X. *On the Reflection and Refraction of Light at the Surface of a Magnetized Medium.*

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1. The object of this investigation is to endeavour to ascertain how far the electromagnetic theory of light, as at present developed, is capable of giving a theoretical explanation of Dr. Kerr's experiments on the effect of magnetism on light.

In the first series of experiments* polarized light was reflected from the polished pole of an electromagnet, and it was found that when the circuit was closed, so that the reflecting surface became magnetized perpendicularly to itself, the reflected light exhibited certain peculiarities, which disappeared when the circuit was broken.

In the second series of experiments† the reflector was a polished plate of soft iron laid upon the poles of a horse-shoe electromagnet, so that the direction of magnetization was parallel, or approximately so, to the reflecting surface; and it was found that the effect of the current was analogous to, though by no means identical with, the effect produced in the first series of experiments. It was also found that when the incidence was normal, or when the plane of incidence was perpendicular to the direction of magnetization, no effect was produced.

In both series of experiments it was found that the effects produced by magnetization materially varied with the angle of incidence. It was also found that these effects were in most cases reversed, when the direction of the magnetizing current was reversed; that is to say, if the intensity of the reflected light was strengthened by a right-handed current, it was weakened by a left-handed one.

Since the effects produced by the current were in most cases reversed when the direction of the current was reversed, it follows that the *first power* of the magnetic force must enter into the expression for the intensity of the reflected light.

It will be noticed that in all these experiments a metallic reflector was employed, and consequently the results were complicated by the influence of metallic reflection. It therefore seems hopeless to attempt to construct a theory which shall furnish a complete explanation of these phenomena, until a perfectly satisfactory electromagnetic theory of metallic reflection has been obtained, and, so far as I am aware,

* 'Phil. Mag.,' May, 1877.

† *Ibid.*, March, 1878.

no such theory has been discovered. Lord RAYLEIGH* has shown that CAUCHY'S formulæ may be derived from GREEN'S theory of elastic media, by assuming that the effect of metallic reflection is represented by a term proportional to the velocity; and in a subsequent paper† he has shown that the same formulæ may be obtained from the electromagnetic theory by taking into account the conductivity of the metallic reflector. The results obtained by either of these theories are open to various objections, which Lord RAYLEIGH has discussed in the papers referred to; and he considers "that much remains to be done before the electrical theory of metallic reflection can be accepted as complete."

There are, however, several non-metallic reflecting media (such as strong solutions of certain chemical compounds of iron), which are capable, when magnetized, of producing an effect upon light; and the theoretical explanation of the magnetic action of such media is accordingly free from the difficulties surrounding metallic reflection. It would be exceedingly desirable that experiments upon magnetic solutions should be made; and in view of the possibility of such experiments I have thought it worth while to develop a theory applicable to them. Whether the results of the present paper will stand the test of experiment cannot be finally decided until the experiments alluded to have been made;‡ but it must not excite surprise if the results of the theory do not agree very well with KERR'S experiments, since the conditions of the problem are materially different.

2. I shall now describe four of KERR'S experiments, which it will be necessary to consider more fully when discussing the results obtained from the mathematical theory; and I shall then proceed to give a brief abstract of the results to which this theory leads, stating the points of agreement and disagreement.

In Experiments I. and II. the magnetization is perpendicular to the reflector.

Experiment I.—Light polarized in or perpendicularly to the plane of incidence is allowed to fall on the polished pole of an electromagnet, and the analyser is placed in the position of extinction. When the circuit is closed, so that the reflector becomes magnetized, the light immediately reappears; when the circuit is broken the light disappears, and again reappears when the current is reversed.

The light reflected whilst the circuit is closed is not plane polarized, since it cannot be extinguished by rotating the analyser; also the analyser's position of extinction before closing is the position of minimum intensity after closing.

Experiment II.—The arrangements are the same as in the last experiment, and the analyser is turned from the position of extinction through a small angle towards the right-hand of the observer,§ giving a faint restoration of light. When the circuit is

* Hon. J. W. STRUTT, 'Phil. Mag.,' May, 1872.

† 'Phil. Mag.,' August, 1881.

‡ See, however, § 16, July, 1891.

§ The terms right-handed and left-handed refer to an observer who is looking at the point of incidence through the analyser.

closed, so that the reflector becomes a negative pole, the intensity is increased; but if the current is reversed, so that the reflector becomes a positive pole, the intensity is diminished. The weakening effect of the second operation is always less than the strengthening effect of the first, and its effect diminishes as the angle through which the analyser is turned is diminished. In this experiment the angle of incidence lay between 60° and 80°. Similar results were also obtained when the incidence was normal.

In Experiments III. and IV. the direction of magnetization is parallel to the reflector, and the plane of incidence is parallel to the lines of magnetic force.

Experiment III.—The incident light is polarized *in* the plane of incidence, and the analyser is placed in the position of extinction, and is then turned through a small angle. The circuit is now closed, and it is found that the light restored from extinction by a small right-handed rotation of the analyser is always strengthened by a right-handed current, and weakened by a left-handed one.

The intensity of these optical effects of magnetization varies with the angle of incidence. The effects increase from grazing incidence up to about 65° to 60°, and then diminish as the angle of incidence decreases.

Experiment IV.—The incident light is polarized *perpendicularly* to the plane of incidence, and the arrangements are the same as in the last experiment. At an incidence of 85° the light restored by a right-handed rotation of the analyser is strengthened by a right-handed current and weakened by a left-handed one, and the effects are undistinguishable from those of the last experiment. As the angle of incidence decreases the effects diminish, and finally disappear at an angle of 75°. As the angle of incidence still further decreases the effects reappear, but are reversed; for the light restored by a right-handed rotation is now weakened by a right-handed current and strengthened by a left-handed one. The effects increase from 75° to about 60°, and then gradually diminish.

I shall now describe the results of the present paper, in which the reflecting surface is supposed to be a magnetized transparent medium.

As regards the points of agreement I find (i.) that the reflected light is elliptically polarized; (ii.) that when the magnetization is parallel to the reflecting surface no effect is produced when the incidence is normal, or when the plane of incidence is perpendicular to the lines of magnetic force; (iii.) when the plane of incidence is parallel to the lines of magnetic force, and the light is polarized *in* the plane of incidence, the magnetic term increases from grazing incidence to a maximum value, and then diminishes to normal incidence.

The points of disagreement are that in all cases the intensity is unchanged when the magnetizing current is reversed; and also that no results analogous to those of the fourth experiment are obtained.

It is not improbable that this disagreement is due to the disturbing effect of metallic reflection, inasmuch as it is easy to see, from Lord RAYLEIGH's proof of

CAUCHY's formulæ for metallic reflection, how a term in the intensity, which is proportional to the first power of the magnetic force, and which would therefore be the most important term, might come in, when the reflector is a metal; but the question is one which can only be decided by experiment, and it is therefore very desirable that experiments on magnetic solutions should be made.

Theory of Magnetic Action on Light.

3. We must now consider the theories which have been proposed to explain the action of magnetism on light.

One of the difficulties of MAXWELL's theory* of magnetic action is, that it is inconsistent with his previous theory of isotropic and doubly refracting media. In the latter theory, he has assumed that the disturbance which produces the sensation of light is represented by the *electric displacement*, whereas, in the theory of magnetic action, he has supposed that light is produced by the vibrations of an elastic medium or ether, and that when this medium is acted upon by magnetic force, a state of vortex motion is set up, which modifies the motion which produces light.

4. Professor FITZGERALD † has modified MAXWELL's theory, by supposing that the effect of magnetic force is to introduce into the kinetic energy an additional term

$$4\pi \mathrm{C} \iiint \left(\frac{d\xi}{d\omega} \dot{f} + \frac{d\eta}{d\omega} \dot{g} + \frac{d\zeta}{d\omega} \dot{h} \right) dx\, dy\, dz \quad \ldots \ldots \quad (1),$$

where

$$\frac{d}{d\omega} = \alpha \frac{d}{dx} + \beta \frac{d}{dy} + \gamma \frac{d}{dz} \quad \ldots \ldots \quad (2),$$

and f, g, h are the components of electric displacement, whilst α, β, γ are the components of the external magnetic force. The quantities ξ, η, ζ are the time integrals of the components of the periodic portions of the magnetic force, and, accordingly, f, g, h are proportional to the curl of ξ, η, ζ, so that

$$4\pi f = \frac{d\zeta}{dy} - \frac{d\eta}{dz} \;\&\mathrm{c}. \quad \ldots \ldots \quad (3).$$

If, therefore, we substitute the values of f, g, h from (3) in (1), and integrate by parts, we shall obtain

* 'Electricity and Magnetism,' vol. 2, ch. xxi.

† 'Phil. Trans.,' 1880, p. 691.

$$\mathrm{C}\iint\left\{(m\dot{\zeta} - n\dot{\eta})\frac{d\xi}{d\omega} + (n\dot{\xi} - l\dot{\zeta})\frac{d\eta}{d\omega} + (l\dot{\eta} - m\dot{\xi})\frac{d\zeta}{d\omega}\right\} d\mathrm{S}$$

$$+ \mathrm{C}\iiint\left\{\dot{\xi}\frac{d}{d\omega}\left(\frac{d\zeta}{dy} - \frac{d\eta}{dz}\right) + \dot{\eta}\frac{d}{d\omega}\left(\frac{d\xi}{dz} - \frac{d\zeta}{dx}\right) + \dot{\zeta}\frac{d}{d\omega}\left(\frac{d\eta}{dx} - \frac{d\xi}{dy}\right)\right\} dx\,dy\,dz \quad . \quad (4).$$

The volume integral part of this expression is of the same form as MAXWELL'S expression, but the similarity between the two theories ends here; for in MAXWELL'S theory ξ, η, ζ are the displacements of an elastic medium, whereas, in FITZGERALD'S theory, they are the time integrals of magnetic force. It also appears that the additional terms in the latter theory cannot be accounted for on MAXWELL'S hypothesis of molecular vortices, inasmuch as this would require us to assume that the components α', β', γ', of the total magnetic force, satisfy the equations

$$\alpha' = \alpha + d\xi/d\omega, \qquad \beta' = \beta + d\eta/d\omega, \qquad \gamma' = \gamma + d\zeta/d\omega,$$

where ξ, η, ζ are the time integrals of the periodic parts of the magnetic force;—a relation which cannot apparently be justified on any physical grounds whatever.

The equations which are satisfied at the surface of two different media are also not deduced in a very satisfactory manner, nor is their physical interpretation fully explained. Professor FITZGERALD has also failed to recognise, that when light is incident upon a magnetic medium, there are always (except in special cases), two refracted waves which are circularly polarized in opposite directions, and whose *directions*, as well as their velocities of propagation, are different. It is quite true, that the angle between the directions of the two waves is very small, and would be difficult of observation; but since it depends upon the *first power* of the external magnetic force, we are not at liberty to neglect it.

5. The theory which I propose to develop in the present paper, depends upon the following considerations:—

It was proved experimentally by HALL,* that when a current passes through a conductor which is placed in a magnetic field, an electromotive force is produced, whose intensity is proportional to the product of the current and the magnetic force, and whose direction is at right angles to the plane containing the current and the force. Now, Professor ROWLAND† has assumed that this result holds good in a dielectric which is under the action of a strong magnetic force; if, therefore, we adopt this hypothesis, we must substitute the time variations of the electric displacement for the current, and the equations of electromotive force become

* 'Phil. Mag.,' March, 1880.

† *Ibid.*, April, 1881, p. 254.

$$\left.\begin{aligned} P &= -\frac{dF}{dt} - C(\gamma\dot{g} - \beta\dot{h}) - \frac{d\psi}{dx} \\ Q &= -\frac{dG}{dt} - C(\alpha\dot{h} - \gamma\dot{f}) - \frac{d\psi}{dy} \\ R &= -\frac{dH}{dt} - C(\beta\dot{f} - \alpha\dot{g}) - \frac{d\psi}{dz} \end{aligned}\right\} \quad . \; . \quad . \; . \; . \quad (5),$$

where α, β, γ are components of the total magnetic force.

When the magnetic field is disturbed by the passage of a wave of light, α, β, γ may be supposed to have the same values as before disturbance, since their variations when multiplied by $\dot{f}$, $\dot{g}$, $\dot{h}$ are terms of the second order, which may be neglected. Since we shall confine our attention to the propagation of light in a uniform magnetic field, α, β, γ may be regarded as constant quantities.

I shall, therefore, assume that when light is transmitted through a medium, which, when under the action of a strong magnetic force, is capable of magnetically affecting light, the equations of electromotive force are represented by (5), where C is called HALL's constant, and is a quantity which depends on the nature of the medium, and possibly, also, on the period of vibration. Since we shall require to use the letters α, β, γ to denote that portion of the magnetic force which is due to optical causes, we shall write these equations in the form

$$\left.\begin{aligned} P &= -\frac{dF}{dt} - p_3\dot{g} + p_2\dot{h} - \frac{d\psi}{dx} \\ Q &= -\frac{dG}{dt} - p_1\dot{h} + p_3\dot{f} - \frac{d\psi}{dy} \\ R &= -\frac{dH}{dt} - p_2\dot{f} + p_1\dot{g} - \frac{d\psi}{dz} \end{aligned}\right\} \quad . \; . \; . \; . \; . \; . \; . \quad (6),$$

where $p_1 = C\alpha$, &c. All the other equations of the field are the same as MAXWELL's, with the exception that we do not suppose that $dF/dx + dG/dy + dH/dz$ is zero.

6. In order to obtain the equations of electric displacement, let us consider a medium which is magnetically isotropic but electrostatically anisotropic. Let k be the magnetic permeability; K_1, K_2, K_3 the three principal specific inductive capacities; also let

$$kK_1 = A^{-2}, \qquad kK_2 = B^{-2}, \qquad kK_3 = C^{-2}.$$

$$\Omega = A^2\frac{df}{dx} + B^2\frac{dg}{dy} + C^2\frac{dh}{dz} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (7),$$

$$\frac{d}{d\omega} = p_1\frac{d}{dx} + p_2\frac{d}{dy} + p_3\frac{d}{dz} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (8).$$

From the last two of (6) we obtain

$$\frac{da}{dt} + p_1 \frac{d\dot{f}}{dy} + p_3 \frac{d\dot{f}}{dz} - p_1 \frac{d\dot{g}}{dy} - p_1 \frac{d\dot{h}}{dz} = \frac{dQ}{dz} - \frac{dR}{dy}.$$

Substituting the values of P, Q, R from the equations $P = 4\pi f/K_1$, &c., and recollecting that

$$\frac{df}{dx} + \frac{dg}{dy} + \frac{dh}{dz} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9),$$

we obtain

$$\frac{da}{dt} = 4\pi k \left(B^2 \frac{dg}{dz} - C^2 \frac{dh}{dy} \right) - \frac{d\dot{f}}{d\omega} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

with two similar equations.

Now,

$$4\pi k \ddot{f} = 4\pi k \dot{u} = \frac{d}{dt} \left(\frac{dc}{dy} - \frac{db}{dz} \right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11);$$

substituting the values of $\dot{a}$, $\dot{b}$, $\dot{c}$ from (10) we obtain

$$\left.\begin{aligned}
\frac{d^2 f}{dt^2} &= A^2 \nabla^2 f - \frac{d\Omega}{dx} + \frac{1}{4\pi k} \frac{d}{d\omega} \left(\frac{d\dot{g}}{dz} - \frac{d\dot{h}}{dy} \right) \\
\frac{d^2 g}{dt^2} &= B^2 \nabla^2 g - \frac{d\Omega}{dy} + \frac{1}{4\pi k} \frac{d}{d\omega} \left(\frac{d\dot{h}}{dx} - \frac{d\dot{f}}{dz} \right) \\
\frac{d^2 h}{dt^2} &= C^2 \nabla^2 h - \frac{d\Omega}{dz} + \frac{1}{4\pi k} \frac{d}{d\omega} \left(\frac{d\dot{f}}{dy} - \frac{d\dot{g}}{dx} \right)
\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12).$$

These are the equations satisfied by the components of electric displacement.

Professor FITZGERALD has not obtained the equations of electric displacement, but he has obtained the equations satisfied by the components of magnetic force (see p. 702); and by forming these equations by means of (10) and (11) it will be found that the results agree. It is not, however, easy to see what is the physical interpretation of those boundary conditions which involve the differential coefficients of the time integrals of the components of magnetic force; and it will be seen later on that they disagree with those given in the present paper, which are based upon physical considerations. A term in the kinetic energy which involves a volume integral, can usually be transformed in a variety of ways by integration by parts, into another volume integral and a surface integral; and although it is immaterial, so far as equations of motion are concerned, which volume integral we use, the boundary conditions will depend upon the surface integral. Accordingly, if we omitted a surface integral term from the expression for the kinetic energy and were to apply the principle of least action, it is evident that we should obtain the correct equations of motion, but the boundary conditions would be erroneous.

The boundary conditions $\xi_1 = \xi$, $\eta_1 = \eta$ (p. 706) appear to be perfectly satisfactory, since they imply that the component of magnetic force parallel to the reflecting surface is continuous; and the rejection of the condition $\zeta_1 = \zeta$ can be justified on the ground that it implies that the component of magnetic *force* perpendicular to the reflecting surface is continuous, whereas it is the corresponding component of magnetic *induction* that is continuous.

7. We shall now confine our attention to isotropic media. In this case $A = B = C = U$, where $U^{-2} = kK$; hence (10) becomes

$$\frac{da}{dt} = 4\pi k U^2 \left(\frac{dg}{dz} - \frac{dh}{dy}\right) - \frac{d\dot{f}}{d\omega}.$$

Let

$$f = S\lambda, \quad g = S\mu, \quad h = S\nu, \quad S = \epsilon^{2\iota\pi/V\tau.(lx + my + nz - Vt)} \quad . \quad . \quad . \quad (13),$$

then

$$\frac{da}{dt} = \frac{8\iota\pi^2 U^2 k}{V\tau}(n\mu - m\nu)\,S - \frac{2\iota\pi}{V\tau}(lp_1 + mp_2 + np_3)\,\dot{S}\lambda\,,$$

whence

$$\alpha = -\frac{4\pi U^2}{V}(n\mu - m\nu)\,S - \frac{2\iota\pi}{Vk\tau}(lp_1 + mp_2 + np_3)\,S\lambda\,.$$

Accordingly if $\mathfrak{H}$ denote the component of the external magnetic force perpendicular to the wave-front, the equations of magnetic force become

$$\left.\begin{aligned} \alpha &= -\frac{4\pi U^2}{V}(ng - mh) - \frac{2\iota\pi}{Vk\tau}\,C\mathfrak{H}f \\ \beta &= -\frac{4\pi U^2}{V}(lh - nf) - \frac{2\iota\pi}{Vk\tau}\,C\mathfrak{H}g \\ \gamma &= -\frac{4\pi U^2}{V}(mf - lg) - \frac{2\iota\pi}{Vk\tau}\,C\mathfrak{H}h \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (14),$$

where C is HALL'S constant.

Propagation of Light.

8. We are now prepared to consider the propagation of light in a magnetized medium.

Let us suppose that plane waves of light are incident upon the surface of separation of air and a magnetized medium. Let the axis of x be the normal, and be drawn into the first medium, and let the axis of z be perpendicular to the plane of incidence; also let the direction of magnetization be parallel to the axis of x. Then $p_2 = p_3 = 0$, and none of the quantities are functions of z; whence the equations of motion become

$$\left.\begin{aligned}\frac{d^2f}{dt^2} &= U^2\nabla^2 f - \frac{p}{4\pi k}\frac{d^2\dot{h}}{dx\,dy}\\ \frac{d^2g}{dt^2} &= U^2\nabla^2 g + \frac{p}{4\pi k}\frac{d^2\dot{h}}{dx^2}\\ \frac{d^2h}{dt^2} &= U^2\nabla^2 h + \frac{p}{4\pi k}\frac{d}{dx}\left(\frac{d\dot{f}}{dy} - \frac{d\dot{g}}{dx}\right)\end{aligned}\right\} \quad \ldots\ldots\ldots \quad (15),$$

where p is written for p_1.

Let

$$f = A'S, \quad g = A''S, \quad h = AS, \quad S = \epsilon^{2\iota\pi/V\tau.(lx + my - Vt)}.$$

Substituting in (15) we obtain

$$(U^2 - V^2)\,A' = \frac{\iota p}{2k\tau}\,lmA,$$

$$(U^2 - V^2)\,A'' = \frac{\iota p}{2k\tau}\,l^2A,$$

$$(U^2 - V^2)\,A = \frac{\iota pl}{2k\tau}(A\,m - A''l).$$

From these equations we deduce

$$V^2 = U^2 \pm \frac{pl}{2k\tau} \quad \ldots\ldots\ldots\ldots \quad (16),$$

whence

$$A' = \pm\,\iota mA, \qquad A'' = \mp\,\iota lA.$$

Hence, if V_1, V_2 denote the two values of V corresponding to the upper and lower signs, we see that two waves are propagated with velocities V_1, V_2.

It must also be borne in mind that the directions of the two refracted waves corresponding to an incident wave are, in general, different. To see this, let the suffixes 1 and 2 refer to the two refracted waves, and let the incident wave be

$$h = \epsilon^{2\iota\pi/V\tau.(lx + my - Vt)},$$

then the displacements in one of the refracted waves will be

$$f_1 = \iota m_1 A_1 S_1, \quad g_1 = -\,\iota l_1 A_1 S_1, \quad h_1 = A_1 S_1,$$

where

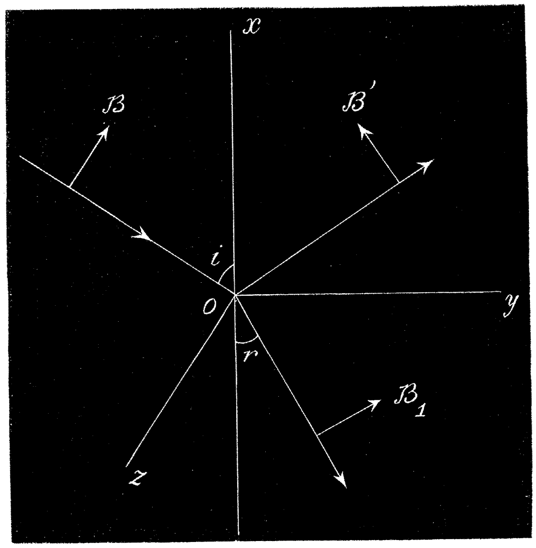
$$S_1 = \epsilon^{2\iota\pi/V_1\tau.(l_1x + m_1y - V_1t)},$$

and the displacements in the other wave will be obtained by changing the suffix from 1 to 2, and changing the signs of f_1, g_1. Now, if r_1, r_2 be the angles of refraction,

$m_1 = \sin r_1$, $m_2 = \sin r_2$; and, since the coefficient of y must be the same in all three waves, we must have

$$\frac{V}{\sin i} = \frac{V_1}{\sin r_1} = \frac{V_2}{\sin r_2},$$

which shows that r_1 is different from r_2.



Let $\mathfrak{B}_1$, $\mathfrak{B}_2$ be the component displacements in the plane $z = 0$, then since $l_1 = -\cos r_1$, it follows that

$$\mathfrak{B}_1 = f_1 \sin r_1 + g_1 \cos r_1 = \iota A_1 S_1.$$

Similarly

$$\mathfrak{B}_2 = -\iota A_2 S_2.$$

The component displacements perpendicular to the wave-fronts are evidently zero; whence, in real quantities, the displacements in the two waves are

$$h_1 = A_1 \cos \frac{2\pi}{V_1 \tau} (l_1 x + m_1 y - V_1 t),$$

$$\mathfrak{B}_1 = -A_1 \sin \frac{2\pi}{V_1 \tau} (l_1 x + m_1 y - V_1 t),$$

and

$$h_2 = A_2 \cos \frac{2\pi}{V_2 \tau} (l_2 x + m_2 y - V_2 t),$$

$$\mathfrak{B}_2 = A_2 \sin \frac{2\pi}{V_2 \tau} (l_2 x + m_2 y - V_2 t),$$

and, consequently, the two waves are circularly polarized in opposite directions.

The Boundary Conditions.

9. When light is reflected and refracted at the surface of separation of two isotropic or crystalline media, the boundary conditions are (i) that the components of the electromotive and magnetic forces *parallel* to the surface of separation must be continuous; (ii) that the components of electric displacement and magnetic induction *perpendicular* to the surface of separation must likewise be continuous. We have, therefore, *six* equations to determine *four* unknown quantities; but inasmuch as two pairs of these equations are identical, the total number reduces to four, which is just sufficient to determine the four unknown quantities. If, however, we were to assume these six conditions in the case of a magnetized medium, we should find that we should be led to inconsistent results, and we shall, therefore, proceed to prove the boundary conditions.

Since the electric displacement and the magnetic induction both satisfy the equation

$$\frac{df}{dx}+\frac{dg}{dy}+\frac{dh}{dz}=0,$$

which is an equation of the same form as the equation of continuity of an incompressible fluid in hydrodynamics, it follows that the components of the electric displacement and magnetic induction perpendicular to the surface of separation must be continuous.

To obtain the other conditions, let us suppose as before, that the plane $x=0$ is the surface of separation, and that the plane $z=0$ contains the direction of propagation. Then, since the coefficients of y and t in the exponential factor must be the same in all four waves, d/dy and d/dt of any continuous function will also be continuous, and conversely. Since none of the quantities are functions of z,

$$4\pi\dot{f}=\frac{d\gamma}{dy},$$

which shows that γ is continuous.

Since the continuity of γ follows from that of f, the conditions of continuity of both these quantities will be expressed by the same equation.

Since

$$a=\frac{d\mathrm{H}}{dy},$$

it follows that H is continuous, whence if the accents refer to the second medium, we obtain from (6)

$$\mathrm{R}'+p_2\dot{f}'-p_1\dot{g}'=\mathrm{R} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17).$$

This equation shows that the electromotive force parallel to z is discontinuous. This circumstance may, at first sight, appear somewhat strange, and may perhaps be

regarded as an objection to the theory; but since the p's are exceedingly small quantities, the discontinuity is also very small. We have, moreover, assumed that the transition from one medium to the other is *abrupt,* whereas, if we were better acquainted with the conditions at the confines of two different media, we should probably find that this was not the case; and under these circumstances there would be a rapid but continuous change in the component of the electromotive force parallel to the boundary, in passing from one medium to the other.

We have, therefore, as yet, only obtained two independent boundary equations. Now, we shall presently see that when plane polarized light is reflected and refracted at the surface of a magnetized medium, the reflected light is elliptically polarized; whilst, as we have already shown, the two refracted waves are circularly polarized in opposite directions. We have, therefore, four unknown quantities to determine, viz., the amplitudes of the two components of the reflected vibration, and the amplitudes of the two refracted waves. We, therefore, require two more equations. To find a third equation, we may assume, as has been done by previous writers, that the component of magnetic force parallel to the axis of y is continuous. A fourth equation will be obtained from the condition of continuity of energy; for since there is no conversion of energy into heat, or any form of energy other than the electrical kind, it follows that the rate of increase of the electrostatic and electrokinetic energies within any closed surface must be equal to the rate at which energy flows in across the boundary.

[We must now obtain an expression for the energy.*

It is a general principle of dynamics, that if equations are given which are sufficient to completely determine the motion of a system, the principle of energy can be deduced from these equations. The proper form of the principle of energy in the case of a dielectric medium is this:—*Describe any closed surface in the medium, then the rate at which energy increases within the surface, is equal to the rate at which energy flows in across the boundary.* If E be the electric energy per unit of volume, the rate at which energy increases within the surface is $\iiint \dot{E}\, dx\, dy\, dz$, and, consequently, this quantity must be capable of being expressed as a surface integral taken over the boundary; and any form of $\dot{E}$ which is not capable of being so expressed must certainly be wrong. If the medium were a conductor, in which there is a conversion of energy into heat, $\iiint \dot{E}\, dx\, dy\, dz$ would not be expressible in the form of a surface integral,† but this case need not be considered, since we are dealing with a transparent dielectric.

Since $P = 4\pi f/K_1 = 4\pi k A^2 f$, equation (10) may be written in the form

$$\frac{da}{dt} = \frac{dQ}{dz} - \frac{dR}{dy} - \frac{d\dot{f}}{d\omega}.$$

* Rewritten, July 1, 1891.

† *See* POYNTING, 'Phil. Trans.,' 1884, p. 343.

Multiply this equation and the two corresponding ones by α, β, γ; then add and integrate throughout any closed surface, and we shall obtain

$$\tfrac{1}{2} k \frac{d}{dt}\iiint(\alpha^2 + \beta^2 + \gamma^2)\, dx\, dy\, dz$$

$$= \iiint\left\{\alpha\left(\frac{dQ}{dz} - \frac{dR}{dy}\right) + \beta\left(\frac{dR}{dx} - \frac{dP}{dz}\right) + \gamma\left(\frac{dP}{dy} - \frac{dQ}{dx}\right)\right\}dx\, dy\, dz$$

$$- \iiint\left(\alpha\frac{d\dot{f}}{d\omega} + \beta\frac{d\dot{g}}{d\omega} + \gamma\frac{d\dot{h}}{d\omega}\right)dx\, dy\, dz \quad . \quad (17\text{A}).$$

Let

$$W = 2\pi k\iiint(A^2 f^2 + B^2 g^2 + C^2 h^2)\, dx\, dy\, dz \quad . \quad . \quad . \quad . \quad (17\text{B}),$$

then

$$\frac{dW}{dt} = 4\pi k\iiint(A^2 f\dot{f} + B^2 g\dot{g} + C^2 h\dot{h})\, dx\, dy\, dz$$

$$= \iiint(P\dot{f} + Q\dot{g} + R\dot{h})\, dx\, dy\, dz.$$

Substituting the values of $\dot{f}$, $\dot{g}$, $\dot{h}$ in terms of α, β, γ, and integrating by parts, we obtain

$$\frac{dW}{dt} = \frac{1}{4\pi}\iint\{l\,(R\beta - Q\gamma) + m\,(P\gamma - R\alpha) + n\,(Q\alpha - P\beta)\}\, dS$$

$$- \frac{1}{4\pi}\iiint\left\{\alpha\left(\frac{dQ}{dz} - \frac{dR}{dy}\right) + \beta\left(\frac{dR}{dx} - \frac{dP}{dz}\right) + \gamma\left(\frac{dP}{dy} - \frac{dQ}{dx}\right)\right\}dx\, dy\, dz \quad . \quad (17\text{C}).$$

If in the identity

$$\dot{f}(p_3\dot{g} - p_2\dot{h}) + \dot{g}\,(p_1\dot{h} - p_3\dot{f}) + \dot{h}\,(p_2\dot{f} - p_1\dot{g}) = 0,$$

we substitute the values of $\dot{f}$, $\dot{g}$, $\dot{h}$ from the equations of $4\pi\dot{f} = d\gamma/dy - d\beta/dz$, &c., in the coefficients of the terms in brackets, and integrate by parts, we shall find that the last volume integral in (17A) is equal to

$$- \iint[l\,\{(p_2\dot{f} - p_1\dot{g})\,\beta - (p_1\dot{h} - p_3\dot{f})\,\gamma\}$$

$$+ m\,\{(p_3\dot{g} - p_2\dot{h})\,\gamma - (p_2\dot{f} - p_1\dot{g})\,\alpha\}$$

$$+ n\,\{(p_1\dot{h} - p_3\dot{f})\,\alpha - (p_3\dot{g} - p_2\dot{h})\,\beta\}]\, dS \quad . \quad . \quad . \quad . \quad (17\text{D}).$$

Accordingly (17A) becomes on substitution from (17B), (17C) and (17D)

$$\frac{d}{dt}\iiint\left\{\frac{k}{8\pi}(\alpha^2+\beta^2+\gamma^2)+2\pi k(\mathrm{A}^2f^2+\mathrm{B}^2g^2+\mathrm{C}^2h^2)\right\}dx\,dy\,dz$$
$$=\frac{1}{4\pi}\iint[l\,\{(\mathrm{R}+p_2\dot{f}-p_1\dot{g})\,\beta-(\mathrm{Q}+p_1\dot{h}-p_3\dot{f})\,\gamma\}$$
$$+m\,\{(\mathrm{P}+p_3\dot{g}-p_2\dot{h})\,\gamma-(\mathrm{R}+p_2\dot{f}-p_1\dot{g})\,\alpha\}$$
$$+n\,\{(\mathrm{Q}+p_1\dot{h}-p_3\dot{f})\,\alpha-(\mathrm{P}+p_3\dot{g}-p_2\dot{h})\,\beta\}]\,d\mathrm{S} \quad . \quad . \quad (18).$$

The physical interpretation of this equation is, that the rate at which something increases within the closed surface, must be equal to the rate at which something flows into the surface. This cannot be anything else but energy; we are therefore led to identify the expression

$$\frac{k}{8\pi}(\alpha^2+\beta^2+\gamma^2)+2\pi k\,(\mathrm{A}^2f^2+\mathrm{B}^2g^2+\mathrm{C}^2h^2)$$

as representing the energy of the electric field per unit of volume. The first term represents the electrokinetic energy, and the second term the electrostatic energy.

The above expressions are the same as those obtained by MAXWELL by a different method, and it thus appears that the expressions for each species of energy are not altered by the additional terms which have been introduced into the general equations of electromotive force.

The right hand side of (18) represents the rate at which work is done by the electric and magnetic forces which act upon the surface of S.]

In the optical problem which we are considering, the bounding surface is the plane $x=0$, whence if the quantities in the magnetized medium be denoted by accented letters, the condition of continuity of energy becomes

$$\mathrm{R}\beta-\mathrm{Q}\gamma=(\mathrm{R}'+p_2\dot{f}'-p_1\dot{g}')\,\beta'-(\mathrm{Q}'+p_1\dot{h}'-p_3\dot{f}')\gamma'.$$

Since $\beta=\beta'$ and $\gamma=\gamma'$, it follows from (17) that this equation reduces to

$$\mathrm{Q}=\mathrm{Q}'+p_1\dot{h}'-p_3\dot{f}' \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19),$$

which shows that the components of the electromotive force *in* the plane of incidence are also discontinuous.

The boundary conditions are therefore the following; (i.) continuity of electric displacement perpendicular to the reflecting surface, which is equivalent to continuity of magnetic force parallel to z; (ii.) continuity of magnetic induction perpendicular to the reflecting surface, which is equivalent to equation (17); (iii.) continuity of magnetic force parallel to y; (iv.) equation (19), which follows partly from (i.), (ii.), and (iii.), and partly from the condition that the flow of energy must be continuous.

We have therefore four equations and no more, to determine the four unknown quantities.

Reflection and Refraction when the Reflecting Surface is Magnetized Normally.

10. We are now prepared to calculate the amplitudes of the reflected and refracted waves.

Let us first suppose that the reflecting surface is magnetized normally; then $p_2 = p_3 = 0$. Also let A, B be the amplitudes of the two components of the incident vibrations perpendicular to, and in the plane of incidence; then the displacements in the four waves may be written

$$\begin{array}{lll} h = AS, & \mathfrak{B} = BS, & \text{incident wave;} \\ h' = A'S', & \mathfrak{B}' = B'S', & \text{reflected wave;} \\ h_1 = A_1S_1, & \mathfrak{B}_1 = \iota A_1S_1, & \text{1st refracted wave;} \\ h_2 = A_2S_2, & \mathfrak{B}_2 = -\iota A_2S_2, & \text{2nd refracted wave.} \end{array}$$

Also

$$\begin{array}{llll} l = -\cos i, & l' = \cos i, & l_1 = -\cos r_1, & l_2 = -\cos r_2; \\ m = \sin i, & m' = \sin i, & m_1 = \sin r_1, & m_2 = \sin r_2. \end{array}$$

The boundary conditions (i.), (ii.), (iii.), (iv.), at the end of § 9, furnish the following equations:—

$$(B + B')\, V = \iota\, (A_1V_1 - A_2V_2) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (20),$$

$$(A + A')\, V^2 = U^2k\, (A_1 + A_2) - \frac{p}{2\tau}\, (A_1 \cos r_1 - A_2 \cos r_2) \quad . \quad . \quad (21),$$

$$(A - A')\, V \cos i = U^2 \left(\frac{A_1}{V_1} \cos r_1 + \frac{A_2}{V_2} \cos r_2\right) - \frac{p}{2k\tau} \left(\frac{A_1}{V_1} \cos^2 r_1 - \frac{A_2}{V_2} \cos^2 r_2\right). \quad (22),$$

$$(B - B')\, V^2 \cos i = \iota U^2k\, (A_1 \cos r_1 - A_2 \cos r_2) - \frac{\iota p}{2\tau}\, (A_1 + A_2) \quad . \quad . \quad (23),$$

where p is written for p_1.

We shall now simplify these equations by introducing an auxiliary angle R, such that

$$\frac{V}{\sin i} = \frac{V_1}{\sin r_1} = \frac{V_2}{\sin r_2} = \frac{U}{\sin R} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (24).$$

Hence, R is the angle of refraction when the second medium is unmagnetized; and accordingly r_1 and r_2 will differ from R by a small quantity which depends upon p.

Since the magnetic effects are small, we shall neglect squares and higher powers of p, and we may, therefore, in the terms multiplied by p, put $r_1 = r_2 = \mathrm{R}$.

Let $q = p/4k\tau$, then from (16)

$$\mathrm{V}_1 = \mathrm{U} - \frac{q\cos\mathrm{R}}{\mathrm{U}}, \quad \mathrm{V}_2 = \mathrm{U} + \frac{q\cos\mathrm{R}}{\mathrm{U}} \quad \ldots \ldots \quad (25);$$

also from (24) and (25) we obtain

$$\cos r_1 = \cos\mathrm{R} + \frac{q\sin^2\mathrm{R}}{\mathrm{U}^2}, \quad \cos r_2 = \cos\mathrm{R} - \frac{q\sin^2\mathrm{R}}{\mathrm{U}^2} \quad \ldots \ldots \quad (26).$$

Substituting these values in equations (20) to (23) and reducing, they finally become

$$\left.\begin{aligned}
(\mathrm{B}-\mathrm{B}')\,\mathrm{V}^2\cos i &= \iota\mathrm{U}^2k\,(\mathrm{A}_1-\mathrm{A}_2)\cos\mathrm{R} + \iota qk\,(\sin^2\mathrm{R}-2)\,(\mathrm{A}_1+\mathrm{A}_2)\\
(\mathrm{B}+\mathrm{B}')\,\mathrm{V} &= \iota\mathrm{U}\,(\mathrm{A}_1-\mathrm{A}_2) - \frac{\iota q\cos\mathrm{R}}{\mathrm{U}}\,(\mathrm{A}_1+\mathrm{A}_2)\\
(\mathrm{A}-\mathrm{A}')\,\mathrm{V}\cos i &= \mathrm{U}\,(\mathrm{A}_1+\mathrm{A}_2)\cos\mathrm{R} - \frac{q\cos 2\mathrm{R}}{\mathrm{U}}\,(\mathrm{A}_1-\mathrm{A}_2)\\
(\mathrm{A}+\mathrm{A}')\,\mathrm{V}^2 &= \mathrm{U}^2k\,(\mathrm{A}_1+\mathrm{A}_2) - 2qk\,(\mathrm{A}_1-\mathrm{A}_2)\cos\mathrm{R}
\end{aligned}\right\} \quad (27).$$

These equations determine the amplitudes of the reflected and refracted waves, when the magnetization is perpendicular to the reflecting surface.

From the first two of (27) we get

$$\iota qk\,(\mathrm{A}_1+\mathrm{A}_2) = \mathrm{BV}\,(\mathrm{U}k\cos\mathrm{R} - \mathrm{V}\cos i) + \mathrm{B'V}\,(\mathrm{U}k\cos\mathrm{R} + \mathrm{V}\cos i),$$

$$\begin{aligned}\iota\mathrm{U}^2k\,(\mathrm{A}_1-\mathrm{A}_2) = \mathrm{BV}\,&\{\mathrm{U}k\,(2-\sin^2\mathrm{R}) - \mathrm{V}\cos i\cos\mathrm{R}\}\\ &+ \mathrm{B'V}\,\{\mathrm{U}k\,(2-\sin^2\mathrm{R}) + \mathrm{V}\cos i\cos\mathrm{R}\}.\end{aligned}$$

Substituting in the last two, we obtain

$$\begin{aligned}
&(\mathrm{A}-\mathrm{A}')\cos i\\
&\quad = -\iota\mathrm{B}\left[\frac{\mathrm{U}\cos\mathrm{R}}{qk}(\mathrm{U}k\cos\mathrm{R} - \mathrm{V}\cos i) - \frac{q\cos 2\mathrm{R}}{\mathrm{U}^3k}\{\mathrm{U}k\,(2-\sin^2\mathrm{R}) - \mathrm{V}\cos i\cos\mathrm{R}\}\right]\\
&\qquad - \iota\mathrm{B}'\left[\frac{\mathrm{U}\cos\mathrm{R}}{qk}(\mathrm{U}k\cos\mathrm{R} + \mathrm{V}\cos i) - \frac{q\cos 2\mathrm{R}}{\mathrm{U}^3k}\{\mathrm{U}k\,(2-\sin^2\mathrm{R}) + \mathrm{V}\cos i\cos\mathrm{R}\}\right] \quad (28)
\end{aligned}$$

and

$$\begin{aligned}
&(\mathrm{A}+\mathrm{A}')\,\mathrm{V}\\
&\quad = -\iota\mathrm{B}\left[\frac{\mathrm{U}^2}{q}(\mathrm{U}k\cos\mathrm{R} - \mathrm{V}\cos i) - \frac{2q\cos\mathrm{R}}{\mathrm{U}^2}\{\mathrm{U}k\,(2-\sin^2\mathrm{R}) - \mathrm{V}\cos i\cos\mathrm{R}\}\right]\\
&\qquad - \iota\mathrm{B}'\left[\frac{\mathrm{U}^2}{q}(\mathrm{U}k\cos\mathrm{R} + \mathrm{V}\cos i) - \frac{2q\cos\mathrm{R}}{\mathrm{U}^2}\{\mathrm{U}k\,(2-\sin^2\mathrm{R}) + \mathrm{V}\cos i\cos\mathrm{R}\}\right] (29).
\end{aligned}$$

Solving these equations, we obtain

$$A' = \frac{A\,(Uk\cos i - V\cos R)}{Uk\cos i + V\cos R} + \frac{2\iota qkBV\cos i}{U\,(Uk\cos R + V\cos i)\,(Uk\cos i + V\cos R)} \quad . \quad (30),$$

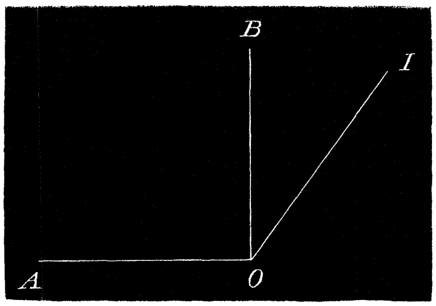
$$B' = -\frac{B\,(Uk\cos R - V\cos i)}{Uk\cos R + V\cos i} + \frac{2\iota qkAV\cos i}{U\,(Uk\cos R + V\cos i)\,(Uk\cos i + V\cos R)} \quad (31).$$

11. We shall now discuss these results.

Equations (30) and (31) give the amplitudes of the two components of the reflected light, and we see that the magnetic terms vanish at grazing incidence, but do not vanish for any other incidence.

The equations may be written in the form.

$$\left.\begin{aligned} A' &= A\alpha + \iota qB\beta \\ B' &= B\gamma + \iota qA\beta \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (32).$$



In the figure let I be the point of incidence, IO the normal to the reflected wave, and let O be the observer; also let OA, OB be drawn at right angles to OI, perpendicular to and in the plane of incidence respectively. Let ξ, η be the displacements along OA, OB; also let $\phi = (2\pi/\lambda)\,(x\cos i + y\sin i - Vt)$. Then by (32)

$$\left.\begin{aligned} \xi &= A\alpha\cos\phi - q\,B\beta\sin\phi \\ \eta &= B\gamma\cos\phi - q\,A\beta\sin\phi \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (33),$$

which shows that the reflected light is elliptically polarized.

Let us now suppose that the incident light is polarized *in* the plane of incidence, so that $B = 0$, and let the principal section of the analyser coincide with OB. Then the intensity of the reflected light after it has passed through the analyser is proportional to $A^2\beta^2q^2$, and is therefore independent of the direction of the magnetizing current, and vanishes when the current is cut off.

These results are in accordance with the first experiment.

Let the analyser be now turned through a small angle ϵ towards the right hand of the observer. From (33) we see that the intensity of the reflected light after emerging from the analyser is proportional to

$$A^2 (\alpha^2\epsilon^2 + \beta^2 q^2),$$

from which it appears that the effect of the current is always to increase the intensity, and that the intensity is independent of the direction of the current.

This result is not in accordance with the second experiment, since in that experiment it was found that if a current in one direction strengthened the reflected light, a current in the opposite direction weakened it. In order that theory and experiment should agree, it would be necessary that the amplitudes of the two reflected vibrations should satisfy equations of the form

$$A' = A(\alpha + q\alpha') + \iota q B\beta, \qquad B' = B(\gamma + q\gamma') + \iota q A\delta \quad . \quad . \quad . \quad (34),$$

where α', γ' are quantities which may be either real or complex, but not purely imaginary. Under these circumstances the principal part of the magnetic effect would be represented by a term proportional to q, which would change sign when the current is reversed.

I do not, however, think that this failure of the theory to agree with KERR's second experiment is a conclusive objection to the theory itself, inasmuch as he used a metallic reflector, and his results were accordingly complicated by the influence of metallic reflection; and it is by no means improbable that the terms α', γ' in equation (34), which are necessary to explain the phenomena observed, are the result of metallic reflection, and disappear when the reflector is a non-metallic substance.

These considerations show the importance of endeavouring to carry out a series of experiments upon the reflection of light from non-metallic media which are capable of magnetically affecting light. If it should be found that the intensity of the reflected light is independent of the direction of the magnetizing current, when such a medium is substituted for the metallic pole, the evidence in favour of the theory will be considerably strengthened; but if it should be found that the effect depends upon the direction of the current, the present theory must be abandoned.

Reflection and Refraction when the Reflecting Surface is Magnetized parallel to itself.

12. We must now consider the case in which the magnetic force is parallel to the reflecting surface.

Let the axes of y coincide with the direction of the lines of magnetic force, then $p_1 = p_3 = 0$; and let us first suppose that the plane of incidence is the plane of xz. Since none of the quantities in this case are functions of y, it follows that the last

terms in equations (12) are zero, and therefore magnetization produces no optical effect.

In the next place suppose that the incidence is normal; then all the quantities are functions of x and t only; whence the last terms in equations (12) vanish in this case also, so that magnetization produces no optical effect.

These results are in accordance with KERR's experiments.

13. We shall now consider the case in which the plane of incidence is parallel to the direction of magnetization.

From (12) it follows that the equations of motion in this case are

$$\left.\begin{aligned} \frac{d^2f}{dt^2} &= U^2\nabla^2 f - \frac{p}{4\pi k}\frac{d^2\dot{h}}{dy^2} \\ \frac{d^2g}{dt^2} &= U^2\nabla^2 g + \frac{p}{4\pi k}\frac{d^2\dot{h}}{dx\,dy} \\ \frac{d^2h}{dt^2} &= U^2\nabla^2 h + \frac{p}{4\pi k}\frac{d}{dy}\left(\frac{d\dot{f}}{dy} - \frac{d\dot{g}}{dx}\right) \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (35),$$

where p is written for p_2. Solving these equations in the same way as (15) we obtain

$$V^2 = U^2 \pm \frac{pm}{2k\tau} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (36),$$

$$A' = \pm\,\iota m A, \qquad A'' = \mp\,\iota l A.$$

Accordingly, the component displacements in the two refracted waves are

$$h_1 = A_1 S_1, \qquad \mathfrak{B}_1 = \iota A_1 S_1;$$
$$h_2 = A_2 S_1, \qquad \mathfrak{B}_2 = -\,\iota A_2 S_2.$$

The boundary conditions (i.), (ii.), (iii.) and (iv.) of § 9 give rise to the following equations, viz.,

$$(B + B')\,V = \iota\,(A_1V_1 - A_2V_2) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (37),$$

$$(A + A')\,V^2 = U^2k\,(A_1 + A_2) + \frac{p}{2\tau}\,(A_1 \sin r_1 - A_2 \sin r_2) \quad . \quad . \quad . \quad . \quad (38),$$

$$(A - A')\,V\cos i = U^2\left(\frac{A_1\cos r_1}{V_1} + \frac{A_2\cos r_2}{V_2}\right) + \frac{p\sin R}{2Uk\tau}\,(A_1\cos r_1 - A_2\cos r_2) \quad (39),$$

$$(B - B')\,V^2\cos i = \iota U^2k\,(A_1\cos r_1 - A_2\cos r_2) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (40).$$

From (36) and (24) we obtain

$$V_1 = U + \frac{q\sin R}{U}, \qquad V_2 = U - \frac{q\sin R}{U} \quad . \quad . \quad . \quad . \quad . \quad . \quad (41),$$

$$\cos r_1 = \cos R - \frac{q\sin^3 R}{U^2\cos R}, \qquad \cos r_2 = \cos R + \frac{q\sin^3 R}{U^2\cos R} \quad . \quad . \quad . \quad (42).$$

Substituting in equations (37) to (40), and rewriting, we obtain

$$\left.\begin{aligned}
(\mathrm{B}-\mathrm{B}')\,\mathrm{V}^2\cos i &= \iota\mathrm{U}^2k\cos\mathrm{R}\,(\mathrm{A}_1-\mathrm{A}_2)-\frac{\iota qk\sin^3\mathrm{R}}{\cos\mathrm{R}}(\mathrm{A}_1+\mathrm{A}_2)\\
(\mathrm{B}+\mathrm{B}')\,\mathrm{V} &= \iota\mathrm{U}\,(\mathrm{A}_1-\mathrm{A}_2)+\frac{\iota q\sin\mathrm{R}}{\mathrm{U}}(\mathrm{A}_1+\mathrm{A}_2)\\
(\mathrm{A}-\mathrm{A}')\,\mathrm{V}\cos i &= \mathrm{U}\cos\mathrm{R}\,(\mathrm{A}_1+\mathrm{A}_2)+\frac{q\cos 2\mathrm{R}\tan\mathrm{R}}{\mathrm{U}}(\mathrm{A}_1-\mathrm{A}_2)\\
(\mathrm{A}+\mathrm{A}')\,\mathrm{V}^2 &= \mathrm{U}^2k\,(\mathrm{A}_1+\mathrm{A}_2)+2qk\sin\mathrm{R}\,(\mathrm{A}_1-\mathrm{A}_2)
\end{aligned}\right\}\ (43\text{–}46).$$

These are the equations which determine the amplitudes of the reflected and refracted light, when the lines of magnetic force are parallel to the reflecting surface, and the plane of incidence is parallel to them.

From the first two of (43–46) we obtain

$$\iota qk\tan\mathrm{R}\,(\mathrm{A}_1+\mathrm{A}_2)=\mathrm{BV}\,(\mathrm{U}k\cos\mathrm{R}-\mathrm{V}\cos i)+\mathrm{B}'\mathrm{V}\,(\mathrm{U}k\cos\mathrm{R}+\mathrm{V}\cos i),$$

$$\begin{aligned}\iota\mathrm{U}^2k\,(\mathrm{A}_1-\mathrm{A}_2)=\mathrm{BV}\,&(\mathrm{U}k\sin^2\mathrm{R}+\mathrm{V}\cos i\cos\mathrm{R})\\&+\mathrm{B}'\mathrm{V}\,(\mathrm{U}k\sin^2\mathrm{R}-\mathrm{V}\cos i\cos\mathrm{R}).\end{aligned}$$

Substituting in the last two of (43–46) we obtain

$$\begin{aligned}(\mathrm{A}-\mathrm{A}')\cos i=-\iota\mathrm{B}&\left\{\frac{\mathrm{U}\cos^2\mathrm{R}}{qk\sin\mathrm{R}}(\mathrm{U}k\cos\mathrm{R}-\mathrm{V}\cos i)\right.\\&\qquad\left.+\frac{q\cos2\mathrm{R}\tan\mathrm{R}}{\mathrm{U}^3k}(\mathrm{U}k\sin^2\mathrm{R}+\mathrm{V}\cos i\cos\mathrm{R})\right\}\\
-\iota\mathrm{B}'&\left\{\frac{\mathrm{U}\cos^2\mathrm{R}}{qk\sin\mathrm{R}}(\mathrm{U}k\cos\mathrm{R}+\mathrm{V}\cos i)\right.\\&\qquad\left.+\frac{q\cos2\mathrm{R}\tan\mathrm{R}}{\mathrm{U}^3k}(\mathrm{U}k\sin^2\mathrm{R}-\mathrm{V}\cos i\cos\mathrm{R})\right\}\end{aligned}$$

$$\begin{aligned}(\mathrm{A}+\mathrm{A}')\,\mathrm{V}=-\iota\mathrm{B}&\left\{\frac{\mathrm{U}^2\cot\mathrm{R}}{q}(\mathrm{U}k\cos\mathrm{R}-\mathrm{V}\cos i)\right.\\&\qquad\left.+\frac{2q\sin\mathrm{R}}{\mathrm{U}^2}(\mathrm{U}k\sin^2\mathrm{R}+\mathrm{V}\cos i\cos\mathrm{R})\right\}\\
-\iota\mathrm{B}'&\left\{\frac{\mathrm{U}^2\cot\mathrm{R}}{q}(\mathrm{U}k\cos\mathrm{R}+\mathrm{V}\cos i)\right.\\&\qquad\left.+\frac{2q\sin\mathrm{R}}{\mathrm{U}^2}(\mathrm{U}k\sin^2\mathrm{R}-\mathrm{V}\cos i\cos\mathrm{R})\right\},\end{aligned}$$

whence

$$A' = \frac{A\,(Uk \cos i - V \cos R)}{Uk \cos i + V \cos R} - \frac{2\iota qk BV \tan R \cos i}{U\,(Uk \cos i + V \cos R)\,(Uk \cos R + V \cos i)} \quad (47),$$

$$B' = -\frac{B\,(Uk \cos R - V \cos i)}{Uk \cos R + V \cos i} + \frac{2\iota qk AV \tan R \cos i}{U\,(Uk \cos i + V \cos R)\,(Uk \cos R + V \cos i)} \quad (48).$$

14. In the third experiment, the incident light is polarized *in* the plane of incidence, so that $B = 0$. We therefore see that the component perpendicular to the plane of incidence is unaffected by magnetization, but that the effect of the latter is to introduce a component in the plane of incidence, which is proportional to the magnetic force, and which is given by (48). Now B' may be put into the form

$$\frac{2\iota qk A\mu \sin 2i \sin^2 i}{V^2\,(\mu^2 - \sin^2 i)^{\frac{1}{2}}\,(\sin i \cos R + k \sin R \cos i)\,(\sin 2i + k \sin 2R)},$$

where μ is the index of refraction; and this vanishes when $i = 0$ or $i = \frac{1}{2}\pi$. Accordingly, the magnetic term increases from grazing incidence to a maximum value and then decreases to normal incidence.

This result is in partial agreement with the third experiment, since it was found that the magnetic effect attained a maximum value when i was equal to about 60°; but it disagrees, owing to the fact that the intensity depends on the square of the magnetic force, and is, therefore, unaffected when the current is reversed.

In the fourth experiment, the incident light was polarized perpendicularly to the plane of incidence, so that $A = 0$; and we see from (47) and (48) that the effects are of the same kind as when the incident light is polarized in the plane of incidence. This result altogether disagrees with the fourth experiment.

The theory which I have attempted to develop, cannot, I think, be considered altogether unsatisfactory, since it certainly explains some of KERR's experimental results; at the same time the theory must be regarded as a somewhat tentative one, until experiments upon magnetic solutions have been made.

[15.* In a recent paper,† I have attempted to show that the rotatory properties of quartz may be accounted for by assuming that the electromotive force and the electric displacement are connected together by equations of the form

$$\left.\begin{aligned} P &= 4\pi f/K_1 + p_3\dot{g} - p_2\dot{h} \\ Q &= 4\pi g/K_2 + p_1\dot{h} - p_3\dot{f} \\ R &= 4\pi h/K_3 + p_2\dot{f} - p_1\dot{g}, \end{aligned}\right\} \quad \ldots\ldots\ldots \quad (49),$$

* Partly rewritten, July 1, 1891.

† 'Phil. Mag.,' August, 1890, p. 152.

where p_1, p_2, p_3 are absolute constants which depend upon the rotatory properties of the medium.

This assumption leads to equations (12) of the present paper; and, consequently, if we introduce the additional terms into (6), the equations of motion given in the paper in the 'Philosophical Magazine'* will be of the same form as those of the present paper.

I am disposed on reconsideration to modify the former theory in the above manner; in which case the properties of quartz would be accounted for by the hypothesis that the molecular structure of quartz produces an effect upon light, which may be explained by supposing that the general equations of electromotive force may be represented by (6). There is, however, an important distinction between the two classes of phenomena, for, in the case of a magnetized medium, p_1, p_2, p_3 are proportional to the components of magnetic force, and consequently change sign when the magnetic force is reversed; but, in the case of quartz, p_1, p_2, p_3 are absolute constants. Moreover, if the axis of z be the axis of the crystal, p_1 and p_2 must each be zero, because a wave of light, which is transmitted perpendicularly to the axis, experiences no rotation. In the case of syrup, turpentine, and other isotropic substances which produce rotatory polarization, p_1, p_2, p_3 must be equal to one another.

Under these circumstances we should anticipate that when light is reflected from a plate of quartz, whose faces are parallel or perpendicular to its axis, the reflected light would exhibit certain peculiarities, which are analogous to those produced by a magnetized transparent medium; but in working out the problem we must recollect that in quartz there are two principal wave velocities, and this circumstance would lead to certain modifications in the final results of equations (30), (31), (47), and (48).

The effect produced by a single reflection would probably be too small to be detected; but possibly some peculiarities might be observed, if (as Dr. KERR has suggested to me) the method of multiple reflections were employed. In any case the subject is one which opens up a wide field for experimentalists.]

[†16. When glass is placed in a strong field of magnetic force, and a ray of plane polarized light is transmitted in the direction of the magnetic force, it is known that the plane of polarization is rotated in the *positive direction;* that is to say, the rotation takes place in the same direction as the amperean currents which would produce the force. It therefore follows that HALL'S constant must be negative for glass. The experiments of KUNDT‡ on the transmission of light through thin metallic films, show that the direction of rotation in iron, cobalt, and nickel is the same as in

* Equations (10) of that paper, in which the factor $(4\pi\mu)^{-1}$, or $(4\pi k)^{-1}$ in the notation of the present paper, was inadvertently omitted from the last term.

† Added July 1, 1891.

‡ 'Berlin Sitzungsberichte,' July 10, 1884; translated, 'Phil. Mag.,' series 5, vol. 18, p. 308.

glass; if, therefore, magnetic rotation by metals could be explained by HALL's effect, it ought to follow that HALL's constant is negative for all three metals. When HALL's *constant* is negative, the additional electromotive force, or HALL's *effect*, is positive; for if under these circumstances the conductor is a plate which coincides with the plane of (xy), and the magnetic force and the primary current are respectively parallel to the axes of z and y, the additional electromotive force will be parallel to the positive direction of the axis of x. It appears from experiment that HALL's effect is positive for iron and cobalt, and negative for nickel.* Sir W. THOMSON† has, however, shown that, under certain conditions, the behaviour of nickel with respect to magnetic force is opposite to that of iron, and it is therefore not surprising that a similar peculiarity should exist with regard to the connection between HALL's constant and the rotation of the plane of polarization.

At the time that this paper was written I was unacquainted with the experiments of KUNDT referred to above. Most of them relate to metals, and cannot, therefore, be compared with the theoretical results of this paper. KUNDT has, however, made one series of experiments which is capable of comparison, in which light was incident upon a plate of glass whose sides were not quite accurately parallel, so that the rays which had undergone two refractions at the anterior surface, and one reflection at the posterior surface, were well separated from those which were reflected at the anterior surface. When the plate was magnetized certain effects were observed, which are described by KUNDT, and which are analogous to those obtained by KERR in the case of metals; and in particular,‡ when the lines of magnetic force are perpendicular to the faces of the plate, and the incident light is polarized in the plane of incidence, the plane of polarization is always rotated in the positive direction (*i.e.*, in the direction of the amperean currents); but when the light is polarized perpendicularly to the plane of incidence, the rotation is positive from 0° to the polarizing angle, and negative up to 90°.

To give a complete mathematical investigation of these experimental results would lead to some rather complicated expressions; I shall therefore confine myself to the case in which the lines of magnetic force are perpendicular to the faces of the plate, and the incidence is sensibly normal, and shall calculate the intensity of the light which has undergone two refractions at the anterior surface, and one reflection at the posterior.

Let the incident vibration be $f = 0$, $g = 0$, $h = 2\text{A}\epsilon^{-2\iota\pi t/\tau}$; we shall find it convenient to resolve this into the two circularly polarized waves,

* 'HALL, 'Phil. Mag.,' series 5, vol. 12, p. 157; VON ETTINGHAUSEN and NERNST, 'Amer. Journ. Science,' series 3, vol. 34, p. 151.

† 'Phil. Trans.,' 1879, p. 55.

‡ 'Phil. Mag.,' series 5, vol. 18, p. 326.

$$h = \text{AS}, \qquad g = \iota\text{AS} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (50),$$

$$h = \text{AS}, \qquad g = -\iota\text{AS} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (51),$$

where $\text{S} = \epsilon^{-2\iota\pi t/\tau}$.

Since we are dealing with glass, we may put $k = 1$; consequently for normal incidence we obtain from (30), (31), and (50)

$$\text{A}' = \text{A}(\alpha - q\beta), \qquad \text{B}' = -\iota\text{A}',$$

where

$$\alpha = \frac{\text{U} - \text{V}}{\text{U} + \text{V}}, \qquad \beta = \frac{2\text{V}}{\text{U}(\text{U} + \text{V})^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (52).$$

If, however, the incident wave is polarized in the opposite direction, and is therefore represented by (51), we shall obtain

$$\text{A}' = \text{A}(\alpha + q\beta), \qquad \text{B}' = \iota\text{A}',$$

in which the sign of q is reversed.

In order to calculate the intensity of the refracted light let us first confine our attention to the incident wave given by (50). Then if we put $i = \text{R} = 0$ in (27), we shall obtain $\text{A}_2 = 0$,

$$\text{A}_1 = \frac{2\text{AV}^2}{\text{U}}\left\{\frac{1}{\text{U} + \text{V}} + \frac{q(2\text{U} + \text{V})}{\text{U}^2(\text{U} + \text{V})^2}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (53).$$

If, however, we considered the other wave (51), we should obtain $\text{A}_1 = 0$,

$$\text{A}_2 = \frac{2\text{AV}^2}{\text{U}}\left\{\frac{1}{\text{U} + \text{V}} - \frac{q(2\text{U} + \text{V})}{\text{U}^2(\text{U} + \text{V})^2}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (54),$$

in which the sign of q is again reversed.

In order to calculate the intensities, when light propagated in glass is reflected at the surface in contact with air, we may use STOKES's principle of reversion,* and apply it separately to each of the two circularly polarized waves, into which the incident wave may be conceived to be resolved. Let Ab, Ac be the amplitudes of the reflected and refracted waves, when the wave (50) passes from air into glass; and let Ae, Af be the amplitudes when the wave passes from glass into air, then

$$b + e = 0, \quad b^2 + cf = 1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (55).$$

Also if we denote by accented letters the corresponding quantities for the other

* 'Math. and Phys. Papers,' vol. 2, p. 91.

wave (51), the values of b', c', e', f' will be obtained from those of b, c, e, f by writing $-q$ for q.

By (52) and (53), the values of b, c, e, f are

$$\left.\begin{aligned} b &= \frac{U-V}{U+V} - \frac{2qV}{U(U+V)^2} = -e \\ c &= \frac{2V^2}{U}\left\{\frac{1}{U+V} + \frac{q(2U+V)}{U^2(U+V)^2}\right\} \\ f &= \frac{2U^2}{V}\left\{\frac{1}{U+V} - \frac{q(2V+U)}{U^2(U+V)^2}\right\} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (56),$$

which can also be verified by independent calculation.

If r be the thickness of the plate, the emergent light is

$$h = A\left(cef\epsilon^{4\iota\pi r/V_1} + c'e'f'\epsilon^{4\iota\pi r/V_2}\right)\epsilon^{-2\iota\pi t/\tau},$$

$$g = \iota A\left(cef\epsilon^{4\iota\pi r/V_1} - c'e'f'\epsilon^{4\iota\pi r/V_2}\right)\epsilon^{-2\iota\pi t/\tau},$$

whence, if

$$\lambda = \alpha(1-\alpha^2), \quad \mu = \beta(3\alpha^2 - 1),$$

the real parts become

$$-h = \quad A(\lambda + q\mu)\cos\frac{2\pi}{\tau}\left(\frac{2r}{V_1} - t\right) + A(\lambda - q\mu)\cos\frac{2\pi}{\tau}\left(\frac{2r}{V_2} - t\right),$$

$$-g = -A(\lambda + q\mu)\sin\frac{2\pi}{\tau}\left(\frac{2r}{V_1} - t\right) + A(\lambda - q\mu)\sin\frac{2\pi}{\tau}\left(\frac{2r}{V_2} - t\right).$$

Let

$$\phi = \frac{2\pi}{\tau}\left(\frac{2r}{V_2} - t\right), \quad \eta = \frac{2\pi r}{\tau}\left(\frac{1}{V_1} - \frac{1}{V_2}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad (57);$$

then

$$-h = \quad 2A\lambda\cos(\phi + \eta)\cos\eta - 2Aq\mu\sin(\phi + \eta)\sin\eta,$$

$$-g = -2A\lambda\cos(\phi + \eta)\sin\eta - 2Aq\mu\sin(\phi + \eta)\cos\eta.$$

Let the analyser be placed in the position of extinction, and be then turned through a small angle ϵ, which will be considered positive when the analyser is turned towards the *right hand* of an observer who is looking through it; then the vibration on emerging from the analyser is

$$2A\lambda\cos(\phi + \eta)\sin(\eta - \epsilon) + 2Aq\mu\sin(\phi + \eta)\cos(\eta - \epsilon)$$

Since q^2 is to be neglected, the intensity of the emergent light is

$$4A^2\lambda^2(\sin\eta - \epsilon\cos\eta)^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (58),$$

and will therefore be approximately zero when

$$\epsilon = \tan \eta.$$

Now from (16),

$$V_1^2 = U^2 - 2q$$
$$V_2^2 = U^2 + 2q$$
$$q = C\alpha/4\tau,$$

where C is HALL's constant, and α is the external magnetic force; whence the last of (57) becomes

$$\eta = \frac{\pi r C\alpha}{U}.$$

Since C is negative for glass, it follows that η, and therefore ϵ, is negative; whence the plane of polarization is rotated in the *same* direction as the amperean currents, which agrees with KUNDT's experimental result.

It also follows from (58) that the effect depends upon the direction of the current; for if the current is flowing in the same direction as that in which the analyser is rotated, its effect is to weaken the light, but if it flows in the opposite direction its effect is to strengthen the light.

Hence the effect produced by the glass plate is the *opposite* of that which is produced by a metallic magnetic pole.]