II. *A Class of Functional Invariants.*

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The investigations herein contained are indirectly connected with some results in an earlier memoir.* In that memoir functions called quotient-derivatives are obtained in the form of certain combinations of differential coefficients of a quantity y dependent on a single independent variable x; and they are there shown to possess the property of invariance for isolated homographic transformations of the dependent and the independent variables. It is evident, however, from their form that they do not constitute the complete aggregate of irreducible invariants for the case of a single independent variable; and the deduction of this aggregate and an investigation of the relation in which they stand to a particular class of reciprocants were made in a subsequent paper.† The present memoir is a continuation of the theory of functional invariants, the invariants herein considered being constituted by combinations of the differential coefficients of a function of more than one independent variable which are such that, when the independent variables are transformed, each combination is reproduced save as to a factor depending on the transformations to which the variables are subjected. The transformations, in the case of which any detailed results are given, are of the general homographic type; and the investigations are limited to invariantive derivatives of a function of two independent variables only, a limitation introduced partly for the sake of conciseness. The characteristic properties, such as the symmetry of the invariants and the forms of the simultaneous linear partial differential equations satisfied by them, can in the case of more than two independent variables be inferred from the properties actually given; but many of the deductions made are necessarily proper to functions of only two independent variables.

In the matter of notation it is convenient here to state that the independent variables are denoted by x and y, and the dependent variable by z. The general differential coefficient $\partial^{m+n}z/\partial x^m\,\partial y^n$ is represented by $z_{m,n}$; but frequently the following modifications for the notation of particular coefficients are made, viz.:

$$\begin{aligned}&p,\ q \text{ replace } z_{10},\ z_{01}:\\&r,\ s,\ t \text{ replace } z_{20},\ z_{11},\ z_{02}:\\&a,\ b,\ c,\ d\ \ .\ .\ .\ z_{30},\ z_{21},\ z_{12},\ z_{03}:\\&e,\ f,\ g,\ h,\ i\ \ .\ .\ .\ z_{40},\ z_{31},\ z_{22},\ z_{13},\ z_{04}:\end{aligned}$$

* "Invariants, Covariants, and Quotient-Derivatives associated with Linear Differential Equations," 'Phil. Trans.,' A, 1888, pp. 377–489.

† "Homographic Invariants and Quotient-Derivatives," 'Mess. of Math.,' vol. 17 (1888), pp. 154–192.

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respectively. The transformed independent variables are denoted by X and Y; and quantities bearing to them the same relation as the foregoing bear to x and y are denoted by $Z_{m,n}$, P, Q, ... And in three different instances it has been necessary, for the sake of uniformity of notation for similar successions of quantities, to use different symbols for the same quantity occurring in different successions; these are $u_4 = A_0$ (§§ 3, 12), $u_8 = -A_1$ (§§ 12, 13), $u_{13} = A_2$ (§§ 16, 17).

The general results of the memoir may be stated as follows:—

Every invariant is explicitly free from the variables themselves, viz., the dependent and the two [m] independent variables; it is homogeneous in the differential coefficients of the dependent variable; it is of uniform grade in differentiations with regard to each of the dependent variables, and it is either symmetric or skew symmetric with regard to such differentiations.

It satisfies six [$m^2 + m$] linear partial differential equations, all of the first order, of which four [m^2] are characteristic equations and determine the form of the invariant, and the remaining two [m] are index equations and are identically satisfied when the form is known and the index is derived by inspection from the form.

Every invariant involves the two [m] differential coefficients of the first order.

The following results relative to irreducible invariants derived from a single dependent variable z are given:—The invariants can be ranged in sets, each set being proper * to a particular rank. There is no invariant proper to the rank 1; there is one proper to the rank 2; there are three invariants proper to the rank 3; and, for a value of n greater than 3, there are $n + 1$ invariants proper to the rank n, which can be chosen so as to be linear in the differential coefficients of order n. Every invariant can be expressed in terms of these irreducible invariants; and the expression involves invariants of rank no higher than the order of the highest differential coefficient which occurs in that invariant.

In the case of irreducible invariants, involving differential coefficients of two dependent variables, it is shown that there is a single one proper to the rank 1, and that there are four proper to the rank 2.

Some eductive operators are given; and in one case the educts are discussed so as to select those of the invariants thus obtained which are evidently reducible. Some general results analogous to reversor operations are derived.

Finally, it is shown how the theory of binary forms can be partly connected with the theory of functional invariants; for functional invariants are expressible in terms of the simultaneous concomitants of a certain set of quantities, viewed as binary quantics of successive orders in q and $-p$ as variables.

[Note added December 5, 1888.†—The invariants in the present memoir are distinct

* An invariant is said to be *proper* to the rank n when the highest differential coefficient of z occurring in it is of order n.

† This addition is due to a desire which has been expressed that some indication should be given of the difference between the functions considered in the present memoir and invariantive functions of

in character from the differential invariants of M. HALPHEN and the ternary reciprocants of Mr. ELLIOTT.

The earliest record of M. HALPHEN's investigations is his well-known thesis,* wherein he considers the invariance of a differential equation $f(x, y, y', y'', \ldots) = 0$, when the single independent variable x and the single dependent variable y are (p. 20, *loc. cit.*) subjected to the transformation

$$\frac{\mathrm{X}}{ax + by + c} = \frac{\mathrm{Y}}{a'x + b'y + c'} = \frac{1}{a''x + b''y + c''}.$$

The only reference in the thesis to the case of three variables is (p. 60) in the concluding paragraph, where it is said that the theory can be extended to the case of one dependent variable z and two dependent variables x and y, the transformation suggested, but not explicitly stated, being

$$\frac{\mathrm{X}}{\alpha x + \beta y + \gamma z + \delta} = \frac{\mathrm{Y}}{\alpha' x + \beta' y + \gamma' z + \delta'} = \frac{\mathrm{Z}}{\alpha'' x + \beta'' y + \gamma'' z + \delta''} = \frac{1}{\alpha''' x + \beta''' y + \gamma''' z + \delta'''}.$$

M. HALPHEN, again,† considers differential invariants, in which the last transformation is effected on functions of the three variables; but in this investigation y and z are taken to be two dependent variables of the single independent variable x.

Mr. ELLIOTT's theory ‡ of ternary reciprocants is closely connected with the concluding paragraph of M. HALPHEN's thesis; the functions are invariantive for interchanges of z, x, y, where z is a variable dependent on x and y; and the pure reciprocants are invariantive for the above-suggested transformations.

The theory in this memoir deals almost entirely with the case of three variables, z, x, y, where z is a dependent variable, and x and y are independent variables. The transformations, through which the invariance is maintained, refer to the independent variables only; they are—

$$\frac{x}{\alpha_1 + \beta_1 \mathrm{X} + \gamma_1 \mathrm{Y}} = \frac{y}{\alpha_2 + \beta_2 \mathrm{X} + \gamma_2 \mathrm{Y}} = \frac{1}{\alpha_3 + \beta_3 \mathrm{X} + \gamma_3 \mathrm{Y}}.$$

The dependent variable is left untransformed; it does not enter into the equations of transformation.

It follows, from the difference between the transformation in the theory here

other classes, such as the differential invariants of M. HALPHEN and the ternary reciprocants of Mr. ELLIOTT.

* 'Sur les Invariants Différentiels,' Paris, 1878.

† "Sur les Invariants Différentiels des Courbes gauches," 'Journ. de l'École Polytechnique,' vol. 28, 1880, pp. 1–102.

‡ "On Ternary and n-ary Reciprocants," 'London Math. Soc. Proc.' vol. 17 (1886), pp. 171–196; "On the Linear Partial Differential Equations satisfied by pure Ternary Reciprocants," *ibid.*, vol. 18 (1887), pp. 142–164; "On pure Ternary Reciprocants and Functions allied to them," *ibid.*, vol. 19 (1888), pp. 6–23.

exposed in which the dependent variable does not enter into the equations of transformation, and the transformations above indicated in which the occurrence of the dependent variable in the equations of transformation is essential, that different results will be obtained. Two examples will suffice. First, a comparison of the characteristic equations of ELLIOTT's reciprocants and of those characteristic of the present functional invariants may be made from the forms expressed in the notations of this memoir :—

Annihilators of ELLIOTT's reciprocants.

$$\Omega_1 = r\frac{\partial}{\partial s} + 2s\frac{\partial}{\partial t} + a\frac{\partial}{\partial b} + 2b\frac{\partial}{\partial c} + 3c\frac{\partial}{\partial d} + \ldots$$

$$\Omega_2 = t\frac{\partial}{\partial s} + 2s\frac{\partial}{\partial r} + d\frac{\partial}{\partial c} + 2c\frac{\partial}{\partial b} + 3c\frac{\partial}{\partial a} + \ldots$$

$$\begin{aligned} -\mathrm{E}_1 = -\mu + 3r\frac{\partial}{\partial r} + 2s\frac{\partial}{\partial s} + t\frac{\partial}{\partial t} \\ + 4a\frac{\partial}{\partial a} + 3b\frac{\partial}{\partial b} + 2c\frac{\partial}{\partial c} + d\frac{\partial}{\partial d} + \ldots \end{aligned}$$

$$\begin{aligned} -\mathrm{E}_2 = -\mu + r\frac{\partial}{\partial r} + 2s\frac{\partial}{\partial s} + 3t\frac{\partial}{\partial t} \\ + a\frac{\partial}{\partial a} + 2b\frac{\partial}{\partial b} + 3c\frac{\partial}{\partial c} + 4d\frac{\partial}{\partial d} + \ldots \end{aligned}$$

$$\begin{aligned} \mathrm{V}_1 = 3r^2\frac{\partial}{\partial a} + 3rs\frac{\partial}{\partial b} + (rt + 2s^2)\frac{\partial}{\partial c} \\ + 3st\frac{\partial}{\partial d} + \ldots \end{aligned}$$

$$\begin{aligned} \mathrm{V}_2 = 3rs\frac{\partial}{\partial a} + (rt + 2s^2)\frac{\partial}{\partial b} + 3st\frac{\partial}{\partial c} \\ + 3t^2\frac{\partial}{\partial d} + \ldots \end{aligned}$$

Annihilators of Invariants in this memoir.

$$\begin{aligned} \Delta_4 = p\frac{\partial}{\partial q} + r\frac{\partial}{\partial s} + 2s\frac{\partial}{\partial t} \\ + a\frac{\partial}{\partial b} + 2b\frac{\partial}{\partial c} + 3c\frac{\partial}{\partial d} + \ldots \end{aligned}$$

$$\begin{aligned} \Delta_3 = q\frac{\partial}{\partial p} + t\frac{\partial}{\partial s} + 2s\frac{\partial}{\partial r} \\ + d\frac{\partial}{\partial c} + 2c\frac{\partial}{\partial b} + 3b\frac{\partial}{\partial a} + \ldots \end{aligned}$$

$$\begin{aligned} \Omega_0 = -3\lambda + 2p\frac{\partial}{\partial p} + q\frac{\partial}{\partial q} \\ + 4r\frac{\partial}{\partial r} + 3s\frac{\partial}{\partial s} + 2t\frac{\partial}{\partial t} \\ + 6a\frac{\partial}{\partial a} + 5b\frac{\partial}{\partial b} + 4c\frac{\partial}{\partial c} + 3d\frac{\partial}{\partial d} + \ldots \end{aligned}$$

$$\begin{aligned} \Omega_1 = -3\lambda + p\frac{\partial}{\partial p} + 2q\frac{\partial}{\partial q} \\ + 2r\frac{\partial}{\partial r} + 3s\frac{\partial}{\partial s} + 4t\frac{\partial}{\partial t} \\ + 3a\frac{\partial}{\partial a} + 4b\frac{\partial}{\partial b} + 5c\frac{\partial}{\partial c} + 6d\frac{\partial}{\partial d} + \ldots \end{aligned}$$

$$\begin{aligned} \Delta_1 = 1\left(q\frac{\partial}{\partial s} + 2p\frac{\partial}{\partial r}\right) \\ + 2\left(t\frac{\partial}{\partial c} + 2s\frac{\partial}{\partial b} + 3r\frac{\partial}{\partial a}\right) + \ldots \end{aligned}$$

$$\begin{aligned} \Delta_2 = 1\left(p\frac{\partial}{\partial s} + 2q\frac{\partial}{\partial t}\right) \\ + 2\left(r\frac{\partial}{\partial b} + 2s\frac{\partial}{\partial c} + 3t\frac{\partial}{\partial d}\right) + \ldots \end{aligned}$$

Second, as an inference from the equations $\Omega_1 = 0$, $\Omega_2 = 0$, in ELLIOTT's theory, it follows that all pure reciprocants are invariants of the binary quantics $(r, s, t \between \xi, \eta)^2$, $(a, b, c, d \between \xi, \eta)^3, \ldots$ —but all invariants are not reciprocants—and that there are no covariants among these reciprocants. From the equations $\Delta_4 = 0$, $\Delta_3 = 0$, in the present theory it follows that all the functional invariants are algebraical covariants of the binary quantics $(r, s, t \between q, -p)^2$, $(a, b, c, d \between q, -p)^3, \ldots$ —but not all algebraical covariants are functional invariants; and, from the other equations, that no algebraical invariants of these quantics are functional invariants. In particular, $rt - s^2$ is a reciprocant, but not a functional invariant; $q^2r - 2pqs + p^2t$ is a functional invariant, but not a reciprocant.]

Isolated Transformations.

1. We may briefly consider functions which are *invariantive for merely isolated changes of the independent variables*, that is, for changes which are effected by one relation between x and X only, and one relation between y and Y only. For such transformations we have

$$\mathrm{P} = p\frac{dx}{d\mathrm{X}}, \qquad \mathrm{Q} = q\frac{dy}{d\mathrm{Y}}, \qquad \mathrm{S} = s\frac{dx}{d\mathrm{X}}\frac{dy}{d\mathrm{Y}},$$

so that $s \div pq$ is an absolute invariant. Again,

$$\frac{\partial}{\partial \mathrm{X}} = \frac{dx}{d\mathrm{X}}\frac{\partial}{\partial x}, \qquad \frac{\partial}{\partial \mathrm{Y}} = \frac{dy}{d\mathrm{Y}}\frac{\partial}{\partial y},$$

so that $1/p\ \partial/\partial x$ and $1/q\ \partial/\partial y$ are absolute invariantive operators, which, when applied to absolute invariants, will produce absolute invariants. We therefore have the series

$$\frac{1}{p}\frac{\partial}{\partial x}\left(\frac{s}{pq}\right), \qquad \frac{1}{q}\frac{\partial}{\partial y}\left(\frac{s}{pq}\right);$$

$$\left(\frac{1}{p}\frac{\partial}{\partial x}\right)^2\frac{s}{pq}, \qquad \left(\frac{1}{p}\frac{\partial}{\partial x}\frac{1}{q}\frac{\partial}{\partial y}\right)\frac{s}{pq}, \qquad \left(\frac{1}{q}\frac{\partial}{\partial y}\frac{1}{p}\frac{\partial}{\partial x}\right)\frac{s}{pq}, \qquad \left(\frac{1}{q}\frac{\partial}{\partial y}\right)^2\frac{s}{pq},$$

and so on. The operators $1/p\ \partial/\partial x$ and $1/q\ \partial/\partial y$ may be applied, any number of times in any order, to the absolute invariant s/pq (or any other invariant which is absolute), and the result will be an absolute invariant.

These invariants possess their property for any general isolated transformations of x and y; but, if special isolated transformations are effected on x and on y, *e.g.*, the homographic transformations of the form

$$x = \frac{a\mathrm{X} + b}{c\mathrm{X} + d}, \qquad y = \frac{a'\mathrm{Y} + b'}{c'\mathrm{Y} + d'},$$

additional invariants will be introduced. For instance, we then have

$$\mathrm{A}(z) = \frac{z_{30}z_{10} - \frac{3}{2}z_{20}^2}{z_{10}^4}, \qquad \mathrm{B}(z) = \frac{z_{03}z_{01} - \frac{3}{2}z_{02}^2}{z_{01}^4},$$

both absolute invariants; in the former the variation of y, and in the latter the variation of x, do not come into consideration. From these we can derive a series of educts by the application of combinations of the absolute invariantive operators $1/p\ \partial/\partial x$ and $1/q\ \partial/\partial y$. When any such educt is invariantive, say

$$\mathrm{A}_m = \left(\frac{1}{p}\frac{\partial}{\partial x}\right)^m \mathrm{A}(z),$$

we may obtain from it other invariants by taking as the function, the differential coefficients of which are to enter, not z, but any educt which is an absolute invariant. All such invariants, however, thus obtained are expressible in terms of the educts obtained from $\mathrm{A}(z)$ and $\mathrm{B}(z)$ by repeated application of $1/p\ \partial/\partial x$ and $1/q\ \partial/\partial y$ in all possible combinations. Thus, it is easy to verify that if I be any absolute invariant, and I_1, I_2, I_3 its first, second, and third educts due to successive operations on I by $1/p\ \partial/\partial x$, the equation

$$\mathrm{A}(\mathrm{I}) = \frac{\mathrm{I}_1\mathrm{I}_3 - \frac{3}{2}\mathrm{I}_2^2}{\mathrm{I}_1^4} - \frac{\mathrm{A}(z)}{\mathrm{I}_1^2}$$

is satisfied; and the law is general.

2. Nor is it necessary to consider in any detail functions of the differential coefficients of z, which are *invariantive for isolated transformation of the dependent variable;* that is, for a transformation which connects z with a new variable ζ, without regard to the dependent variables. Such a transformation can be effected by means of an equation,

$$\phi(z, \zeta) = 0;$$

and we then have

$$z_{01} = \zeta_{01}\frac{dz}{d\zeta}, \qquad z_{10} = \zeta_{10}\frac{dz}{d\zeta}.$$

Since $dz/d\zeta$ is determinate from the transforming equation, it follows that

$$\frac{z_{01}}{z_{10}}$$

is an absolute invariant for the transformations at present under consideration. Moreover, in the present case $\partial/\partial x$ and $\partial/\partial y$ are absolute invariantive operators; and therefore

$$\frac{\partial^{m+n}}{\partial x^m\, \partial y^n}\left(\frac{z_{01}}{z_{10}}\right)$$

will, for all values of m and n, be an absolute invariant. Thus, taking in succession $m = 1$ and $n = 0$, and $m = 0$ and $n = 1$, we have

$$\frac{z_{10}z_{11} - z_{01}z_{20}}{z_{10}^2} \quad \text{and} \quad \frac{z_{10}z_{02} - z_{01}z_{11}}{z_{10}^2},$$

the former of which, multiplied by z_{01}/z_{10} and subtracted from the latter, gives

$$\frac{z_{01}^2 z_{20} - 2z_{01}z_{10}z_{11} + z_{10}^2 z_{02}}{z_{10}^3}$$

as an absolute invariant, or $z_{10}^2 z_{20} - 2z_{01} z_{10} z_{11} + z_{10}^2 z_{02}$ as a relative invariant, for the present transformation.

General Transformation.

3. We now proceed to the consideration of functions which are invariantive for the general simultaneous homographic transformation of the independent variables represented by

$$\frac{x}{\alpha_1 + \beta_1 X + \gamma_1 Y} = \frac{y}{\alpha_2 + \beta_2 X + \gamma_2 Y} = \frac{1}{\alpha_3 + \beta_3 X + \gamma_3 Y}.$$

As it will be convenient to have some one invariant at least, a relative invariant for these transformations can be obtained as follows. An integral relation given by

$$z = \frac{a + bx + cy}{a' + b'x + c'y} = \frac{u}{v}$$

reproduces itself in form when the independent variables are subjected to the above transformation; and the differential equation which is the equivalent of this integral relation will, therefore, also reproduce itself, and so will furnish an invariant.

Now, both u and v satisfy the three equations

$$\frac{\partial^2}{\partial x^2} = 0, \qquad \frac{\partial^2}{\partial x\, \partial y} = 0, \qquad \frac{\partial^2}{\partial y^2} = 0\,;$$

and therefore, substituting vz as the value of u in these, we have

$$\begin{aligned} 0 &= z_{20}v + 2z_{10}v_{10}, \\ 0 &= z_{11}v + z_{10}v_{01} + z_{01}v_{10}, \\ 0 &= z_{02}v + 2z_{01}v_{01}. \end{aligned}$$

The elimination of v, v_{10}, v_{01} between these leads to the result

$$0 = \begin{vmatrix} z_{20}, & 2z_{10}, & 0 \\ z_{11}, & z_{01}, & z_{10} \\ z_{02}, & 0, & 2z_{01} \end{vmatrix} = 2\left(z_{01}{}^2 z_{20} - 2z_{10}z_{01}z_{11} + z_{10}{}^2 z_{02}\right);$$

and, therefore,

$$A_0 = z_{01}{}^2 z_{20} - 2z_{10}z_{01}z_{11} + z_{10}{}^2 z_{02} = (z_{20}, z_{11}, z_{02} \between z_{01}, -z_{10})^2$$

is an invariant.

The integral of a partial differential equation of the second order, which is most general so far as concerns the number of arbitrary constants, contains five such independent arbitrary constants; and, therefore, a general integral of

$$A_0 = 0$$

is

$$z = \frac{a + bx + cy}{a' + b'x + c'y}.$$

It has already appeared that A_0 is an invariant for arbitrary change of z; and therefore, an immediate corollary is that

$$z = \phi\left(\frac{a + bx + cy}{a' + b'x + c'y}\right),$$

where ϕ is arbitrary, is a general integral of the equation $A_0 = 0$.

4. As an invariant is self-reproductive after transformations have been effected, save as to a factor, it is necessary to obtain the form of this factor. For this purpose it will be sufficient to consider a simple case.

Let z_1 and z_2 be two functions, and suppose the transformations of the variables to be any whatever, say of the form

$$x = \phi(X, Y), \qquad y = \psi(X, Y).$$

Then we have

$$P_1 = p_1 \frac{\partial x}{\partial X} + q_1 \frac{\partial y}{\partial X}, \qquad P_2 = p_2 \frac{\partial x}{\partial X} + q_2 \frac{\partial y}{\partial X},$$

$$Q_1 = p_1 \frac{\partial x}{\partial Y} + q_1 \frac{\partial y}{\partial Y}, \qquad Q_2 = p_2 \frac{\partial x}{\partial Y} + q_2 \frac{\partial y}{\partial Y};$$

and therefore

$$\begin{vmatrix} P_1, & Q_1 \\ P_2, & Q_2 \end{vmatrix} = \begin{vmatrix} p_1, & q_1 \\ p_2, & q_2 \end{vmatrix} \frac{\partial(x, y)}{\partial(X, Y)}.$$

Hence, in the present case, the factor is $\partial(x, y)/\partial(X, Y) = J$; and, by the analogy of all invariants, the factor for any one will be some power of J.

The invariants at present under consideration may, therefore, be defined as follows:—

A function ϕ of the partial differential coefficients of z with regard to x and to y is called an invariant if, when the independent variables are changed to X *and* Y *and the same function Φ of the new variables is formed, the equation*

$$\Phi = \mathrm{J}^m \phi$$

is satisfied, where

$$\mathrm{J} = \frac{\partial(x, y)}{\partial(\mathrm{X}, \mathrm{Y})}.$$

5. The following properties of *irreducible* invariants are easily obtained :—

(i.) An invariant does not contain the dependent variable, nor either of the independent variables.

(ii.) An invariant is homogeneous in the differential coefficients.

(iii.) An invariant is of uniform grade,* equal to its index m, in differentiation with regard to x; and of uniform grade, also equal to its index m, in differentiation with regard to y.

(iv.) An invariant is either symmetric or skew symmetric in differentiation with regard to the independent variables.

All these properties hold of A_0, the index of which is easily seen to be 2; it is a symmetric invariant, that is, it is unchanged if x and y be interchanged.

The index of a symmetric invariant is an even integer; the index of a skew symmetric invariant is an odd integer.

These properties hold for functions which are invariants for any general transformation, and not merely for the homographic transformations to be adopted; but the forms of possible functions, as well as the value of J, will be determined by the character of the transformation. And, in particular, for the homographic transformation it is easy to prove that

$$\mathrm{J} = \begin{vmatrix} \alpha_1, & \alpha_2, & \alpha_3 \\ \beta_1, & \beta_2, & \beta_3 \\ \gamma_1, & \gamma_2, & \gamma_3 \end{vmatrix} (\alpha_3 + \beta_3\mathrm{X} + \gamma_3\mathrm{Y})^{-3}.$$

6. The method adopted for the determination of the forms of invariants will be to obtain the partial differential equations satisfied by them; these equations can be obtained, as in a similar case,† by using the principle of complete infinitesimal variation. For this purpose it will be necessary to have the formulæ expressing the relations between differential coefficients of z when the variables are transformed. This relation is given in the following proposition, the transformations being supposed any whatever. The special application to the homographic transformation will afterwards be made.

* The grade of a term is the sum of the orders of differentiation with regard to one variable of the factors; thus, the x-grade of A_0 is 2; the y-grade is 2.

† "Homographic Invariants and Quotient-Derivatives," 'Mess. of Math.,' vol. 17 (1888), pp. 154–192.

Let $z = \theta(x, y)$, and let the variables be transformed by the equations

$$x = \phi(\mathrm{X}, \mathrm{Y}), \qquad y = \psi(\mathrm{X}, \mathrm{Y}).$$

Let

$$\Phi = \phi(\mathrm{X} + \rho,\ \mathrm{Y} + \sigma) - x,$$
$$\Psi = \psi(\mathrm{X} + \rho,\ \mathrm{Y} + \sigma) - y,$$

so that Φ and Ψ vanish with ρ and σ. Then, by the generalised form of TAYLOR's Theorem,

$$\frac{1}{m!\,n!}\frac{\partial^{m+n} z}{\partial \mathrm{X}^m\,\partial \mathrm{Y}^n}$$

is the coefficient of $\rho^m\sigma^n$ in the expansion in ascending powers of

$$\begin{aligned}&\theta\{\phi(\mathrm{X} + \rho,\ \mathrm{Y} + \sigma),\ \psi(\mathrm{X} + \rho,\ \mathrm{Y} + \sigma)\}\\ &= \theta(x + \Phi,\ y + \Psi),\end{aligned}$$

where ρ and σ occur only in Φ and Ψ. Now,

$$\theta(x + \Phi,\ y + \Psi) = \sum_{m'=0}\sum_{n'=0}\frac{1}{m'!\,n'!}\frac{\partial^{m'+n'}z}{\partial x^{m'}\,\partial y^{n'}}\Phi^{m'}\Psi^{n'};$$

and therefore

$$\frac{1}{m!\,n!}\frac{\partial^{m+n}z}{\partial \mathrm{X}^m\,\partial \mathrm{Y}^n} = \sum_{m'=0}\sum_{n'=0}\frac{1}{m'!\,n'!}\frac{\partial^{m'+n'}z}{\partial x^{m'}\,\partial y^{n'}}\mathrm{C}_{m,n}(\Phi^{m'}\Psi^{n'}),$$

where $\mathrm{C}_{m,n}(\Phi^{m'}\Psi^{n'})$ denotes the coefficient of $\rho^m\sigma^n$ in the expansion of $\Phi^{m'}\Psi^{n'}$ in ascending powers of ρ and σ. When m' and n' both vanish, or when $m'+n' > m+n$, the coefficient $\mathrm{C}_{m,n}(\Phi^{m'}\Psi^{n'})$ is zero.

The form of the corresponding theorem for the case of any number of independent variables is evident.

Homographic Transformation: Characteristic Equations.

7. When we consider the general homographic transformation, we may take α_1 and α_2 to be zero, for the invariants do not explicitly contain x and y, but only differential coefficients with regard to them, and so they may be modified by the subtraction of the respective constants α_1/α_3, α_2/α_3; and then the general forms are equivalent to

$$\frac{x}{\mathrm{X} + \alpha\mathrm{Y}} = \frac{y}{\mathrm{Y} + \beta\mathrm{X}} = \frac{1}{\alpha_3 + \beta_3\mathrm{X} + \gamma_3\mathrm{Y}}.$$

In order to apply the method of infinitesimal variation, it is sufficient to make the factor J nearly equal to unity, or, what is the same thing, to make x nearly equal to

X and y nearly equal to Y. Hence, we take α_3 to be unity, and β_3 and γ_3 small, say, $-\epsilon$ and $-\theta$ respectively; and α and β are to be considered small, quantities of the first order being retained. Thus, we have

$$\mathrm{J} = \begin{vmatrix} 0, & 0, & 1 \\ 1, & \beta, & -\epsilon \\ \alpha, & 1, & -\theta \end{vmatrix} (1 - \epsilon \mathrm{X} - \theta \mathrm{Y})^{-3},$$

$$= 1 + 3\epsilon \mathrm{X} + 3\theta \mathrm{Y},$$

so far as quantities of the first order. Also

$$x = \frac{\mathrm{X} + \alpha \mathrm{Y}}{1 - \epsilon \mathrm{X} - \theta \mathrm{Y}} = \mathrm{X} + \alpha \mathrm{Y} + \epsilon \mathrm{X}^2 + \theta \mathrm{XY} = \phi(\mathrm{X}, \mathrm{Y}),$$

$$y = \frac{\mathrm{Y} + \beta \mathrm{X}}{1 - \epsilon \mathrm{X} - \theta \mathrm{Y}} = \beta \mathrm{X} + \mathrm{Y} + \epsilon \mathrm{XY} + \theta \mathrm{Y}^2 = \psi(\mathrm{X}, \mathrm{Y}),$$

to the same order; and therefore

$$\begin{aligned} \Phi &= \phi(\mathrm{X} + \rho, \ \mathrm{Y} + \sigma) - \phi(\mathrm{X}, \mathrm{Y}) \\ &= \rho + \alpha\sigma + \epsilon(\rho^2 + 2\mathrm{X}\rho) + \theta(\rho\sigma + \rho\mathrm{Y} + \sigma\mathrm{X}), \\ \Psi &= \psi(\mathrm{X} + \rho, \ \mathrm{Y} + \sigma) - \psi(\mathrm{X}, \mathrm{Y}) \\ &= \sigma + \beta\rho + \epsilon(\rho\sigma + \rho\mathrm{Y} + \sigma\mathrm{X}) + \theta(\sigma^2 + 2\mathrm{Y}\sigma). \end{aligned}$$

Hence, to the first order inclusive, we have

$$\begin{aligned} \Phi^{m'}\Psi^{n'} = \rho^{m'}\sigma^{n'} &+ m'\rho^{m'-1}\sigma^{n'}\{\alpha\sigma + \epsilon(\rho^2 + 2\mathrm{X}\rho) + \theta(\rho\sigma + \rho\mathrm{Y} + \sigma\mathrm{X})\} \\ &+ n'\rho^{m'}\sigma^{n'-1}\{\beta\rho + \epsilon(\rho\sigma + \rho\mathrm{Y} + \sigma\mathrm{X}) + \theta(\sigma^2 + 2\mathrm{Y}\sigma)\}, \end{aligned}$$

and therefore

$$\begin{gathered} \mathrm{C}_{m', n'} = 1 + m'(2\epsilon\mathrm{X} + \theta\mathrm{Y}) + n'(\epsilon\mathrm{X} + 2\theta\mathrm{Y}), \\ \mathrm{C}_{m'-1, n'+1} = m'(\alpha + \theta\mathrm{X}), \quad \mathrm{C}_{m'+1, n'-1} = n'(\beta + \epsilon\mathrm{Y}), \\ \mathrm{C}_{m'+1, n'} = (m' + n')\,\epsilon, \qquad \mathrm{C}_{m', n'+1} = (m' + n')\,\theta. \end{gathered}$$

All other coefficients are negligible, being of a higher order of small quantities or zero (non-occurring); and these give all the combinations of values of m and n for $\mathrm{C}_{m, n}(\Phi^{m'}\Psi^{n'})$. Therefore, for all values of m and n, we have

$$\begin{aligned} \frac{\partial^{m+n} z}{\partial \mathrm{X}^m \partial \mathrm{Y}^n} = \frac{\partial^{m+n} z}{\partial x^m \partial y^n} &\{1 + m(2\epsilon\mathrm{X} + \theta\mathrm{Y}) + n(\epsilon\mathrm{X} + 2\theta\mathrm{Y})\} \\ &+ \frac{\partial^{m+n} z}{\partial x^{m+1} \partial y^{n-1}}\, n(\alpha + \theta\mathrm{X}) + \frac{\partial^{m+n} z}{\partial x^{m-1} \partial y^{n+1}}\, n(\beta + \epsilon\mathrm{Y}) \\ &+ \frac{\partial^{m+n-1} z}{\partial x^{m-1} \partial y^n}\, m(m + n - 1)\,\epsilon + \frac{\partial^{m+n-1} z}{\partial x^m \partial y^{n-1}}\, n(m + n - 1)\,\theta. \end{aligned}$$

8. If, then, we have an invariant f of index λ, such that

$$F(\ldots, Z_{m,n}, \ldots) = J^{\lambda} f(\ldots, z_{m,n}, \ldots),$$

and we substitute for J and for all differential coefficients $Z_{m,n}$, and then expand, retaining all small quantities of the first order, we have the following equations derived from a comparison of corresponding terms.

From the terms which are multiplied by ϵX

$$\Sigma\Sigma\,(2m+n)\,z_{m,n}\frac{\partial f}{\partial z_{m,n}} = 3\lambda f \quad \ldots\ldots\ldots \quad \text{(i.)};$$

from the terms in θY

$$\Sigma\Sigma\,(2n+m)\,z_{m,n}\frac{\partial f}{\partial z_{m,n}} = 3\lambda f \quad \ldots\ldots\ldots \quad \text{(ii.)};$$

from the terms in ϵ

$$\Delta_1 f = \Sigma\Sigma m\,(m+n-1)\,z_{m-1,n}\frac{\partial f}{\partial z_{m,n}} = 0 \quad \ldots\ldots \quad \text{(iii.)};$$

from the terms in θ

$$\Delta_2 f = \Sigma\Sigma n\,(m+n-1)\,z_{m,n-1}\frac{\partial f}{\partial z_{m,n}} = 0 \quad \ldots\ldots \quad \text{(iv.)};$$

from the terms in $\beta + \epsilon Y$

$$\Delta_3 f = \Sigma\Sigma m z_{m-1,n+1}\frac{\partial f}{\partial z_{m,n}} = 0 \quad \ldots\ldots\ldots \quad \text{(v.)};$$

and from the terms in $\alpha + \theta X$

$$\Delta_4 f = \Sigma\Sigma n z_{m+1,n-1}\frac{\partial f}{\partial z_{m,n}} = 0 \quad \ldots\ldots\ldots \quad \text{(vi.)}.$$

Equations (iii.)–(vi.) determine the form of the function f; when the form is obtained, the index is derivable by inspection, and equations (i.) and (ii.) are then identically satisfied.

9. Before considering these equations, characteristic of the invariants, one remark should be made. If the quantities ϵ and θ are absolutely zero so that the transformations are

$$x = X + \alpha Y, \qquad y = \beta X + Y,$$

that is, transformations to which a binary form is subject, the terms which, in what precedes, give rise to equations (i.)–(iv.) do not exist, and, therefore, these equations do not exist; but there are terms in β and α, and, therefore, equations (v.) and (vi.) survive, being in fact the partial differential equations determining those covariants which can be expressed in terms of partial differential coefficients of the form with regard to the variables.

Invariants in the Second Order.

10. First, let us consider invariants which involve no partial differential coefficients of order higher than the second. The differential equations to be satisfied are, in the non-subscript notation,

$$\text{(iii.)}\quad q\frac{\partial f}{\partial s} + 2p\frac{\partial f}{\partial r} = 0,$$

$$\text{(iv.)}\quad p\frac{\partial f}{\partial s} + 2q\frac{\partial f}{\partial t} = 0,$$

$$\text{(v.)}\quad q\frac{\partial f}{\partial p} + 2s\frac{\partial f}{\partial r} + t\frac{\partial f}{\partial s} = 0,$$

$$\text{(vi.)}\quad p\frac{\partial f}{\partial q} + 2s\frac{\partial f}{\partial t} + r\frac{\partial f}{\partial s} = 0,$$

so far as concerns the form of the function. From these equations we have

$$\frac{\dfrac{\partial f}{\partial r}}{q^2} = \frac{\dfrac{\partial f}{\partial s}}{-2pq} = \frac{\dfrac{\partial f}{\partial t}}{p^2} = \frac{\dfrac{\partial f}{\partial p}}{2(pt-qs)} = \frac{\dfrac{\partial f}{\partial q}}{2(qr-sp)}.$$

When Θ is taken to be the common value of these fractions, it follows that

$$\begin{aligned} df &= \frac{\partial f}{\partial r}dr + \frac{\partial f}{\partial s}ds + \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial p}dp + \frac{\partial f}{\partial q}dq \\ &= \Theta\, d\,(q^2r - 2pqs + p^2t). \end{aligned}$$

Now, df is a perfect differential, and therefore Θ is some function of $q^2r - 2pqs + p^2t$; hence, f also is some function of $q^2r - 2pqs + p^2t$, and therefore the *only irreducible invariant which contains differential coefficients of order not higher than the second is*

$$q^2r - 2pqs + p^2t.$$

This is the function A_0, already (§ 3) considered; the integral equation corresponding to the vanishing of this invariant is known.

Invariants in the Third Order.

11. When we come to consider invariants which involve differential coefficients of higher order, the method just used is no longer available, because the four differential equations are not sufficient to determine the ratios of the differential coefficients which

enter. If the determination of the invariants be made from the point of view that they are simultaneous solutions of the four equations, one method of proceeding will be to adopt JACOBI's process.

Any function f, which satisfies (iii.)–(vi.), must also satisfy each of the equations

$$[\Delta_r, \Delta_s] = 0$$

for $r, s = 1, 2, 3, 4$. Forming these, it is easy to show that

$$\begin{aligned}[\Delta_4, \Delta_3] &= \Sigma\,\Sigma \left\{ \frac{\partial(\Delta_4 f)}{\partial\left(\dfrac{\partial f}{\partial z_{m,n}}\right)} \frac{\partial(\Delta_3 f)}{\partial z_{m,n}} - \frac{\partial(\Delta_4 f)}{\partial z_{m,n}} \frac{\partial(\Delta_3 f)}{\partial\left(\dfrac{\partial f}{\partial z_{m,n}}\right)} \right\} \\ &= \Sigma\Sigma\,(m-n)\, z_{m,n} \frac{\partial f}{\partial z_{m,n}}\end{aligned}$$

after substitution and collection. Since this does not vanish in virtue of any one of the given equations, we must have a new equation

$$\Delta_5 f = \Sigma\Sigma\,(m-n)\, z_{m,n} \frac{\partial f}{\partial z_{m,n}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(vii.)},$$

to be associated with the rest. But this is the only additional equation; for

$$\begin{aligned}&[\Delta_4, \Delta_2] = 0\,; \quad [\Delta_4, \Delta_1] = -\Delta_2 f = 0\,; \quad [\Delta_3, \Delta_2] = -\Delta_1 f = 0\,; \\ &[\Delta_3, \Delta_1] = 0\,; \quad [\Delta_2, \Delta_1] = 0\,; \quad [\Delta_5, \Delta_4] = 2\Delta_4 f = 0\,; \\ &[\Delta_5, \Delta_3] = -2\Delta_3 f = 0\,; \quad [\Delta_5, \Delta_2] = 0\,; \quad [\Delta_5, \Delta_1] = 0.\end{aligned}$$

It is easy to verify that $f = A_0$ satisfies (vii.). Every invariant will be a simultaneous solution of (iii.)–(vii.).

It may be noticed that the equation (vii.) can be otherwise obtained; it arises by equating the left-hand sides of equations (i.) and (ii.) to one another, for, on re-arrangement of this, we have

$$\Sigma\Sigma\,(m-n)\, z_{m,n} \frac{\partial f}{\partial z_{m,n}} = 0.$$

Hence, equation (vii.) may be considered as replacing either (i.) or (ii.); and any function, which is a simultaneous solution of (iii.)–(vi.) and for the same value of λ satisfies (i.) and (ii.), will also satisfy (vii.).

One deduction as to the character of the invariants can at once be made from the form of the equations.

Every Irreducible Invariant must involve z_{01} and z_{10}.

For every irreducible invariant f satisfies the five equations; if then it be independent of z_{01}, we have $\partial f/\partial z_{01} = 0$. Since there is no term in it which involves z_{01}, and since there is a single term involving z_{01} in $\Delta_3 f = 0$, viz., $z_{01}(\partial f/\partial z_{10})$, we must therefore have $\partial f/\partial z_{10}$, *i.e.*, the function must be independent of z_{10}. From $\Delta_1 f = 0$, it then follows that $\partial f/\partial z_{20}$ and $\partial f/\partial z_{11}$ both vanish; from $\Delta_2 f = 0$, it then follows that $\partial f/\partial z_{11}$ and $\partial f/\partial z_{02}$ both vanish, and, therefore, that f involves no differential coefficients of the second order. Proceeding in this way to the successive orders, it appears that f involves no differential coefficients whatever; so that it cannot be an invariant, other than a constant or z.

12. Proceeding now to the consideration of invariants which involve differential coefficients of the third order as the highest, and denoting them for convenience by a, b, c, d ($= z_{30}, z_{12}, z_{21}, z_{30}$, respectively), we have, as the subsidiary equations necessary for the construction of the general solution of $\Delta_3 f = 0$, the set

$$\frac{dp}{q} = \frac{dq}{0} = \frac{dr}{2s} = \frac{ds}{t} = \frac{dt}{0} = \frac{da}{3b} = \frac{db}{2c} = \frac{dc}{d} = \frac{dd}{0}.$$

To deduce that general solution, eight independent integrals of the subsidiary set must be obtained; bearing in mind the character of the invariants (§ 11) ultimately to be arrived at, we take these integrals in the form

$$\begin{aligned}
u_1 &= q, \\
u_2 &= t, \\
u_3 &= qs - pt, \\
u_4 &= q^2 r - 2pqs + p^2 t, \\
u_5 &= d, \\
u_6 &= pd - qc, \\
u_7 &= p^2 d - 2pqc + q^2 b, \\
u_8 &= p^3 d - 3p^2 qc + 3pq^2 b - q^3 a.
\end{aligned}$$

Any solution of the equation $\Delta_3 f = 0$ can be expressed as a functional combination of $u_1, u_2, \ldots, u_8$; thus

$$rt - s^2 = \frac{u_2 u_4 - u_3^2}{u_1^2},$$

$$bd - c^2 = \frac{u_5 u_7 - u_6^2}{u_1^2},$$

$$(ad - bc)^2 - 4(ac - b^2)(bd - c^2) = \frac{(u_5 u_8 - u_6 u_7)^2 - 4(u_5 u_7 - u_6^2)(u_6 u_8 - u_7^2)}{u_1^6},$$

and so on.

In order to obtain the most general solution which simultaneously satisfies $\Delta_3 f = 0$ and $\Delta_1 f = 0$, it will be sufficient to obtain the irreducible functional combinations of $u_1, u_2, \ldots, u_8$, which satisfy $\Delta_1 f = 0$. Now,

$$\Delta_1 u_1 = 0, \quad \Delta_1 u_2 = 0, \quad \Delta_1 u_4 = 0, \quad \Delta_1 u_5 = 0\,;$$

and

$$\begin{aligned} \Delta_1 u_3 &= u_1^2, \\ \Delta_1 u_6 &= -\,2u_1 u_2, \\ \Delta_1 u_7 &= 4u_1 u_3, \\ \Delta_1 u_8 &= -\,6u_1 u_4\,; \end{aligned}$$

so that

$$\begin{aligned} \Delta_1 (u_1 u_6 + 2u_2 u_3) &= 0, \\ \Delta_1 (u_1 u_7 - 2u_3^2) &= 0, \\ \Delta_1 (u_1 u_8 + 6u_3 u_4) &= 0. \end{aligned}$$

Hence, the most general simultaneous solution of $\Delta_3 f = 0$, and $\Delta_1 f = 0$, can be expressed as a functional combination of

$$\begin{aligned} &u_1,\ u_2,\ u_4,\ u_5, \\ v_6 &= u_1 u_6 + 2u_2 u_3, \\ v_7 &= u_1 u_7 - 2u_3^2, \\ v_8 &= u_1 u_8 + 6u_3 u_4. \end{aligned}$$

In order to obtain the most general simultaneous solution of $\Delta_3 f = 0$, $\Delta_1 f = 0$, $\Delta_2 f = 0$, it will be sufficient to obtain the irreducible functional combinations of $u_1, u_2, u_4, u_5, v_6, v_7, v_8$ which satisfy $\Delta_2 f = 0$. Now, it is easy to show that

$$\Delta_2 u_1 = 0, \quad \Delta_2 u_4 = 0, \quad \Delta_2 v_6 = 0, \quad \Delta_2 v_8 = 0\,;$$

and

$$\begin{aligned} \Delta_2 u_2 &= 2u_1, \\ \Delta_2 u_5 &= 6u_2, \\ \Delta_2 v_7 &= 2u_1 u_4\,; \end{aligned}$$

so that

$$\begin{aligned} \Delta_2 (u_1 u_5 - \tfrac{2}{3} u_2^2) &= 0, \\ \Delta_2 (v_7 - u_2 u_4) &= 0. \end{aligned}$$

Hence, the most general simultaneous solution of $\Delta_3 f = 0$, $\Delta_1 f = 0$, $\Delta_2 f = 0$ can be expressed as a functional combination of

$$\begin{aligned} &u_1, \quad u_4, \quad v_6, \quad v_8, \\ v_5 &= u_1 u_5 - \tfrac{3}{2} u_2^2, \\ w_7 &= v_7 - u_2 u_4 = u_1 u_7 - u_2 u_4 - 2u_3^2. \end{aligned}$$

In order to obtain the most general simultaneous solution of $\Delta_3 f = 0$, $\Delta_1 f = 0$, $\Delta_2 f = 0$, $\Delta_4 f = 0$, it will be sufficient to obtain the irreducible functional combinations of u_1, u_4, v_5, v_6, w_7, v_8 which satisfy $\Delta_4 f = 0$. Now it is easy enough to show that

$$\Delta_4 u_4 = 0\ ;$$

and that

$$\begin{aligned} \Delta_4 u_1 &= p, \\ u_1 \Delta_4 v_5 &= 4pv_5 - 3v_6, \\ u_1 \Delta_4 v_6 &= 3pv_6 - 2w_7, \\ u_1 \Delta_4 w_7 &= 2pw_7 - v_8, \\ u_1 \Delta_4 v_8 &= pv_8 + 6u_4^2. \end{aligned}$$

If, then, we write

$$\frac{v_8}{u_1} = \mathrm{P}_8, \quad \frac{w_7}{u_1^2} = \mathrm{P}_7, \quad \frac{v_6}{u_1^3} = \mathrm{P}_6, \quad \frac{v_5}{u_1^4} = \mathrm{P}_5,$$

these equations become

$$\begin{aligned} u_1^2 \Delta_4 \mathrm{P}_8 &= 6u_4^2, \\ u_1^2 \Delta_4 \mathrm{P}_7 &= -\mathrm{P}_8, \\ u_1^2 \Delta_4 \mathrm{P}_6 &= -2\mathrm{P}_7, \\ u_1^2 \Delta_4 \mathrm{P}_5 &= -3\mathrm{P}_6\ ; \end{aligned}$$

and therefore, bearing in mind that $\Delta_4 u_4 = 0$, we have

$$\begin{aligned} &\Delta_4 (\mathrm{P}_8^2 + 12u_4^2 \mathrm{P}_7) = 0, \\ &\Delta_4 (\mathrm{P}_8^3 + 18u_4^2 \mathrm{P}_7 \mathrm{P}_8 + 54u_4^4 \mathrm{P}_6) = 0, \\ &\Delta_4 (\mathrm{P}_8 \mathrm{P}_6 - \mathrm{P}_7^2 + 2u_4^2 \mathrm{P}_5) = 0. \end{aligned}$$

Hence the most general simultaneous solution of $\Delta_1 f = 0$, $\Delta_2 f = 0$, $\Delta_3 f = 0$, $\Delta_4 f = 0$ can be expressed as a functional combination of u_4, and

$$\begin{aligned} \mathrm{Q}_7 &= \mathrm{P}_8^2 + 12u_4^2 \mathrm{P}_7, \\ \mathrm{Q}_6 &= \mathrm{P}_8^3 + 18u_4^2 \mathrm{P}_7 \mathrm{P}_8 + 54u_4^4 \mathrm{P}_6, \\ \mathrm{Q}_5 &= \mathrm{P}_8 \mathrm{P}_6 - \mathrm{P}_7^2 + 2u_4^2 \mathrm{P}_5. \end{aligned}$$

13. Before considering the question as to whether these functions satisfy (i.) and (ii.), and, therefore, also (vii.), it is desirable to modify their expressions.

We have already had the quantity u_4; it is the same as A_0, so that we write

$$u_4 = \mathrm{A}_0\ ;$$

and it will be convenient to write

$$-u_8 = \mathrm{A}_1 = (z_{30},\ z_{21},\ z_{12},\ z_{03} \between z_{01},\ -z_{10})^3.$$

Then we have

$$Q_7 = \frac{1}{u_1^2}(v_8^2 + 12u_4^2 w_7)$$
$$= \frac{1}{u_1^2}\{u_1^2 u_8^2 + 12u_1 u_4 (u_3 u_8 + u_4 u_7) + 12u_4^2 (u_3^2 - u_2 u_4)\}.$$

But

$$\begin{aligned} u_3 u_8 + u_4 u_7 &= (qs - pt)(p^3 d - 3p^2 qc + 3pq^2 b - q^3 a) \\ &\qquad + \{q(qr - ps) - p(qs - pt)\}(p^2 d - 2pqc + q^2 b) \\ &= q(qr - ps)(p^2 d - 2pqc + q^2 b) + q(qs - pt)(-p^2 c + 2pqb - q^2 a) \\ &= \tfrac{1}{6} q \left(\frac{\partial A_0}{\partial p}\frac{\partial A_1}{\partial q} - \frac{\partial A_0}{\partial q}\frac{\partial A_1}{\partial p}\right) = \tfrac{1}{6} u_1 J_{01}, \end{aligned}$$

where J_{01} denotes the Jacobian of A_0, A_1 with regard to u_{10} and u_{01}. Similarly,

$$u_3^2 - u_2 u_4 = u_1^2 (s^2 - rt) = -u_1^2 H_0;$$

so that

$$Q_7 = A_1^2 + 2A_0 J_{01} + 12A_0^2 H_0,$$

H_0 being the discriminant of A_0.

For the modification of the expression of Q_6 we have, on substituting for the quantities P in terms of the quantities u,

$$Q_6 = u_8^3 + 18\frac{u_4 u_8}{u_1}(u_3 u_8 + u_4 u_7) + 18\frac{u_4^2}{u_1^2}(4u_3^2 u_8 - u_2 u_4 u_8 + 6u_3 u_4 u_7 + 3u_4^2 u_6).$$

The modification of the second term has already been given; for the third we have

$$\begin{aligned} 4u_3^2 u_8 - 4u_2 u_4 u_8 &= 4u_1^2 A_1 H_0; \\ u_2 u_8 + 2u_3 u_7 + u_4 u_6 &= u_{12}\{r(pd - qc) - 2s(pc - qb) + t(pb - qa)\}; \\ &= \tfrac{1}{12} u_1^2 \left\{\frac{\partial^2 A_0}{\partial q^2}\frac{\partial^2 A_1}{\partial p^2} - 2\frac{\partial^2 A_0}{\partial p\,\partial q}\frac{\partial^2 A_1}{\partial p\,\partial q} + \frac{\partial^2 A_0}{\partial p^2}\frac{\partial^2 A_1}{\partial q^2}\right\}; \\ &= -\tfrac{1}{12} u_1^2 H_{01}, \end{aligned}$$

where H_{01} denotes the simultaneous Hessian of A_0 and A_1 with regard to u_{01} and u_{10}. Hence

$$Q_6 = -A_1^3 - 3A_0 A_1 J_{01} + 72\,A_0^2 A_1 H_0 - \tfrac{9}{2}\,A_0^3 H_{01}.$$

For Q_5 we have, after substitution for P_5, P_6, P_7, P_8, the form

$$\begin{aligned} Q_5 &= \frac{1}{u_1^2}(u_6 u_8 - u_7^2) \\ &+ \frac{2}{u_1^3}(u_2 u_3 u_8 + u_4^2 u_5 + 3u_3 u_4 u_6 + u_2 u_4 u_7 + 2u_3^2 u_7) \\ &- \frac{1}{u_1^4}(4u_2^2 u_4^2 - 8u_2 u_4 u_3^2 + 4u_3^4). \end{aligned}$$

Now, for the first set of terms

$$u_6 u_8 - u_7^2 = q^2 \{q^2 (ac - b^2) - pq (ad - bc) + p^2 (bd - c^2)\}$$
$$= u_1^2 \mathrm{H}_1$$

where H_1 is the Hessian of A_1 considered as a ground-form in q and $-p$; and the third set of terms is

$$-\frac{4}{u_1^4}(u_2 u_4 - u_3^2)^2 = -4\mathrm{H}_0^2.$$

For the middle set of terms it is easily found, by the results already proved, that the terms within the bracket can be expressed in the form

$$u_1 [q \{r^2 d - 3rsc + (2s^2 + rt) b - sta\} + p \{t^2 a - 3tsb + (2s^2 + rt) c - rsd\}]$$
$$= u_1^3 \mathrm{L}_2,$$

say; so that we have

$$\mathrm{Q}_5 = \mathrm{H}_1 + 2\mathrm{L}_2 - 4\mathrm{H}_0^2.$$

14. Considering now the question as to whether each of the functions thus obtained will satisfy (i.) and (ii.), for one and the same numerical value of the index λ probably associated with it, we may proceed as follows. Writing the equations in the form

$$\Omega_0 f = 3\lambda f \quad \ldots \ldots \ldots \quad \text{(i'.)},$$

$$\Omega_1 f = 3\lambda f \quad \ldots \ldots \ldots \quad \text{(ii'.)},$$

we have the following result:—

$f =$	$\Omega_0 f =$	$\Omega_1 f =$	$\lambda =$
A_0	$6\mathrm{A}_0$	$6\mathrm{A}_0$	2
A_1	$9\mathrm{A}_1$	$9\mathrm{A}_1$	3
J_{01}	$12\mathrm{J}_{01}$	$12\mathrm{J}_{01}$	4
L_2	$12\mathrm{L}_2$	$12\mathrm{L}_2$	4
H_{01}	$9\mathrm{H}_{01}$	$9\mathrm{H}_{01}$	3
H_0	$6\mathrm{H}_0$	$6\mathrm{H}_0$	2
H_1	$12\mathrm{H}_1$	$12\mathrm{H}_1$	4

By means of these results we at once find

$$
\begin{aligned}
\Omega_0 A_0 &= 6A_0 = \Omega_1 A_0\,;\\
\Omega_0 Q_7 &= 18Q_7 = \Omega_1 Q_7\,;\\
\Omega_0 Q_6 &= 27Q_6 = \Omega_1 Q_6\,;\\
\Omega_0 Q_5 &= 12Q_5 = \Omega_1 Q_5.
\end{aligned}
$$

Hence A_0, Q_5, Q_6, Q_7 satisfy all the necessary equations, and they are therefore invariants; their respective indices are 2, 4, 9, 6; and, therefore, *every invariant which involves differential coefficients of z of order not higher than* 3 *can be expressed as an algebraical function of* A_0, Q_5, Q_6, Q_7, *where* (changing the sign of Q_6 from § 13)

$$
\begin{aligned}
Q_5 &= H_1 + 2L_2 - 4H_0^2,\\
Q_6 &= A_1^3 + 3A_0A_1J_{01} - 72A_0^2A_1H_0 + \tfrac{9}{2}A_0^3H_{01},\\
Q_7 &= A_1^2 + 2A_0J_{01} + 12A_0^2H_0,
\end{aligned}
$$

and the quantities A_0, A_1, J_{01}, H_0, H_1, H_{01}, L_2 *are given by the equations*

$$
\begin{aligned}
A_0 &= (z_{20}, z_{11}, z_{02} \between z_{01}, -z_{10})^2\,;\\
A_1 &= (z_{30}, z_{21}, z_{12}, z_{03} \between z_{01}, -z_{10})^3\,;\\
J_{01} &= \frac{\partial A_0}{\partial z_{10}}\frac{\partial A_1}{\partial z_{01}} - \frac{\partial A_0}{\partial z_{01}}\frac{\partial A_1}{\partial z_{10}}\,;\\
H_0 &= z_{20}z_{02} - z_{11}^2\,;\\
H_{01} &= 12\,\{z_{01}(z_{20}z_{12} - 2z_{11}z_{21} + z_{02}z_{30}) - z_{10}(z_{20}z_{03} - 2z_{11}z_{12} + z_{02}z_{21})\}\\
&= \frac{\partial^2 A_0}{\partial z_{10}^2}\frac{\partial^2 A_1}{\partial z_{01}^2} - 2\,\frac{\partial^2 A_0}{\partial z_{10}\,\partial z_{01}}\,\frac{\partial^2 A_1}{\partial z_{10}\,\partial z_{01}} + \frac{\partial^2 A_0}{\partial z_{01}^2}\frac{\partial_z A_1}{\partial z_{10}^2}\,;\\
H_1 &= (z_{30}z_{12} - z_{21}^2,\ z_{30}z_{03} - z_{21}z_{12},\ z_{21}z_{03} - z_{12}^2 \between z_{01}, -z_{10})^2\,;\\
L_2 &= z_{01}\,\{z_{20}^2z_{03} - 3z_{20}z_{11}z_{12} + (2z_{11}^2 + z_{20}z_{02})\,z_{21} - z_{11}z_{02}z_{30}\}\\
&\qquad - z_{10}\,\{z_{20}z_{11}z_{03} - (2z_{11}^2 + z_{20}z_{02})\,z_{12} + 3z_{11}z_{02}z_{21} - z_{02}^2z_{30}\}\\
&= \tfrac{1}{24}\left(\frac{\partial A_0}{\partial z_{10}}\frac{\partial H_{01}}{\partial z_{01}} - \frac{\partial A_0}{\partial z_{01}}\frac{\partial H_{01}}{\partial z_{10}}\right).
\end{aligned}
$$

The invariant A_0 is that which was obtained before, and it may be called the irreducible invariant of the second order; the invariants Q_5, Q_6, Q_7 may be called the irreducible invariants of the third order.

15. It may be remarked that the quantities additional to A_1 and A_0 which are necessary for the expression of Q_5, Q_6, Q_7 all belong to the simultaneous concomitant

system of A_0 and A_1 regarded as binary ground-forms in z_{01}, $-z_{10}$ as variables.* To this we shall return (§ 34).

A Special Series of Invariants.

16. There is a succession of invariants of consecutive orders, comparatively simple in form, which can be derived by using the remark made in § 9. A set of invariants of the form suggested by the covariants of a binary quantic, which involve only differential coefficients of the quantic with respect to the variables, is derivable by considering the functions in z analogous to HERMITE's "associated covariants" which may be taken to be

$$A_{m-2} = (z_{m,0},\quad z_{m-1,1},\quad z_{m-2,2}, \ldots\ldots, \quad z_{2,m-2},\quad z_{1,m-1},\quad z_{0,m} \between z_{01}, -z_{10})^m$$

for values 2, 3, 4, . . . of m.

It is easy to see that each of these functions satisfies the equations (v.) and (vi.), viz., $\Delta_3 f = 0$ and $\Delta_4 f = 0$; these, in fact, are the equations which suggest the functions.

But, when we consider the operators Δ_1 and Δ_2 which do not arise in connexion with covariants of binary forms, we have

$$\begin{aligned} &\Delta_1 A_{m-2} \\ &= (m-1)\left(z_{0,m-1}\frac{\partial}{\partial z_{1,m-1}} + 2z_{1,m-2}\frac{\partial}{\partial z_{2,m-2}} + 3z_{2,m-3}\frac{\partial}{\partial z_{3,m-3}} + \ldots + mz_{m-1,0}\frac{\partial}{\partial z_{m,0}}\right)A_{m-2} \\ &= (m-1)\,\{mz_{m-1,0}\,z_{01}{}^m - (m-1)\,z_{m-2,1}\,mz_{01}{}^{m-1}z_{10} + \ldots\} \\ &= m\,(m-1)\,z_{10}A_{m-3}\,; \end{aligned}$$

and, similarly,

$$\Delta_2 A_{m-2} = -m\,(m-1)\,z_{10}A_{m-3}.$$

Again, in regard to the operators which occur in (i.) and (ii.), it is easy to show that

$$\Omega_0 A_{m-2} = 3mA_{m-2},$$
$$\Omega_1 A_{m-2} = 3mA_{m-2}.$$

If, then, we can obtain combinations of A_0, A_1, A_2, . . . which are homogeneous and of uniform grade, such as to satisfy $\Delta_1 f = 0$ and $\Delta_2 f = 0$, these combinations will be invariants; and it follows from the effect of the linear operators Δ_1 and Δ_2 on the quantities A that any combination of the A's which satisfies $\Delta_1 f = 0$ will also satisfy $\Delta_2 f = 0$.

* SALMON, 'Higher Algebra' (3rd edition), § 198; CLEBSCH, 'Theorie der binären Formen,' § 59; GORDAN, 'Vorlesungen über Invariantentheorie,' vol. 2, § 31. The quantities A_1, A_0, J_{01}, L_2, H_0, H_{01}, H_1 are, save as to numerical factors, respectively the same as SALMON's symbols u, v, $(1, 1)$, L_2, Δ, L_1, $(2, 0)$; as CLEBSCH's symbols ϕ, f, ϑ, q, D, p, Δ; as GORDAN's symbols ϕ, f, ϑ, q, A_{ff}, p, Δ.

Combinations of this kind, which are of uniform grade and are homogeneous, are A_0, $A_0A_2 - k_1A_1^2$, $A_0^2A_3 - l_1A_0A_1A_2 + l_2A_1^3$, $A_0A_4 - m_1A_1A_3 + m_2A_2^2$, and so on. When these are substituted in $\Delta_1 f = 0$ and the coefficients $k, l, m, \ldots$ are determined so that the equation is satisfied, we find the following set of invariants :—

$$\begin{aligned} U_0 &= A_0, \\ U_2 &= A_0A_2 - A_1^2, \\ U_3 &= A_0^2A_3 - \tfrac{10}{3}A_0A_1A_2 + \tfrac{20}{9}A_1^3, \\ U_4 &= A_0A_4 - 5A_1A_3 + \tfrac{25}{6}A_2^2, \\ &\cdots\cdots\cdots\cdots \end{aligned}$$

These combinations suggest an analogy with the coefficients of the principal irreducible covariants of a quantic. If we change the symbols by the relation

$$A_{m-2} = m!\,(m-1)\,C_{m-2},$$

then, except as to numerical factors, the functions are

$$\begin{aligned} &C_0, \\ &C_0C_2 - C_1^2, \\ &C_0^2C_3 - 3C_0C_1C_2 + 2C_1^3, \\ &C_0C_4 - 4C_1C_3 + 3C_2^2, \\ &\cdots\cdots\cdots\cdots \end{aligned}$$

that is, they follow the same law of formation as the leading terms of the covariants referred to; and they can therefore be expressed in terms of the quantities C and can thence be deduced in terms of the quantities A.

All these functions satisfy the equation

$$C_0\frac{\partial U}{\partial C_1} + 2C_1\frac{\partial U}{\partial C_2} + 3C_2\frac{\partial U}{\partial C_3} + \ldots \quad = 0,$$

that is, they satisfy the equation

$$\sum_{n=1}(n+2)(n+1)A_{n-1}\frac{\partial U}{\partial A_n} = 0,$$

by means of which the numerical coefficients in U can be directly determined.

It is evident from the form of U_{m-2} that the highest order of differential coefficient which enters is the mth, that all the differential coefficients of the mth order enter linearly and into only one set of terms, and that the remaining terms all involve coefficients of lower order of differentiation.

Invariants in the Fourth Order.

17. To obtain the irreducible invariants which involve no differential coefficient of order higher than four, we may proceed as in § 11 by forming the irreducible functions which satisfy the differential equations; among these functions the invariants already obtained will occur.

For convenience, let the differential coefficients of the fourth order be denoted by e, f, g, h, i ($= z_{40}, z_{31}, z_{22}, z_{13}, z_{04}$ respectively). Then, beginning as before with $\Delta_3 f = 0$, the subsidiary equations additional to those already (§ 12) considered are

$$\left[\frac{dp}{q} = \frac{dq}{0}\right] = \frac{de}{4f} = \frac{df}{3g} = \frac{dg}{2h} = \frac{dh}{i} = \frac{di}{0};$$

of which the irreducible independent integrals are

$$\begin{aligned}
u_9 &= i, \\
u_{10} &= pi - qh, \\
u_{11} &= p^2 i - 2pqh + q^2 g, \\
u_{12} &= p^3 i - 3p^2 qh + 3pq^2 g - q^3 f, \\
u_{13} &= p^4 i - 4p^3 qh + 6p^2 q^2 g - 4pq^3 f + q^4 e;
\end{aligned}$$

and any solution of $\Delta_3 f = 0$ is expressible as a function of $u_1, u_2, \ldots, u_{13}$.

The remainder of the analysis is very similar to that which has been used for the earlier question, and so it is not here reproduced; the following are the results:—

(i) The functional combinations of the thirteen quantities u which satisfy $\Delta_1 f = 0$ (and which are, therefore, the irreducible simultaneous solutions of $\Delta_3 f = 0 = \Delta_1 f$) are

$$\begin{aligned}
&u_1, u_2, u_4, u_5, u_9; \\
v_6 &= u_1 u_6 + 2u_2 u_3, \\
v_7 &= u_1 u_7 - 2u_3^2, \\
v_8 &= u_1 u_8 + 6u_3 u_4; \\
v_{10} &= u_1 u_{10} + 3u_3 u_5, \\
v_{11} &= u_1 u_{11} + 6u_3 u_6 + 3u_2 u_7, \\
v_{12} &= u_1^2 u_{12} + 9u_1 u_3 u_7 - 12u_3^3, \\
v_{13} &= u_1^2 u_{13} + 12u_1 u_3 u_8 + 36u_3^2 u_4.
\end{aligned}$$

(ii) The functional combinations of these twelve quantities which satisfy $\Delta_2 f = 0$ (and which are therefore the irreducible simultaneous solutions of $\Delta_3 f = \Delta_1 f = \Delta_2 f = 0$) are

$$u_1,\ u_4,\ v_6,\ v_8,\ v_{13}\ ;$$
$$v_5 = u_1u_5 - \tfrac{3}{2}u_2^2,$$
$$w_7 = v_7 - u_2u_4\ ;$$
$$v_9 = u_1^2u_9 - 6u_1u_2u_5 + 6u_2^3,$$
$$w_{10} = u_1v_{10} - \tfrac{9}{2}u_2v_6,$$
$$w_{11} = u_1v_{11} - 6u_2u_7 + \tfrac{3}{2}u_2^2u_4,$$
$$w_{12} = v_{12} - \tfrac{3}{2}u_2v_8.$$

(iii) When these functional combinations are substituted in turn for f in $\Delta_4 f$, the equations additional to those in § 12 can be transformed to

$$\begin{aligned}
u_1\,\Delta_4 v_9 &= 6pv_9 - 4w_{10},\\
u_1\,\Delta_4 w_{10} &= 5pw_{10} - 3w_{11} + 3u_4v_5,\\
u_1\,\Delta_4 w_{11} &= 4pw_{11} - 2w_{12} + 6u_4v_6,\\
u_1\,\Delta_4 w_{12} &= 3pw_{12} - v_{13} + 9u_4w_7,\\
u_1\,\Delta_4 v_{13} &= 2pv_{13} + 12u_4v_8\ ;
\end{aligned}$$

and therefore, if we write

$$\frac{v_{13}}{u_1^2} = \mathrm{P}_{13},\quad \frac{w_{12}}{u_1^3} = \mathrm{P}_{12},\quad \frac{w_{11}}{u_1^4} = \mathrm{P}_{11},\quad \frac{w_{10}}{u_1^5} = \mathrm{P}_{10},\quad \frac{v_9}{u_1^6} = \mathrm{P}_9,$$

these equations become

$$\begin{aligned}
u_1^2\Delta_4\mathrm{P}_9 &= -4\mathrm{P}_{10},\\
u_1^2\Delta_4\mathrm{P}_{10} &= -3\mathrm{P}_{11} + 3u_4\mathrm{P}_5,\\
u_1^2\Delta_4\mathrm{P}_{11} &= -2\mathrm{P}_{12} + 6u_4\mathrm{P}_6,\\
u_1^2\Delta_4\mathrm{P}_{12} &= -\mathrm{P}_{13} + 9u_4\mathrm{P}_7,\\
u_1^2\Delta_4\mathrm{P}_{13} &= 12u_4\mathrm{P}_8.
\end{aligned}$$

In addition to the former irreducible solutions, Q_7, Q_6, Q_5, which were obtained from equations in § 13, the following irreducible solutions can be obtained :—

$$\begin{aligned}
\mathrm{Q}_{13} &= u_4\mathrm{P}_{13} - \mathrm{P}_8^2,\\
\mathrm{Q}_{12} &= 18u_4^3\mathrm{P}_{12} + 3u_4\mathrm{P}_8\mathrm{P}_{13} + 81u_4^4\mathrm{P}_6 - 2\mathrm{P}_8^3,\\
\mathrm{Q}_{11} &= 72u_4^5\mathrm{P}_{11} + 24u_4^3\mathrm{P}_8\mathrm{P}_{12} + 2u_4\mathrm{P}_8^2\mathrm{P}_{13} + 144u_4^6\mathrm{P}_5 + 108u_4^4\mathrm{P}_7^2 - \mathrm{P}_8^4,\\
\mathrm{Q}_{10} &= 216u_4^7\mathrm{P}_{10} + 108u_4^5\mathrm{P}_8\mathrm{P}_{11} + 18u_4^3\mathrm{P}_8^2\mathrm{P}_{12} + u_4\mathrm{P}_8^3\mathrm{P}_{13}\\
&\qquad - (108u_4^6\mathrm{P}_8\mathrm{P}_5 + 81u_4^4\mathrm{P}_8^2\mathrm{P}_6 + 18u_4^2\mathrm{P}_8^3\mathrm{P}_7 + \mathrm{P}_8^5),\\
\mathrm{Q}_9 &= 1296u_4^9\mathrm{P}_9 + 864u_4^7\mathrm{P}_8\mathrm{P}_{10} + 216u_4^5\mathrm{P}_8^2\mathrm{P}_{11} + 24u_4^3\mathrm{P}_8^3\mathrm{P}_{12} + u_4\mathrm{P}_8^4\mathrm{P}_{13}\\
&\qquad - (216u_4^6\mathrm{P}_8^2\mathrm{P}_5 + 108u_4^4\mathrm{P}_8^3\mathrm{P}_6 + 18u_4^2\mathrm{P}_8^4\mathrm{P}_7 + \tfrac{5}{6}\mathrm{P}_8^6).
\end{aligned}$$

These are not necessarily the simplest forms obtainable, but every simultaneous solution of $\Delta_1 f = \Delta_2 f = \Delta_3 f = \Delta_4 f = 0$ can be expressed as a functional combination of u_4, Q_5, Q_6, Q_7, Q_9, . . ., Q_{13}.

It will be seen that the new irreducible functions Q_9, . . ., Q_{13} are linear in the quantities P_9, P_{10}, . . ., P_{13}, and are therefore linear in the partial differential coefficients of the fourth order. In this respect they apparently differ from Q_5, Q_6, Q_7, which are the irreducible invariants of the third rank in differentiation; but, if we take instead of Q_5 an equivalent invariant $144u_4^2Q_5 + Q_7^2$, which is

$$288u_4^6P_5 + 144u_4^4P_6P_8 + 24u_4^2P_7P_8^2 + P_8^4,$$

the law of successive formation of the invariants (the new) Q_5, Q_6, Q_7 is similar to that for the functions Q_9, Q_{10}, . . ., Q_{13}.

18. But, before it can be asserted that Q_9, Q_{10}, . . . Q_{13} are invariants, it must be shown that they severally for a common value of λ satisfy the equations (i′.) and (ii′.). Now the following results are easily obtained :—

$f =$	$\Omega_0 f =$	$\Omega_1 f =$	$\lambda =$
P_5	0	0	0
P_6	$3P_6$	$3P_6$	1
P_7	$6P_7$	$6P_7$	2
P_8	$9P_8$	$9P_8$	3
P_9	0	0	0
P_{10}	$3P_{10}$	$3P_{10}$	1
P_{11}	$6P_{11}$	$6P_{11}$	2
P_{12}	$9P_{12}$	$9P_{12}$	3
P_{13}	$12P_{12}$	$12P_{12}$	4

From this Table it at once follows that

$$\begin{aligned}
\Omega_0 Q_{13} &= 18 Q_{13} = \Omega_1 Q_{13}, \\
\Omega_0 Q_{12} &= 27 Q_{12} = \Omega_1 Q_{12}, \\
\Omega_0 Q_{11} &= 36 Q_{11} = \Omega_1 Q_{11}, \\
\Omega_0 Q_{10} &= 45 Q_{10} = \Omega_1 Q_{10}, \\
\Omega_0 Q_{9} &= 54 Q_{9} \ = \Omega_1 Q_{9}.
\end{aligned}$$

Hence, Q_9, Q_{10}, Q_{11}, Q_{12}, Q_{13} *are invariants of indices* 18, 15, 12, 9, 6 *respectively; and every invariant which involves no differential coefficient of order higher than the fourth can be expressed as a function of* u_4, Q_5, Q_6, Q_7, Q_9, Q_{10}, Q_{11}, Q_{12}, Q_{13}.

General Inferences.

19. And if, among the sets of irreducible invariants thus obtained, those invariants which involve the partial differential coefficients of the nth order as the highest that occur, and which are linear in those partial differential coefficients of highest order, are called irreducible invariants *proper to the rank* n, then we have the following propositions relating to the complete aggregate of invariants :—

(i) The irreducible invariants can be ranged in sets, each set being proper to a particular rank;

(ii) There is no irreducible invariant proper to the rank unity;

(iii) There is a single irreducible invariant ($= u_4 = A_0$) proper to the rank 2;

(iv) There are three irreducible invariants ($= Q_5, Q_6, Q_7$) proper to the rank 3;

(v) For every value of n greater than 3, there are $n + 1$ irreducible invariants proper to the rank n, and they can be so chosen as to be linear in the differential coefficients of order n;

(vi) Every invariant can be expressed as a function of the irreducible invariants; and, if such an invariant have differential coefficients of order r as those of highest order occurring in it, the functional equivalent involves some or all of the aggregate of irreducible invariants proper to ranks not greater than r; it involves some of the irreducible invariants proper to the rank r, but no irreducible invariant proper to a rank greater than r.

Simultaneous Invariants of Two Functions.

20. Hitherto we have considered invariants of only a single dependent variable which is a function of the two independent variables; but we may consider a second dependent variable, say z', which is also a function of x and y. The two quantities

z and z' are independent of one another; but, if a third dependent variable be introduced, it can, by the elimination of x and y, be expressed in terms of z and z' alone, and its invariants will be expressible partly in terms of the invariants of z and of z', and partly in terms of functions arising in connexion with the transformation of z and z'. It is thus sufficient to consider two, and not more than two, dependent variables when there are two independent variables.

21. In addition to the invariants possessed by each of the dependent variables separately, there will be simultaneous invariants which involve differential coefficients of both the variables; such a simultaneous invariant is

$$\mathrm{J} = pq' - p'q,$$

which we have already obtained in § 4.

When the characteristic differential equations of simultaneous invariants are formed by the method already (§ 8) adopted for invariants of a single function, they are as follows:—Let F′ generally denote the same function associated with z' that F denotes associated with z. Then the equations satisfied by a simultaneous invariant ψ of two functions z and z' are

$$\begin{aligned}
\Phi_0\psi &= (\Omega_0 + \Omega'_0)\,\psi = 3\lambda\psi,\\
\Phi_1\psi &= (\Omega_1 + \Omega'_1)\,\psi = 3\lambda\psi,\\
\Theta_1\psi &= (\Delta_1 + \Delta'_1)\,\psi = 0,\\
\Theta_2\psi &= (\Delta_2 + \Delta'_2)\,\psi = 0,\\
\Theta_3\psi &= (\Delta_3 + \Delta'_3)\,\psi = 0,\\
\Theta_4\psi &= (\Delta_4 + \Delta'_4)\,\psi = 0.
\end{aligned}$$

It is easy to verify that J satisfies these equations, its index λ being unity; and it is evident that the invariants of z alone, and those of z' alone, all satisfy these equations.

As in § 11, it is easy to prove that *every simultaneous invariant must involve p and q; or p' and q'; or p, q, p', and q'.*

22. The only simultaneous invariant so far obtained is J; we proceed to obtain all the simultaneous invariants which involve no differential coefficients of z and of z' which are of order higher than the second, using for this purpose the method adopted in §§ 12, 17. Among these invariants there must evidently occur

$$\begin{aligned}
\mathrm{J} &= pq' - p'q,\\
\mathrm{A}_0 &= q^2r - 2pqs + p^2t,\\
\mathrm{A}'_0 &= q'^2r' - 2p'q's' + p'^2t',
\end{aligned}$$

the two latter being invariants each in one dependent variable only, which necessarily satisfy all the equations.

Taking the equations in the order adopted before and beginning with $(\Delta_3 + \Delta'_3)\psi = 0$, we have the set of subsidiary equations

$$\frac{dp}{q} = \frac{dq}{0} = \frac{dr}{2s} = \frac{ds}{t} = \frac{dt}{0} = \frac{dp'}{q'} = \frac{dq'}{0} = \frac{dr'}{2s'} = \frac{ds'}{t'} = \frac{dt'}{0},$$

nine in number. It is necessary to obtain nine independent integrals of the set; and we may take these in the form

$$\left.\begin{aligned} u_1 &= q \\ u'_1 &= q' \end{aligned}\right\}, \quad \left.\begin{aligned} u_2 &= t \\ u'_2 &= t \end{aligned}\right\}, \quad \left.\begin{aligned} u_3 &= qs' - pt' \\ u'_3 &= q's - p't \end{aligned}\right\}, \quad \left.\begin{aligned} u_4 &= A_0 \\ u_4' &= A'_0 \end{aligned}\right\}, \quad u_5 = J,$$

bearing in mind the property above proved. By the theory of linear partial differential equations of the first order, it follows that every solution of the equation $(\Delta_3 + \Delta'_3)\psi = 0$ can be expressed as a functional combination of u_1, u'_1, u_2, u'_2, u_3, u'_3, u_4, u'_4, u_5. Thus,

$$\begin{aligned} qs - pt &= \frac{u_1 u'_3 - u_2 u_5}{u'_1}, \\ st' - s't &= \frac{u_1 u'_2 u'_3 - u'_1 u_2 u_3 - u_2 u'_2 u_5}{u_1 u'_1}, \\ q^2 r' - 2pqs' + p^2 t &= \frac{u_1^2 u'_4 - 2u'_1 u_3 u_5 - u'_2 u_5^2}{u'^2_1}, \\ q'^2 r - 2p'q's + p'^2 t &= \frac{u'^2_1 u_4 + 2u_1 u'_3 u_5 - u_2 u_5^2}{u_1^2}, \end{aligned}$$

and so on.

To obtain the most general solution of $\Theta_3\psi = 0 = \Theta_1\psi$, it will be sufficient to form the irreducible combinations of $u_1, \ldots, u_5$ which satisfy $\Theta_1\psi = 0$. Now,

$$\Theta_1 u_1 = 0,\ \Theta_1 u'_1 = 0; \quad \Theta_1 u_2 = 0,\ \Theta_1 u'_2 = 0; \quad \Theta_1 u_4 = 0,\ \Theta_1 u'_4 = 0; \quad \Theta_1 u_5 = 0;$$

and

$$\Theta_1 u_3 = qq', \qquad \Theta_1 u'_3 = q'q;$$

so that

$$\Theta_1 (u_3 - u'_3) = 0;$$

and, therefore, the irreducible combinations which satisfy $\Theta_1\psi = 0$ are u_1, u'_1, u_2, u'_2, v_3, u_4, u'_4, u_5, where

$$v_3 = u_3 - u_3 = qs' - q's - pt' + p't.$$

To obtain the most general solution of $\Theta_3\psi = 0 = \Theta_1\psi = \Theta_2\psi$, it will be sufficient to form the irreducible combinations of the preceding eight quantities which satisfy $\Theta_2\psi = 0$. Now

$$\Theta_2 u_1 = 0, \quad \Theta_2 u'_1 = 0; \qquad \Theta_2 u_4 = 0, \quad \Theta_2 u'_4 = 0; \qquad \Theta_2 u_5 = 0;$$

and

$$\begin{aligned} \Theta_2 u_2 &= \quad 2u_1, \\ \Theta_2 u'_2 &= \quad 2u'_1, \\ \Theta_2 v_3 &= -3u_5. \end{aligned}$$

Hence

$$\begin{aligned} \Theta_2 (u_1 u'_2 - u_2 u'_1) &= 0, \\ \Theta_2 (3u_2 u_5 + 2u_1 v_3) &= 0; \end{aligned}$$

and therefore, if

$$\begin{aligned} \mathrm{P} &= u_1 u'_2 - u_2 u'_1 = qt' - q't, \\ \mathrm{Q} &= 3u_2 u_5 + 2u_1 v_3, \end{aligned}$$

the irreducible combinations which satisfy $\Theta_1\psi = 0 = \Theta_2\psi = \Theta_3\psi$, are u_1, u'_1, P, u_4, u'_4, u_5, Q.

It is now necessary to obtain the irreducible combinations of these seven quantities which satisfy $\Theta_4\psi = 0$. We have

$$\Theta_4 u_1 = p, \qquad \Theta_4 u'_1 = p';$$

so that

$$u'_1 \Theta_4 u_1 - u_1 \Theta_4 u'_1 = u_5.$$

Again,

$$\Theta_4 \mathrm{P} = pt' - p't + 2qs' - 2q's.$$

Now,

$$q(pt' - p't) = p(\mathrm{P} + q't) - p'qt = p\mathrm{P} + u_5 u_2;$$

so that

$$\begin{aligned} \Theta_4 \mathrm{P} &= 3(pt' - p't) + 2v_3 \\ &= \frac{3}{q}(p\mathrm{P} + u_2 u_5) + 2v_3; \end{aligned}$$

and hence

$$u_1 \Theta_4 \mathrm{P} - 3p\mathrm{P} = \mathrm{Q}.$$

Again,

$$\tfrac{1}{2}\Theta_4 \mathrm{Q} = 3su_5 + pv_3 + q\Theta_4 v_3;$$

but

$$\Theta_4 v_3 = qr' - ps' - (q'r - p's);$$

and

$$\begin{aligned} qr' - ps' &= \frac{1}{q}(q^2 r' - 2pqs' + p^2 t') + \frac{p}{q}(qs' - pt') \\ &= \frac{1}{u_1 u'^2_1}(u_1^2 u'_4 - 2u_1' u_3 u_5 - u'_2 u_5^2) + \frac{pu'_3}{u_1}, \end{aligned}$$

$$q'r - p's = \frac{1}{u_1' u_1^2}(u'^2_1 u_4 + 2u'_1 u_3 u_5 - u'_2 u_5^2) + \frac{p'u'_3}{u'_1};$$

so that

$$\Theta_4 v_3 = \frac{p}{u_1} u_3 - \frac{p' u'_3}{u'_1} + \frac{u_1 u'_4}{u_1^2} - \frac{u'_1 u_4}{u_1^2} - \frac{2u_5}{u_1 u'_1}(u_3 + u_3') - \frac{u_5^2}{u_1^2 u'^2_1} \mathrm{P}.$$

Hence

$$pv_3 + q\Theta_4 v_3 = 2pv_3 + \frac{u_1^2}{u'^2_1} u'_4 - \frac{u'_1}{u_1} u_4 - \frac{u_5^2}{u_1 u'^2_1} \mathrm{P} - \frac{u_5}{u'_1}(2u_3 + u'_3).$$

Hence

$$\begin{aligned} \tfrac{1}{2} u_1 \Theta_4 \mathrm{Q} - p\mathrm{Q} &= u_1\left(\frac{u_1^2}{u'^2_1} u_4' - \frac{u'_1}{u_1} u_4\right) - \frac{u_5^2}{u'^2_1} \mathrm{P} - \frac{u_5 u_1}{u'_1}\left(2u_3 + u'_3 - 3q's + 3\frac{pq'}{q} t\right) \\ &= u_1\left(\frac{u_1^2}{u'^2_1} u_4' - \frac{u'_1}{u_1} u_4\right) - \frac{u_5^2}{u'^2_1} \mathrm{P} - \frac{u_5 \mathrm{Q}}{u'_1}. \end{aligned}$$

If now we write

$$\frac{u_1}{u'_1} = \mathrm{C},$$

$$\frac{\mathrm{P}}{u_1^3} = \mathrm{B},$$

$$\frac{\mathrm{Q}}{u_1^2} = \mathrm{A},$$

the three results can be put into the forms

$$\begin{aligned} u'^2_1 \Theta_4 \mathrm{C} &= u_5, \\ u'^2_1 \Theta_4 \mathrm{B} &= \mathrm{AC}^{-4}, \\ u'^2_1 \Theta_4 \mathrm{A} &= 2\mathrm{C}^2 u'_4 - 2\mathrm{C}^{-1} u_4 - 2\mathrm{BC}^2 u_5^2. \end{aligned}$$

And, further, we have

$$\Theta_4 u_4 = 0, \qquad \Theta_4 u_4' = 0, \qquad \Theta_4 u_5 = 0.$$

Since the result of operating with Θ_4 on B gives a quantity into which A enters linearly, and the result of operating with Θ_4 on A gives another quantity into which B enters linearly, we are led to assume that the irreducible solution (or solutions) of $\Theta_4 \psi = 0$ are of the form

$$\mathrm{RA} + \mathrm{SB} + \mathrm{T},$$

where R, S, T are independent of A and B. If this be a solution, we have

$$u'^2_1 \Theta_4 \mathrm{T} = \mathrm{R}(2\mathrm{C}^2 u'_4 - 2\mathrm{C}^{-1} u_4 - 2\mathrm{BC}^2 u_5^2) + \mathrm{SAC}^{-4} + \mathrm{A} u'^2_1 \Theta_4 \mathrm{R} + \mathrm{B} u'^2_1 \Theta_4 \mathrm{S};$$

and we suppose R and S so determined that

$$\mathrm{SC}^{-4} = -u'^2_1 \Theta_4 \mathrm{R},$$

$$2\mathrm{RC}^2 u_5^2 = u'^2_1 \Theta_4 \mathrm{S}.$$

But, if $R = C^n$, then

$$SC^{-4} = -nu'^2_1 C^{n-1}\Theta_4 C = -nu_5 C^{n-1};$$

so that

$$S = -nu_5 C^{n+3};$$

and therefore

$$\begin{aligned} 2u_5^2 C^{n+2} = 2RC^2 u_5^2 &= -n(n+3)\, u_5 C^{n+2} u'^2_1 \Theta_4 C \\ &= -n(n+3)\, u_5^2 C^{n+2}, \end{aligned}$$

whence

$$n = -1 \quad \text{or} \quad -2.$$

First, taking $n = -1$, we have

$$u'^2_1 \Theta_4 T = 2Cu'_4 - 2C^{-2}u_4,$$

and therefore

$$Tu_5 = C^2 u'_4 + 2u_4 C^{-1}.$$

Hence we may take as one irreducible solution

$$X = \frac{A}{C} + BC^2 u_5 - \frac{2u_4 + C^3 u'_4}{u_5 C}.$$

Second, taking $n = -2$, we have

$$u'^2_1 \Theta_4 T = 2u'_4 - 2C^{-3}u_4;$$

and therefore

$$Tu_5 = 2Cu'_4 + C^{-2}u_4.$$

Hence we may take as another irreducible solution

$$Y = \frac{A}{C^2} + 2BCu_5 - \frac{u_4 + 2C^3 u'_4}{u_5 C^2}.$$

And it follows from the method of derivation, and by an application of the theory of linear partial differential equations, that every simultaneous solution of the equations $\Theta_1\psi = 0 = \Theta_2\psi = \Theta_3\psi = \Theta_4\psi$ which involves no quantity of order higher than r, s, t, r', s', t' can be expressed as a functional combination of u_5; u_4, u'_4; X, Y.

23. It is now necessary to consider the index equations. We have for $u_5 \,(= J)$

$$\Phi_0 J = 3J,$$
$$\Phi_1 J = 3J;$$

so that J is an invariant of index unity. For $u_4 \,(= A_0)$ we have

$$\Phi_0 A_0 = 6A_0,$$
$$\Phi_1 A_0 = 6A_0;$$

so that A_0 is an invariant of index 2 ; and similarly for $u'_4 (= A'_0)$,

$$\Phi_0 A'_0 = 6A'_0,$$
$$\Phi_1 A_0' = 6A'_0 ;$$

so that A'_0 is an invariant of index 2.

It is desirable to modify the forms of X and Y, so as to express them explicitly in terms of the quantities $p, p', \ldots$. When the values of A, B, C, u_4, u'_4, u_5 are substituted in X, and it is multiplied by $-u_5$, it takes the form

$$q^2 r' - 2pqs' + p^2 t' + 2\{qq'r - (pq' + p'q)s + pp't\},$$

which may be denoted by $\mathfrak{A}_0$; and when exactly the same operations are applied to Y, it takes the form

$$q'^2 r - 2p'q's + p'^2 t + 2\{qq'r' - (pq' + p'q)s' + pp't'\},$$

which may be denoted by $\mathfrak{A}'_0$.

It is now easy to verify that

$$\Phi_0 \mathfrak{A}_0 = 6\mathfrak{A}_0,$$
$$\Phi_1 \mathfrak{A}_0 = 6\mathfrak{A}_0 ;$$

so that $\mathfrak{A}_0$ is an invariant of index 2 ; and similarly that

$$\Phi_0 \mathfrak{A}'_0 = 6\mathfrak{A}'_0,$$
$$\Phi_1 \mathfrak{A}'_0 = 6\mathfrak{A}'_0 ;$$

so that $\mathfrak{A}'_0$ is an invariant of index 2.

24. The general result of the preceding investigation can be enunciated as follows :—

Every simultaneous invariant of two functions z and z' of two independent variables, which involves no differential coefficients of order higher than the second, can be expressed in terms of the five irreducible invariants J (*of index* 1) *and* A_0, A'_0, $\mathfrak{A}_0$, $\mathfrak{A}'_0$ (*each of index* 2) *where*

$$
\begin{aligned}
J &= pq' - p'q, \\
A_0 &= q^2 r - 2pqs + p^2 t, \\
A'_0 &= q'^2 r' - 2p'q's' + p'^2 t', \\
\mathfrak{A}_0 &= q^2 r - 2pqs' + p^2 t' + 2\{qq'r - (pq' + p'q)s + pp't\}, \\
\mathfrak{A}'_0 &= q'^2 r - 2p'q's + p'^2 t + 2\{qq'r' - (pq' + p'q)s' + pp't'\},
\end{aligned}
$$

and p, q, r, s, t; p', q', r', s', t'; *have their ordinary significations as partial differential coefficients of z and of z'**.

* It is easy to see that the invariant A_0, formed for $z + \lambda z'$ is

$$A_0 + \lambda \mathfrak{A}_0 + \lambda^2 \mathfrak{A}'_0 + \lambda^3 A'_0.$$

This remark is practically due to Professor CAYLEY.

Theory of Eduction.

25. It has already appeared from § 4 that the operator

$$q\frac{\partial}{\partial x} - p\frac{\partial}{\partial y}$$

operating on z' produces an invariant of index unity. But for the purposes of this operation z' may be regarded merely as an unchanging quantity, and, therefore, it may be replaced by an absolute invariant (of index zero); and, when the operator acts upon an absolute invariant, there results a new invariant, of the next higher rank in the differential coefficient of the variable and of index unity.

We can, however, make the operator an absolute invariant, for the index of A_0 is 2; and, therefore,

$$A_0^{-\frac{1}{2}}\left(q\frac{\partial}{\partial x} - p\frac{\partial}{\partial y}\right)$$

is an absolute invariantive operator which, when it operates on an absolute invariant, generates a new absolute invariant of next higher rank.

The operator can evidently be applied any number of times in succession, so that, if I be an absolute invariant,

$$\left\{A_0^{-\frac{1}{2}}\left(q\frac{\partial}{\partial x} - p\frac{\partial}{\partial y}\right)\right\}^r \mathrm{I}$$

is an absolute invariant for all values of the index r.

Similarly, the result of operating upon any absolute invariant with the operator

$$q'\frac{\partial}{\partial x} - p'\frac{\partial}{\partial y}$$

is to give a relative invariant of index unity; and, if we are considering simultaneous invariants in two variables z and z', then

$$\frac{1}{\mathrm{J}}\left(q\frac{\partial}{\partial x} - p\frac{\partial}{\partial y}\right),$$

$$\frac{1}{\mathrm{J}}\left(q'\frac{\partial}{\partial x} - p'\frac{\partial}{\partial y}\right)$$

are absolute invariantive operators, which, when applied to absolute invariants, produce absolute invariants.

26. Thus, in the case of a single dependent variable, we have

$$A = Q_5 A_0^{-2}, \qquad B = Q_6 A_0^{-\frac{5}{2}}, \qquad C = Q_7 A_0^{-3}$$

as the three irreducible invariants proper to the rank three, and they form the complete system of irreducible absolute invariants within this rank. Hence

$$\mathrm{A}' = \mathrm{A}_0^{-\frac{1}{2}} \left(q \frac{\partial}{\partial x} - p \frac{\partial}{\partial y} \right) \mathrm{A},$$

$$\mathrm{B}' = \mathrm{A}_0^{-\frac{1}{2}} \left(q \frac{\partial}{\partial x} - p \frac{\partial}{\partial y} \right) \mathrm{B},$$

$$\mathrm{C}' = \mathrm{A}_0^{-\frac{1}{2}} \left(q \frac{\partial}{\partial x} - p \frac{\partial}{\partial y} \right) \mathrm{C},$$

are absolute invariants proper to the rank four. But it is not to be inferred that A, B, C, A′, B′, C′ constitute the complete system of irreducible absolute invariants within the rank four.

Again, in the operator the quantities p and q are first differential coefficients of an unchanging quantity z; they can be replaced by first differential coefficients of any absolute invariant I, and then

$$\mathrm{A}_0^{-\frac{1}{2}} \left(\frac{\partial \mathrm{I}}{\partial y} \frac{\partial}{\partial x} - \frac{\partial \mathrm{I}}{\partial x} \frac{\partial}{\partial y} \right)$$

is an absolute invariantive operator, the operation of which on absolute invariants produces other absolute invariants. Hence

$$\mathrm{D} = \mathrm{A}_0^{-\frac{1}{2}} \left(\frac{\partial \mathrm{B}}{\partial y} \frac{\partial}{\partial x} - \frac{\partial \mathrm{B}}{\partial x} \frac{\partial}{\partial y} \right) \mathrm{C},$$

$$\mathrm{E} = \mathrm{A}_0^{-\frac{1}{2}} \left(\frac{\partial \mathrm{C}}{\partial y} \frac{\partial}{\partial x} - \frac{\partial \mathrm{C}}{\partial x} \frac{\partial}{\partial y} \right) \mathrm{A},$$

$$\mathrm{F} = \mathrm{A}_0^{-\frac{1}{2}} \left(\frac{\partial \mathrm{A}}{\partial y} \frac{\partial}{\partial x} - \frac{\partial \mathrm{A}}{\partial x} \frac{\partial}{\partial y} \right) \mathrm{B},$$

are absolute invariants proper to the rank four, and they are of the second degree in the differential coefficients of the fourth order. Among the six quantities A′, B′, C′, D, E, F there is the relation,

$$\mathrm{A}'\mathrm{D} + \mathrm{B}'\mathrm{E} + \mathrm{C}'\mathrm{F} = 0;$$

so that only five of them can be independent.

27. In any higher rank n let I, J, K, . . . be the invariants, absolute and irreducible, proper to that rank; let I′ denote the absolute invariant educed from I by the operator

$$\mathrm{A}_0^{-\frac{1}{2}} \left(q \frac{\partial}{\partial x} - p \frac{\partial}{\partial y} \right),$$

and I_m the absolute invariant educed from I by the operator

$$\mathrm{A}_0^{-\frac{1}{2}} \left(\frac{\partial \mathrm{M}}{\partial y} \frac{\partial}{\partial x} - \frac{\partial \mathrm{M}}{\partial x} \frac{\partial}{\partial y} \right),$$

M being an absolute invariant. Then, by means of all the eductive operators associated with absolute invariants of successive ranks, we can obtain from I the set of educed invariants

$$\begin{array}{l} \mathrm{I}' ; \\ \mathrm{I}_a,\ \mathrm{I}_b,\ \mathrm{I}_c ; \\ \mathrm{I}_{a'},\ \mathrm{I}_{b'},\ \mathrm{I}_{c'},\ \mathrm{I}_d,\ \mathrm{I}_e,\ \mathrm{I}_f ; \\ .\ \ .\ \ .\ \ .\ \ .\ \ .\ \ . ; \\ \mathrm{I}_j,\ \mathrm{I}_k,\ \ldots , \end{array}$$

all proper to the rank $n+1$; and there is a similar set from each of the other invariants J, K, . . .

This number, however, can be at once reduced; for, if I_m be any educed invariant other than I′ and I_a, we have

$$\begin{vmatrix} \mathrm{I}', & q, & p \\ \mathrm{I}_a, & \dfrac{\partial \mathrm{A}}{\partial y}, & \dfrac{\partial \mathrm{A}}{\partial x} \\ \mathrm{I}_m, & \dfrac{\partial \mathrm{M}}{\partial y}, & \dfrac{\partial \mathrm{M}}{\partial x} \end{vmatrix} = 0 ;$$

and therefore

$$\mathrm{I}_a \mathrm{M}' = \mathrm{I}' \mathrm{M}_a + \mathrm{I}_m \mathrm{A}',$$

which shows that I_m can be expressed in terms of I′ and I_a, and of invariants proper to lower ranks if M be different from J, K, . . ., and that, if M coincide with one of the invariants J, K, . . ., the invariant I_m can be expressed in terms of the set I′, J′, . . ., the set I_a, J_a, . . ., and of invariants proper to lower ranks.

It therefore follows that the invariants, educed from the absolute irreducible invariants I, J, K, . . . proper to the rank n, can be expressed in terms of I′, J′, K′, . . . ; I_a, J_a, K_a, . . . proper to the rank $n+1$, and of invariants proper to lower ranks. All these educed invariants are, if n be greater than 3, linear in the partial differential coefficients, which are of order $n+1$, and so determine the rank of the invariants.

We know that, for values of n greater than 3, the number of irreducible invariants proper to the rank n is $n+1$, all of which can be made absolute on division by an appropriate power of A_0; hence, the number of invariants educed as above is $2(n+1)$, which must all be expressible in terms of the $n+2$ irreducible invariants proper to the rank $n+1$. But so far there is nothing to indicate which of them, or how many of them, are equivalent to irreducible invariants proper to the rank to which they belong.

28. Again, we have seen that there are four simultaneous invariants of two functions proper to the rank 2, and that there is a single invariant proper to the rank 1; so that

$$C_0 = A_0 J^{-2}, \quad C'_0 = A'_0 J^{-2}, \quad \mathfrak{C}_0 = \mathfrak{A}_0 J^{-2}, \quad \mathfrak{C}'_0 = \mathfrak{A}'_0 J^{-2},$$

are absolute irreducible invariants proper to the rank 2. Let

$$\frac{1}{J}\left(q\frac{\partial}{\partial x} - p\frac{\partial}{\partial y}\right) C_0 = F, \qquad \frac{1}{J}\left(q'\frac{\partial}{\partial x} - p'\frac{\partial}{\partial y}\right) C_0 = F';$$

$$\frac{1}{J}\left(q\frac{\partial}{\partial x} - p\frac{\partial}{\partial y}\right) C'_0 = G, \qquad \frac{1}{J}\left(q'\frac{\partial}{\partial x} - p'\frac{\partial}{\partial y}\right) C'_0 = G';$$

$$\frac{1}{J}\left(q\frac{\partial}{\partial x} - p\frac{\partial}{\partial y}\right) \mathfrak{C}_0 = \mathfrak{F}, \qquad \frac{1}{J}\left(q'\frac{\partial}{\partial x} - p'\frac{\partial}{\partial y}\right) \mathfrak{C}_0 = \mathfrak{F}';$$

$$\frac{1}{J}\left(q\frac{\partial}{\partial x} - p\frac{\partial}{\partial y}\right) \mathfrak{C}'_0 = \mathfrak{G}, \qquad \frac{1}{J}\left(q'\frac{\partial}{\partial x} - p'\frac{\partial}{\partial y}\right) \mathfrak{C}'_0 = \mathfrak{G}';$$

then F, F', G, G', $\mathfrak{F}$, $\mathfrak{F}'$, $\mathfrak{G}$, $\mathfrak{G}'$, are eight educed invariants proper to the rank three. But instead of q and p, or q' and p', we can substitute the first differential coefficients of any unchanging quantity, say of any one of the absolute invariants C_0, C'_0, $\mathfrak{C}_0$, $\mathfrak{C}'_0$, and thus educe new invariants. All these, however, can be expressed in terms of the set of eight already retained; for we at once have

$$\frac{\partial C_0}{\partial x} = p F' - p' F,$$

$$\frac{\partial C_0}{\partial y_3} = q F' - q' F;$$

and therefore

$$\frac{1}{J}\left(\frac{\partial C_0}{\partial y}\frac{\partial}{\partial x} - \frac{\partial C_0}{\partial x}\frac{\partial}{\partial y}\right) C'_0 = J\,(F' G - F G'),$$

which proves the statement.*

Hence, through the present class of eductive operators we are able to derive from the simultaneous invariants proper to a rank n double the number of educed invariants proper to the next higher rank; but it is not to be inferred that they are all irreducible, or that they form the complete system of irreducible invariants proper to that rank.

29. The foregoing linear operators are not the only eductive operators; in fact, each new invariant suggests a new eductive operator. For the fundamental property of non-variation on the part of z, the differential coefficients of which are combined into invariants, enables us to substitute for z any other unchanging quantity, such as an absolute invariant. Thus, for instance, if I be any absolute invariant, then

$$\left(\frac{\partial I}{\partial y}\right)^2 \frac{\partial^2 I}{\partial x^2} - 2\frac{\partial I}{\partial x}\frac{\partial I}{\partial y}\frac{\partial^2 I}{\partial x\,\partial y} + \left(\frac{\partial I}{\partial x}\right)^2 \frac{\partial^2 I}{\partial y^2},$$

* Similarly for functions of $z + \lambda z'$; thus

$$F(z + \lambda z') = F + \lambda(\mathfrak{F} + F') + \lambda^2(\mathfrak{G} + \mathfrak{F}') + \lambda^3(G + \mathfrak{G}') + \lambda^4 G'.$$

the same function of I as A'_0 is of z', is an invariant of index 2, and

$$q^2 \frac{\partial^2 \mathrm{I}}{\partial x^2} - 2pq \frac{\partial^2 \mathrm{I}}{\partial x \, \partial y} + p^2 \frac{\partial^2 \mathrm{I}}{\partial y^2} + 2 \left\{ (qr - ps) \frac{\partial \mathrm{I}}{\partial y} - (qs - pt) \frac{\partial \mathrm{I}}{\partial x} \right\},$$

the same function of z and I as $\mathfrak{A}_0$ is of z and z', is an invariant of index 2.

30. The expressions for educed invariants can be applied as follows to obtain the expressions for the effects of the *operation of* Θ_1, Θ_2, Θ_3, Θ_4 (§ 20) *on differential coefficients of the invariants with regard to the variables.*

Let V be an invariant of index m, and let the operators $q \frac{\partial}{\partial x} - p \frac{\partial}{\partial y}$, $q' \frac{\partial}{\partial x} - p' \frac{\partial}{\partial y}$ be denoted by δ and δ' respectively. Then

$$\mathrm{V}_1 = \mathrm{J}\delta\mathrm{V} - m\mathrm{V}\delta\mathrm{J},$$
$$\mathrm{V}'_1 = \mathrm{J}\delta'\mathrm{V} - m\mathrm{V}\delta'\mathrm{J}$$

are invariants of index $m + 2$; they must satisfy the equations

$$\Theta_1 f = 0 = \Theta_2 f = \Theta_3 f = \Theta_4 f.$$

Hence

$$\mathrm{J}\Theta\delta\mathrm{V} = m\mathrm{V}\Theta\delta\mathrm{J},$$
$$\mathrm{J}\Theta\delta'\mathrm{V} = m\mathrm{V}\Theta\delta'\mathrm{J},$$

are satisfied for each of the operators Θ, because J and V are themselves invariants. Now, actual substitution gives

$$\begin{aligned} \Theta_1\delta\mathrm{J} &= 3q\mathrm{J}, & \Theta_1\delta'\mathrm{J} &= 3q'\mathrm{J}; \\ \Theta_2\delta\mathrm{J} &= -3p\mathrm{J}, & \Theta_2\delta'\mathrm{J} &= -3p'\mathrm{J}; \\ \Theta_3\delta\mathrm{J} &= 0, & \Theta_3\delta'\mathrm{J} &= 0; \\ \Theta_4\delta\mathrm{J} &= 0, & \Theta_4\delta'\mathrm{J} &= 0; \end{aligned}$$

and therefore

$$\begin{aligned} \Theta_1\delta\mathrm{V} &= 3q\mathrm{V}m, & \Theta_1\delta'\mathrm{V} &= 3q'\mathrm{V}m; \\ \Theta_2\delta\mathrm{V} &= -3p\mathrm{V}m, & \Theta_2\delta'\mathrm{V} &= -3p'\mathrm{V}m; \\ \Theta_3\delta\mathrm{V} &= 0, & \Theta_3\delta'\mathrm{V} &= 0; \\ \Theta_4\delta\mathrm{V} &= 0, & \Theta_4\delta'\mathrm{V} &= 0. \end{aligned}$$

Now, since

$$\Theta_1 (\delta\mathrm{V}) = q\Theta_1 \frac{\partial \mathrm{V}}{\partial x} - p\Theta_1 \frac{\partial \mathrm{V}}{\partial y},$$

and

$$\Theta_1\delta'\mathrm{V} = q'\Theta_1 \frac{\partial \mathrm{V}}{\partial x} - p'\Theta_1 \frac{\partial \mathrm{V}}{\partial y},$$

it follows from the first pair of equations that

$$\left.\begin{aligned}\Theta_1 \frac{\partial V}{\partial x} &= 3mV \\ \Theta_1 \frac{\partial V}{\partial y} &= 0\end{aligned}\right\}.$$

Similarly, from the second pair

$$\left.\begin{aligned}\Theta_2 \frac{\partial V}{\partial x} &= 0 \\ \Theta_2 \frac{\partial V}{\partial y} &= 3mV\end{aligned}\right\};$$

from the third pair

$$\left.\begin{aligned}\Theta_3 \frac{\partial V}{\partial x} &= \frac{\partial V}{\partial y} \\ \Theta_3 \frac{\partial V}{\partial y} &= 0\end{aligned}\right\};$$

and from the fourth pair

$$\left.\begin{aligned}\Theta_4 \frac{\partial V}{\partial x} &= 0 \\ \Theta_4 \frac{\partial V}{\partial y} &= \frac{\partial V}{\partial x}\end{aligned}\right\}.$$

And the general laws, of which these are particular examples, and which can be established by means of the successive educts of the invariant V, are

$$\left.\begin{aligned}\Theta_1 \frac{\partial^n V}{\partial x^s\, \partial y^{n-s}} &= s\,(3m + n - 1)\frac{\partial^{n-1} V}{\partial x^{s-1}\, \partial y^{n-s}} \\ \Theta_2 \frac{\partial^n V}{\partial x^{n-s}\, \partial y^{s}} &= s\,(3m + n - 1)\frac{\partial^{n-1} V}{\partial x^{n-s}\, \partial y^{s-1}} \\ \Theta_3 \frac{\partial^n V}{\partial x^r\, \partial y^{n-r}} &= r\,\frac{\partial^{n} V}{\partial x^{r-1}\, \partial y^{n-r+1}} \\ \Theta_4 \frac{\partial^n V}{\partial x^{n-r}\, \partial y^{r}} &= r\,\frac{\partial^{n} V}{\partial x^{n-r+1}\, \partial y^{r-1}}\end{aligned}\right\}.$$

From these the effect on V of any combinations in any order of the operators $\partial/\partial x$, $\partial/\partial y$, Θ_1, Θ_2, Θ_3, Θ_4, can be deduced.

31. The following is another application of the theory of eduction. The index of U_2 (§ 16) is 6, so that $U_2 U_0^{-3}$ is an absolute invariant, and therefore

$$\left(q\frac{\partial}{\partial x} - p\frac{\partial}{\partial y}\right) U_2 U_0^{-3}$$

is an invariant, say

$$V = V_0\, \delta U_2 - 3U_2\, \delta U_0.$$

Now the quantities U are expressed in terms of the quantities A; and from the values of those quantities it at once follows that

$$\begin{aligned}\delta A_m &= A_{m+1} + (qr - sp)\frac{\partial A_m}{\partial p} + (qs - tp)\frac{\partial A_m}{\partial q} \\ &= A_{m+1} - \tfrac{1}{2}\left(\frac{\partial A_0}{\partial p}\frac{\partial A_m}{\partial q} - \frac{\partial A_0}{\partial q}\frac{\partial A_m}{\partial p}\right) \\ &= A_{m+1} - \tfrac{1}{2} J_{0m};\end{aligned}$$

and therefore in particular

$$\begin{aligned}\delta A_0 &= A_1, \\ \delta A_1 &= A_2 - \tfrac{1}{2}J_{01}, \\ \delta A_2 &= A_3 - \tfrac{1}{2}J_{02}.\end{aligned}$$

Hence

$$\begin{aligned}\delta U_2 &= \delta(A_0A_2 - A_1^2) \\ &= A_0A_3 - A_1A_2 - \tfrac{1}{2}A_0J_{02} + A_1J_{01};\end{aligned}$$

and therefore

$$V = A_0^2A_3 - 4A_0A_1A_2 + 3A_1^3 - \tfrac{1}{2}A_0^2J_{02} + A_0A_1J_{01},$$

an invariant proper to the rank 5. But

$$U_3 = A_0^2A_3 - \tfrac{10}{3}A_0A_1A_2 + \tfrac{20}{9}A_1^3$$

is an invariant proper to the rank 5; hence

$$Z = V - U_3 = \tfrac{7}{9}A_1^3 - \tfrac{2}{3}A_0A_1A_2 - \tfrac{1}{2}A_0^2J_{02} + A_0A_1J_{01}$$

is an invariant, and it is evidently proper to the rank 4. It must, therefore, be expressible in terms of the irreducible invariants within the rank 4 given by u_4, Q_5, Q_6, Q_7, Q_9, . . . , Q_{13}. The verification of this inference is as follows.

32. We have

$$Q_{13} = u_4P_{13} - P_8^2;$$

when the values of P_{13} and of P_8—viz. :

$$\frac{1}{u'^2_1}(u_1^2u_{13} + 12u_1u_3u_8 + 36u_3^2u_4) \quad \text{and} \quad \frac{1}{u_1}(u_1u_8 + 6u_3u_4)$$

respectively—are substituted, we at once have

$$\begin{aligned}Q_{13} &= u_4u_{13} - u_8^2 \\ &= A_0A_2 - (-A_1)^2 = A_0A_2 - A_1^2,\end{aligned}$$

thus identifying Q_{13} with U_2.

Again, we have

$$\begin{aligned} Q_{12} &= 18u_4^3P_{12} + 3u_4P_8P_{13} + 81u_4^4P_6 - 2P_8^3 \\ &= 3u_4X + 81u_4^4P_6 - 2P_8^3; \end{aligned}$$

so that X, denoting $6u_4^2P_{12} + P_8P_{13}$, includes all terms proper to the rank 4. When we substitute for P_8, P_{12}, and P_{13} their values we have

$$\begin{aligned} X &= \frac{6u_4^2}{u_1^3}(u_1^2u_{12} + 9u_1u_3u_7 - 12u_3^3 - \tfrac{3}{2}u_2v_8) + \frac{v_8}{u_1^3}(u_1^2u_{13} + 12u_1u_3u_8 + 36u_3^2u_4) \\ &= \frac{6u_4^2u_{12} + v_8u_{13}}{u_1} + X', \end{aligned}$$

where

$$X' = \frac{18u_4^2}{u_{13}}(3u_1u_3u_7 - 4u_3^3 - \tfrac{1}{2}u_2v_8) + \frac{12v_8}{u_1^3}(u_1u_3u_8 + 3u_3^2u_4).$$

Now, for the first part of X we have

$$\frac{6u_4u_{12} + v_8u_{13}}{u_1} = u_8u_{13} + \frac{6u_4}{u_1}(u_4u_{12} + u_3u_{13});$$

and

$$u_4u_{12} + u_3u_{13} = \tfrac{1}{8}\left\{\left(p\frac{\partial u_4}{\partial p} + q\frac{\partial u_4}{\partial q}\right)\frac{\partial u}{\partial p} + \left(q\frac{\partial u_{13}}{\partial q} + p\frac{\partial u_{13}}{\partial p}\right)\left(-\frac{\partial u_4}{\partial p}\right)\right\} = -\tfrac{1}{8}u_1J_{02},$$

the former of the two last lines being obtained partly from the forms of u_3 and u_{12} and partly because u_4 and u_{13} are homogeneous in p and q. Hence

$$\begin{aligned} X &= X' + u_8u_{13} + \tfrac{3}{4}u_4J_{02} \\ &= X' - A_1A_2 - \tfrac{3}{4}A_0J_{02}, \end{aligned}$$

and X′ includes terms of rank not greater than 3. It thus appears that the aggregate of the terms proper to the rank 4 are functionally the same in Z as in Q_{12}; and we have

$$\begin{aligned} \frac{Q_{12}}{3u_4} - \frac{3Z}{2u_4} &= X' + 27u_4^3P_6 - \tfrac{2}{3}\frac{P_8^3}{u_4} - \tfrac{7}{6}\frac{A_1^3}{u_4} - \tfrac{3}{2}A_1J_{01} \\ &= X' + 27u_4^3P_6 - \tfrac{2}{3}\frac{P_8^3}{u_4} + \tfrac{7}{6}\frac{u_8^3}{u_4} + \tfrac{3}{2}u_8J_{01}. \end{aligned}$$

Now, from § 13 we have

$$J_{01} = \frac{6}{u_1}(u_3u_8 + u_4u_7);$$

and from the values of P_7 and P_8 it follows that

$$\begin{aligned} u_8 &= P_8 - \frac{6u_3u_4}{u_1}, \\ u_7 &= u_1P_7 + \frac{u_2u_4}{u_1} + \frac{2u_3^2}{u_1}. \end{aligned}$$

Substituting now in X', in J_{01}, and for u_8 the values as given for u_7 and u_8 in the last two equations (so as to express all the aggregates of coefficients proper to the rank 3 in terms of P_6, P_7, P_8 and to leave the residue of terms—if there be such a residue—as a function of u_1, u_2, u_3, u_4) and gathering together like terms, we find

$$\begin{aligned} X' + 27u_4^{\,3}P_6 - \tfrac{2}{3}\frac{P_8^{\,3}}{u_4} + \tfrac{7}{6}\frac{u_8^{\,3}}{u_4} + \tfrac{3}{2}u_8J_{01} &= 27u_4^{\,3}P_6 + \tfrac{1}{2}\frac{P_8^{\,3}}{u_4} + 9u_4P_7P_8 \\ &= \frac{Q_6}{2u_4}. \end{aligned}$$

Hence we have

$$Z = \tfrac{2}{9}Q_{12} - \tfrac{1}{3}Q_6\,;$$

and it follows that *the first educt of* U_2 $(= Q_{13})$ *when reduced by means of* U_3 *is functionally equivalent to the invariant* Q_{12}.

33. In the preceding investigation the Jacobian of the function A_0 and any other function A_m of the series in § 16 entered. The following formulæ, interesting in themselves, are of use in a verification that Z actually satisfies all the differential equations which are characteristic of an invariant :—

For $m > 2$,

$$\left.\begin{aligned} \Delta_1\frac{\partial A_{m-2}}{\partial p} &= m(m-1)\,q\,\frac{\partial A_{m-3}}{\partial p} \\ \Delta_1\frac{\partial A_{m-2}}{\partial q} &= m(m-1)\left(q\,\frac{\partial A_{m-3}}{\partial q} + A_{m-3}\right) \end{aligned}\right.$$

and, for $m = 2$,

$$\left.\begin{aligned} \Delta_1\frac{\partial A_0}{\partial p} &= -\,2q^2 \\ \Delta_1\frac{\partial A_0}{\partial q} &= 2pq \end{aligned}\right\}\,;$$

and, for $m > 2$,

$$\left.\begin{aligned} \Delta_2\frac{\partial A_{m-2}}{\partial p} &= -\,m(m-1)\left(p\,\frac{\partial A_{m-3}}{\partial p} + A_{m-3}\right) \\ \Delta_2\frac{\partial A_{m-2}}{\partial q} &= -\,m(m-1)\,p\,\frac{\partial A_{m-3}}{\partial q} \end{aligned}\right.$$

and, for $m = 2$,

$$\left.\begin{aligned} \Delta_2\frac{\partial A_0}{\partial p} &= 2pq \\ \Delta_2\frac{\partial A_0}{\partial q} &= -\,2p^2 \end{aligned}\right\}\,;$$

and, for all values of m,

$$\left.\begin{aligned} \Delta_3 \frac{\partial \mathrm{A}_{m-2}}{\partial p} &= 0 \\ \Delta_3 \frac{\partial \mathrm{A}_{m-2}}{\partial q} &= -\frac{\partial \mathrm{A}_{m-2}}{\partial p} \end{aligned}\right\} ;$$

$$\left.\begin{aligned} \Delta_4 \frac{\partial \mathrm{A}_{m-2}}{\partial p} &= -\frac{\partial \mathrm{A}_{m-2}}{\partial q} \\ \Delta_4 \frac{\partial \mathrm{A}_{m-2}}{\partial q} &= 0 \end{aligned}\right\} .$$

From these it follows that, if neither l nor m be zero,

$$\left.\begin{aligned} \Delta_1 \mathrm{J}_{l,m} &= (l+1)(l+2)\left(q\mathrm{J}_{l-1,m} - \mathrm{A}_{l-1}\frac{\partial \mathrm{A}_m}{\partial p}\right) + (m+1)(m+2)\left(q\mathrm{J}_{l,m-1} + \mathrm{A}_{m-1}\frac{\partial \mathrm{A}_l}{\partial p}\right) \\ \Delta_2 \mathrm{J}_{l,m} &= -(l+1)(l+2)\left(p\mathrm{J}_{l-1,m} + \mathrm{A}_{l-1}\frac{\partial \mathrm{A}_m}{\partial q}\right) - (m+1)(m+2)\left(p\mathrm{J}_{l,m-1} - \mathrm{A}_{m-1}\frac{\partial \mathrm{A}_l}{\partial q}\right) \\ \Delta_3 \mathrm{J}_{l,m} &= 0 = \Delta_4 \mathrm{J}_{l,m} \end{aligned}\right\} ;$$

and

$$\left.\begin{aligned} \Delta_1 \mathrm{J}_{0,m} &= -2q(m+2)\mathrm{A}_m + (m+2)(m+1)\left(q\mathrm{J}_{0,m-1} + \frac{\partial \mathrm{A}_0}{\partial p}\mathrm{A}_{m-1}\right) \\ \Delta_2 \mathrm{J}_{0,m} &= 2p(m+2)\mathrm{A}_m - (m+2)(m+1)\left(p\mathrm{J}_{0,m-1} - \frac{\partial \mathrm{A}_0}{\partial q}\mathrm{A}_{m-1}\right) \\ \Delta_3 \mathrm{J}_{0,m} &= 0 = \Delta_4 \mathrm{J}_{0,m} \end{aligned}\right\} .$$

Connexion with Theory of Binary Forms.

34. In connexion with the fact (§ 15) that the irreducible invariants proper to the rank 3 are expressible in terms of the simultaneous concomitants of A_0 and A_1, viewed as binary forms (quadratic and cubic) in q and $-p$ as variables, it is important to remark that the equations $\Delta_3 f = 0$ and $\Delta_4 f = 0$ are in fact the differential equations satisfied by all concomitants of binary forms which have q and $-p$ for their variables, and have

$$\begin{array}{lllll} r, & s, & t\,; & & \\ a, & b, & c, & d\,; & \\ e, & f, & g, & h, & i\,; \\ \cdot & \cdot & \cdot & \cdot & \cdot \quad \cdot \quad \cdot \quad \cdot \end{array}$$

for their coefficients, that is, of A_0, A_1, A_2, . . . , viewed as binary forms. Each form of a concomitant-system satisfies the differential equations characteristic of its con-

comitants; and it thus appears how $A_0, A_1, A_2, \ldots$ are (§§ 9, 16) simultaneous solutions of the two characteristic equations in question. Moreover, since the Jacobian of two binary forms occurs in their concomitant-system, and therefore satisfies the characteristic equations, it is now evident that the quantities denoted by $J_{l,m}$, being

$$\frac{\partial A_l}{\partial p}\frac{\partial A_m}{\partial q} - \frac{\partial A_l}{\partial q}\frac{\partial A_m}{\partial p},$$

must satisfy the equations $\Delta_3 f = 0 = \Delta_4 f$.

Hence, it appears that one method of obtaining the irreducible invariants, which are proper to the rank n and are additional to those proper to ranks less than n, is as follows:—(1) to obtain the concomitants of A_{n-2}, and the simultaneous concomitants of A_{n-2}, and of the concomitant-system of $A_{n-3}, A_{n-4}, \ldots, A_1, A_0$, viewed as binary forms; (2) to frame the combinations of these concomitants which will satisfy the remaining characteristic equations $\Delta_1 f = 0 = \Delta_2 f$; (3) to select from among these combinations such as are, from the supposed known algebraical relations among the concomitants, found to be irreducible.

35. Again, in the case of binary forms in two systems of variables, q and $-p$, q' and $-p'$, and with coefficients

$$\begin{array}{cccc} r, & s, & t, & \\ a, & b, & c, & d, \\ \cdot & \cdot & \cdot & \cdot \end{array} \qquad \begin{array}{cccc} r', & s', & t', & \\ a', & b', & c', & d', \\ \cdot & \cdot & \cdot & \cdot \end{array}$$

the characteristic equations satisfied by their simultaneous concomitants are of the form

$$(\Delta_3 + \Delta'_3)\psi = 0 = (\Delta_4 + \Delta'_4)\psi,$$

that is,

$$\Theta_3\psi = 0 = \Theta_4\psi.$$

And every solution of these equations, with proper limitations as to degree and grade, is a concomitant. Hence, every functional invariant of the two dependent variables z and z' already considered can be expressed in terms of simultaneous concomitants of the set of quantities $A_0, A'_0; A_1, A'_1; \ldots,$ viewed as binary forms in variables q and $-p$, q' and $-p'$.

Thus, for example, we have seen the simultaneous functional invariants, proper to the rank 2, are five in number, and they are—one, J, being the covariant $pq' - p'q$ in the variables alone; two, A_0 and A'_0, being the quadratic forms; and two, $\mathfrak{A}_0$ and $\mathfrak{A}'_0$, which can be exhibited in the respective forms

$$\tfrac{1}{2}\left(q\frac{\partial}{\partial q'} + p\frac{\partial}{\partial p'}\right)^2 A'_0 + \left(q'\frac{\partial}{\partial q} + p'\frac{\partial}{\partial p}\right)A_0,$$

and

$$\tfrac{1}{2}\left(q'\frac{\partial}{\partial q} + p'\frac{\partial}{\partial p}\right)^2 A_0 + \left(q\frac{\partial}{\partial q'} + p\frac{\partial}{\partial p'}\right)A'_0,$$

which are combinations of polar emanants of A_0 and A'_0, the fundamental quadratic forms. And, from the note to § 24, it follows that they can also be represented in the forms

$$\left(q'\frac{\partial}{\partial q} + p'\frac{\partial}{\partial p} + r'\frac{\partial}{\partial r} + s'\frac{\partial}{\partial s} + t'\frac{\partial}{\partial t}\right)A_0,$$

$$\left(q\frac{\partial}{\partial q'} + p\frac{\partial}{\partial p'} + r\frac{\partial}{\partial r'} + s\frac{\partial}{\partial s'} + t\frac{\partial}{\partial t'}\right)A'_0,$$

and also in the forms

$$\tfrac{1}{2}\left(q\frac{\partial}{\partial q'} + p\frac{\partial}{\partial p'} + r\frac{\partial}{\partial r'} + s\frac{\partial}{\partial s'} + t\frac{\partial}{\partial t'}\right)^2 A'_0,$$

$$\tfrac{1}{2}\left(q'\frac{\partial}{\partial q} + p'\frac{\partial}{\partial p} + r'\frac{\partial}{\partial r} + s'\frac{\partial}{\partial s} + t'\frac{\partial}{\partial t}\right)^2 A_0.$$

36. Returning now to the functional invariants of only a single dependent variable, we have seen that they are combinations of the simultaneous covariants of A_0, A_1, A_2, . . . , considered as binary forms in q and $-p$; and all these simultaneous covariants satisfy the equations $\Delta_3 f = 0$, and must, therefore, be expressible in terms of $u_1, u_2, \ldots, u_5, \ldots, u_9, \ldots$. The actual expressions may be obtained as follows:—

From the values of the quantities u we have

$$\begin{cases} t = u_2, \\ u_1 s = u_3 + pu_2, \\ u_1^2 r = u_4 + 2pu_3 + p^2 u_2\,; \end{cases}$$

$$\begin{cases} d = u_5, \\ u_1 c = -u_6 + pu_5, \\ u_1^2 b = u_7 - 2pu_6 + p^2 u_5, \\ u_1^3 a = -u_8 + 3pu_7 - 3p^2 u_6 + p^3 u_5\,; \end{cases}$$

$$\begin{cases} i = u_9, \\ u_1 h = -u_{10} + pu_9, \\ u_1^2 g = u_{11} - 2pu_{10} + p^2 u_9, \\ u_1^3 f = -u_{12} + 3pu_{11} - 3p^2 u_{10} + p^3 u_9, \\ u_1^4 e = u_{13} - 4pu_{12} + 6p^2 u_{11} - 4p^3 u_{10} + p^4 u_{10}\,; \end{cases}$$

and so on. It thus appears that any differential coefficient of z, when multiplied by a power of u_1 equal to the x-grade of the differential coefficient, is linearly expressible in terms of the quantities u proper to its rank, the coefficients of these quantities u in the expression being powers of u.

But in the case of the function A_{n-2}, which is

$$(z_{n,0},\ z_{n-1,1},\ z_{n-2,2},\ \ldots \between q,\ -p)^n,$$

the weights of its concomitants are estimated by assigning to $z_{n,0}$, $z_{n-1,1}$, $z_{n-2,2}$, ... the weights 0, 1, 2, ... in succession, that is, the integers which represent the y-grade of these coefficients $z_{n-s,s}$.

If we have a covariant Ψ of order m which is simultaneous to A_{n-2}, $A_{n'-2}$, $A_{n''-2}$, ..., and of degrees l, l', l'', ... in their respective coefficients, and its leading term be $C_0 q^m$, then the weight of C_0 is

$$\tfrac{1}{2}(nl + n'l' + n''l'' + \ldots - m),$$

which is, therefore, the number representing the y-grade of C_0, considered as the leading term of a functional invariant. Since the grade of each of the coefficients of A_{n-2} is n, it follows that the grade of C_0, so far as it involves the coefficients of A_{n-2} is ln, and, therefore, the grade of C_0 is, in the aggregate,

$$nl + n'l' + n''l'' + \ldots.$$

But the aggregate grade of C_0 is the sum of the x-grade and the y-grade; hence, the x-grade of C_0 is

$$\tfrac{1}{2}(nl + n'l' + n''l'' + \ldots + m).$$

In order, then, to express Ψ in terms of the quantities u, we should proceed to substitute for the coefficients r, s, t, ... the values above obtained, and assuming

$$\Psi = C_0 q^m - C_1 q^{m-1} p + \ldots,$$

it is evident that the only term in Ψ from which terms independent of p can come is the first term. Moreover, since Ψ is expressible as a function of the quantities u alone, it follows that when these substitutions are carried out the terms involving p must disappear, for p is the only non-u quantity which enters into the expressions substituted; and the value of Ψ is, therefore, the aggregate of terms which survive, that is, the aggregate of terms independent of p arising from $C_0 q^m$.

Now, in C_0 this aggregate is obtained by replacing a coefficient, $z_{m,n}$, by a quantity $\pm u_\theta u_1^{-m}$; since C_0 is isobaric *qua* seminvariant, it is of uniform x-grade *qua* part of functional invariant; and therefore the result of these substitutions is to give a function of u_2, u_3, u_4, ..., divided by a power of u_1 equal to the x-grade of C_0, that is, divided by

$$u_1^{\frac{1}{2}(nl + n'l' + n''l'' + \ldots + m)}.$$

If, then, Γ_0 denote this function of u_2, u_3, u_4, ..., we have

$$\Psi = u_1^m \,.\, u_1^{-\frac{1}{2}(nl + n'l' + n''l'' + \ldots + m)},$$

or

$$u_1^{\frac{1}{2}(nl + n'l' + n''l'' + \ldots - m)}\,\Psi = \Gamma_0.$$

37. Hence we have the following theorem:

To express any simultaneous concomitant Ψ of

$$
\begin{aligned}
&(r, s, t)(q, -p)^2,\\
&(a, b, c, d)(q, -p)^3,\\
&(e, f, k, h, i)(q, -p)^4,\\
&\quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot
\end{aligned}
$$

in terms of the quantities $u_1, u_2, u_3, u_4, \ldots$, which are the irreducible solutions of $\Delta_3 f = 0$, the equation characteristic of all these concomitants, it is sufficient to take the coefficient C_0 of the highest power of q in Ψ, to construct a new function Γ_0, which is the same combination of the coefficients of

$$
\begin{aligned}
&(u_4, u_3, u_2)(q, -p)^2,\\
&(-u_8, u_7, -u_6, u_5)(q, -p)^3,\\
&(u_{13}, -u_{12}, u_{11}, -u_{10}, u_9)(q, -p)^4,\\
&\quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot
\end{aligned}
$$

as C_0 is of the coefficients of the binary quantics, and to divide Γ_0 by $u_1^{\frac{1}{2}(nl + n'l' + n''l'' + \ldots - m)}$, where m is the degree of Ψ in q and $-p$, and $l, l', l'', \ldots$ are the degrees of C_0 in the coefficients of $A_{n-2}, A_{n'-2}, A_{n''-2}, \ldots$ respectively.

The theorem is illustrated by one or two examples which have already occurred in the reduction of Q_5, Q_6, Q_7. Thus, for $H_0 = rt - s^2$, we have only one quantic entering into its composition, viz., A_0; so that $n = 2$, $l = 2$; $l' = 0 = l'' \ldots$, and $m = 0$ hence,

$$u_1^{\frac{1}{2}\cdot 2\cdot 2} H_0 = u_4 u_2 - u_3^2,$$

that is,

$$H_0 = (u_4 u_2 - u_3^2)\, u_1^{-2}.$$

Again

$$
\begin{aligned}
-\tfrac{1}{12} H_{01} &= r\,(pd - qc) - 2s\,(pc - qb) + t\,(pb - qa)\\
&= q\,(-cr + 2bs - at) + p\,(dr - 2cs + bt);
\end{aligned}
$$

so that two quantics enter. Thus, we have, for H_{01},

$$
\begin{aligned}
&n = 2,\ \ l = 1;\\
&n' = 3,\ \ l' = 1; \qquad l'' = l''' = \ldots = 0;\\
&m = 1.
\end{aligned}
$$

Hence

$$
\begin{aligned}
-\tfrac{1}{12} u_1^{\frac{1}{2}(2.1+3.1-1)} H_{01} &= -u_4(-u_6) + 2u_7 u_3 - u_2(-u_8)\\
&= u_4 u_6 + 2u_3 u_7 + u_2 u_8
\end{aligned}
$$

as before; and so for others.

38. We might, if we pleased, carry the theorem further, for every simultaneous concomitant satisfies the two characteristic equations $\Delta_3 f = 0$, $\Delta_4 f = 0$, and is therefore expressible in terms of the simultaneous irreducible solutions of these two equations. Such irreducible solutions are necessarily functional combinations of $u_1, u_2, u_3, \ldots$ such as satisfy $\Delta_4 f = 0$; and, as in the earlier cases of §§ 14, 17, it is easy to show that these irreducible solutions within the rank three are equivalent to the set

$$\begin{aligned} u_4 &= \mathrm{A}_0, \\ (u_2 u_4 - u_3^2)\, u_1^{-2} &= \mathrm{H}_0, \\ u_8 &= -\mathrm{A}_1, \\ (u_4 u_7 + u_3 u_8)\, u_1^{-1} &= \mathrm{J}_{01}, \quad \text{(dropping a factor 6)} \\ (u_6 u_8 - u_7^2)\, u_1^{-2} &= \mathrm{H}_1, \\ (u_5 u_8^2 - 3u_6 u_7 u_8 + 2u_7^3)\, u_1^{-3} &= \mathrm{Q}, \end{aligned}$$

which are f, A_{ff}, ϕ, ϑ, Δ, Q respectively in GORDAN's notation. Thus, every simultaneous concomitant within the rank three can be expressed algebraically—though not necessarily rationally—in terms of these six quantities.

The actual expression can be obtained by a development of the method adopted in the preceding theorem. It is first necessary to replace the covariant by its value in terms of $u_1, u_2, u_3, \ldots$; then to substitute by means of the equations

$$\begin{aligned} u_4 &= \mathrm{A}_0, \\ u_2 u_4 &= u_3^2 + u_1^2 \mathrm{H}_0, \\ u_8 &= \mathrm{A}_1, \\ u_4 u_7 &= u_1 \mathrm{J}_{01} + u_3 \mathrm{A}_1, \\ u_4^2 u_6 u_8 &= u_1^2 u_4^2 \mathrm{H}_1 + (u_1 \mathrm{J}_{01} + u_3 \mathrm{A}_1)^2, \\ u_5 u_4^3 u_8^2 &= u_1^3 u_4^3 \mathrm{Q} + (u_1 \mathrm{J}_{01} + u_3 \mathrm{A}_1)^3 + 3u_1^2 u_4^2 \mathrm{H}_1 (u_1 \mathrm{J}_{01} + u_3 \mathrm{A}_1), \end{aligned}$$

for $u_2, u_4, u_5, u_6, u_7, u_8$. The result, we know, must appear as a function of A_0, H_0, A_1, J_{01}, H_1, Q; and, therefore, the terms involving u_3 will disappear, and the factors u_1 will cancel.

For example, in the case of H_{01}, $= p$ (GORDAN) $= \mathrm{L}_1$ (SALMON), we have, dropping the factor 12 and using L_1, which is $\frac{1}{2}\mathrm{H}_{01}$,

$$\begin{aligned} u_1^2 \mathrm{L}_1 &= -u_2 u_8 - 2u_3 u_7 - u_4 u_6 \\ &= \frac{\mathrm{A}_1}{\mathrm{A}_0}(u_3^2 + u_1^2 \mathrm{H}_0) - 2\frac{u_3}{\mathrm{A}_0}(u_1 \mathrm{J}_{01} + u_3 \mathrm{A}_1) + \frac{1}{\mathrm{A}_0 \mathrm{A}_1}\{u_1^2 \mathrm{A}_0^2 \mathrm{H}_1 + (u_1 \mathrm{J}_{01} + u_3 \mathrm{A}_1)^2\} \\ &= \left(\frac{\mathrm{A}_1}{\mathrm{A}_0}\mathrm{H}_0 + \frac{\mathrm{A}_0}{\mathrm{A}_1}\mathrm{H}_1 + \frac{\mathrm{J}_{01}^2}{\mathrm{A}_0 \mathrm{A}_1}\right); \end{aligned}$$

so that

$$\mathrm{L}_1 \mathrm{A}_0 \mathrm{A}_1 = \mathrm{A}_1^2 \mathrm{H}_0 + \mathrm{A}_0^2 \mathrm{H}_1 + \mathrm{J}_{01}^2.$$

Similarly for others of the simultaneous concomitants of A_0 and A_1. And it is not difficult to show that all functional invariants within the rank 4, or, what is the equivalent, all the simultaneous concomitants of A_0, A_1, A_2, considered as three binary forms, can be expressed in terms of the foregoing six quantities A_0, H_0, A_1, J_{01}, H_1, Q, and the succeeding five, viz.:—

$$
\begin{aligned}
u_{13} &= A_2, \\
(u_{12}u_4 + u_3u_{13})\,u_1^{-1} &= J_{02}, \\
(u_{11}u_{13} - u_{12}^2)\,u_1^{-2} &= H_2, \\
(u_{10}u_{13}^2 - 3u_{11}u_{12}u_{13} + 2u_{12}^3)\,u_1^{-3} &= \Phi_2, \\
(u_9u_{13} - 4u_{10}u_{12} + 3u_{11}^2)\,u_1^{-4} &= I_2.
\end{aligned}
$$

Inferences can also be deduced as to the expressibility of the simultaneous concomitants of A_0 and A_2 alone as simultaneous quantics, and of the simultaneous concomitants of A_1 and A_3 alone, as simultaneous concomitants; but all such results are chiefly interesting from the point of view of the theory of binary forms, and are more useful in that theory than in the theory of functional invariants.