IV. *A Certain Class of Generating Functions in the Theory of Numbers.*

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Introductory Abstract.

The present investigation arose from my "Memoir on the Compositions of Numbers," recently read before the Royal Society and now in course of publication in the 'Philosophical Transactions.' The main theorem may be stated as follows:—

If $X_1, X_2, \ldots, X_n$ be linear functions of quantities $x_1, x_2, \ldots, x_n$ given by the matricular relation

$$(X_1, X_2, \ldots\ldots X_n) = \left(\begin{array}{cccc} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{n} & a_{22} & \ldots & a_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ a_{n1} & a_{n2} & {}_{n3} & a_{nn} \end{array}\right)(x_1, x_2, \ldots\ldots, x_n),$$

that portion of the algebraic fraction

$$\frac{1}{(1 - s_1X_1)(1 - s_2X_2)\ldots\ldots(1 - s_nX_n)}$$

which is a function of the products

$$s_1x_1, s_2x_2, \ldots\ldots, s_nx_n,$$

only, is $1/V_n$, where (putting $s_1 = s_2 = \ldots\ldots = s_n = 1$)

$$V_n = (-)^n x_1x_2\ldots\ldots x_n \left|\begin{array}{cccc} a_{11} - 1/x_1, & a_{12}, & \ldots & a_{1n} \\ a_{21}, & a_{22} - 1/x_2 & \ldots & a_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \\ a_{n1} & a_{n2} & \ldots & a_{nn} - 1/x_n \end{array}\right|.$$

The proof of this theorem rests upon an identity which, for order 3, is

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$$\begin{vmatrix} a_{11}s_1x_1 - 1, & a_{12}s_1x_1, & a_{13}s_1x_1, \\ a_{21}s_2x_2, & a_{22}s_2x_2 - 1, & a_{23}s_2x_2, \\ a_{31}s_3x_3, & a_{32}s_3x_3, & a_{33}s_3x_3 - 1, \end{vmatrix}$$

$$= \begin{vmatrix} 1 - s_1X_1, & 0, & 0, \\ 0, & 1 - s_2X_2, & 0, \\ 0, & 0, & 1 - s_3X_3, \end{vmatrix}$$

$$\times \begin{vmatrix} \dfrac{s_1(a_{11}x_1 - X_1)}{1 - s_1X_1} - 1, & \dfrac{a_{12}s_1x_1}{1 - s_1X_1}, & \dfrac{a_{13}s_1x_1}{1 - s_1X_1}, \\ \dfrac{a_{21}s_2x_2}{1 - s_2X_2}, & \dfrac{s_2(a_{22}x_2 - X)}{1 - s_2X_2} - 1, & \dfrac{a_{23}s_2x_2}{1 - s_2X_2}, \\ \dfrac{a_{31}s_3x_3}{1 - s_3X_3}, & \dfrac{a_{32}s_3x_3}{1 - s_3X_3}, & \dfrac{s_3(a_{33}x_3 - X_3)}{1 - s_3X_3} - 1, \end{vmatrix}$$

and is very easily established.

An instantaneous deduction of the general theorem is the result that the generating function for the coefficients of $x_1^{\xi_1}x_2^{\xi_2} \ldots x_n^{\xi_n}$ in the product

$$X_1^{\xi_1}X_2^{\xi_2} \ldots X_n^{\xi_n}$$

is

$$1/V_n.$$

The expression V_n involves the several coaxial minors of the determinant of the linear functions. Thus

$$V_3 = 1 - a_{11}x_1 - a_{22}x_2 - a_{33}x_3 + | a_{11}a_{22} | x_1x_2 + | a_{11}a_{33} | x_1x_3 - | a_{22}a_{33} | x_2x_3 \\ - | a_{11}a_{22}a_{33} | x_1x_2x_3.$$

The theorem is of considerable arithmetical importance and is also of interest in the algebraical theories of determinants and matrices.

The product

$$X_1^{\xi_1}X_2^{\xi_2} \ldots X_n^{\xi_n},$$

often appears in arithmetic as a redundant form of generating function. The theorem above supplies a condensed or exact form of generating function.

Ex. gr. It is clear that the number of permutations of the $\Sigma\xi$ symbols in the product

$$x_1^{\xi_1}x_2^{\xi_2} \ldots x_n^{\xi_n}$$

which are such that every symbol is displaced, is obviously the coefficient of

$$x_1^{\xi_1}x_2^{\xi_2} \ldots x_n^{\xi_n}$$

in the product

$$(x_2 + \ldots + x_n)^{\xi_1}(x_1 + x_3 + \ldots + x_n)^{\xi_2} \ldots (x_1 + x_2 + \ldots + x_{n-1})^{\xi_n},$$

and thence we easily pass to the true generating function

$$\frac{1}{1 - \Sigma x_1 x_2 - 2\,\Sigma x_1 x_2 x_3 - 3\,\Sigma x_1 x_2 x_3 x_4 - \ldots - (n-1)\,x_1 x_2 \ldots x_n}.$$

In the paper many examples are given.

Frequently the redundant and condensed generating functions are differently interpretable; we then obtain an arithmetical correspondence, two cases of which presented themselves in the "Memoir on the Compositions of Numbers."

A more important method of obtaining arithmetical correspondences is developed in the researches which follow the statement and proof of the theorem.

The general form of V_n is such that the equation

$$V_n = 0$$

gives each quantity x_s as a homographic function of the remaining $n-1$ quantities, and it is interesting to enquire whether, assuming the coefficients of V_n arbitrarily, it is possible to pass to a corresponding redundant generating function.

I find that the coefficients of V_n must satisfy

$$2^n - n^2 + n - 2$$

conditions, and, assuming the satisfaction of these conditions, a redundant form can be constructed which involves

$$n - 1$$

undetermined quantities. In fact, when a redundant form exists at all, it is necessarily of a $(n-1)$-tuply infinite character.

We are now able to pass from any particular redundant generating function to an equivalent generating function which involves $n-1$ undetermined quantities. Assuming these quantities at pleasure, we obtain a number of different algebraic products, each of which may have its own meaning in arithmetic, and thus the number of arithmetical correspondences obtainable is subject to no finite limit.

This portion of the theory is given at length in the paper, with illustrative examples.

Incidentally interesting results are obtained in the fields of special and general determinant theory. The special determinant, which presents itself for examination, provisionally termed "inversely symmetric," is such that the constituents symmetrically placed in respect to the principal axis have, each pair, a product unity, whilst the constituents on the principal axis itself are all of them equal to unity. The determinant possesses many elegant properties which are of importance to the principal investigation of the paper. The theorems concerning the general determinant are connected entirely with the co-axial minors.

I find that the general determinant of even order, greater than two, is expressible

in precisely two ways as an irrational function of its co-axial minors, whilst no determinant of uneven order is so expressible at all.

Of order superior to 3, it is not possible to assume arbitrary values for the determinant itself and all of its co-axial minors. In fact of order n the values assumed must satisfy

$$2^n - n^2 + n - 2$$

conditions, but, these conditions being satisfied, the determinant can be constructed so as to involve $n - 1$ undetermined quantities.

§ 1.

ART. 1. In a Memoir on "The Theory of the Composition of Numbers," recently communicated to the Royal Society (as above-mentioned), there occurred certain generating functions which admitted important transformations to redundant forms.

I proceed to the general theory of these transformations, and subsequently discuss the algebraical and arithmetical consequences. The main theorem is, in reality, a theorem in determinants, of considerable interest, as will appear.

Art. 2. Consider the algebraic fraction

$$\frac{1}{(1 - s_1X_1)(1 - s_2X_2)\ldots(1 - s_nX_n)},$$

wherein $X_1, X_2, \ldots X_n$ are linear functions, of n quantities $x_1, x_2, \ldots x_n$, as given by the matricular relation

$$(X_1, X_2, \ldots X_n) = \begin{vmatrix} a_1, a_2, \ldots a_n \\ b_1, b_2, \ldots b_n \\ \ldots\ldots \\ n_1, n_2, \ldots n_n \end{vmatrix} (x_1, x_2, \ldots x_n).$$

I assume the quantities involved to have such values that the fraction is capable of expansion in ascending powers, and products of $x_1, x_2, \ldots x_n$ by a convergent series.

Art. 3. A certain portion of this expansion is a function of $s_1x_1, s_2x_2, \ldots s_nx_n$, and of the coefficients of the linear functions $X_1, X_2, \ldots X_n$ only. One object of this investigation is the isolation of this portion of the expansion which, for some purposes, in the Theory of Numbers is the only portion of importance.*

* It will occur to mathematicians, who are familiar with the Theory of Invariants, that generating functions not unfrequently present themselves in a redundant form. In particular, it is frequently necessary to isolate that portion of a generating function which includes the whole of the positive terms of the expansion, the negative terms, though admitting of interpretation, being of little moment.

Without specifying at present the arithmetical meaning of the generating function, I will call the portion above-written the "redundant form," and the essential portion, to which reference has been made, the "condensed form."

Art. 4. As typical of the general case, put $n = 3$.

It will be shown that the condensed form is $1/\mathrm{N}$, where

$$\mathrm{N} = 1 - a_1s_1x_1 - b_2s_2x_2 - c_3s_3x_3$$
$$+ \mid a_1b_2 \mid s_1s_2x_1x_2 + \mid a_1c_3 \mid s_1s_3x_1x_3 + \mid b_2c_3 \mid s_2s_3x_2x_3 - \mid a_1b_2c_3 \mid s_1s_2s_3x_1x_2x_3.$$

The notation is that in use in the Theory of Determinants, the coefficients of N being the several co-axial minors of the determinant $\mid a_1b_2c_3 \mid$; this determinant is the content of the matrix which occurs in the definition of the linear quantics X_1, X_2, X_3.

Art. 5. In determinant form N may be written

$$\begin{vmatrix} 1 - a_1s_1x_1, & -a_2s_1x_1, & -a_3s_1x_1 \\ -b_1s_2x_2, & 1 - b_2s_2x_2, & -b_3s_2x_3 \\ -c_1s_3x_3, & -c_2s_3x_3, & 1 - c_3s_3x_3 \end{vmatrix}$$

and also in the important symbolic form

$$\mid (1 - a_1s_1x_1)(1 - b_2s_2x_2)(1 - c_3s_3x_3) \mid ,$$

wherein, after multiplication, the a, b, c products are to be written in determinant brackets. Such symbolic multiplication will be denoted by external determinant brackets as shown.

Art. 6. We have now

$$\frac{\mathrm{N}}{(1 - s_1\mathrm{X}_1)(1 - s_2\mathrm{X}_2)(1 - s_3\mathrm{X}_3)}$$
$$= \frac{\mid (1 - a_1s_1x_1)(1 - b_2s_2x_2)(1 - c_3s_3x_3) \mid}{(1 - s_1\mathrm{X}_1)(1 - s_2\mathrm{X}_2)(1 - s_3\mathrm{X}_3)}$$
$$= \frac{\mid (1 - s_1\mathrm{X}_1 + s_1\mathrm{X}_1 - a_1s_1x_1)(1 - s_2\mathrm{X}_2 + s_2\mathrm{X}_2 - b_2s_2x_2)(1 - s_3\mathrm{X}_3 + s_3\mathrm{X}_3 - c_3s_3x_3) \mid}{(1 - s_1\mathrm{X}_1)(1 - s_2\mathrm{X}_2)(1 - s_3\mathrm{X}_3)}$$
$$= 1 + \frac{s_1(\mathrm{X}_1 - a_1x_1)}{1 - s_1\mathrm{X}_1} + \frac{s_2(\mathrm{X}_2 - b_2x_2)}{1 - s_2\mathrm{X}_2} + \frac{s_3(\mathrm{X}_3 - c_3x_3)}{1 - s_3\mathrm{X}_3} + \frac{s_2s_3 \mid (\mathrm{X}_2 - b_2x_2)(\mathrm{X}_3 - c_3x_3) \mid}{(1 - s_2\mathrm{X}_2)(1 - s_3\mathrm{X}_3)}$$
$$+ \frac{s_3s_1 \mid (\mathrm{X}_3 - c_3x_3)(\mathrm{X}_1 - a_1x_1) \mid}{(1 - s_3\mathrm{X}_3)(1 - s_1\mathrm{X}_1)} + \frac{s_1s_2 \mid (\mathrm{X}_1 - a_1x_1)(\mathrm{X}_2 - b_2x_2) \mid}{(1 - s_1\mathrm{X}_1)(1 - s_2\mathrm{X}_2)},$$

since, as will be seen presently, the determinant

$$| (X_1 - a_1x_1)(X_2 - b_2x_2)(X_3 - c_3x_3) |$$

vanishes identically.

The right-hand side of their identity does not, on expansion, contain any terms which are functions of s_1x_1, s_2x_2, s_3x_3 and of the coefficients a, b, c only.

Art. 7. Before proceeding to establish this, it may be remarked that the above identity may be written in the determinant form :—

$$\frac{\begin{vmatrix} a_1s_1x_1 - 1, & a_2s_1x_1, & a_3s_1x_1 \\ b_1s_2x_2, & b_2s_2x_2 - 1, & b_3s_2x_2 \\ c_1s_3x_3, & c_2s_3x_3, & c_3s_3x_3 - 1 \end{vmatrix}}{\begin{vmatrix} 1 - s_1X_1, & 0, & 0 \\ 0, & 1 - s_2X_2, & 0 \\ 0, & 0, & 1 - s_3X_3 \end{vmatrix}}$$

$$= \begin{vmatrix} \dfrac{s_1(a_1x_1 - X_1)}{1 - s_1X_1} - 1, & \dfrac{a_2s_1x_1}{1 - s_1X_1}, & \dfrac{a_3s_1x_1}{1 - s_1X_1} \\ \dfrac{b_1s_2x_2}{1 - s_2X_2}, & \dfrac{s_2(b_2x_2 - X_2)}{1 - s_2X_2} - 1, & \dfrac{b_3s_2x_2}{1 - s_2X_2} \\ \dfrac{c_1s_3x_3}{- s_3X_3}, & \dfrac{c_2s_3x_3}{1 - s_3X_3}, & \dfrac{s_3(c_3x_3 - X_3)}{1 - s_3X_3} - 1 \end{vmatrix},$$

and, in this form, is very easily established.

Art. 8. Consider, in regard to the order n, the algebraic fraction

$$\frac{s_1s_2 \ldots s_t \, | (X_1 - a_1x_1)(X_2 - b_2x_2) \ldots (X_t - t_tx_t) |}{(1 - s_1X_1)(1 - s_2X_2) \ldots (1 - s_tX_t)},$$

wherein t has an integer value not superior to n. This fraction is specified by the first t natural numbers, but this is merely for convenience, as what follows can be readily modified to meet the case of a fraction specified by any selection of t natural numbers, which are unequal and not superior to n.

To show that this fraction contains, on expansion, no terms which are functions of s_1x_1, s_2x_2, $\ldots$ s_nx_n only, it is merely necessary to show that every term in the development of the determinant

$$| (X_1 - a_1x_1)(X_2 - b_2x_2) \ldots (X_t - t_tx_t) |,$$

contains either $x_{t+1}, x_{t+2}, \ldots x_n$; viz., that every term contains an x with a suffix that does not occur in the s-product

$$s_1 s_2 \ldots s_t;$$

for visibly the fraction contains neither

$$s_{t+1}, \; s_{t+2}, \ldots \text{nor } s_n;$$

or, the same thing, the quantities s, occurring in the product

$$s_1 s_2 \ldots s_t,$$

are the only ones that are found in the fraction, the determinant should therefore vanish by putting

$$x_{t+1} = x_{t+2} = \ldots = x_n = 0.$$

The determinant is

$$\begin{vmatrix} X_1 - a_1 x_1, & -a_2 x_1, & \ldots & -a_t x_1 \\ -b_1 x_2, & X_2 - b_2 x_2, & \ldots & -b_t x_2 \\ \cdot & \cdot & \ldots & \cdot \\ \cdot & \cdot & \ldots & \cdot \\ -t_1 x_t, & -t_2 x_t, & \ldots & X_t - t_t x_t \end{vmatrix};$$

putting

$$x_{t+1} = x_{t+2} = \ldots = x_n = 0,$$

the first row is

$$a_2 x_2 + a_3 x_3 + \ldots + a_t x_t, \; -a_2 x_1, \; -a_3 x_1, \ldots -a_t x_1,$$

and adding together, x_1 times the first element, x_2 times the second, . . . , &c., x_t times the t^{th} element, we obtain zero.

A similar operation, performed on the elements of all the other rows, likewise results in zero.

Hence the determinant vanishes on the supposition

$$x_{t+1} = x_{t+2}, = \ldots = x_n = 0,$$

and accordingly every term, in its development, contains as factor one at least of the quantities

$$x_{t+1}, \; x_{t+2}, \ldots x_n.$$

This proves the proposition and also shows that the determinant

$$|(\mathrm{X}_1 - a_1x_1)(\mathrm{X}_2 - b_2x_2)\ldots(\mathrm{X}_n - n_nx_n)|,$$

of the n^{th} order, vanishes identically.

Art. 9. Hence, of order 3, we have the identity

$$\frac{1}{(1 - s_1\mathrm{X}_1)(1 - s_2\mathrm{X}_2)(1 - s_3\mathrm{X}_3)} = \frac{1}{|(1 - a_1s_1x_1)(1 - b_2s_2x_2)(1 - c_3s_3x_3)|},$$

multiplied by

$$1 + \frac{s_1(\mathrm{X}_1 - a_1x_1)}{1 - s_1\mathrm{X}_1} + \frac{s_2(\mathrm{X}_2 - b_2x_2)}{1 - s_2\mathrm{X}_2} + \frac{s_3(\mathrm{X}_3 - c_3x_3)}{1 - s_3\mathrm{X}_3} + \frac{s_2s_3|(\mathrm{X}_2 - b_2x_2)(\mathrm{X}_3 - c_3x_3)|}{(1 - s_2\mathrm{X}_2)(1 - s_3\mathrm{X}_3)}$$
$$+ \frac{s_3s_1|(\mathrm{X}_3 - c_3x_3)(\mathrm{X}_1 - a_1x_1)|}{(1 - s_3\mathrm{X}_3)(1 - s_1\mathrm{X}_1)} + \frac{s_1s_2|(\mathrm{X}_1 - a_1x_1)(\mathrm{X}_2 - b_2x_2)|}{(1 - s_1\mathrm{X}_1)(1 - s_2\mathrm{X}_2)};$$

and, of order n, the identity

$$\frac{1}{(1 - s_1\mathrm{X}_1)(1 - s_2\mathrm{X}_2)\ldots(1 - s_n\mathrm{X}_n)} = \frac{1}{|(1 - a_1s_1x_1)(1 - b_2s_2x_2)\ldots(1 - n_ns_nx_n)|},$$

multiplied by

$$1 + \Sigma\frac{s_1(\mathrm{X}_1 - a_1x_1)}{1 - s_1\mathrm{X}_1} + \Sigma\frac{s_1s_2\,|\,(\mathrm{X}_1\; a_1x_1)(\mathrm{X}_2 - b_2x_2)\,|}{(1 - s_1\mathrm{X}_1)(1 - s_2\mathrm{X}_2)}$$
$$+ \ldots + \Sigma\frac{s_1s_2\ldots s_t\,|\,(\mathrm{X}_1 - a_1x_1)(\mathrm{X}_2 - b_2x_2)\ldots(\mathrm{X}_t - t_tx_t)\,|}{(1 - s_1\mathrm{X}_1)(1 - s_2\mathrm{X}_2)\ldots(1 - s_t\mathrm{X}_t)} + \ldots,$$

the last batch of fractions involving, each, $n - 1$ denominator factors, and the numbers of fractions, under the summation signs, being in order

$$\binom{n}{1}, \binom{n}{2}, \ldots \binom{n}{t}, \ldots \binom{n}{n-1}.$$

Moreover, it has been shown that the fraction

$$\frac{1}{|\,(1 - a_1s_1x_1)(1 - b_2s_2x_2)\ldots(1 - n_ns_nx_n)\,|}$$

is the condensed form of the fraction

$$\frac{1}{(1 - s_1\mathrm{X}_1)(1 - s_2\mathrm{X}_2)\ldots(1 - s_n\mathrm{X}_n)},$$

or we may regard the latter as a redundant form of the former.

Art. 10. The coefficients of the terms

$$(s_1x_1)^{\xi_1}(s_2x_2)^{\xi_2}\ldots(s_nx_n)^{\xi_n},$$

in the expansions of both fractions, are the same.

Hence, the coefficient of the product

$$x_1^{\xi_1}x_2^{\xi_2}\ldots x_n^{\xi_n},$$

in the expansion of algebraic fraction

$$\frac{1}{|(1-a_1x_1)(1-b_2x_2)\ldots(1-n_nx_n)|},$$

is equal to the same coefficient in the product

$$(a_1x_1+\ldots+a_nx_n)^{\xi_1}(b_1x_1+\ldots+b_nx_n)^{\xi_2}\ldots(n_1x_1+\ldots+n_nx_n)^{\xi_n},$$

where this product is a "particular redundant generating function," the use of which renders the quantities $s_1, s_2, \ldots s_n$ unnecessary to the statement of the theorem.

Art. 11. The theorem regarded as a proposition concerning the coaxial minors of a general determinant is very remarkable; for it will be observed that we are able to exhibit the coefficient of

$$x_1^{\xi_1}x_2^{\xi_2}\ldots x_n^{\xi_n}$$

in the "particular redundant generating function" as a function of the coaxial minors of the determinant of the n quantities.

§ 2. *Arithmetical Interpretations.*

Art. 12. Most of the arithmetical results that can be deduced arise from duality of interpretation from algebra to arithmetic in particular cases. In the memoir to which reference has been made two particular cases presented themselves.

Art. 13. The first one was connected with the matricular relation

$$(X_1, X_2, X_3 \ldots X_n) = \left(\begin{array}{ccccccc} k, & 1, & 1, & . & . & . & 1 \\ k, & k, & 1, & . & . & . & 1 \\ k, & k, & k, & . & . & . & 1 \\ . & . & . & . & . & . & 1 \\ . & . & . & . & . & . & 1 \\ . & . & . & . & . & . & 1 \\ k, & k, & k, & . & . & . & k \end{array}\right) x_1, x_2, x_3 \ldots x_n).$$

and the condensed form, thence derivable, which has the form

$$\frac{1}{1-k\Sigma x_1+k(k-1)\,\Sigma x_1x_2-k(k-1)^2\,\Sigma x_1x_2x_3+\ldots+(-)^n\,k(k-1)^{n-1}\,x_1x_2\ldots x_n}.$$

The latter generating function occurs in the Theory of the Composition of Numbers. The corresponding redundant form is not unique (this will appear in the sequel, but that given above is one of the most useful.

Art. 14. The second one was founded on the relation

$$(\mathrm{X}_1, \mathrm{X}_2, \mathrm{X}_3 \ldots \mathrm{X}_n) = \begin{pmatrix} 1, & \lambda_{21}, & \lambda_{31}, & . & . & . & \lambda_{n1} \\ 1, & 1, & \lambda_{32}, & . & . & . & \lambda_{n2} \\ 1, & 1, & 1, & . & . & . & \lambda_{n3} \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ 1, & 1, & 1, & 1, & . & . & 1 \end{pmatrix} (x_1, x_2, x_3 \ldots x_n)$$

leading to the condensed form

$$\frac{1}{\left[\begin{array}{l}1-\Sigma x_1-\Sigma(\lambda_{\beta\alpha}-1)\,x_\alpha x_\beta-\Sigma(\lambda_{\beta\alpha}-1)(\lambda_{\gamma\beta}-1)\,x_\alpha x_\beta x_\gamma \\ \qquad -\ldots-(\lambda_{21}-1)(\lambda_{32}-1)(\lambda_{43}-1)\ldots(\lambda_{n,\,n-1}-1)\,x_1x_2x_3x_4\ldots x_{n-1}\,x_n\end{array}\right]}$$

wherein the numbers $\alpha, \beta, \gamma, \ldots$ are in ascending order of magnitude.

These particular cases gave rise to dual interpretations in arithmetic.

Art. 15. The general theorem, as so far developed, apparently only admits of a single interpretation.

Regarding the product

$$(a_1x_1+a_2x_2+\ldots+a_nx_n)^{\xi_1}(b_1x_1+b_2x_2+\ldots+b_nx_n)^{\xi_2}\ldots(n_1x_1+n_2x_2+\ldots+n_nx_n)^{\xi_n},$$

the coefficient of

$$a_1^{\alpha_1}b_1^{\beta_1}\ldots n_1^{\nu_1}a_2^{\alpha_2}b_2^{\beta_2}\ldots n_2^{\nu_2}\ldots a_n^{\alpha_n}b_n^{\beta_n}\ldots n_n^{\nu_n}x_1^{\xi_1}x_2^{\xi_2}\ldots x_n^{\xi_n}$$

may be interpreted in the theory of permutations.

Considering the permutations of the $\Sigma\xi$ quantities which form the product

$$x_1^{\xi_1}x_2^{\xi_2}\ldots x_n^{\xi_n},$$

the coefficient indicates the number of permutations which possess the property that

x_1	occurs	α_1	times in places	originally occupied	by an	x_1
,,	,,	β_1	,,	,,	,,	x_2
.	.	.	.	.	.	.
,,	,,	ν_1	,,	,,	,,	x_n
x_2	,,	α_2	,,	,,	,,	x_1
,,	,,	β_2	,,	,,	,,	x_2
.	.	.	.	.	.	.
,,	,,	ν_2	,,	,,	,,	x_n
		:	:	:	:	:
x_n	,,	α_n	,,	,,	,,	x_1
,,	,,	β_n	,,	,,	,,	x_2
.	.	.	.	.	.	.
,,	,,	ν_n	,,	,,	,,	x_n.

Accordingly the proper generating function for the enumeration of the permutations possessing this property is

$$\frac{1}{|\,(1 - a_1x_1)\,(1 - b_2x_2)\ldots(1 - n_nx_n)\,|}\,.$$

Art. 16. As an interesting particular case we can find the generating function for the enumeration of those permutations of the quantities in

$$x_1^{\xi_1}\,x_2^{\xi_2}\ldots x_n^{\xi_n}$$

which possess the property that no quantity is in the place originally occupied; that is, in the permutation, no x_s is to occupy a position formerly occupied by an x_s, s having all values from 1 to n.

Clearly we have merely to put

$$a_1 = b_2 = c_3 = \ldots = n_n = 0,$$

and the remaining letters, $a, b, c, \ldots n$ equal to unity. The generating function involves the coaxial minors of the determinant of the n^{th} order

$$\begin{vmatrix} 0, & 1, & 1, & \ldots & 1 \\ 1, & 0, & 1, & \ldots & 1 \\ 1, & 1, & 0, & \ldots & 1 \\ \cdot & \cdot & \cdot & \cdot\,\cdot & \cdot \\ 1, & 1, & 1, & \ldots & 0 \end{vmatrix}.$$

This determinant has the value

$$(-)^n\,(n-1),$$

while its first coaxial minors have each the value

$$(-)^{n-1}(n-2),$$

and its s^{th} coaxial minors each the value

$$(-)^{n-s}(n-s-1).$$

Hence the generating function is

$$\frac{1}{\{1-\Sigma x_1x_2-2\Sigma x_1x_2x_3-3\Sigma x_1x_2x_3x_4-\ldots-s\Sigma x_1x_2\ldots x_{s+1}-\ldots-(n-1)\,x_1x_2\ldots x_n\}},$$

or writing

$$(x-x_1)(x-x_2)\ldots(x-x_n)=x^n-a_1x^{n-1}+a_2x^{n-2}-\ldots,$$

this is

$$\frac{1}{1-a_2-2a_3-3a_4-\ldots-(n-1)\,a_n}.$$

Art. 17. As another example, again consider the permutations of the quantities in

$$x_1^{\xi_1}\,x_2^{\xi_2}\ldots x_n^{\xi_n}.$$

Divide the places occupied by the quantities into compartments

$$A_1\,A_2\ldots A_n,$$

such that the first ξ_1 places are in compartment A_1
next ξ_2 ,, ,, A_2
⋮
last ξ_n ,, ,, A_n,

and let us find the number of the permutations which have the property that no quantity with an uneven suffix is in a compartment with an uneven suffix, and no quantity with an even suffix is in a compartment with an even suffix.

In the "particular redundant generating function" we have merely to put

$$\begin{gathered}a_1=a_3=a_5=\ldots=0,\\ b_2=b_4=b_6=\ldots=0,\\ c_1=c_3=c_5=\ldots=0,\\ \&c.,\qquad \&c.,\end{gathered}$$

and the remaining $a, b, c, \ldots$ letters equal to unity.

For the true general (or condensed) generating function we have thus to evaluate the coaxial minors of the chess-board pattern determinant of the n^{th} order,

$$\begin{vmatrix} 0, & 1, & 0, & 1, & 0 & \ldots \\ 1, & 0, & 1, & 0, & 1 & \ldots \\ 0, & 1, & 0, & 1, & 0 & \ldots \\ 1, & 0, & 1, & 0, & 1 & \ldots \\ 0, & 1, & 0, & 1, & 0 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \end{vmatrix}$$

Here, all the minors of Order 1 are zero.

A minor (coaxial) of Order 2 has either the value zero or negative unity. If the minor be formed by deletion of all rows except the p^{th} and q^{th} and all columns except the p^{th} and q^{th} $(q > p)$ the value will be zero, if $q - p \equiv 0 \bmod 2$, and will be negative unity in all other cases.

Coaxial minors of Order > 2 as well as the whole determinant vanish, because in every case two rows are found to be identical.

Hence the true generating function is

$$\frac{1}{1 - x_1(x_2 + x_4 + \ldots) - x_2(x_3 + x_5 + \ldots) - x_3(x_4 + x_6 + \ldots) - \ldots - x_{n-1}x_n},$$

which may be written

$$\frac{1}{1 - \sum^{\alpha}\sum^{m} x_\alpha x_{\alpha+2m+1}}.$$

Art. 18. Again for the enumeration of the permutations which are such that no quantity with an uneven suffix is in a compartment with an even suffix, and also no quantity with an even suffix is in a compartment with an uneven suffix, we are led to the complementary chess-board pattern determinant:—

$$\begin{vmatrix} 1, & 0, & 1, & 0, & 1 & \ldots \\ 0, & 1, & 0, & 1, & 0 & \ldots \\ 1, & 0, & 1, & 0, & 1 & \ldots \\ 0, & 1, & 0, & 1, & 0 & \ldots \\ 1, & 0, & 1, & 0, & 1 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \end{vmatrix}$$

and thence to the true generating function

$$\frac{1}{[1-x_1-x_2-x_3-\ldots-x_n+x_1(x_2+x_4+\ldots)+x_2(x_3+x_5+\ldots)+x_3(x_4+x_6+\ldots)+\ldots+x_{n-1}x_n]}$$

which may be written

$$\frac{1}{1-\Sigma x_1+\overset{\alpha}{\Sigma}\overset{m}{\Sigma} x_\alpha x_{\alpha+2m+1}}.$$

Art. 19. Again, if it be necessary to enumerate the permutations of

$$x_1^{\xi_1}x_2^{\xi_2}\ldots x_n^{\xi_n},$$

in which x_1 occurs α_1 times in the compartment A_1,

,,	β_1	,,	,,	,,	A_2,
,,	γ_1	,,	,,	,,	A_3,
,,	$\vdots$				

we are led to the true generating function

$$\frac{1}{1-a_1x_1-x_2-x_3-\ldots-x_n+(a_1-b_1)x_1x_2+(a_1-c_1)x_1x_3+\ldots+(a_1-n_1)x_1x_n},$$

in which we have to seek the coefficient of

$$a_1^{\alpha_1}b_1^{\beta_1}c_1^{\gamma_1}\ldots n_1^{\nu_1}x_1^{\xi_1}x_2^{\xi_2}\ldots x_n^{\xi_n}.$$

Art. 20. Again consider the general problem of "Derangements in the Theory of Permutations."

In regard to the permutations of

$$x_1^{\xi_1}x_2^{\xi_2}\ldots x_n^{\xi_n}$$

it is necessary to determine the number of permutations such that exactly m of the symbols are in the places they originally occupied.

We have the particular redundant product

$$(ax_1+x_2+\ldots+x_n)^{\xi_1}(x_1+ax_2+\ldots+x_n)^{\xi_2}\ldots(x_1+x_2+\ldots+ax_n)^{\xi_n},$$

in which the number sought is the coefficient of

$$a^m x_1^{\xi_1}x_2^{\xi_2}\ldots x_n^{\xi_n}.$$

The true generating function (*i.e.*, condensed form) is derived from the coaxial minors of the determinant of order n :—

$$\begin{vmatrix} a & 1 & 1 & 1 & \dots \\ 1 & a & 1 & 1 & \dots \\ 1 & 1 & a & 1 & \dots \\ 1 & 1 & 1 & a & \dots \\ \vdots & \vdots & \vdots & \vdots & \end{vmatrix} \begin{aligned} &= (a-1)^n + n(a-1)^{n-1} \\ &= (a-1)^{n-1}(a+n-1). \end{aligned}$$

Thence the true generating function

$$\frac{1}{\{1 - a\Sigma x_1 + (a-1)(a+1)\Sigma x_1x_2 - (a-1)^2(a+2)\Sigma x_1x_2x_3 + \dots + (-)^n(a-1)^{n-1}(a+n-1)x_1x_2\dots x_n\}},$$

which constitutes a *perfect* solution of the problem of "derangement."

§ 3. *The General Theory Resumed.*

Art. 21. The denominator of a perfect generating function, of the type under consideration, is the most general function linear in each of n variables $x_1, x_2, \dots x_n$.

Let V_n be the most general linear function of the n quantities, involving $2^n - 1$ independent coefficients.

Art. 22. I enquire, irrespective of arithmetical interpretation or correspondence, into the possibility of expressing the fraction

$$V_n^{-1}$$

in a factorized redundant form.

Art. 23. The coefficients of V_n must be the several coaxial minors of some determinant, and the question arises: Can a determinant be constructed such that its coaxial minors assume given values?

The redundant form of order n involves n^2 coefficients. In general, in order that the fraction

$$V_n^{-1}$$

may be expressible in a redundant form, its coefficients must satisfy

$$\sigma_n$$

conditions, and, assuming the satisfaction of these conditions, a redundant form involving

$$n^2 - (2^n - 1 - \sigma_n)$$

arbitrary coefficients can be constructed.

Art. 24. The relation

$$n^2 - (2^n - 1 - \sigma_n) = n - 1$$

will be established, and this leads to the conclusion that the redundant form, when possible, is always of a

$$(n-1)^{\text{tuply}}$$

infinite character.

Art. 25. The fact, subject to the above-mentioned conditions, that there is an infinite flexibility in the redundant forms is of great importance in the Theory of Numbers, because the potentiality of arithmetical interpretation would appear to have no finite limit.

Art. 26. Observe that

$$\sigma_n$$

denotes the number of identical relations or syzygies connecting the coaxial minors of a general determinant of order n.

Art. 27. The discussion of the theory of the first few orders forms a convenient method of approaching the general theory.

I take the general form of V_n as

$$1 - p_1 s_1 x_1 - p_2 s_2 x_2 - \ldots + p_{12} s_1 s_2 x_1 x_2 + \ldots + (-)^n p_{12\ldots n} s_1 s_2 \ldots s_n x_1 x_2 \ldots x_n.$$

Art. 28. *The case* $n = 1$.

This case is trivial because the perfect form

$$V_1^{-1} = \frac{1}{1 - p_1 s_1 x_1}$$

coincides with the redundant form

$$\sigma_1 = 0;$$

$$n^2 - (2^n - 1 - \sigma_1) = 0.$$

Art. 29. *The case* $n = 2$.

In order that

$$\frac{1}{\{1 - s_1(a_{11}x_1 + a_{12}x_2)\}\{1 - s_2(a_{21}x_1 + a_{22}x_2)\}}$$

may be a redundant form of

$$V_2^{-1} = \frac{1}{1 - p_1 s_1 x_1 - p_2 s_2 x_2 + p_{12} s_1 s_2 x_1 x_2},$$

we have

$$a_{11} = p_1, \qquad a_{22} = p_2,$$
$$|\, a_{11},\ a_{22} \,| = p_{12},$$

and thence $a_{12}a_{21} = p_1 p_2 - p_{12} = q_{12}$ (suppose); introducing an undetermined quantity α_{12}, we may put :—

$$a_{12} = \alpha_{12} q_{12},$$
$$a_{21} = 1/\alpha_{12},$$

where α_{12} may be a *certain* function of the quantities

$$p_1,\ p_2,\ p_{12},\ x_1,\ x_2\ ;$$

but, numerically, may not be either zero or infinity.

The matricular relation is

$$(\mathrm{X}_1,\ \mathrm{X}_2) = \begin{pmatrix} a_{11}, & a_{12} \\ a_{21}, & a_{22} \end{pmatrix} (x_1,\ x_2) = \begin{pmatrix} p_1, & \alpha_{12} q_{12} \\ 1/\alpha_{12}, & p_2 \end{pmatrix} (x_1, x_2)$$

and the redundant form

$$\frac{1}{\{1 - s_1 (p_1 x_1 + \alpha_{12} q_{12} x_2)\}\ \{1 - s_2 (1/\alpha_{12} x_1 + p_2 x_2)\}}$$

of a singly infinite character.

$$\sigma_2 = 0\ ;$$
$$n^2 - (2^n - 1 - \sigma_2) = 1.$$

Art. 30. *The case* $n = 3$.

The matrix being that connected with the determinant

$$|\, a_{13} \,|\ ,$$

we have the following relations

$$a_{11} = p_1, \qquad a_{22} = p_2, \qquad a_{33} = p_3,$$
$$|\, a_{11},\ a_{22} \,| = p_{12}, \qquad |\, a_{11},\ a_{33} \,| = p_{13}, \qquad |\, a_{22},\ a_{33} \,| = p_{23},$$
$$|\, a_{11},\ a_{22},\ a_{33} \,| = p_{123}\ ;$$

and thence

$$a_{12}a_{21} = q_{12}, \qquad a_{13}a_{31} = q_{13}, \qquad a_{23}a_{32} = q_{23},$$

where

$$(q_{12},\ q_{13},\ q_{23}) = (p_1 p_2 - p_{12},\ p_1 p_3 - p_{13},\ p_2 p_3 - p_{23})\ ;$$

introducing the undetermined quantities

$$\alpha_{12}, \quad \alpha_{13}, \quad \alpha_{23},$$

write

$$a_{12} = \alpha_{12} q_{12}, \qquad a_{13} = \alpha_{13} q_{13}, \qquad a_{23} = \alpha_{23} q_{23},$$

$$a_{21} = \frac{1}{\alpha_{12}}, \qquad a_{31} = \frac{1}{\alpha_{13}}, \qquad a_{32} = \frac{1}{\alpha_{23}},$$

and thence by substitution

$$\begin{vmatrix} p_1 & \alpha_{12} q_{12} & \alpha_{13} q_{13} \\ \frac{1}{\alpha_{12}} & p_2 & \alpha_{23} q_{23} \\ \frac{1}{\alpha_{13}} & \frac{1}{\alpha_{23}} & p_3 \end{vmatrix} = p_{123},$$

which may be written

$$\begin{vmatrix} p_1 & q_{12} & \frac{\alpha_{13}}{\alpha_{12}\alpha_{23}} q_{13} \\ 1 & p_2 & q_{23} \\ \frac{\alpha_{12}\alpha_{23}}{\alpha_{13}} & 1 & p_3 \end{vmatrix} = p_{123};$$

this is a quadratic equation for the evaluation of $\alpha_{13}/\alpha_{12}\alpha_{23}$, which may be written

$$\left(\frac{\alpha_{13}}{\alpha_{12}\alpha_{23}} - \frac{1}{c_{13}}\right)\left(\frac{\alpha_{13}}{\alpha_{12}\alpha_{23}} - \frac{1}{c_{31}}\right) = 0.$$

Thus two of the three quantities α_{12}, α_{13}, α_{23} remain undetermined, and the coefficients of V_3 are not subject to any condition.

The matricular relation is either

$$(X_1, X_2, X_3) = \begin{pmatrix} p_1 & \alpha_{12} q_{12} & \frac{\alpha_{12}\alpha_{23}}{c_{13}} q_{13} \\ \frac{1}{\alpha_{12}} & p_2 & \alpha_{23} q_{23} \\ \frac{c_{13}}{\alpha_{12}\alpha_{23}} & \frac{1}{\alpha_{23}} & p_3 \end{pmatrix} (x_1, x_2, x_3)$$

or the one involving the matrix similar to the above with c_{31} written for c_{13}.

α_{12}, α_{23} are undetermined quantities, and c_{13}^{-1}, c_{31}^{-1} are the roots of the above-

given quadratic equation, which are expressible as irrational functions of the coefficients of V_3. The redundant form is

$$\frac{1}{(1 - s_1X_1)(1 - s_2X_2)(1 - s_3X_3)},$$

of a doubly infinite character.

Also

$$\sigma_3 = 0,$$
$$n^2 - (2^n - 1 - \sigma_n) = 2, \text{ for } n = 3.$$

Art. 31. *The case $n = 4$.*

The matrix being that connected with the determinant $| a_{14} |$ we have the relations :—

$$a_{11} = p_1, \qquad a_{22} = p_2, \qquad a_{33} = p_3, \qquad a_{44} = p_4,$$

$$| a_{11}a_{22} | = p_{12}, \qquad | a_{11}a_{33} | = p_{13}, \qquad | a_{22}a_{33} | = p_{23},$$
$$| a_{11}a_{44} | = p_{14}, \qquad | a_{22}a_{44} | = p_{24}, \qquad | a_{33}a_{44} | = p_{34},$$
$$| a_{11}a_{22}a_{33} | = p_{123}, \qquad | a_{11}a_{22}a_{44} | = p_{124},$$
$$| a_{11}a_{33}a_{44} | = p_{134}, \qquad | a_{22}a_{33}a_{44} | = p_{234},$$
$$| a_{11}a_{22}a_{33}a_{44} | = p_{1234};$$

and thence

$$a_{12}a_{21} = q_{12}, \qquad a_{13}a_{31} = q_{13}, \qquad a_{23}a_{32} = q_{23},$$
$$a_{14}a_{41} = q_{14}, \qquad a_{24}a_{42} = q_{24}, \qquad a_{34}a_{43} = q_{34};$$

and introducing six undetermined quantities,

$$a_{12} = \alpha_{12}q_{12}, \quad a_{13} = \alpha_{13}q_{13}, \quad a_{14} = \alpha_{14}q_{14}, \quad a_{23} = \alpha_{23}q_{23}, \quad a_{24} = \alpha_{24}q_{24}, \quad a_{34} = \alpha_{34}q_{34},$$
$$a_{21} = \frac{1}{\alpha_{12}}, \quad a_{31} = \frac{1}{\alpha_{13}}, \quad a_{41} = \frac{1}{\alpha_{14}}, \quad a_{32} = \frac{1}{\alpha_{23}}, \quad a_{42} = \frac{1}{\alpha_{24}}, \quad a_{43} = \frac{1}{\alpha_{34}},$$

and thence by substitution in the remaining relations,

$$\begin{vmatrix} p_1, & \alpha_{12}q_{12}, & \alpha_{13}q_{13} \\ \frac{1}{\alpha_{12}}, & p_2, & \alpha_{23}q_{23} \\ \frac{1}{\alpha_{13}}, & \frac{1}{\alpha_{23}}, & p_3 \end{vmatrix} = p_{123}, \qquad \begin{vmatrix} p_1, & \alpha_{12}q_{12}, & \alpha_{14}q_{14} \\ \frac{1}{\alpha_{12}}, & p_2, & \alpha_{24}q_{24} \\ \frac{1}{\alpha_{14}}, & \frac{1}{\alpha_{24}}, & p_4 \end{vmatrix} = p_{124},$$

$$\begin{vmatrix} p_1, & \alpha_{13}q_{13}, & \alpha_{14}q_{14} \\ \frac{1}{\alpha_{13}}, & p_3, & \alpha_{34}q_{34} \\ \frac{1}{\alpha_{14}}, & \frac{1}{\alpha_{34}}, & p_4 \end{vmatrix} = p_{134}, \qquad \begin{vmatrix} p_2, & \alpha_{23}q_{23}, & \alpha_{24}q_{24} \\ \frac{1}{\alpha_{23}}, & p_3, & \alpha_{34}q_{34} \\ \frac{1}{\alpha_{24}}, & \frac{1}{\alpha_{34}}, & p_4 \end{vmatrix} = p_{234},$$

$$\begin{vmatrix} p_1, & \alpha_{12}q_{12}, & \alpha_{13}q_{13}, & \alpha_{14}q_{14} \\ \frac{1}{\alpha_{12}}, & p_2, & \alpha_{23}q_{23}, & \alpha_{24}q_{24} \\ \frac{1}{\alpha_{13}}, & \frac{1}{\alpha_{23}}, & p_3, & \alpha_{34}q_{34} \\ \frac{1}{\alpha_{14}}, & \frac{1}{\alpha_{24}}, & \frac{1}{\alpha_{34}} & p_4 \end{vmatrix} = p_{1234}.$$

The six undetermined quantities that have been introduced must satisfy these five equations. However, the six quantities only enter the equations in three combinations; for, writing

$$\gamma_{13} = \frac{\alpha_{13}}{\alpha_{12}\alpha_{23}}, \qquad \gamma_{14} = \frac{\alpha_{14}}{\alpha_{12}\alpha_{23}\alpha_{34}}, \qquad \gamma_{24} = \frac{\alpha_{24}}{\alpha_{23}\alpha_{34}},$$

the five equations are easily transformed into the following five—

$$\begin{vmatrix} p_1, & q_{12}, & \gamma_{13}q_{13} \\ 1, & p_2, & q_{23} \\ \frac{1}{\gamma_{13}}, & 1, & p_3 \end{vmatrix} = p_{123}, \qquad \begin{vmatrix} p_1, & q_{12}, & \frac{\gamma_{14}}{\gamma_{24}}q_{14} \\ 1, & p_2, & q_{24} \\ \frac{\gamma_{24}}{\gamma_{14}}, & 1, & p_4 \end{vmatrix} = p_{124},$$

$$\begin{vmatrix} p_1, & q_{13}, & \frac{\gamma_{14}}{\gamma_{13}}q_{14} \\ 1, & p_3, & q_{34} \\ \frac{\gamma_{13}}{\gamma_{14}}, & 1, & p_4 \end{vmatrix} = p_{134}, \qquad \begin{vmatrix} p_2, & q_{23}, & \gamma_{24}q_{24} \\ 1, & p_3, & q_{34} \\ \frac{1}{\gamma_{24}}, & 1, & p_4 \end{vmatrix} = p_{234},$$

$$\begin{vmatrix} p_1, & q_{12}, & \gamma_{13}q_{13} & \gamma_{14}q_{14} \\ 1, & p_2, & q_{23}, & \gamma_{24}q_{24} \\ \frac{1}{\gamma_{13}}, & 1, & p_3, & q_{34} \\ \frac{1}{\gamma_{14}}, & \frac{1}{\gamma_{24}}, & 1, & p_4 \end{vmatrix} = p_{1234},$$

which involve only the three undetermined quantities

$$\gamma_{13}, \gamma_{14}, \gamma_{24}.$$

From these five equations we can eliminate the three quantities

$$\gamma_{13}, \qquad \gamma_{14}, \qquad \gamma_{24},$$

and thus obtain two independent relations between the coefficients of V_4. These are the two conditions that the coefficients must satisfy in order that a redundant form may be possible.

Since also these coefficients are the several co-axial minors of the determinant

$$| \alpha_{14} |$$

we establish the fact that these co-axial minors are connected by two relations or syzygies. Thus

$$\sigma_4 = 2 ;$$

and assuming the satisfaction of these two conditions we can solve the equations so as to express

$$\gamma_{13}, \qquad \gamma_{14}, \qquad \gamma_{24}$$

as functions of the coefficients of V_4.

Solving these equations and writing

$$P_{123} = p_{123} - p_1 p_{23} - p_2 p_{13} - p_3 p_{12} + 2 p_1 p_2 p_3,$$

we find

$$\gamma_{13} = \frac{1}{2q_{13}} \{ P_{123} \pm \surd (P^2_{123} - 4 q_{12} q_{13} q_{23}) \} ,$$

$$\gamma_{24} = \frac{1}{2q_{24}} \{ P_{234} \pm \surd (P^2_{234} - 4 q_{23} q_{24} q_{34}) \} ,$$

$$\frac{\gamma_{14}}{\gamma_{13}} = \frac{1}{2q_{14}} \{ P_{134} \pm \surd (P^2_{134} - 4 q_{13} q_{34} q_{14}) \} ,$$

$$\frac{\gamma_{14}}{\gamma_{24}} = \frac{1}{2q_{14}} \{ P_{124} \pm \surd (P^2_{124} - 4 q_{12} q_{24} q_{14}) \} ;$$

and assuming these four equations, as well as the fifth equation, consistent, there are just two systems of values of

$$\gamma_{13}, \qquad \gamma_{14}, \qquad \gamma_{24},$$

which satisfy all the equations.

Let the two values of γ_{13} be

$$1/c_{13} \quad \text{and} \quad 1/c_{31},$$

corresponding to the positive and negative signs respectively, and further taking the signs *all* positive, let γ_{xy} have the value

$$1/c_{xy},$$

and taking all the signs negative, let the value be

$$1/c_{yx}.$$

We have the solutions

$$(\gamma_{13},\ \gamma_{14},\ \gamma_{24}) = \left(\frac{1}{c_{13}},\ \frac{1}{c_{14}},\ \frac{1}{c_{24}}\right)$$

$$(\gamma_{13},\ \gamma_{14},\ \gamma_{24}) = \left(\frac{1}{c_{31}},\ \frac{1}{c_{41}},\ \frac{1}{c_{42}}\right)$$

and we may write either

$$(\alpha_{13},\ \alpha_{14},\ \alpha_{24}) = \left(\frac{\alpha_{12}\alpha_{23}}{c_{13}},\ \frac{\alpha_{12}\alpha_{23}\alpha_{34}}{c_{14}}\ \ \frac{\alpha_{23}\alpha_{34}}{c_{24}}\right),$$

or

$$(\alpha_{13},\ \alpha_{14},\ \alpha_{24}) = \left(\frac{\alpha_{12}\alpha_{23}}{c_{31}},\ \frac{\alpha_{12}\alpha_{23}\alpha_{34}}{c_{41}}\ \ \frac{\alpha_{23}\alpha_{34}}{c_{42}}\right).$$

The undetermined quantities are thus reduced to the three

$$\alpha_{12},\ \alpha_{23},\ \alpha_{34}.$$

Writing for brevity,

$$(\alpha_{12}\alpha_{23},\ \alpha_{23}\alpha_{34},\ \alpha_{12}\alpha_{23}\alpha_{34}) = (\beta_{13},\ \beta_{24},\ \beta_{14}),$$

and also

$$\alpha_{x,\,x+1} = \beta_{x,\,x+1},$$

the matrix that defines $X_1,\ X_2,\ X_3,\ X_4$ is either

$$\left(\begin{array}{cccc} p_1 & \beta_{12}q_{12} & \frac{\beta_{13}}{c_{13}}q_{13} & \frac{\beta_{14}}{c_{14}}q_{14} \\ \frac{1}{\beta_{12}} & p_2 & \beta_{23}q_{23} & \frac{\beta_{24}}{c_{24}}q_{24} \\ \frac{c_{13}}{\beta_{13}} & \frac{1}{\beta_{23}} & p_3 & \beta_{34}q_{34} \\ \frac{c_{14}}{\beta_{14}} & \frac{c_{24}}{\beta_{24}} & \frac{1}{\beta_{34}} & p_4 \end{array}\right)$$

or the same matrix with the substitution of c_{yx} for c_{xy}.

The redundant form is

$$\frac{1}{(1-s_1X_1)\,(1-s_2X_2)\,(1-s_3X_3)\,(1-s_4X_4)}$$

of a triply infinite character and of two forms.

Also for $n = 4$,

$$n^2 - (2^n - 1 - \sigma_n) = 3.$$

Art. 32. In order to proceed to the general case it is necessary to make a digression for the purpose of establishing certain properties of a determinant of special form.

§ 4. *Digression on the Theory of Inversely Symmetrical Determinants.*

Art. 33. The determinant of special form which I have provisionally termed "inversely symmetrical" is

$$\begin{vmatrix} 1, & \alpha_{12}, & \alpha_{13} & . & . & . & \alpha_{1n} \\ \frac{1}{\alpha_{12}}, & 1, & \alpha_{23} & . & . & . & \alpha_{2n} \\ \frac{1}{\alpha_{13}}, & \frac{1}{\alpha_{23}}, & 1 & . & . & . & \alpha_{3n} \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ \frac{1}{\alpha_{1n}}, & \frac{1}{\alpha_{2n}}, & \frac{1}{\alpha_{3n}} & . & . & . & 1 \end{vmatrix},$$

which involves $\binom{n}{2}$ different quantities α, and is such that the elements on the principal axis are all unity, and is inversely axi-symmetric in the sense that elements, symmetrically placed in regard to the principal axis, have a product equal to unity.

Art. 34. The property of this determinant, which is of vital import to the present investigation, may be stated as follows :—

"The determinant, as well as all of its co-axial minors, may be exhibited as functions of $\binom{n-1}{2}$ combinations of the $\binom{n}{2}$ quantities α_{xy}."

To establish this, first, consider the determinant itself, and put

$$\beta_{xy} = \alpha_{x,x+1}\,\alpha_{x+1,x+2}\,.\,.\,.\,\alpha_{y-1,y},\ (x < y),$$
$$\gamma_{xy} = \alpha_{xy}/\beta_{xy},$$

so that

$$\beta_{x,x+1} = \alpha_{x,x+1},$$
$$\gamma_{x,x+1} = 1.$$

Observe that the combinations

$$\gamma_{x,y} \quad (x < y - 1)$$

are $\binom{n-1}{2}$ in number; it will be shown that the quantities $\gamma_{x,y}$ are those to which reference has been made in the above statement of theorem.

Art. 35. With the new symbols the determinant may be written :—

$$\left|\begin{array}{cccccccc}
1 & \beta_{12} & \beta_{13}\gamma_{13} & \beta_{14}\gamma_{14} & . & . & \beta_{1,n-1}\gamma_{1,n-1} & \beta_{1n}\gamma_{1n} \\
\frac{1}{\beta_{12}} & 1 & \beta_{23} & \beta_{24}\gamma_{24} & . & . & \beta_{2,n-1}\gamma_{2,n-1} & \beta_{2n}\gamma_{2n} \\
\frac{1}{\beta_{13}\gamma_{13}} & \frac{1}{\beta_{23}} & 1 & \beta_{34} & . & . & \beta_{3,n-1}\gamma_{3,n-1} & \beta_{3n}\gamma_{3n} \\
\frac{1}{\beta_{14}\gamma_{14}} & \frac{1}{\beta_{24}\gamma_{24}} & \frac{1}{\beta_{34}} & 1 & . & . & \beta_{4,n-1}\gamma_{4,n-1} & \beta_{4n}\gamma_{4n} \\
. & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . \\
\frac{1}{\beta_{1,n-1}\gamma_{1,n-1}} & \frac{1}{\beta_{2,n-1}\gamma_{2,n-1}} & \frac{1}{\beta_{3,n-1}\gamma_{3,n-1}} & \frac{1}{\beta_{4,n-1}\gamma_{4,n-1}} & . & . & 1 & \beta_{n-1,n} \\
\frac{1}{\beta_{1n}\gamma_{1n}} & \frac{1}{\beta_{2n}\gamma_{2n}} & \frac{1}{\beta_{3n}\gamma_{3n}} & \frac{1}{\beta_{4n}\gamma_{4n}} & . & . & \frac{1}{\beta_{n-1,n}} & 1
\end{array}\right|$$

and may be transformed, without alteration of value, by the following operations performed successively.

Multiply

$$\begin{array}{lll}
1^{\text{st}} & \text{column by} & \beta_{12} \\
,, & \text{row} & \frac{1}{\beta_{12}} \\
3^{\text{rd}} & \text{column} & \frac{1}{\beta_{23}} \\
,, & \text{row} & \beta_{23} \\
4^{\text{th}} & \text{column} & \frac{1}{\beta_{24}} \\
,, & \text{row} & \beta_{24} \\
,, & ,, & ,, \\
s^{\text{th}} & \text{column} & \frac{1}{\beta_{23}} \\
,, & \text{row} & \beta_{23} \\
,, & ,, & ,, \\
n^{\text{th}} & \text{column} & \frac{1}{\beta_{2n}} \\
,, & \text{row} & \beta_{2n}
\end{array}$$

it then assumes the form—

$$\begin{vmatrix}
1 & 1 & \gamma_{13} & \gamma_{14} & . & . & \gamma_{1,n-2} & \gamma_{1,n-1} & \gamma_{1n} \\
1 & 1 & 1 & \gamma_{24} & . & . & \gamma_{2,n-2} & \gamma_{2,n-1} & \gamma_{2n} \\
\frac{1}{\gamma_{13}} & 1 & 1 & 1 & . & . & \gamma_{3,n-2} & \gamma_{3,n-1} & \gamma_{3n} \\
\frac{1}{\gamma_{14}} & \frac{1}{\gamma_{24}} & 1 & 1 & . & . & \gamma_{4,n-2} & \gamma_{4,n-1} & \gamma_{4n} \\
. & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . \\
\frac{1}{\gamma_{1,n-2}} & \frac{1}{\gamma_{2,n-2}} & \frac{1}{\gamma_{3,n-2}} & \frac{1}{\gamma_{4,n-2}} & . & . & 1 & 1 & \gamma_{n-2,n} \\
\frac{1}{\gamma_{1,n-1}} & \frac{1}{\gamma_{2,n-1}} & \frac{1}{\gamma_{3,n-1}} & \frac{1}{\gamma_{4,n-1}} & . & . & 1 & 1 & 1 \\
\frac{1}{\gamma_{1,n}} & \frac{1}{\gamma_{2,n}} & \frac{1}{\gamma_{3,n}} & \frac{1}{\gamma_{4,n}} & . & . & \frac{1}{\gamma_{n-2,n}} & 1 & 1
\end{vmatrix}$$

which involves only the $\binom{n-1}{2}$ combinations $\gamma_{x,y}$ of the $\binom{n}{2}$ quantities $\alpha_{x,y}$.

Art. 36. The determinant is also inversely symmetrical, and not only the principal diagonals, but also the adjacent minor diagonals consist wholly of units. In regard to the occurrence of three diagonals of units, we have here the normal form of inversely symmetrical determinant.

Art. 37. We have next to consider the coaxial minor of order $n-1$ obtained by deletion of the s^{th} row and s^{th} column.

The following successive operations, which do not alter the value, have then to be performed—

Multiply

$$
\begin{array}{llcllll}
1^{\text{st}} \text{ column by} & & \beta_{12} & \text{and} & 1^{\text{st}} & \text{row by} & \dfrac{1}{\beta_{12}}. \\[2ex]
3^{\text{rd}} & ,, & \dfrac{1}{\beta_{23}} & ,, & 3^{\text{rd}} & ,, & \beta_{23}. \\[2ex]
\cdot & & \cdot & & \cdot & & \cdot \\[2ex]
(s-1)^{\text{th}} & ,, & \dfrac{1}{\beta_{2,s-1}} & ,, & (s-1)^{\text{th}} & ,, & \beta_{2,s-1}. \\[2ex]
(s+1)^{\text{th}} & ,, & \dfrac{1}{\gamma_{s-1,s+1}\beta_{2,s+1}} & ,, & (s+1)^{\text{th}} & ,, & \gamma_{s-1,s+1}\beta_{2,s+1}. \\[2ex]
(s+2)^{\text{th}} & ,, & \dfrac{1}{\gamma_{s-1,s+1}\beta_{2,s+2}} & ,, & (s+2)^{\text{th}} & ,, & \gamma_{s-1,s+1}\beta_{2,s+2}. \\[2ex]
\cdot & & \cdot & & \cdot & & \cdot \\[2ex]
n^{\text{th}} & ,, & \dfrac{1}{\gamma_{s-1,s+1}\beta_{2,n}} & ,, & n^{\text{th}} & ,, & \gamma_{s-1,s+1}\beta_{2,n}.
\end{array}
$$

Art. 38. To represent the result conveniently, suppose the determinant divided into four compartments by the lines of deletion, thus—

$$
\left|\begin{array}{c|c}
\text{I.} & \text{II.} \\
\hline
\text{III.} & \text{IV.}
\end{array}\right|.
$$

We then obtain—

$$
\begin{array}{r|ccccccc|ccccccc|l}
 & 1 & 1 & \gamma_{13} & \gamma_{14} & \cdots & \gamma_{1,s-2} & \gamma_{1,s-1} & \frac{\gamma_{1,s+1}}{\gamma_{s-1,s+1}} & \frac{\gamma_{1,s+2}}{\gamma_{s-1,s+1}} & \cdots & \frac{\gamma_{1,n-3}}{\gamma_{s-1,s+1}} & \frac{\gamma_{1,n-2}}{\gamma_{s-1,s+1}} & \frac{\gamma_{1,n-1}}{\gamma_{s-1,s+1}} & \frac{\gamma_{1,n}}{\gamma_{s-1,s+1}} & \\
 & 1 & 1 & 1 & \gamma_{24} & \cdots & \gamma_{2,s-2} & \gamma_{2,s-1} & \frac{\gamma_{2,s+1}}{\gamma_{s-1,s+1}} & \frac{\gamma_{2,s+2}}{\gamma_{s-1,s+1}} & \cdots & \frac{\gamma_{2,n-3}}{\gamma_{s-1,s+1}} & \frac{\gamma_{2,n-2}}{\gamma_{s-1,s+1}} & \frac{\gamma_{2,n-1}}{\gamma_{s-1,s+1}} & \frac{\gamma_{2,n}}{\gamma_{s-1,s+1}} & \\
 & \frac{1}{\gamma_{13}} & 1 & 1 & 1 & \cdots & \gamma_{3,s-2} & \gamma_{3,s-1} & \frac{\gamma_{3,s+1}}{\gamma_{s-1,s+1}} & \frac{\gamma_{3,s+2}}{\gamma_{s-1,s+1}} & \cdots & \frac{\gamma_{3,n-3}}{\gamma_{s-1,s+1}} & \frac{\gamma_{3,n-2}}{\gamma_{s-1,s+1}} & \frac{\gamma_{3,n-1}}{\gamma_{s-1,s+1}} & \frac{\gamma_{3,n}}{\gamma_{s-1,s+1}} & \\
\mathrm{I.} = & \frac{1}{\gamma_{14}} & \frac{1}{\gamma_{24}} & 1 & 1 & \cdots & \gamma_{4,s-2} & \gamma_{4,s-1} & \frac{\gamma_{4,s+1}}{\gamma_{s-1,s+1}} & \frac{\gamma_{4,s+2}}{\gamma_{s-1,s+1}} & \cdots & \frac{\gamma_{4,n-3}}{\gamma_{s-1,s+1}} & \frac{\gamma_{4,n-2}}{\gamma_{s-1,s+1}} & \frac{\gamma_{4,n-1}}{\gamma_{s-1,s+1}} & \frac{\gamma_{4,n}}{\gamma_{s-1,s+1}} & = \mathrm{II.} \\
 & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \\
 & \frac{1}{\gamma_{1,s-2}} & \frac{1}{\gamma_{2,s-2}} & \frac{1}{\gamma_{3,s-2}} & \frac{1}{\gamma_{4,s-2}} & \cdots & 1 & 1 & \frac{\gamma_{s-2,s+1}}{\gamma_{s-1,s+1}} & \frac{\gamma_{s-2,s+2}}{\gamma_{s-1,s+1}} & \cdots & \frac{\gamma_{s-2,n-3}}{\gamma_{s-1,s+1}} & \frac{\gamma_{s-2,n-2}}{\gamma_{s-1,s+1}} & \frac{\gamma_{s-2,n-1}}{\gamma_{s-1,s+1}} & \frac{\gamma_{s-2,n}}{\gamma_{s-1,s+1}} & \\
 & \frac{1}{\gamma_{1,s-1}} & \frac{1}{\gamma_{2,s-1}} & \frac{1}{\gamma_{3,s-1}} & \frac{1}{\gamma_{4,s-1}} & \cdots & 1 & 1 & 1 & \frac{\gamma_{s-1,s+2}}{\gamma_{s-1,s+1}} & \cdots & \frac{\gamma_{s-1,n-3}}{\gamma_{s-1,s+1}} & \frac{\gamma_{s-1,n-2}}{\gamma_{s-1,s+1}} & \frac{\gamma_{s-1,n-1}}{\gamma_{s-1,s+1}} & \frac{\gamma_{s-1,n}}{\gamma_{s-1,s+1}} & \\
\hline
 & \frac{\gamma_{s-1,s+1}}{\gamma_{1,s+1}} & \frac{\gamma_{s-1,s+1}}{\gamma_{2,s+1}} & \frac{\gamma_{s-1,s+1}}{\gamma_{3,s+1}} & \frac{\gamma_{s-1,s+1}}{\gamma_{4,s+1}} & \cdots & \frac{\gamma_{s-1,s+1}}{\gamma_{s-2,s+1}} & 1 & 1 & 1 & \cdots & \gamma_{s+1,n-3} & \gamma_{s+1,n-2} & \gamma_{s+1,n-1} & \gamma_{s+1,n} & \\
 & \frac{\gamma_{s-1,s+1}}{\gamma_{1,s+2}} & \frac{\gamma_{s-1,s+1}}{\gamma_{2,s+2}} & \frac{\gamma_{s-1,s+1}}{\gamma_{3,s+2}} & \frac{\gamma_{s-1,s+1}}{\gamma_{4,s+2}} & \cdots & \frac{\gamma_{s-1,s+1}}{\gamma_{s-2,s+2}} & \frac{\gamma_{s-1,s+1}}{\gamma_{s-1,s+2}} & 1 & 1 & \cdots & \gamma_{s+2,n-3} & \gamma_{s+2,n-2} & \gamma_{s+2,n-1} & \gamma_{s+2,n} & \\
 & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \\
\mathrm{III.} = & \frac{\gamma_{s-1,s+1}}{\gamma_{1,n-3}} & \frac{\gamma_{s-1,s+1}}{\gamma_{2,n-3}} & \frac{\gamma_{s-1,s+1}}{\gamma_{3,n-3}} & \frac{\gamma_{s-1,s+1}}{\gamma_{4,n-3}} & \cdots & \frac{\gamma_{s-1,s+1}}{\gamma_{s-2,n-3}} & \frac{\gamma_{s-1,s+1}}{\gamma_{s-1,n-3}} & \frac{1}{\gamma_{s+1,n-3}} & \frac{1}{\gamma_{s+2,n-3}} & \cdots & 1 & 1 & \gamma_{n-3,n-1} & \gamma_{n-3,n} & = \mathrm{IV.} \\
 & \frac{\gamma_{s-1,s+1}}{\gamma_{1,n-2}} & \frac{\gamma_{s-1,s+1}}{\gamma_{2,n-2}} & \frac{\gamma_{s-1,s+1}}{\gamma_{3,n-2}} & \frac{\gamma_{s-1,s+1}}{\gamma_{4,n-2}} & \cdots & \frac{\gamma_{s-1,s+1}}{\gamma_{s-2,n-2}} & \frac{\gamma_{s-1,s+1}}{\gamma_{s-1,n-2}} & \frac{1}{\gamma_{s+1,n-2}} & \frac{1}{\gamma_{s+2,n-2}} & \cdots & 1 & 1 & 1 & \gamma_{n-2,n} & \\
 & \frac{\gamma_{s-1,s+1}}{\gamma_{1,n-1}} & \frac{\gamma_{s-1,s+1}}{\gamma_{2,n-1}} & \frac{\gamma_{s-1,s+1}}{\gamma_{3,n-1}} & \frac{\gamma_{s-1,s+1}}{\gamma_{4,n-1}} & \cdots & \frac{\gamma_{s-1,s+1}}{\gamma_{s-2,n-1}} & \frac{\gamma_{s-1,s+1}}{\gamma_{s-1,n-1}} & \frac{1}{\gamma_{s+1,n-1}} & \frac{1}{\gamma_{s+2,n-1}} & \cdots & \frac{1}{\gamma_{n-3,n-1}} & 1 & 1 & 1 & \\
 & \frac{\gamma_{s-1,s+1}}{\gamma_{1,n}} & \frac{\gamma_{s-1,s+1}}{\gamma_{2,n}} & \frac{\gamma_{s-1,s+1}}{\gamma_{3,n}} & \frac{\gamma_{s-1,s+1}}{\gamma_{4,n}} & \cdots & \frac{\gamma_{s-1,s+1}}{\gamma_{s-2,n}} & \frac{\gamma_{s-1,s+1}}{\gamma_{s-1,n}} & \frac{1}{\gamma_{s+1,n}} & \frac{1}{\gamma_{s+2,n}} & \cdots & \frac{1}{\gamma_{n-3,n}} & \frac{1}{\gamma_{n-2,n}} & 1 & 1 &
\end{array}
$$

Art. 39. This is an inversely symmetrical determinant of normal form involving the $\binom{n-1}{2}$ quantities γ_{xy}. In the compartment II, the elements, other than the units, have the denominator $\gamma_{s-1,\,s+1}$. The transformed of the minor is derived from the transformed complete determinant by deletion of the s^{th} row and s^{th} column, and the subsequent division of each γ element in the compartment II by $\gamma_{s-1,\,s+1}$ and multiplication of each γ element in compartment III by $\gamma_{s-1,\,s+1}$.

It is now obvious that if a minor be formed from the untransformed determinant by deletion of the

$$s^{\text{th}}\ (s+1)^{\text{th}}\ .\ .\ .\ (s+\sigma)^{\text{th}} \text{ rows}$$

and the

$$s^{\text{th}}\ (s+1)^{\text{th}}\ .\ .\ .\ (s+\sigma)^{\text{th}} \text{ columns,}$$

the transformed minor will be obtained from the transformed complete determinant by deletion of the aforesaid rows and columns, and subsequent division of all γ elements which are at once above the s^{th} row and to the right of the $(s+\sigma)^{\text{th}}$ column by $\gamma_{s-1,\,s+\sigma+1}$ and corresponding multiplication of the inversely symmetrical elements by the same quantity. Or, as before, we may suppose the minor divided into four compartments and state the rule with reference to them. It will be convenient to allude to these compartments as I_s, II_s, III_s, IV_s.

In addition to the aforesaid rows and columns, suppose the $t^{\text{th}}\ (t+1)^{\text{th}}\ .\ .\ .\ (t+\tau)^{\text{th}}$ rows and columns deleted.

In correspondence we have other four compartments, I_t, II_t, III_t, IV_t; and there will be a certain extent of overlapping of compartments.

Art. 40. The rule is (after deletion from transformed complete determinant):—

Divide γ elements in II_s by $\gamma_{s-1,\,s+\sigma+1}$,

,, ,, II_t ,, $\gamma_{t-1,\,t+\tau+1}$,

with corresponding multiplication of the inversely symmetrical elements.

If this be carried out it will be found that those γ elements which are in both II_s and II_t will be divided by $\gamma_{s-1,\,s+\sigma+1}\,\gamma_{t-1,\,t+\tau+1}$.

The general rule guiding the formation of the minor when there are any number of sets of compartments arising from the deletions will be now perfectly clear.

Art. 41. We are thus enabled to exhibit all the co-axial minors of the determinant as functions of the $\binom{n-1}{2}$ quantities γ.

So much of the theory of these interesting determinants suffices for present purposes.

§ 5.

Art. 42. *The general case.*

The matrix being that connected with the determinant

$$| a_{1n} | ,$$

we have the relations

$$a_{xx} = p_x,$$

$$| a_{xx} a_{yy} | = p_{xy},$$

as well as

$$2^n - 1 - n - \binom{n}{2}$$

other relations

$$| a_{xx} a_{yy} a_{zz} \ldots | = p_{xyz} \ldots,$$

connected with the co-axial minors of order greater than 2.

From the relation

$$| a_{xx} a_{yy} | = p_{xy}$$

is derived

$$a_{xy} a_{yx} = p_x p_y - p_{xy} = q_{xy} \text{ (suppose).}$$

We now introduce $\binom{n}{2}$ undetermined quantities α_{xy} such that

$$a_{xy} = \alpha_{xy} q_{xy},$$

$$a_{yx} = 1/\alpha_{xy},$$

and substitute in the remaining

$$2^n - 1 - n - \binom{n}{2}$$

relations.

The typical relation

$$| a_{xx} a_{yy} a_{zz} \ . \ \ . \ | = p_{xyz} \ldots$$

then becomes

$$\begin{vmatrix} p_x, & \alpha_{xy}q_{xy}, & \alpha_{xz}q_{xz} & \cdot & \cdot & \cdot \\ \frac{1}{\alpha_{xy}}, & p_y, & \alpha_{yz}q_{yz} & \cdot & \cdot & \cdot \\ \frac{1}{\alpha_{xz}}, & \frac{1}{\alpha_{yz}}, & p_z & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{vmatrix} = p_{xyz}\ldots$$

In the determinant the quantities α occur in an inversely symmetrical manner, and the determinant becomes inversely symmetrical on putting the quantities p and q equal to unity.

Art. 43. The determinant is transformable in the same manner as the corresponding inversely symmetrical form, and the foregoing "Digression" establishes the fact that the quantities α will then occur in only some or all of $\binom{n-1}{2}$ combinations $\gamma_{x,\,y}$, where

$$\gamma_{x,\,y} = \frac{\alpha_{xy}}{\alpha_{x,\,x+1}\alpha_{x+1,\,x+2}\ldots\alpha_{y-1,\,y}} = \frac{\alpha_{xy}}{\beta_{xy}}.$$

Hence we are presented with

$$2^n - 1 - n - \binom{n}{2} \text{ equations}$$

involving $\binom{n-1}{2}$ quantities $\gamma_{x,\,y}$.

Art. 44. Eliminating these $\binom{n-1}{2}$ quantities, we find

$$2^n - 1 - n - \binom{n}{2} - \binom{n-1}{2} = 2^n - n^2 + n - 2$$

relations or syzygies between the coaxial minors

of the determinant
$$p_{xyz}\ldots$$
$$|\,\alpha_{1n}\,|.$$

Art. 45. This shows that the coefficients of V_n must satisfy

$$2^n - n^2 + n - 2$$

independent conditions.

Art. 46. Assuming the satisfaction of these conditions we can solve the equations so as to express the $\binom{n-1}{2}$ quantities $\gamma_{x,y}$ in terms of the coefficients of V_n.

Hence we can express the $\frac{1}{2}(n-1)(n-4)$ quantities

$$\alpha_{x,y} \qquad (y > x + 1),$$

in terms of the $n - 1$ quantities

$$\alpha_{x,x+1},$$

thus reducing the number of undetermined quantities to

$$n - 1.$$

Art. 47. We have

$$\sigma_n = 2^n - n^2 + n - 2,$$

while the matrix, which defines

$$X_1, X_2, \ldots X_n$$

of the redundant form, is :—

$$\begin{pmatrix} p_1 & \beta_{12}q_{12} & \frac{\beta_{13}}{c_{13}}q_{13} & \frac{\beta_{14}}{c_{14}}q_{14} & \cdot\ \cdot & \frac{\beta_{1n}}{c_{1n}}q_{1n} \\ \frac{1}{\beta_{12}} & p_2 & \beta_{23}q_{23} & \frac{\beta_{24}}{c_{24}}q_{24} & \cdot\ \cdot & \frac{\beta_{2n}}{c_{2n}}q_{2n} \\ \frac{c_{13}}{\beta_{13}} & \frac{1}{\beta_{23}} & p_3 & \beta_{34}q_{34} & \cdot\ \cdot & \frac{\beta_{3n}}{c_{3n}}q_{3n} \\ \frac{c_{14}}{\beta_{14}} & \frac{c_{24}}{\beta_{24}} & \frac{1}{\beta_{34}} & p_4 & \cdot\ \cdot & \frac{\beta_{4n}}{c_{4n}}q_{4n} \\ \cdot & \cdot & \cdot & \cdot & \cdot\ \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot\ \cdot & \cdot \\ \frac{c_{1n}}{\beta_{1n}} & \frac{c_{2n}}{\beta_{2n}} & \frac{c_{3n}}{\beta_{3n}} & \frac{c_{4n}}{\beta_{4n}} & \cdot\ \cdot & p_n \end{pmatrix}$$

or the matrix similar to this with c_{yx} written for c_{xy}.

Postponing particular explanation in regard to the quantities c_{xy} I merely remark that c_{xy}^{-1} is a value of $\gamma_{x,y}$ deduced from the equations.

The quantity β_{xy} has been defined to be

$$\alpha_{x,x+1}\,\alpha_{x+1,x+2}\,\ldots\,\alpha_{y-1,y}.$$

The matrix involves $n - 1$ undetermined quantities

$$\alpha_{12}, \alpha_{23}, \ldots \alpha_{n-1,n},$$

or since

$$\beta_{xy} = \beta_{1y}/\beta_{1x},$$

we may take the undetermined quantities to be

$$\beta_{12}, \beta_{13}, \ldots \beta_{1,n}.$$

Each redundant form is thus of the nature

$$\infty^{n-1}$$

as was to be shown.

Art. 48. The equations for the determination of the $\binom{n-1}{2}$ quantities $\gamma_{x,y}$ can be taken from amongst the $\binom{n}{3}$ equations connected with the co-axial minors of Order 3.

One such equation is

$$| a_{xx}a_{yy}a_{zz} | = p_{x,y,z},$$

which may be written

$$\begin{vmatrix} p_x & q_{xy} & \dfrac{\gamma_{xz}}{\gamma_{xy}\gamma_{yz}} q_{xz} \\ 1 & p_y & q_{yz} \\ \dfrac{\gamma_{xy}\gamma_{yz}}{\gamma_{xz}} & 1 & p_z \end{vmatrix} = p_{xyz},$$

and this is a quadratic equation for $\gamma_{xz}/\gamma_{xy}\gamma_{yz}$.

If x, y, z be consecutive integers, this is simply a quadratic equation for γ_{xz}. Hence, the $n-2$ quantities $\gamma_{x,x+2}$ are at once determined. The $n-3$ quantities $\gamma_{x,x+3}$ are found by the aid of $\gamma_{x,x+1}$, which is unity, and $\gamma_{x+1,x+3}$. Thence, $\gamma_{x,x+s}$ is found in terms of $\gamma_{x+1,x+s}$, and all the quantities γ_{xy} are easily found.

Assuming the coefficients of V_n to satisfy the above-mentioned

$$2^n - n^2 + n - 2$$

conditions, we have to find systems of values of the quantities γ_{xy} which satisfy the

$$2^n - 1 - n - \binom{n}{2} \text{ equations}$$

in which they appear.

I find that there are only two such systems, obtained respectively by taking the positive and the negative signs in the solutions of the quadratic equations. In the one solution the signs are all taken positive and in the other all negative.

Let $c_{xy}{}^{-1}$ be the value of γ_{xy} obtained by always taking positive signs and $c_{yx}{}^{-1}$ that value obtained by always taking negative signs.

We have the system $c_{xy}{}^{-1}$ and the system $c_{yx}{}^{-1}$. There are thus two representations of the redundant form, each involving $n - 1$ undetermined quantities.

Art. 49. Given a redundant form of order n, involving the matrix

$$\left(a_{1n} \right),$$

we may exhibit its two representations, each involving $n - 1$ undetermined quantities.

The coefficients of the condensed form now necessarily satisfy the proper conditions, and passing through the condensed form we must, in the matrix of Art. 48, write

$$\begin{aligned} p_x &= a_{xx} \\ q_{xy} &= a_{xx}a_{yy} - \mid a_{xx}a_{yy} \mid = a_{xy}a_{yx}, \end{aligned}$$

and then it only remains to find the values of c_{xy} and c_{yx} in terms of the elements of the determinant

$$\mid a_{1n} \mid .$$

Solving the quadratic equation

$$\begin{vmatrix} a_{xx} & a_{xy}a_{yx} & \dfrac{\gamma_{xz}}{\gamma_{xy}\,\gamma_{yx}}\,a_{xz}a_{zx} \\ 1 & a_{yy} & a_{yz}a_{zy} \\ \dfrac{\gamma_{xy}\,\gamma_{yz}}{\gamma_{xz}} & 1 & a_{zz} \end{vmatrix} = \mid a_{xx}a_{yy}a_{zz} \mid ,$$

transformed from Art. 48, we find

$$\frac{\gamma_{xz}}{\gamma_{xy}\,\gamma_{yz}} = \frac{(a_{xy}a_{yz}a_{zx} + a_{yx}a_{zy}a_{xz}) \pm (a_{xy}a_{yz}a_{zx} - a_{yx}a_{zy}a_{xz})}{2a_{xz}a_{zx}},$$

or taking the positive sign

$$\frac{\gamma_{xz}}{\gamma_{xy}\,\gamma_{yz}} = \frac{a_{xy}\,a_{yz}}{a_{xz}},$$

and taking the negative sign

$$\frac{\gamma_{xz}}{\gamma_{xy}\,\gamma_{yz}} = \frac{a_{yx}\,a_{zy}}{a_{zx}}.$$

Hence, if $c_{xy}{}^{-1}$, be the value of γ_{xy} deduced by always taking positive signs and $c_{yx}{}^{-1}$ that value arising from the negative signs, we find

$$c_{xy} = \frac{a_{xy}}{a_{x,\,x+1}\,a_{x+1,\,x+2}\cdots a_{y-1,\,y}} = \frac{a_{xy}}{b_{xy}},$$

$$c_{yx} = \frac{a_{yx}}{a_{y,\,y-1}\,a_{y-1,\,y-2}\cdots a_{x+1,\,x}} = \frac{a_{yx}}{b_{yx}},$$

where the symbols b_{xy} have been introduced, so that now

$$a_{xy},\ b_{xy},\ c_{xy}$$

in regard to the elements of the matrix of the fundamental form are analogous to

$$\alpha_{xy},\ \beta_{xy},\ \gamma_{xy}$$

in regard to the undetermined quantities.

It is easy to verify that the two systems of values

$$c_{xy}{}^{-1},\ c_{yx}{}^{-1},$$

of the quantities γ_{xy}, satisfy the whole of the $2^n - 1 - n - \binom{n}{2}$ equations, but I do not stop to prove that these are the only systems of values of γ_{xy}.

Substituting in the matrix of Art. 47 we obtain the two representations

$$\begin{pmatrix} a_{11} & \beta_{12}a_{21}b_{12} & \beta_{13}a_{31}b_{13} & \beta_{14}a_{41}b_{14} & \cdot\ \cdot & \beta_{1n}a_{n1}b_{1n} \\ \dfrac{1}{\beta_{12}} & a_{22} & \beta_{23}a_{32}b_{23} & \beta_{24}a_{42}b_{24} & \cdot\ \cdot & \beta_{2n}a_{n2}b_{2n} \\ \dfrac{a_{13}}{\beta_{13}\,b_{13}} & \dfrac{1}{\beta_{23}} & a_{33} & \beta_{34}a_{43}b_{34} & \cdot\ \cdot & \beta_{3n}a_{n3}b_{3n} \\ \dfrac{a_{14}}{\beta_{14}\,b_{14}} & \dfrac{a_{24}}{\beta_{24}b_{24}} & \dfrac{1}{\beta_{34}} & a_{44} & \cdot\ \cdot & \beta_{4n}a_{n4}b_{4n} \\ \cdot & \cdot & \cdot & \cdot & \cdot\ \cdot & \cdot \\ \dfrac{a_{1n}}{\beta_{1n}\,b_{1n}} & \dfrac{a_{2n}}{\beta_{2n}\,b_{2n}} & \dfrac{a_{3n}}{\beta_{3n}\,b_{3n}} & \dfrac{a_{4n}}{\beta_{4n}\,b_{4n}} & \cdot\ \cdot & a_{nn} \end{pmatrix}$$

$$\begin{pmatrix} a_{11} & \beta_{12}a_{12}b_{21} & \beta_{13}a_{13}b_{31} & \beta_{14}a_{14}b_{41} & \cdot\ \cdot & \beta_{1n}a_{1n}b_{n1} \\ \dfrac{1}{\beta_{12}} & a_{22} & \beta_{23}a_{23}b_{32} & \beta_{24}a_{24}b_{42} & \cdot\ \cdot & \beta_{2n}a_{2n}b_{n2} \\ \dfrac{a_{31}}{\beta_{13}\,b_{31}} & \dfrac{1}{\beta_{23}} & a_{33} & \beta_{34}a_{34}b_{43} & \cdot\ \cdot & \beta_{3n}a_{3n}b_{n3} \\ \dfrac{a_{41}}{\beta_{14}\,b_{41}} & \dfrac{a_{42}}{\beta_{24}\,b_{42}} & \dfrac{1}{\beta_{34}} & a_{44} & \cdot\ \cdot & \beta_{4n}a_{4n}b_{n4} \\ \cdot & \cdot & \cdot & \cdot & \cdot\ \cdot & \cdot \\ \dfrac{a_{n1}}{\beta_{1n}\,b_{n1}} & \dfrac{a_{n2}}{\beta_{2n}\,b_{n2}} & \dfrac{a_{n3}}{\beta_{3n}\,b_{n3}} & \dfrac{a_{n4}}{\beta_{4n}\,b_{n4}} & \cdot\ \cdot & a_{nn} \end{pmatrix}$$

and the second is obtainable from the first by writing

$$(a_{xy}, b_{xy}) = (a_{yx}, b_{yx}).$$

These redundant forms all lead to the same condensed form, viz. :—that derivable from the matrix

$$\left(\begin{vmatrix} a_{1n} \end{vmatrix}\right).$$

Further we have here the most general forms of determinants such that their co-axial minors coincide with those of the determinant

$$| a_{1n} | .$$

The matrix reverts to its primary form on putting

$$\beta_{xy} = a_{xy}/a_{yx}b_{xy}$$

in the first representation, or, on putting

$$\beta_{xy} = 1/b_{yx}$$

in the second representation.

The transverse matrix is obtained, from the first representation, by putting

$$\beta_{xy} = 1/b_{xy}.$$

Art. 50. The function V which has entered in such a fundamentally important manner into the foregoing analysis appears to have a place in the general theory of matrices. Confining ourselves, for simplicity, to the third order, it may be recalled that SYLVESTER terms the function

$$\begin{vmatrix} a_{11} - x & a_{12} & a_{13} \\ a_{21} & a_{22} - x & a_{23} \\ a_{31} & a_{32} & a_{33} - x \end{vmatrix}$$

the latent function of the matrix

$$\left(\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}\right)$$

This function appears very frequently in pure mathematics, and also in applications to physics. From it can be derived a function of three variables, viz. :—

$$\begin{vmatrix} a_{11}-x_1 & a_{12} & a_{13} \\ a_{21} & a_{22}-x_2 & a_{23} \\ a_{31} & a_{32} & a_{33}-x_3 \end{vmatrix}$$

and herein writing $1/x_1$ for x_1, &c., and multiplying by $x_1x_2x_3$ and by -1 when the order is uneven, we get

$$V = \begin{vmatrix} 1-a_{11}x_1 & -a_{12}x_1 & -a_{31}x_1 \\ -a_{21}x_2 & 1-a_{22}x_2 & -a_{32}x_2 \\ -a_{31}x_3 & -a_{32}x_3 & 1-a_{33}x_3 \end{vmatrix}$$

Thus the latent function is a particular case of the function V.

In the discussion of the roots of the latent function we are concerned with the order of vacuity of the matrix which may be any integer of the series $0, 1, 2, \ldots n$. In the case of the function V, which may be called the homographic function of the matrix, it is evident that a more refined nature of vacuity is pertinent to the discussion. We have to consider not merely the vanishing of the sum of all the co-axial minors whose order exceeds a given integer, but rather the vanishing of each separate co-axial minor.

It may be remarked that the homographic function V vanishes for the system of values of x_1, x_2, x_3, which satisfies the equations

$$X_1 = X_2 = X_3 = 1.$$

§ 6. *Digression on the General Theory of Determinants.*

Art. 51. The foregoing investigation has established the fact that the co-axial minors, of a general determinant of Order n, are connected by $2^n - n^2 + n - 2$ relations, or in other words, that but $n^2 - n + 1$ of them can assume given values.

Of these relations a certain number are connected in a special manner with the determinant of Order n, in that they are not relations merely between the coaxial minors of one of the principal coaxial minors of the determinant.

Let this number be

$$\psi(n),$$

and put

$$2^n - n^2 + n - 2 = \phi(n).$$

Then

$$\phi(n) = \psi(n) + \binom{n}{1}\psi(n-1) + \binom{n}{2}\psi(n-2) + \ldots + \binom{n}{n-4}\psi(4);$$

whence

$$\psi(4) = \phi(4) = 2,$$

and

$$\psi(n) = \phi(n) - \binom{n}{1}\phi(n-1) + \binom{n}{2}\phi(n-2) - \ldots (-)^{n-4}\binom{n}{n-4}\phi(4);$$

and, by summation, we obtain the result

$$\psi(n) = 1 + (-1)^n; \ (n \neq 2)$$

shewing that

$$\psi(2m) = 2. \quad (m > 1)$$
$$\psi(2m+1) = 0.$$

Hence, when the determinant is of even order greater than two, there are two special relations between the coaxial minors and these two relations can each be thrown into a form which exhibits the determinant as an irrational function of its coaxial minors.

In the case of a determinant of uneven order no *special* relations exist between the coaxial minors, and it is not possible to express the determinant as a function of its coaxial minors.*

Art. 52. In the investigation we met with $\binom{n}{3}$ equations

$$\begin{vmatrix} p_x & q_{xy} & \dfrac{\gamma_{xz}}{\gamma_{xy}\gamma_{yz}} q_{xz} \\ 1 & p_y & q_{yz} \\ \dfrac{\gamma_{xy}\gamma_{yz}}{\gamma_{xz}} & 1 & p_z \end{vmatrix} = p_{xyz},$$

involving the $\binom{n-1}{2}$ quantities γ_{xy} and the coaxial minors of the first three orders of the determinant $| a_{1n} |$. Hence, by elimination, we find $\binom{n-1}{3}$ identical relations between such coaxial minors.

Also we found

$$\binom{n}{3} + \binom{n}{4} + \ldots + \binom{n}{s}$$

* It is evident that these relations must occur in pairs in accordance with the 'Law of Complementaries' which is so important in the general theory of determinants.

equations involving the $\binom{n-1}{2}$ quantities γ_{xy} and the co-axial minors of the first s orders of the determinant $| a_{1n} |$. Hence, by elimination, we find

$$\binom{n-1}{3} + \binom{n}{4} + \ldots + \binom{n}{s}$$

relations between such coaxial minors.

Special to the coaxial minors of order s, we thus find $\binom{n}{s}$ relations if n be greater than 3. The one relation, special (from this standpoint) to the determinant of even order (greater than two), is obtained by eliminating the determinant itself from the two special identical relations above referred to.

Art. 53. I take this opportunity of verifying the statements made in Art. 49 in regard to the systems of values of the quantities

$$\gamma_{xy}$$

which satisfy the

$$2^n - 1 - n - \binom{n}{2} \text{ equations.}$$

It is, in reality, a question concerning the properties of determinants.

To ensure that the coefficients of the condensed form satisfy the requisite conditions, assume them to be derived from the determinant

$$| a_{1n} | .$$

We have $\binom{n}{2}$ equations of the type

$$\begin{vmatrix} a_{xx} & a_{xy}a_{yx} & \dfrac{\gamma_{xz}}{\gamma_{xy}\gamma_{yz}} a_{xz}a_{zx} \\ 1 & a_{yy} & a_{yz}a_{zy} \\ \dfrac{\gamma_{xy}\gamma_{yz}}{\gamma_{xz}} & 1 & a_{zz} \end{vmatrix} = | a_{xx}a_{yy}a_{zz} |$$

This equation, being a quadratic for $\gamma_{xz} / \gamma_{xy}\gamma_{yz}$, has only two roots, and it is easy to verify that the equation is satisfied by the values

$$\frac{a_{xy}a_{yz}}{a_{xz}}, \quad \frac{a_{yx}a_{zy}}{a_{zx}}.$$

In Art. 49, these values have been obtained by solving the quadratic, and it was found that the values corresponded to the positive and negative sign respectively.

Taking always the positive sign, let c_{xy}^{-1} be the value deduced for γ_{xy}. Then

$$c_{xy} = a_{xy}/b_{xy},$$

and

$$\gamma_{xz}/\gamma_{xy}\gamma_{yz} = c_{xy}c_{yz}/c_{xz}.$$

Hence, the $\binom{n}{2}$ equations are all satisfied by the system

$$\gamma_{xy} = c_{xy}^{-1}.$$

Similarly, they are all satisfied by the system

$$\gamma_{xy} = c_{yx}^{-1},$$

where

$$c_{yx} = a_{yx}/b_{yx}.$$

Art. 54. To show that each of these systems satisfies the remaining equations, it suffices to consider the typical determinant equation of the fourth order.

We have—

$$\begin{vmatrix} a_{xx} & a_{xy}a_{yx} & \dfrac{\gamma_{xz}}{\gamma_{xy}\gamma_{yz}}a_{xz}a_{zx} & \dfrac{\gamma_{xw}}{\gamma_{xy}\gamma_{yz}\gamma_{zw}}a_{xw}a_{wx} \\ 1 & a_{yy} & a_{yz}a_{zy} & \dfrac{\gamma_{yw}}{\gamma_{yz}\gamma_{zw}}a_{yw}a_{wy} \\ \dfrac{\gamma_{xy}\gamma_{yz}}{\gamma_{xz}} & 1 & a_{zz} & a_{zw}a_{wz} \\ \dfrac{\gamma_{xy}\gamma_{yz}\gamma_{zw}}{\gamma_{xw}} & \dfrac{\gamma_{yz}\gamma_{zw}}{\gamma_{yw}} & 1 & a_{ww} \end{vmatrix}$$

$$= |\, a_{xx}\,a_{yy}\,a_{zz}\,a_{ww}\,|\,.$$

On the left-hand side put

$$\gamma_{xy} = c_{xy}^{-1} = b_{xy}/a_{xy},$$

and the determinant becomes

$$\begin{vmatrix} a_{xx} & a_{xy}a_{yx} & a_{xy}a_{yz}a_{zx} & a_{xy}a_{yz}a_{xw}a_{wx} \\ 1 & a_{yy} & a_{yz}a_{zy} & a_{yz}a_{zw}a_{wy} \\ \dfrac{a_{xz}}{a_{xy}a_{yz}} & 1 & a_{zz} & a_{zw}a_{wz} \\ \dfrac{a_{xw}}{a_{xy}a_{yz}a_{zw}} & \dfrac{a_{yw}}{a_{yz}a_{zw}} & 1 & a_{ww} \end{vmatrix}$$

In succession, multiply the first column by a_{xy}, divide the first row by a_{xy}; multiply

the third row by a_{yz}, divide the third column by a_{yz}; multiply the fourth row by a_{zw}, divide the fourth column by a_{zw}; the determinant is then $|\ a_{xx}a_{yy}a_{zz}a_{ww}\ |$.

Similarly it is shown that the equation is satisfied by the system

$$\gamma_{xy} = c_{yx}^{-1} = b_{yx}/a_{yx}.$$

The equations, involving determinants of higher order, can similarly be shown to be satisfied by both systems of values, and since the $\binom{n}{3}$ quadratic equations have each but two roots, it follows at once that

$$c_{xy}^{-1},\ c_{yx}^{-1}$$

are the only systems.

§ 7. *Arithmetical Interpretations resumed.*

Art. 55. The arithmetical interpretations drawn from the theory have been so far of two kinds. In the examples taken from the "Memoir on the Compositions of Numbers" we had a redundant form of generating function and an exact or condensed form; the redundant form and the exact form could be differently interpreted, and this led to an arithmetical correspondence which was duly noted in the memoir quoted. The interpretations, subsequently considered in this paper, were single, and there was no arithmetical correspondence; the condensed forms did not admit of easy and useful interpretations, but only the redundant forms. The redundant forms were not considered in the most general form which, as we have seen, involves $n-1$ undetermined quantities, but each of these quantities was given a special numerical value; this process led to simple and useful arithmetical results but it will be obvious that the possibility of interpretation does not stop here.

Art. 56. In proceeding from the condensed form to the redundant form we met with $n-1$ undetermined quantities

$$\alpha_{12},\ \alpha_{23},\ \ldots\ \alpha_{n-1,n}.$$

As before remarked, we may, if we please, put these quantities equal to certain functions of the quantities

$$x_1,\ x_2,\ \ldots\ x_n.$$

We are not at liberty to choose *any* functions. The functions must satisfy certain conditions, otherwise the coefficient of

$$x_1^{\xi_1}x_2^{\xi_2}\ \ldots\ x_n^{\xi_n}$$

in the particular redundant product will not remain unchanged.

I propose to examine this question.

Art. 57. Of order 2 we have the product

$$(p_1x_1 + \alpha_{12}q_{12}x_2)^{\xi_1}\left(\frac{1}{\alpha_{12}}x_1 + p_2x_2\right)^{\xi_2},$$

and in performing the multiplication we find a term involving

$$(p_1x_1)^m(\alpha_{12}q_{12}x_2)^{\xi_1-m}\left(\frac{x_1}{\alpha_{12}}\right)^{\xi_1-m}(p_2x_2)^{\xi_2-\xi_1+m} = p_1^{\,m}p_2^{\,\xi_2-\xi_1+m}q_{12}^{\,\xi_1-m}x_1^{\,\xi_1}x_2^{\,\xi_2},$$

and if α_{12} be not a function of x_1 and x_2 the terms involving $x_1^{\xi_1}x_2^{\xi_2}$ can only arise in a manner similar to this.

If, however, α_{12} be such that $\alpha_{12}x_2$ is a multiple of x_1, and consequently x_1/α_{12} a multiple of x_2, we at once get an addition to the coefficient of $x_1^{\xi_1}x_2^{\xi_2}$. In the present case the coefficient becomes

$$(p_1 + cq_{12})^{\xi_1}\left(\frac{1}{c} + p_2\right)^{\xi_2}$$

Hence, considering monomial values of α_{12} only, the inequality

$$\frac{\alpha_{12}x_2}{cx_1} \neq 1$$

must be satisfied in assigning to α_{12} a function of x_1 and x_2.

We may put α_{12}, subject to the above condition, equal to any monomial integral or fractional function of x_1 and x_2.

We may *not* put

$$\alpha_{12} = c\frac{x_1}{x_2},$$

where c is any function of p_1, p_2.

We may not, in fact, realize a portion of the coefficient of $x_1^{\xi_1}x_2^{\xi_2}$ as

$$(p_1x_1)^m(\alpha_{12}q_{12}x_2)^{\xi_1-m}\left(\frac{x_1}{\alpha_{12}}\right)^{\xi_1-n}(p_2x_2)^{\xi_2-\xi_1+n},$$

wherein n differs from m.

Art. 58. Of Order 3, the particular redundant product is

$$(p_1x_1 + \alpha_{12}q_{12}x_2 + c_{13}\alpha_{12}\alpha_{23}q_{13}x_3)^{\xi_1}\left(\frac{x_1}{\alpha_{12}} + p_2x_2 + \alpha_{23}q_{23}\right)^{\xi_2}\left(\frac{c_{13}x_1}{\alpha_{12}\alpha_{23}} + \frac{x_2}{\alpha_{23}} + p_3x_3\right)^{\xi_3},$$

and we must realize the coefficient of

$$x_1^{\xi_1}x_2^{\xi_2}x_3^{\xi_3}$$

in the manner

$$(p_1x_1)^{m_1}\left(\frac{x_1}{\alpha_{12}}\right)^{n_1}\left(\frac{x_1}{\gamma_{13}\alpha_{12}\alpha_{23}}\right)^{\xi_1-m-n} \times (\alpha_{12}q_{12}x_2)^{m_2}(p_2x_2)^{n_2}\left(\frac{x_2}{\alpha_{23}}\right)^{\xi_2-m_2-n_2}$$
$$\times \text{ a multiple of } x_3^{\xi_3},$$

where, of the three portions, the first accounts wholly for $x_1^{\xi_1}$, the second wholly for $x_2^{\xi_2}$, and so on; and not in any other manner.

Put

$$(\alpha_{12},\ \alpha_{23}) = (\phi_1,\ \phi_2),$$

where ϕ_1, ϕ_2 are fractions of x_1, x_2, x_3, and consider the simplified matrix,

$$\left(\begin{array}{ccc} x_1, & \phi_1x_2, & \phi_1\phi_2x_3 \\ x_1\ \ , & x_2, & \phi_2x_3 \\ \dfrac{x_1}{\phi_1\phi_2}, & \dfrac{x_2}{\phi_2}, & x_3 \end{array}\right|,$$

in which unnecessary quantities are omitted.

Further, omitting a multiplier, independent of x_1, x_2, x_3, on the right-hand sides, the following six inequalities must be satisfied,

$$\phi_1^2\phi_2 \neq \frac{x_1^2}{x_2x_3}, \qquad \frac{\phi_2}{\phi_1} \neq \frac{x_2^2}{x_1x_3}, \qquad \frac{1}{\phi_1\phi_2^2} \neq \frac{x_3^2}{x_1x_2},$$

$$\phi_2 \neq \frac{x_2}{x_3}, \qquad \frac{1}{\phi_1\phi_2} \neq \frac{x_3}{x_1}, \qquad \phi_1 \neq \frac{x_1}{x_2},$$

putting

$$\Phi_1 = \phi_1\frac{x_2}{x_1}, \qquad \Phi_2 = \phi_2\frac{x_3}{x_2};$$

these conditions are representable by the single inequality

$$\Phi_1^3\Phi_2 + \frac{1}{\Phi_1^3\Phi_2} + \frac{\Phi_2^2}{\Phi_1} + \frac{\Phi_1}{\Phi_2^2} + \Phi_1^2\Phi_2^3 + \frac{1}{\Phi_1^2\Phi_2^3}$$
$$\neq \Phi_2^3\Phi_1 + \frac{1}{\Phi_2^3\Phi_1} + \frac{\Phi_1^2}{\Phi_2} + \frac{\Phi_2}{\Phi_1^2} + \Phi_2^2\Phi_1^3 + \frac{1}{\Phi_2^2\Phi_1^3}.$$

As regards functions of x_1, x_2, x_3, this inequality being satisfied, ϕ_1 and ϕ_2 may be put equal to any functions that may be desired. Like inequalities may be obtained in respect of the fourth and higher orders.

Art. 59. The important point to notice is that it is legitimate to put the undetermined quantities equal to any *integral* functions of $x_1, x_2, \ldots x_n$—a fact, for the general order, that becomes obvious on examination of the above processes.

As subsequently appears, it is such integral functions that usually present themselves in arithmetical applications.

Art. 60. As an example of the applications to arithmetic which swarm about the theory, consider the important condensed form (*vide* Art. 14):—

$$\frac{1}{\left[\begin{array}{l} 1 - \Sigma x_1 - \Sigma (\lambda_{\beta\alpha} - 1) x_\alpha x_\beta - \Sigma (\lambda_{\beta\alpha} - 1)(\lambda_{\gamma\beta} - 1) x_\alpha x_\beta x_\gamma \\ \quad - \ldots - (\lambda_{21} - 1)(\lambda_{32} - 1)(\lambda_{23} - 1) \ldots (\lambda_{n,n-1} - 1) x_1 x_2 x_3 \ldots x_n \end{array}\right]}$$

and, at first, consider the form of Order 3.

The matrix of the redundant form is easily found to be either

$$\begin{pmatrix} 1 & \alpha_{12}\lambda_{21} & \dfrac{\beta_{13}\lambda_{31}}{c_{13}} \\ \dfrac{1}{\alpha_{12}} & 1 & \alpha_{23}\lambda_{32} \\ \dfrac{c_{13}}{\beta_{13}} & \dfrac{1}{\alpha_{23}} & 1 \end{pmatrix}$$

or the similar matrix with c_{31} written for c_{13}. Since

$$c_{13} = \frac{\lambda_{31}}{\lambda_{21}\lambda_{32}}, \qquad c_{31} = 1,$$

we have, taking c_{31} and putting $(\alpha_{12}, \alpha_{23}) = (1, 1)$ a particular redundant product

$$(x_1 + \lambda_{21}x_2 + \lambda_{31}x_3)^{\xi_1} (x_1 + x_2 + \lambda_{32}x_3)^{\xi_2} (x_1 + x_2 + x_3)^{\xi_3}.$$

In this, the coefficient of $x_1^{\xi_1}x_2^{\xi_2}x_3^{\xi_3}$ (which is equal to the coefficient of the same term in the condensed form) is arithmetically interpretable as in Art. 15.

Art. 61. If, however, we put (*vide* Art. 59)

$$(\alpha_{12}, \alpha_{23}\,;\, c_{13}{}^{-1}) = \left(x_1, x_2\,;\, \frac{\lambda_{32}\lambda_{21}}{\lambda_{31}}\right),$$

we obtain a form which may be written:—

$$(x_1 + \lambda_{21}x_2x_1 + \lambda_{32}\lambda_{21}x_3x_2x_1)^{\xi_1} (1 + x_2 + \lambda_{32}x_3x_2)^{\xi_2} \left(\frac{\lambda_{31}x_3x_1}{\lambda_{32}\lambda_{21}x_3x_2x_1} + 1 + x_3\right)^{\xi_3},$$

and herein we see that the coefficient of

$$\lambda_{21}^{s_{21}}\lambda_{31}^{s_{31}}\lambda_{32}^{s_{32}}x_1^{\xi_1}x_2^{\xi_2}x_3^{\xi_3}$$

represents the number of permutations of the symbols in

$$x_1^{\xi_1}x_2^{\xi_2}x_3^{\xi_3},$$

which possess exactly

$$\begin{array}{lll} s_{21}, & x_2x_1 & \text{contacts} \\ s_{31}, & x_3x_1 & \quad ,, \\ s_{32}, & x_3x_2 & \quad ,, \end{array}$$

Here is an entirely new interpretation and we see that the true generating function for the enumeration of the indicated permutations is

$$\frac{1}{1 - x_1 - x_2 - x_3 - (\lambda_{21} - 1)\,x_1x_2 - (\lambda_{31} - 1)\,x_1x_3 - (\lambda_{32} - 1)\,x_2x_3 - (\lambda_{21} - 1)\,(\lambda_{32} - 1)\,x_1x_2x_3},$$

a result which does not lie by any means on the surface.

The arithmetical correspondence should also be noted.

Art. 62. For the order n we have the matrix

$$\left(\begin{array}{cccccc} 1 & \alpha_{12}\lambda_{21} & \dfrac{\beta_{13}\lambda_{31}}{c_{13}} & \dfrac{\beta_{14}\lambda_{41}}{c_{14}} & \ldots & \dfrac{\beta_{1n}\lambda_{n1}}{c_{1n}} \\ \dfrac{1}{\alpha_{12}} & 1 & \alpha_{23}\lambda_{32} & \dfrac{\beta_{24}\lambda_{42}}{c_{24}} & \ldots & \dfrac{\beta_{2n}\lambda_{n2}}{c_{2n}} \\ \dfrac{c_{13}}{\beta_{13}} & \dfrac{1}{\alpha_{23}} & 1 & \alpha_{34}\lambda_{43} & \ldots & \dfrac{\beta_{3n}\lambda_{n3}}{c_{3n}} \\ \dfrac{c_{14}}{\beta_{14}} & \dfrac{c_{24}}{\beta_{24}} & \dfrac{1}{\alpha_{34}} & 1 & \ldots & \dfrac{\beta_{4n}\lambda_{n4}}{c_{4n}} \\ \cdot & \cdot & \cdot & \cdot & \ldots & \cdot \\ \cdot & \cdot & \cdot & \cdot & \ldots & \cdot \\ \dfrac{c_{1n}}{\beta_{1n}} & \dfrac{c_{2n}}{\beta_{2n}} & \dfrac{c_{3n}}{\beta_{3n}} & \dfrac{c_{4n}}{\beta_{4n}} & \ldots & 1 \end{array}\right)$$

and we obtain another form by writing c_{yx} for c_{xy}.

Moreover $(y > x)$ we have

$$c_{yx} = 1, \quad c_{xy} = \lambda_{yx}/\mu_{yx},$$

where

$$\mu_{yx} = \lambda_{y,\,y-1}\lambda_{y-1,\,y-2}\ldots\lambda_{x+1,\,x};$$

whence writing

$$(\alpha_{xy},\ c_{yx}) = (1,\ 1)$$

we obtain the matrix

$$\begin{pmatrix} 1 & \lambda_{21} & \lambda_{31} & . & . & \lambda_{n1} \\ 1 & 1 & \lambda_{32} & . & . & \lambda_{n2} \\ 1 & 1 & 1 & . & . & \lambda_{n3} \\ . & . & . & . & . & . \\ 1 & 1 & 1 & . & . & 1 \end{pmatrix}$$

and we can interpret the coefficient of $x_1^{\xi_1} x_2^{\xi_2} \ldots x^{\xi_n}$ in the corresponding particular redundant product as in Art. 15.

Again, writing

$$(\alpha_{p,p+1},\ c_{xy}) = \left(x_p,\ \frac{\lambda_{yx}}{\mu_{yx}}\right),$$

which, as far as $\alpha_{p,p+1}$ is concerned, Art. 59 shows to be legitimate, we have

$$\beta_{p,q} = x_p x_{p+1} \ldots x_{q-1} = \mathrm{X}_{p,q-1} = \mathrm{X}_{q-1,p} \text{ suppose,}$$

and the matrix

$$\begin{pmatrix} 1 & \lambda_{21}\, x_1 & \mu_{31}\, \mathrm{X}_{12} & \mu_{41}\, \mathrm{X}_{13} & . & . & \mu_{n1}\, \mathrm{X}_{1,\,n-1} \\ \dfrac{1}{x_1} & 1 & \lambda_{32}\, x_2 & \mu_{42}\, \mathrm{X}_{23} & . & . & \mu_{n2}\, \mathrm{X}_{2,\,n-1} \\ \dfrac{\lambda_{31}}{\mu_{31}\, \mathrm{X}_{12}} & \dfrac{1}{x_2} & 1 & \lambda_{43}\, x_3 & . & . & \mu_{n3}\, \mathrm{X}_{3,\,n-1} \\ \dfrac{\lambda_{41}}{\mu_{41}\, \mathrm{X}_{13}} & \dfrac{\lambda_{42}}{\mu_{42}\, \mathrm{X}_{23}} & \dfrac{1}{x_3} & 1 & . & . & \mu_{n4}\, \mathrm{X}_{4,\,n-1} \\ . & . & . & . & . & . & . \\ \dfrac{\lambda_{n1}}{\mu_{n1}\, \mathrm{X}_{1,\,n-1}} & \dfrac{\lambda_{n2}}{\mu_{n2}\, \mathrm{X}_{2,\,n-1}} & \dfrac{\lambda_{n3}}{\mu_{n3}\, \mathrm{X}_{3,\,n-1}} & \dfrac{\lambda_{n4}}{\mu_{n4}\, \mathrm{X}_{4,\,n-1}} & . & . & 1 \end{pmatrix}$$

and the new particular redundant product is :—

$$\begin{array}{l} \left(x_1 + \lambda_{21}\, x_2 x_1 + \mu_{31}\, \mathrm{X}_{31} + \mu_{41}\, \mathrm{X}_{41} + \ldots + \mu_{n1}\, \mathrm{X}_{n1} \right)^{\xi_1} \\ \left(1 + x_2 + \lambda_{32}\, x_3 x_2 + \mu_{42}\, \mathrm{X}_{42} + \ldots + \mu_{n2}\, \mathrm{X}_{n2} \right)^{\xi_2} \\ \left(\dfrac{\lambda_{31}}{\mu_{31}\, x_2} + 1 + x_3 + \lambda_{43}\, x_4 x_3 + \ldots + \mu_{n3}\, \mathrm{X}_{n3} \right)^{\xi_3} \\ \left(\dfrac{\lambda_{41}}{\mu_{41}\, \mathrm{X}_{32}} + \dfrac{\lambda_{42}}{\mu_{42}\, x_3} + 1 + x_4 + \ldots + \mu_{n4}\, \mathrm{X}_{n4} \right)^{\xi_4} \\ \left(\quad . \qquad . \qquad . \qquad . \qquad \ldots \qquad . \quad \right) \\ \left(\dfrac{\lambda_{n1}}{\mu_{n1}\, \mathrm{X}_{n-1,\,2}} + \dfrac{\lambda_{n2}}{\mu_{n2}\, \mathrm{X}_{n-1,\,3}} + \dfrac{\lambda_{n3}}{\mu_{n3}\, \mathrm{X}_{n-1,\,4}} + \dfrac{\lambda_{n4}}{\mu_{n4}\, \mathrm{X}_{n-1,\,5}} + \ldots + x_n \right)^{\xi_n} \end{array}$$

Art. 63. In this product we may interpret the coefficient of

$$x_1^{\xi_1}x_2^{\xi_2}\ldots x_n^{\xi_n}.$$

From the nature of the condensed form we know that this coefficient is an integral function of the quantities λ_{xy}. We may prove that if a portion of the expansion be

$$c\lambda_{21}^{s_{21}}\lambda_{32}^{s_{32}}\ldots\lambda_{qp}^{s_{qp}}\,x_1^{\xi_1}x_2^{\xi_2}\ldots x_n^{\xi_n},$$

the number c indicates the number of permutations of the $\Sigma\xi$ quantities in

$$x_1^{\xi_1}x_2^{\xi_2}\ldots x_n^{\xi_n},$$

which possess exactly s_{21} contacts x_2x_1

$$\begin{array}{ccc} s_{32} & ,, & x_3x_2 \\ \vdots & ,, & \vdots \\ s_{qp} & ,, & x_qx_p \end{array}$$

Regard the above product, as written, as being a square form of n rows and n columns involving n^2 elements.

Observe that if $s \nless t$ the element common to the s^{th} row and t^{th} column is

$$\frac{\lambda_{st}}{\mu_{st}}\cdot\frac{x_t}{X_{t,\,s-1}}$$

while the element common to the t^{th} row and s^{th} column is

$$\mu_{st}X_{st},$$

and that the product of these two elements is

$$\lambda_{st}x_sx_t.$$

Now, take a particular permutation of the $\Sigma\xi$ quantities and observe how it may be considered to arise in the multiplication. Let a portion of the permutation be

$$x_2 \mid x_4x_2x_1 \mid x_8x_5x_3x_2x_1 \mid x_5 \mid x_5\ldots$$

divided off by bars into compartments in such wise that in any compartment the suffixes are in descending order.

The portion is a permutation of

$$\ldots x_1^2 x_2^3 x_3 x_4 x_5^3 x_8 \ldots$$

and we can obtain this portion by selecting for multiplication

2	elements	from	the row	appertaining	to	the exponent		ξ_1
3	,,	,,	,,	,,	,,	,,		ξ_2
1	,,	,,	,,	,,	,,	,,		ξ_3
1	,,	,,	,,	,,	,,	,,		ξ_4
3	,,	,,	,,	,,	,,	,,		ξ_5
1	,,	,,	,,	,,	,,	,,		ξ_8

The permutation is divided into five compartments as shown.

In the first compartment we have simply x_2 which is to be taken from the 2nd row 2nd column. In the second compartment we have

$$x_4 x_2 x_1$$

which is obtainable by multiplication of elements taken from the 4th, 2nd, and 1st rows, as follows :—

In row 4, column 2, we take $\dfrac{\lambda_{42}}{\mu_{42}} \cdot \dfrac{1}{x_3}$

,, 2, ,, 1, ,, 1

,, 1, ,, 4, ,, $\mu_{41} x_4 x_3 x_2 x_1$;

multiplication gives

$$\lambda_{42} \lambda_{21} x_4 x_2 x_1.$$

In the third compartment we find

$$x_8 x_5 x_3 x_2 x_1$$

From row 8, column 5, we take $\dfrac{\lambda_{85}}{\mu_{85}} \dfrac{1}{x_7 x_6}$.

,, 5, ,, 3, ,, $\dfrac{\lambda_{53}}{\mu_{53}} \cdot \dfrac{1}{x_4}$.

,, 3, ,, 2, ,, 1.

,, 2, ,, 1, ,, 1.

,, 1, ,, 8, ,, $\mu_{81} x_8 x_7 x_6 x_5 x_4 x_3 x_2 x_1$.

Multiplication of these five elements yields

$$\lambda_{85} \lambda_{53} \lambda_{32} \lambda_{21} x_8 x_5 x_3 x_2 x_1.$$

In the fourth and fifth compartments we have simply x_5, and in each case the element selected is that in the 5^{th} row and 5^{th} column. Altogether we have obtained the product

$$\lambda_{85}\lambda_{53}\lambda_{42}\lambda_{32}\lambda^2_{21}x_2x_4x_2x_1x_8x_5x_3x_2x_1x_5x_5,$$

and we observe that the contacts

$$x_qx_p \quad (q > p)$$

are correctly indicated by the quantities

$$\lambda_{qp}.$$

Art. 64. The process is obviously a general one, and the rule of element selection to demonstrate the desired result may be set forth as follows :—

If a compartment of the permutation be

$$x_ax_bx_cx_dx_e,$$

a, b, c, d, e being in descending order of magnitude, we take elements in

$$\begin{array}{llll} \text{row } a, & \text{column } b, \\ \;,, \; b, & \;\;,, \; c, \\ \;,, \; c, & \;\;,, \; d, \\ \;,, \; d, & \;\;,, \; e, \\ \;,, \; e, & \;\;,, \; a, \end{array}$$

and thus obtain the product,

$$\lambda_{ab}\lambda_{bc}\lambda_{cd}\lambda_{de}x_ax_bx_cx_dx_e,$$

wherein the contacts are correctly represented by the quantities λ.

If a compartment contain the single quantity x_s, we take the element in the s^{th} row and s^{th} column.

By the above process

$$\begin{array}{lllll} \xi_1 & \text{elements are taken from row} & & & 1, \\ \xi_2 & ,, & ,, & ,, & 2, \\ \vdots & & & & \vdots \\ \xi_n & ,, & ,, & ,, & n, \end{array}$$

to form the product

$$x_1^{\xi_1}x_2^{\xi_2}\,.\,.\,.\,x_n^{\xi_n}.$$

Art. 65. Hence it has been established that the coefficient of the term

$$\lambda_{21}^{s_{21}} \lambda_{32}^{s_{32}} \ldots \lambda_{qp}^{s_{qp}} x_1^{\xi_1} x_2^{\xi_2} \ldots x_n^{\xi_n},$$

in the product, enumerates the permutations of the $\Sigma\xi$ quantities in

$$x_1^{\xi_1} x_2^{\xi_2} \ldots x_n^{\xi_n},$$

which possess exactly

$$\begin{array}{lcl} s_{21} & \text{contacts} & x_2 x_1, \\ s_{32} & ,, & x_3 x_2, \\ \cdot & \cdot & \cdot \\ s_{qp} & ,, & x_q x_p ; \end{array}$$

and since the redundant product can assume the appearance derived from the matrix

$$\left(\begin{array}{cccc} 1 & \lambda_{21} & \ldots & \lambda_{n1} \\ 1 & 1 & \ldots & \lambda_{n2} \\ \cdot & \cdot & \ldots & \cdot \\ 1 & 1 & \ldots & 1 \end{array}\right)$$

we find that the enumeration is identical with that of the permutations which are such that the quantity x_q occurs s_{qp} times in places originally occupied by the quantity x_p, when $q > p$, and, as before, we take the coefficient of

$$\lambda_{21}^{s_{21}} \lambda_{32}^{s_{32}} \ldots \lambda_{qp}^{s_{qp}} x_1^{\xi_1} x_2^{\xi_2} \ldots x_n^{\xi_n}.$$

Hence, an arithmetical correspondence, and, also, the fact that the true generating function for the enumeration of these permutations is

$$\frac{1}{\left[\begin{array}{l} 1 - \Sigma x_1 - \Sigma(\lambda_{\beta\alpha} - 1) x_\alpha x_\beta - \Sigma(\lambda_{\beta\alpha} - 1)(\lambda_{\gamma\beta} - 1) x_\alpha x_\beta x_\gamma \\ - \ldots - (\lambda_{21} - 1)(\lambda_{32} - 1) \ldots (\lambda_{n,n-1} - 1) x_1 x_2 x_3 \ldots x_{n-1} x_n \end{array}\right]}.$$

The above example is only a solitary one of a large number that might be furnished. An advantageous method for procedure appears to be to take some simple interpretable redundant product, and to then pass through the condensed form to the general redundant product, involving $n - 1$ undetermined quantities as well as quantities c_{xy}, which admit of a choice of values. The assignment of these quantities then leads to

a variety of arithmetical correspondences which, as before remarked, is absolutely limitless.

The theory, moreover, includes an exhaustive Theory of Permutations, and gives in every case the true condensed Generating Functions. Its importance in the General Theory of Determinants has been touched upon.

In conclusion, the paper will have achieved its object if it is successful in indicating the arithmetical and algebraical power of the main theorem considered.