of order n, then, as is well known,

$$\mathrm{Q} = \frac{4\pi a}{2n+1}, \quad \text{and} \quad \kappa = \frac{2n+1}{4\pi a}\,\frac{\sigma}{h}.$$

28. If now a conducting shell S be placed in any varying magnetic field, and if the system of currents induced on it by variation of the field be always self-inductive, the state of the system at any time, t, can be found, if the law of time variation of the external field be given.

For let Ω_0 be the magnetic potential of the external field, Ω that of the induced system. Then we have

(1.) Due to induction alone

$$\frac{d\Omega_0}{dt} + \frac{d\Omega}{dt} = 0.$$

(2.) Due to resistance alone

$$\frac{d\Omega}{dt} = -\kappa\Omega.$$

Therefore, generally,

$$\frac{d\Omega_0}{dt} + \frac{d\Omega}{dt} + \kappa\Omega = 0;$$

from which Ω can be found as a function of t, if $d\Omega_0/dt$ be given

29. For example, let $d\Omega_0/dt$ be constant as regards time, but have different values at different points on the surface. At any point let $d\Omega_0/dt = -\mathrm{C}$. Then we have $d\Omega/dt + \kappa\Omega = \mathrm{C}$ to determine the value of Ω at the point.

Whence $\Omega = \frac{\mathrm{C}}{\kappa}(1 - \epsilon^{-\kappa t})$.

If now C be so great that we may make κt infinitely small while Ct remains finite, this represents the ideal case of a system of so called *impulsive currents;* that is, finite currents supposed to be created in an infinitely short time, and Ct represents the *impulse.* In this case $\Omega = \mathrm{C}t$, and is independent of resistance.

If, on the other hand, we make κt very great compared with unity, which we always may do by sufficiently increasing the resistance or the time, the result becomes $\Omega = \mathrm{C}/\kappa$; that is, Ω varies inversely as the resistance. This is a particular case of the result obtained by Lord RAYLEIGH ("On Forced Harmonic Oscillations of Various Periods," 'Phil. Mag.,' May, 1886).

30. Again, let the external system vary according to a simple harmonic law, so that $\Omega_0 = \mathrm{C} \cos \lambda t$, where C is constant as regards time, but a function of position in space. Then our equation becomes, either on the surface or at any internal point,

$$\frac{d\Omega}{dt} + \kappa\Omega = \mathrm{C}\lambda \sin \lambda t,$$

where C has different values according to the point selected, and λ and κ are constant.

We may assume as a solution

$$\Omega = \mathrm{C}' (\cos \lambda t + q \sin \lambda t).$$

Then

$$\mathrm{C}'(\kappa + q\lambda) \cos \lambda t = (\mathrm{C}\lambda + \mathrm{C}'\lambda - \kappa \mathrm{C}'q) \sin \lambda t.$$

And, therefore,

$$q = -\frac{\kappa}{\lambda},$$

and

$$\mathrm{C}'\left\{\lambda + \frac{\kappa^2}{\lambda}\right\} = -\mathrm{C}\lambda, \quad \text{or} \quad \mathrm{C}' = -\mathrm{C} \cdot \frac{\lambda^2}{\kappa^2 + \lambda^2}.$$

And

$$\Omega = -\mathrm{C}\frac{\lambda^2}{\kappa^2 + \lambda^2}\left(\cos \lambda t - \frac{\kappa}{\lambda} \sin \lambda t\right)$$

$$= -\frac{\mathrm{C}\lambda}{\kappa^2 + \lambda^2}(\lambda \cos \lambda t - \kappa \sin \lambda t).$$

And, if

$$\frac{\kappa}{\lambda} = \cot \alpha,$$

$$\Omega = -\mathrm{C} \sin \alpha \sin (\lambda t - \alpha),$$

and

$$\Omega_0 + \Omega = C(\cos\lambda t - \sin\alpha \sin \overline{\lambda t - \alpha})$$
$$= C\cos\alpha\cos\overline{\lambda t - \alpha}. \quad \ldots \ldots \ldots \quad (D)$$

The internal field is therefore diminished in intensity in the proportion $\cos\alpha : 1$, and retarded in phase by $\alpha/2\pi$ of a complete period.

This result agrees with that obtained by Professor LARMOR in case of a spherical sheet ('Phil. Mag.,' January, 1884).

31. The above results are obtained on the tacit hypothesis that the shell, whatever its thickness, is to be regarded for our purpose as a single shell in which all the currents would decay *pari passu,* no allowance being made for variations along the normal. On that hypothesis we may, if σ/h be finite, have finite superficial currents in the shell; and, as a consequence of their being finite, we have a finite difference of phase, and intensity of the field diminished in a finite ratio, between the outer and inner surfaces.

This method cannot give accurate results except in the case of very thin shells. Another way of treating the subject would be to regard the shell as made up of a number of separate layers or subsidiary shells, successively enclosing one another and separated, suppose, by non-conducting surfaces. Then we might apply the formula (D) to each separate layer, and finally proceeding to the limit, make all the functions vary continuously throughout the thickness of the solid shell. We will consider the question in this aspect later. In the meantime we will point out certain consequences which follow from the formula (D), when σ/h becomes infinite, and the superficial currents infinitely small, namely, since $\cot\alpha = \kappa/\lambda$, $\cos\alpha = \kappa/\sqrt{(\kappa^2+\lambda^2)}$, $\sin\alpha = \lambda/\sqrt{(\kappa^2+\lambda^2)}$.

When σ/h becomes infinite, κ becomes infinite compared with λ. Hence, $\cos\alpha = 1$ and $\sin\alpha = \alpha = \lambda/\kappa$.

The formula (D) becomes then

$$\Omega = -C\alpha\sin\lambda t;$$

or, since Ω and α are infinitesimal, we may write

$$\frac{d\Omega}{d\alpha} = -C\sin\lambda t.$$

Again,

$$\alpha = \frac{\lambda}{\kappa} = \frac{\lambda Q h}{\sigma};$$

and, writing $d\nu$, an element of the normal, for h,

$$\frac{d\Omega}{d\nu} = -C\sin\lambda t\,\frac{\lambda Q}{\sigma}.$$

Again, any given phase of the disturbance occurs at a later time in the inner than in the outer field, the difference of time being $dt = \alpha/\lambda$. In the case we are now treating $\alpha = \lambda/\kappa$, and $\alpha/\lambda = 1/\kappa = Q/\sigma h$. The ratio of h to this time, or σ/Q, is the *velocity with which the disturbance penetrates the shell.* Since it is independent of λ, it must be the same for all systems of currents of the type ϕ on the surface S.

We have thus obtained an answer to the question suggested in (17), so far as regards a self-inductive system of currents. The velocity, namely, with which they initially penetrate a solid, or, which is the same thing, the thickness of the stratum which they may be supposed to occupy at a very short time after the commencement of the induction, is proportioned to the thickness of the self-inductive shell at any point.

32. The energy dissipated in the shell per unit of time is $2\kappa T$.

We see then that, comparing similar self-inductive systems with different values of κ, but the same mean energy, the heat generated on average per unit of time varies as κ, or, as this heat must all be drawn from the batteries of the primary system, the *cost of maintenance* of the system varies as κ.

Examples of Self-inductive Systems.

33. A spherical current sheet. (See the works cited above.)

Every spherical current sheet is self-inductive with σ/h constant, if ϕ be a spherical surface harmonic of any one order as $A_n Y_n$. For, the sheet being spherical, $\psi = 0$; and, by a known property of the sphere,

$$F = \frac{4\pi a}{2n+1} U, \quad \text{or} \quad \frac{F}{U} = \frac{4\pi a}{2n+1}$$

when a is the radius.

Similarly,

$$\frac{G}{V} = \frac{4\pi a}{2n+1}, \quad \frac{H}{W} = \frac{4\pi a}{2n+1}.$$

The condition for ϕ is then satisfied; and, as $4\pi a/(2n+1)$ is constant, σ/h has constant value over the surface, or the shell, if of uniform material, must in order that the system may be self-inductive be of uniform thickness. In this case $Q = 4\pi a/(2n+1)$ and $\kappa = \dfrac{2n+1}{4\pi a} \cdot \dfrac{\sigma}{h}$, if h be the uniform thickness of the shell.

34. If S be a solid of revolution about the axis of z then any system of currents on it, determined by an arbitrary function of z as current function, satisfies the conditions $F/U = G/V$ at every point with $\psi = 0$, $H = 0$, provided $d\phi/dz$ be of the same sign throughout S; and, therefore, any such system may be made self-inductive by suitably choosing σ/h.

For the lines of resultant current are circles round points in the axis as centres, and the lines of vector potential are also circles round points in the axis. Therefore

($d\phi/dz$ being always of the same sign) at any point in the surface the resultant of F and G coincides in direction with that of U and V, and, therefore, $F/U = G/V$. Also $lF + mG + nH = 0$ at every point, and, therefore, $\psi = 0$.

35. Again, if ϕ be a function of z only, and if χ, derived from it by the methods above explained, do not contain z, the equations of condition reduce to two, namely :

$$\frac{\sigma}{h} U = \kappa \left(F - \frac{d\chi}{dx} \right),$$

$$\frac{\sigma}{h} V = \kappa \left(G - \frac{d\chi}{dy} \right);$$

and these must necessarily be satisfied at every point, because the resultant of U and V is the intersection of the tangent plane with a plane parallel to that of xy. And this line is also the resultant of $F - d\chi/dx$ and $G - d\chi/dy$. We can then determine σ/h in terms of κ. As an example, let us take for our sheet the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1,$$

and make

$$\phi = Az$$

(see Professor LAMB's paper, 'Phil. Trans.,' A., 1887, above referred to).

We have then

$$U = A \frac{\varpi y}{b^2},$$

$$V = -A \frac{\varpi x}{a^2},$$

$$W = 0,$$

where ϖ is the perpendicular from the centre on the tangent plane. Also, at any internal point,

$$F = A_1 \frac{y}{b^2}, \qquad G = -A_2 \frac{x}{a^2};$$

and at any external point,

$$F = A_1 \frac{y}{b^2} \frac{\dfrac{dq}{db^2}}{\dfrac{dq_0}{db^2}}, \qquad G = -A_2 \frac{x}{a^2} \frac{\dfrac{dq}{da^2}}{\dfrac{dq_0}{da^2}},$$

where

$$q = \int_\lambda^\infty \frac{d\lambda}{\sqrt{\{(a^2 + \lambda)(b^2 + \lambda)(c^2 + \lambda)\}}}$$

and

$$q_0 = \int_0^\infty \frac{d\lambda}{\sqrt{\{(a^2 + \lambda)(b^2 + \lambda)(c^2 + \lambda)\}}}$$

and λ is the positive root of

$$\frac{x^2}{a^2+\lambda}+\frac{y^2}{b^2+\lambda}+\frac{z^2}{c^2+\lambda}=1;$$

and therefore on the sheet

$$\mathrm{F}=\mathrm{A}_1\frac{y}{b^2}, \qquad \mathrm{G}=-\mathrm{A}_2\frac{x}{a^2}.$$

Now, χ is the function which satisfies $d\chi/d\nu=l\mathrm{F}+m\mathrm{G}=(\mathrm{A}_1-\mathrm{A}_2)\,\varpi xy/a^2b^2$ on S, and $\nabla^2\chi=0$ within S.

That is, $\chi=\mathrm{C}xy$, when $\mathrm{C}=(\mathrm{A}_1-\mathrm{A}_2)/(a^2+b^2)$, and this is independent of z. We can therefore make the shell self-inductive.

Again, on the sheet

$$\frac{d\mathrm{F}}{d\nu}-\frac{d\mathrm{F}}{d\nu'}=-4\pi\mathrm{U};$$

that is,

$$-\frac{\mathrm{A}_1 y}{ab^5c}\frac{\varpi}{\frac{dq_0}{db^2}}=-4\pi\mathrm{A}\frac{\varpi\, y}{b^2};$$

and, therefore,

$$\mathrm{A}_1=\mathrm{A}\,.\,4\pi ab^3c\frac{dq_0}{db^2}.$$

Similarly,

$$\mathrm{A}_2=\mathrm{A}\,.\,4\pi a^3bc\frac{dq_0}{da^2}.$$

We have then to satisfy the equations

$$\frac{\sigma}{h}\mathrm{U}=\frac{\sigma}{h}\mathrm{A}\frac{\varpi y}{b^2}=\kappa\left(\mathrm{A}\;4\pi ab^3c\frac{dq_0}{db^2}\frac{y}{b^2}-\frac{d\chi}{dx}\right),$$

or

$$\frac{\sigma}{h}\mathrm{A}\frac{\varpi y}{b^2}=\kappa\left(\mathrm{A}\;4\pi abc\frac{dq_0}{db^2}y-\frac{d\chi}{dx}\right),$$

$$\frac{\sigma}{h}\mathrm{A}\frac{\varpi x}{a^2}=\kappa\left(\mathrm{A}\;4\pi abc\frac{dq_0}{da^2}x-\frac{d\chi}{dy}\right).$$

If we now make $\sigma/h=1/(p\varpi)$, where p is constant, we have, by differentiation,

$$\frac{\mathrm{A}}{p}\cdot\left(\frac{1}{b^2}-\frac{1}{a^2}\right)=4\pi\kappa\mathrm{A}\,.\,abc\,.\left(\frac{dq_0}{db^2}-\frac{dq_0}{da^2}\right),$$

which determines κ in terms of (1) a, b, c, (2) the absolute thickness of the shell, (3) σ, the specific resistance of the material of which the shell is composed. It is independent of the absolute value of the current function.

Case of an Infinite Plane Sheet.

36. The case of an infinite plane sheet differs somewhat in its practical treatment from that of a sphere, and requires independent investigation. It has been fully treated by MAXWELL, MASCART and JOUBERT, and other writers. We here regard it from a somewhat different point of view.

Let the plane be that of xy.

If then for any system of currents in it the condition $F/V = G/V$ be satisfied, we can always make the plane sheet *self-inductive* by suitably choosing σ/h.

For instance, let the system of currents in the plane be induced by the variation of an infinitely small circular current i, parallel to the plane, and of radius a, and distant z from the plane. In that case we see at once by symmetry that the induced currents flow in circles round O, the foot of the perpendicular from the centre of the circular current on the plane. The same is the case with the vector potential of all the currents; and therefore the resultant of F and G coincides with the resultant current. Also $\psi = 0$ in this case. We have then $F/U = G/V$ at every point.

But

$$\frac{dF}{dz} = -2\pi U, \qquad \frac{dG}{dz} = -2\pi V.$$

Therefore, in order that the sheet may be self-inductive, we must make

$$\frac{\sigma}{h} = -\kappa \frac{2\pi F}{\frac{dF}{dz}} = -\kappa \frac{2\pi G}{\frac{dG}{dz}}.$$

But, if r be the distance of a point on the sheet from the circular current

$$F = -2ai \cdot \frac{d}{dy}\frac{1}{r},$$

$$G = -2ai \cdot \frac{d}{dx}\frac{1}{r};$$

and therefore

$$\frac{dF}{dz} = -6ai \cdot \frac{yz}{r^5},$$

$$\frac{dG}{dz} = +6ai \cdot \frac{xz}{r^5};$$

and

$$\frac{\sigma}{h} = \frac{2\pi\kappa}{3}\frac{r^2}{z},$$

or $h \propto 1/r^2$.

That is the condition that the currents excited in a plane sheet under the influence of an external field of the kind in question may be self-inductive.

37. It is usual to treat the sheet with σ/h constant, and therefore without making it self-inductive. We have generally at any point on the sheet

$$\frac{dF}{dz} = -2\pi U, \qquad \frac{dG}{dz} = -2\pi V.$$

And, therefore, on the sheet

$$\frac{dF}{dt} = \frac{\sigma}{2\pi}\frac{dF}{dz}, \qquad \frac{dG}{dt} = \frac{\sigma}{2\pi}\frac{dG}{dz},$$

and

$$\begin{aligned}\frac{d}{dt}\frac{d\Omega}{d\nu} &= \frac{d}{dt}\left(\frac{dG}{dx} - \frac{dF}{dy}\right)\\ &= \frac{d}{dx}\frac{dG}{dt} - \frac{d}{dy}\frac{dF}{dt}\\ &= \frac{\sigma}{2\pi}\frac{d}{dz}\left(\frac{dG}{dx} - \frac{dF}{dy}\right) = \frac{\sigma}{2\pi}\frac{d}{dz}\frac{d\Omega}{d\nu}.\end{aligned}$$

We can now treat the problem directly by the method of (16).

For we have by induction in the absence of resistance

$$\frac{d}{dt}\frac{d\Omega_0}{d\nu} + \frac{d}{dt}\frac{d\Omega}{d\nu} = 0,$$

if Ω_0 be the magnetic potential of the field.

Therefore, generally, to determine the value of Ω on the sheet,

$$\frac{d}{dt}\frac{d\Omega_0}{d\nu} + \frac{d}{dt}\frac{d\Omega}{d\nu} = \frac{\sigma}{2\pi}\frac{d}{dz}\frac{d\Omega}{d\nu}.$$

For example, if the external field is constant as regards time, and the sheet rotates with uniform angular velocity ω round the axis of z,

$$\frac{d}{dt}\frac{d\Omega_0}{d\nu} = \omega\frac{d}{d\theta}\frac{d\Omega_0}{d\nu};$$

and, when the motion has become steady,

$$\frac{d}{dt}\frac{d\Omega}{d\nu} = \omega\frac{d}{d\theta}\frac{d\Omega}{d\nu},$$

where θ is the angle between a plane through the axis and the point considered, and a fixed plane through the axis.

And therefore we obtain a solution for steady motion in the form

$$\omega \frac{d}{d\theta}\left(\frac{d\Omega_0}{d\nu} + \frac{d\Omega}{d\nu}\right) = \frac{\sigma}{2\pi} \frac{d}{dz} \frac{d\Omega}{d\nu}.$$

This is the problem of ARAGO's disc. The result agrees with MAXWELL, § 668 (24).

Concerning Similar Shells Enclosing One Another.

38. Any such shell may be conceived as made up of a series of infinitely thin shells successively enclosing one another. We will consider the case of *similar* shells, which shall be defined as follows:—

Let S be a homogeneous function of x, y, and z of positive degree, the nth. Then we may form about the origin a series of surfaces whose equations are $S = c^n$, where c denotes the *linear dimensions.*

We may call the shells *concentric.*

Points at which the same radius vector from the origin cuts the surfaces are *corresponding points.*

The space between two neighbouring surfaces of the series is *a shell.* Two shells are *similar* when dc varies as c; that is, the thickness of the shell at corresponding points varies as c.

If in two similar shells of a series all similarly situated the current at every point in one is parallel to, and bears a given ratio to, the current at the corresponding point in the other, the systems of currents are *similar* or *corresponding.*

The ratio last mentioned may be any power of c. If it be given, then ϕ in the one shell is equal to ϕ at the corresponding point in the other multiplied by a power of c.

39. We can now compare the value of certain functions in corresponding systems of currents.

Firstly,

$$Q = \frac{F - d\chi/dx}{U} \text{ in all cases varies as the linear dimensions.}$$

Secondly,

$$\kappa = \frac{\sigma}{Qh} \text{ varies inversely as the square of the linear dimensions.}$$

These results are independent of the form of S or ϕ, or the ratio between the currents at corresponding points.

Thirdly, if we make

$$u, v, w \text{ vary as } c^n,$$

then

$$U, V, W \text{ will vary as } c^{n+1}.$$

$$\Omega \text{ and } \phi \text{ will vary as } c^{n+2}.$$

40. If any one of a series of similar shells with similar currents be self-inductive, every one of the shells is self-inductive.

If all the shells within S be filled with similar currents, as above defined, the components of current per unit of area must be

$$u = \frac{dS}{dz}\frac{d\phi}{dy} - \frac{dS}{dy}\frac{d\phi}{dz},$$

$$\&c. = \&c.$$

Let us now find a condition that, if a system of currents of the type ϕ be generated in the outer shell, any inner shell of the series, if a conductor and exposed to the influence of the outer, shall have the corresponding system of currents of the same type excited in it.

At any point on the outer shell S, since the shell is self-inductive, we have

$$\kappa\left(F - \frac{d\chi}{dx}\right) = \sigma u = \sigma\left(\frac{dS}{dz}\frac{d\phi}{dy} - \frac{dS}{dy}\frac{d\phi}{dz}\right)$$

where F is the component of vector potential of the currents in the outer shell S, and χ the associated function. Now, $\nabla^2(F - d\chi/dx) = 0$ at all points within S. If, therefore, $\nabla^2 u = 0$, u being a harmonic of positive degree, at all points within S, it follows that

$$\kappa\left(F - \frac{d\chi}{dx}\right) = \sigma u = \sigma\left(\frac{dS}{dz}\frac{d\phi}{dy} - \frac{dS}{dy}\frac{d\phi}{dz}\right)$$

at all points within S. The same is true of $G - d\chi/dy$ if $\frac{dS}{dx}\frac{d\phi}{dz} - \frac{dS}{dz}\frac{d\phi}{dx}$ is a solid harmonic of positive degree, &c. Let S be any one of the shells within S, and suppose it to become a conductor. If it be self-inductive to the system of currents u, v, w, it follows that this system with reversed sign, and no other, will be excited in it by induction when the corresponding system is excited in the shell S. For instance, in the case of the ellipsoid above treated,

$$S = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2},$$

$$\phi = Az,$$

$$\frac{dS}{dz}\frac{d\phi}{dy} - \frac{dS}{dy}\frac{d\phi}{dz} = -2\frac{Ay}{b^2},$$

and

$$\nabla^2\left(\frac{dS}{dz}\frac{d\phi}{dy} - \frac{dS}{dy}\frac{d\phi}{dz}\right) = 0.$$

If, therefore, the ellipsoid $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$ be divided into similar, similarly-situated, and concentric ellipsoidal shells, each shell is, as we have proved above, self-

inductive to a system of currents determined by $\phi = \mathrm{A}z$. And, if this system be excited in an outer shell of the series, it excites the corresponding system in an inner one.

Similarly, if $\frac{d\mathrm{S}}{dz}\frac{d\phi}{dy} - \frac{d\mathrm{S}}{dy}\frac{d\phi}{dz}$, &c., were spherical harmonics of negative degree, we should prove that the system of currents of the type ϕ, excited in an inner shell of the series, would generate by induction a system of the corresponding type in any outer one.

Of Shells of Finite Thickness.

41. If the superficial currents induced and maintained in a shell be very small, or the currents per unit of area finite, the inductive effect of the shell itself is inappreciable compared with that of the original or inducing system.

Suppose, then, we have a series of similar, similarly-situated, and concentric shells successively enclosing one another, so as to form one solid shell of finite thickness. Let each be separately self-inductive to the currents excited by the external field. If the thickness of the solid shell, though finite, be small, we may, without great error, neglect the inductive effect of any inner shell of the series of which it is composed upon the outer ones.

Let then S be any shell of the series, Ω the magnetic potential on the outer surface of S due to the whole field outside of it; then we shall have, by (30) and (31), if Ω_0 vary according to the simple harmonic law,

$$\frac{d\Omega}{d\nu} = -\mathrm{A}\frac{\lambda \mathrm{Q}}{\sigma}\sin \lambda t, \text{ A being a constant};$$

and, if p be the phase,

$$\frac{dp}{d\nu} = -\frac{\lambda \mathrm{Q}}{\sigma}.$$

Let c denote the linear dimensions of any shell, and let c_1, c_2 be the values of c at the outer and inner surfaces respectively of the solid shell. Then

$$h = \frac{d\nu}{dc},$$

and these equations become

$$\frac{d\Omega}{dc} = h\frac{d\Omega}{d\nu} = -\mathrm{A}\frac{\lambda \mathrm{Q} h}{\sigma}\sin \lambda t = -\mathrm{A}\frac{\lambda}{\kappa}\sin \lambda t,$$

$$\frac{dp}{dc} = -\frac{\lambda}{\kappa};$$

or

$$\Omega_1 - \Omega_2 = \mathrm{A}\lambda \int \frac{1}{\kappa}\, dc \sin \lambda t,$$

$$p_1 - p_2 = \lambda \int \frac{1}{\kappa}\, dc;$$

and, since κ varies inversely as c^2,

$$\Omega_1 - \Omega_2 \propto A\lambda\,(c_1^3 - c_2^3)\sin\lambda t,$$
$$p_1 - p_2 \propto \lambda\,(c_1^3 - c_2^3).$$

42. In any case of induction, if the primary or external system become after a while constant, the induced currents will decay by resistance, and the electrostatic charge, whose potential is ψ, will disappear also. Both the induced currents and the electrostatic charge in disappearing generate heat in the conductor; and this heat is obtained at the expense of the chemical energy of the batteries of the primary system. We know that the closed currents on the conductor, coming into existence and decaying, cause on the whole more chemical energy to be spent in the batteries of the primary system than is accounted for by heat generated in that system, the excess being equal to the energy dissipated in the induced system of currents. The energy of the electrostatic charge, if such exist, must also be obtained at the expense of the batteries. We should then expect to find that charging a conductor electrostatically in the neighbourhood of a closed battery circuit, or moving a charged body in the neighbourhood of the circuit, tends to retard or accelerate the current; that is, to increase or diminish the chemical energy spent per unit of time in maintaining the current constant.