VIII. *The Rotation of an Elastic Spheroid.*

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Introduction.

IT is well known that, if a rigid body whose principal moments of inertia are $\mathfrak{A}$, $\mathfrak{A}$, $\mathfrak{C}$ be set rotating about its axis of symmetry, and then be subjected to a slight disturbance, it will execute oscillations about its mean position, in consequence of which the axis of rotation will undergo periodic displacements relatively to the body in a period which bears to the period of rotation the ratio $\mathfrak{A} : \mathfrak{C} - \mathfrak{A}$. The object of the present investigation is to determine to what extent this period will be modified if the body, instead of being perfectly rigid, is capable of elastic deformations.

The problem has important bearings in connection with the theory of the Earth's rotation. The remarkable researches of Dr. S. C. CHANDLER, published in a series of papers in the 'Astronomical Journal,'* have placed it almost beyond a doubt that the axis of rotation of the Earth is subject to a series of displacements, the most important of which consists of a periodic motion in all respects similar in character to the oscillation mentioned above, but having a period considerably in excess of that which theory would require if the Earth could be regarded as perfectly rigid. It is natural to suppose that this motion has its origin in the same cause, but that the theory by which the period has previously been assigned is in some respects defective. The most plausible attempt which has yet been made to correct this theory is that given by NEWCOMB,† who shows, by an elegant geometrical method, that the elasticity of the solid portions of the Earth and the mobility of the ocean will each have the effect of prolonging the period. He then proceeds to obtain a numerical estimate of this extension, basing his calculations on certain results given by THOMSON and TAIT‡ with reference to the deformation of an elastic sphere. In order to make these results applicable, several assumptions have to be made which do not appear to me

* Vol. 11, *et seq.* For a summary of CHANDLER'S results, *vide* 'Science,' May 3, 1895.

† 'Monthly Notices of the Royal Astronomical Society,' March, 1892.

‡ 'Natural Philosophy,' Part II., § 837.

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to be well founded; and it is with a view to examining these assumptions that I have attempted to exhibit the solution of the problem in an analytical form.

The analysis in the present paper is confined to the case of a homogeneous spheroid of revolution, composed of isotropic, incompressible, gravitating material, while no account is taken of the surface waters. For further simplification we have also supposed that the figure conforms to that required for hydrostatic equilibrium, so that when the body is undisturbed we may suppose it free from strain in its interior. We have reason to suppose that this condition is approximately realized in the case of the Earth.

In §§ 1–2, I have obtained the rigorous dynamical equations for the oscillations of such a system. The form of these equations is, however, such as to render an approximate solution necessary. Hence it has been assumed in the subsequent sections that the ellipticity of the spheroid and, consequently, the angular velocity of rotation, are small quantities. This assumption leads to a considerable reduction in the differential equations which express the elastic displacements, and, in fact, reduces them to the familiar equations for the equilibrium of a strained elastic body The boundary conditions are also similar in form to the surface equations which are obtained in treating of the problem of the deformations of an elastic sphere, and as the solution of this problem is well known, we are in a position to obtain a solution of our differential equations applicable to the problem in hand. §§ 3–4 are devoted to the transformation of the equations into a form convenient for solution, while the actual solution is given in § 5.

The method of approximation followed up to this point fails to lead to a determination of the period. When, however, the body is supposed perfectly rigid, we are able to determine the period accurately by means of the equations of angular momentum for the whole system. This suggests the use of an analogous process, which is employed in § 6, to determine the period when elastic deformations are taken into account.

The principal results of this paper will be found in § 8, where they are compared with the hypotheses made by NEWCOMB. It is found that the general character of the motion agrees with that assumed by NEWCOMB, but that his quantitative law as to the displacement of the pole, due to elastic distortion, is slightly in error. The bearing of these results on the theory of the Earth's constitution is discussed in the final section (§ 9).

§ 1. *Differential Equations of Motion of Isotropic, Incompressible, Elastic Solid, referred to axes rotating uniformly.*

Take as axis of z the axis of rotation, and let the system be rotating with angular velocity ω about this axis. Let $x_0 + u$, $y_0 + v$, $z_0 + w$ be the coordinates at the time

t of the particle, which, when the body is unstrained, is at the point x_0, y_0, z_0. The velocity components of this particle will be

$$
\begin{aligned}
\mathrm{U} &= \frac{d}{dt}(x_0 + u) - \omega(y_0 + v) = \dot{u} - \omega(y_0 + v), \\
\mathrm{V} &= \frac{d}{dt}(y_0 + v) + \omega(x_0 + u) = \dot{v} + \omega(x_0 + u), \\
\mathrm{W} &= \frac{d}{dt}(z_0 + w) \qquad\qquad\qquad = \dot{w}.
\end{aligned}
$$

The components of acceleration will be

$$
\begin{aligned}
\dot{\mathrm{U}} - \mathrm{V}\omega &= \ddot{u} - \dot{v}\omega - \{\dot{v} + \omega(x_0 + u)\}\,\omega = \ddot{u} - 2\omega\dot{v} - \omega^2 u - \omega^2 x_0, \\
\dot{\mathrm{V}} + \mathrm{U}\omega &= \ddot{v} + \dot{u}\omega + \{\dot{u} - \omega(y_0 + v)\}\,\omega = \ddot{v} + 2\omega\dot{u} - \omega^2 v - \omega^2 y_0, \\
\dot{\mathrm{W}} &\qquad\qquad\qquad\qquad\qquad\qquad\qquad = \ddot{w}.
\end{aligned}
$$

If P, Q, R, S, T, U denote the six components of stress at the point x, y, z (where $x = x_0 + u$, &c.), X, Y, Z the components of bodily force at this point, and ρ the density of the material, the equations of motion may be written down in the same manner as if the axes were fixed, provided that we replace the accelerations d^2u/dt^2, d^2v/dt^2, d^2w/dt^2 by the values we have found above. Thus we have*

$$
\begin{aligned}
\frac{\partial \mathrm{P}}{\partial x} + \frac{\partial \mathrm{U}}{\partial y} + \frac{\partial \mathrm{T}}{\partial z} + \rho\mathrm{X} &= \rho(\ddot{u} - 2\omega\dot{v} - \omega^2 u - \omega^2 x_0), \\
\frac{\partial \mathrm{U}}{\partial x} + \frac{\partial \mathrm{Q}}{\partial y} + \frac{\partial \mathrm{S}}{\partial z} + \rho\mathrm{Y} &= \rho(\ddot{v} + 2\omega\dot{u} - \omega^2 v - \omega^2 y_0), \\
\frac{\partial \mathrm{T}}{\partial x} + \frac{\partial \mathrm{S}}{\partial y} + \frac{\partial \mathrm{R}}{\partial z} + \rho\mathrm{Z} &= \rho\ddot{w}.
\end{aligned}
$$

We have here employed two different sets of independent variables. The quantities involved on the left have been supposed to be expressed as functions of the variables x, y, z, t, that is to say, the coordinates of a definite point on which we fix our attention and the time; these variables are analogous to the Eulerian system in Hydrodynamics. On the other hand, the quantities u, v, w on the right have been regarded as functions of x_0, y_0, z_0, t, which correspond to the Lagrangian system of independent variables in Hydrodynamics. It is desirable for us to retain only one set of variables; we propose to select the former set and proceed to examine the modified form of the equations of small motion.

If the symbol d/dt be used as above to denote partial differentiation with respect

* LOVE, 'Elasticity,' vol. 1, p. 60.

to the time, on the supposition that x_0, y_0, z_0, remain constant, and $\partial/\partial t$ be used when x, y, z, are the other independent variables, we have

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \frac{du}{dt}\frac{\partial}{\partial x} + \frac{dv}{dt}\frac{\partial}{\partial y} + \frac{dw}{dt}\frac{\partial}{\partial z}.$$

Hence, if we neglect squares and products of the small quantities u, v, w, we obtain

$$\dot{u} = du/dt = \partial u/\partial t, \qquad \ddot{u} = d^2u/dt^2 = \partial^2 u/\partial t^2, \text{ \&c.}$$

Thus, the only modification necessary on the right will be to replace $x_0 + u$ by x, and $y_0 + v$ by y, and the equations of motion become

$$\left.\begin{aligned} \frac{\partial P}{\partial x} + \frac{\partial U}{\partial y} + \frac{\partial T}{\partial z} + \rho X &= \rho\,(\ddot{u} - 2\omega\dot{v} - \omega^2 x) \\ \frac{\partial U}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial S}{\partial z} + \rho Y &= \rho\,(\ddot{v} + 2\omega\dot{u} - \omega^2 y) \\ \frac{\partial T}{\partial x} + \frac{\partial S}{\partial y} + \frac{\partial R}{\partial z} + \rho Z &= \rho\ddot{w} \end{aligned}\right\} \quad . \; . \; . \; . \; . \; . \quad (1).$$

We have retained the fluxional notation, it being understood that the dots now denote differentiation with regard to the time on the supposition that x, y, z remain constant.

Since the stress-strain relations do not involve differentiation with regard to the time, they may be written down in the same manner as when the axes are fixed. To the degree of approximation to which we are going, we may replace $\partial u/\partial x_0$, &c., by $\partial u/\partial x$, &c., and therefore when the material is isotropic and incompressible, the components of stress are given by

$$\left.\begin{aligned} P &= -p + 2\mathfrak{n}\frac{\partial u}{\partial x_0} = -p + 2\mathfrak{n}\frac{\partial u}{\partial x} \\ Q &= -p + 2\mathfrak{n}\frac{\partial v}{\partial y_0} = -p + 2\mathfrak{n}\frac{\partial v}{\partial y} \\ R &= -p + 2\mathfrak{n}\frac{\partial w}{\partial z_0} = -p + 2\mathfrak{n}\frac{\partial w}{\partial z} \\ S &= \mathfrak{n}\left(\frac{\partial w}{\partial y_0} + \frac{\partial v}{\partial z_0}\right) = \mathfrak{n}\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right) \\ T &= \mathfrak{n}\left(\frac{\partial u}{\partial z_0} + \frac{\partial w}{dx_0}\right) = \mathfrak{n}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) \\ U &= \mathfrak{n}\left(\frac{\partial v}{\partial x_0} + \frac{\partial u}{\partial y_0}\right) = \mathfrak{n}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) \end{aligned}\right\} \quad . \; . \; . \; . \; . \; . \; . \quad (2),$$

where p denotes the hydrostatic pressure at x, y, z, and $\mathfrak{n}$ the rigidity.

If further, the bodily forces be derivable from a potential function V, so that

$$X = \partial V/\partial x, \quad Y = \partial V/\partial y, \quad Z = \partial V/\partial z \quad . \quad . \quad . \quad . \quad . \quad . \quad (3),$$

on substituting the values (2), (3), in (1), we obtain

$$\frac{\mathfrak{n}}{\rho}\nabla^2 u + \frac{\partial}{\partial x}\left\{V + \tfrac{1}{2}\omega^2(x^2+y^2) - \frac{p}{\rho}\right\} = \ddot{u} - 2\omega\dot{v},$$

$$\frac{\mathfrak{n}}{\rho}\nabla^2 v + \frac{\partial}{\partial y}\left\{V + \tfrac{1}{2}\omega^2(x^2+y^2) - \frac{p}{\rho}\right\} = \ddot{v} + 2\omega\dot{u},$$

$$\frac{\mathfrak{n}}{\rho}\nabla^2 w + \frac{\partial}{\partial z}\left\{V + \tfrac{1}{2}\omega^2(x^2+y^2) - \frac{p}{\rho}\right\} = \ddot{w}.$$

Putting $\mathfrak{n}/\rho = n, \quad V + \tfrac{1}{2}\omega^2(x^2+y^2) - p/\rho = \psi$, these equations take the form

$$\left.\begin{aligned} n\nabla^2 u + \partial\psi/\partial x &= \ddot{u} - 2\omega\dot{v}, \\ n\nabla^2 v + \partial\psi/\partial y &= \ddot{v} + 2\omega\dot{u}, \\ n\nabla^2 w + \partial\psi/\partial z &= \ddot{w}. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4).$$

We must in addition express the fact that the material is incompressible; this is done by the equation

$$\partial u/\partial x + \partial v/\partial y + \partial w/\partial z = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5).$$

The equations (4), (5), which are the rigorous equations for the vibrations of a rotating, incompressible, elastic solid, are theoretically sufficient to determine u, v, w, ψ, subject to certain boundary conditions. Eliminating v, u in turn from the first two of equations (4), we obtain

$$\left[\left(n\nabla^2 - \frac{\partial^2}{\partial t^2}\right)^2 + 4\omega^2\frac{\partial^2}{\partial t^2}\right]u = -\left(n\nabla^2 - \frac{\partial^2}{\partial t^2}\right)\frac{\partial\psi}{\partial x} + 2\omega\frac{\partial}{\partial t}\left(\frac{\partial\psi}{\partial y}\right),$$

$$\left[\left(n\nabla^2 - \frac{\partial^2}{\partial t^2}\right)^2 + 4\omega^2\frac{\partial^2}{\partial t^2}\right]v = -\left(n\nabla^2 - \frac{\partial^2}{\partial t^2}\right)\frac{\partial\psi}{\partial y} - 2\omega\frac{\partial}{\partial t}\left(\frac{\partial\psi}{\partial x}\right).$$

Applying the operators $\partial/\partial x, \partial/\partial y$, and making use of (5),

$$\left[\left(n\nabla^2 - \frac{\partial^2}{\partial t^2}\right)^2 + 4\omega^2\frac{\partial^2}{\partial t^2}\right]\frac{\partial w}{\partial z} = \left(n\nabla^2 - \frac{\partial^2}{\partial t^2}\right)\left(\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2}\right).$$

Hence, on eliminating w by means of the third of (4), we obtain the following equation for ψ:—

$$\left[\nabla^2\left(n\nabla^2 - \frac{\partial^2}{\partial t^2}\right)^2 + 4\omega^2\frac{\partial^4}{\partial t^2.\partial z^2}\right]\psi = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6).$$

In the case where n is zero, it will be noticed that the last equation reduces to POINCARÉ's differential equation for the oscillations of a rotating mass of liquid.* It may easily be shown by retaining any one of the four quantities u, v, w, ψ, and eliminating the other three, that each one of these quantities is a solution of the equation (6).

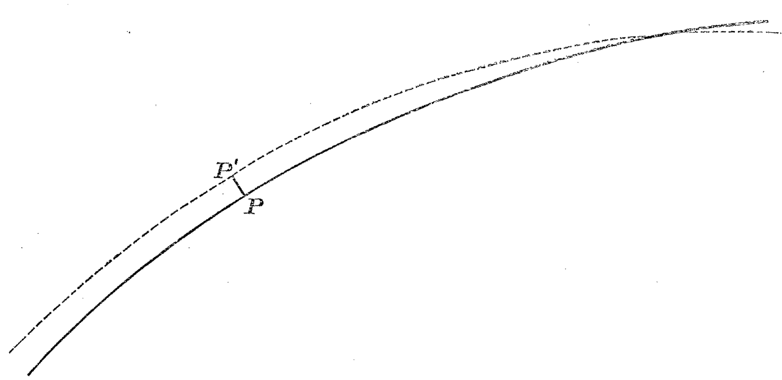
If the motion relatively to the moving axes consists of a simple harmonic vibration in period $2\pi/\lambda$, we may suppose u, v, w, ψ each proportional to $e^{i\lambda t}$, and the equations (4), (5), (6) will then become

$$\left.\begin{array}{l} (n\nabla^2 + \lambda^2)\,u + 2\omega i\lambda v = -\,\partial\psi/\partial x, \\ (n\nabla^2 + \lambda^2)\,v - 2\omega i\lambda u = -\,\partial\psi/\partial y, \\ (n\nabla^2 + \lambda^2)\,w \qquad\quad\;\; = -\,\partial\psi/\partial z, \\ \partial u/\partial x + \partial v/\partial y + \partial w/\partial z = 0, \end{array}\right\} \quad . \quad . \quad . \quad . \quad (7),$$

$$[\nabla^2 (n\nabla^2 + \lambda^2)^2 - 4\omega^2\lambda^2\,\partial^2/\partial z^2]\,\psi = 0 \quad . \quad . \quad . \quad . \quad (8).$$

§ 2. *Boundary Conditions.*

The conditions to be satisfied at the boundary are that the components of surface-traction should vanish at all points of the displaced surface. We proceed to replace these conditions by certain analytical conditions at the mean surface.



Take a point P on the mean surface and let the normal at P meet the surface of the distorted body in P′.

Let $\cos\alpha$, $\cos\beta$, $\cos\gamma$ be the direction-cosines of the normal PP′ and $\cos\alpha + l_1$, $\cos\beta + m_1$, $\cos\gamma + n_1$ the direction-cosines of the normal at P′ to the displaced surface; also let PP′ $= \zeta$. Then l_1, m_1, n_1, ζ will be small quantities of the order of the displacements u, v, w.

Let P, Q, R, S, T, U denote the components of stress at P and the same letters

* 'Acta Mathematica,' vol. 7, p. 356.

accented the components of stress at P′, and let dn' denote an element of the normal to the mean surface. Then to our order of approximation we have

$$P' = P + \zeta\, \partial P/\partial n', \quad Q' = Q + \zeta\, \partial Q/\partial n', \text{ \&c.}$$

Thus the component of surface-traction at P′ parallel to the axis of x is

$$P'(\cos\alpha + l_1) + U'(\cos\beta + m_1) + T'(\cos\gamma + n_1),$$

$$= \left(P + \zeta\frac{\partial P}{\partial n'}\right)(\cos\alpha + l_1) + \left(U + \zeta\frac{\partial U}{\partial n'}\right)(\cos\beta + m_1) + \left(T + \zeta\frac{\partial T}{\partial n'}\right)(\cos\gamma + n_1),$$

$$= P\cos\alpha + U\cos\beta + T\cos\gamma + \zeta\left(\frac{\partial P}{\partial n'}\cos\alpha + \frac{\partial U}{\partial n'}\cos\beta + \frac{\partial T}{\partial n'}\cos\gamma\right)$$
$$+ l_1 P + m_1 U + n_1 T,$$

if we neglect small quantities of the second order.

Now in the small terms we may replace P, Q, R, &c., by their values in the steady motion. Since we have supposed the body when undisturbed to be free from tangential stress in its interior, we have in this case

$$P = Q = R = -p, \quad S = T = U = 0$$

throughout, while at the surface also $p = 0$.

Therefore $l_1 P + m_1 U + n_1 T$, $\partial U/\partial n'$, $\partial T/\partial n'$ may be put equal to zero, and $\dfrac{\partial P}{\partial n'} = -\dfrac{\partial p}{\partial n'} = -\rho\dfrac{\partial}{\partial n'}\{V + \tfrac{1}{2}\omega^2(x^2 + y^2)\} = +\rho g$ say, where g denotes the value of gravity (inclusive of centrifugal force) at the surface. Thus the x-component of surface-traction at P′ is

$$P\cos\alpha + U\cos\beta + T\cos\gamma + \rho g\zeta\cos\alpha.$$

Introducing the values of P, U, T in terms of the displacements and equating this expression to zero, we obtain

$$-p\cos\alpha + \mathfrak{n}\left(\frac{\partial u}{\partial x}\cos\alpha + \frac{\partial u}{\partial y}\cos\beta + \frac{\partial u}{\partial z}\cos\gamma\right) + \mathfrak{n}\left(\frac{\partial u}{\partial x}\cos\alpha + \frac{\partial v}{\partial x}\cos\beta + \frac{\partial w}{\partial x}\cos\gamma\right)$$
$$= -\rho g\zeta\cos\alpha\,;$$

and, in like manner, by considering the components of surface-traction in the directions of the axes of y and z,

$$-p\cos\beta+\mathfrak{n}\left(\frac{\partial v}{\partial x}\cos\alpha+\frac{\partial v}{\partial y}\cos\beta+\frac{\partial v}{\partial z}\cos\gamma\right)+\mathfrak{n}\left(\frac{\partial u}{\partial y}\cos\alpha+\frac{\partial v}{\partial y}\cos\beta+\frac{\partial w}{\partial y}\cos\gamma\right)$$
$$=-\rho g\zeta\cos\beta,$$
$$-p\cos\gamma+\mathfrak{n}\left(\frac{\partial w}{\partial x}\cos\alpha+\frac{\partial w}{\partial y}\cos\beta+\frac{\partial w}{\partial z}\cos\gamma\right)+\mathfrak{n}\left(\frac{\partial u}{\partial z}\cos\alpha+\frac{\partial v}{\partial z}\cos\beta+\frac{\partial w}{\partial z}\cos\gamma\right)$$
$$=-\rho g\zeta\cos\gamma.$$

These equations express the fact that the surface-tractions at the mean surface are equivalent to a normal stress equal to the weight of the harmonic inequality, and they might have been written down at once from this consideration. It has, however, been thought preferable to verify them at length; it will be seen that the shorter procedure is only justifiable in the case where the material is initially in a state of hydrostatic equilibrium. If in the zero configuration this condition is not satisfied the form of the boundary equations will be much more complicated.

Let us now replace p by the function ψ of the previous section. By the definition of ψ we have

$$\begin{aligned} p/\rho &= \mathrm{V}+\tfrac{1}{2}\omega^2(x^2+y^2)-\psi \\ &= \text{non-periodic terms} + v'-\psi, \end{aligned}$$

where v' denotes the potential due to the harmonic inequalities.

The non-periodic terms vanish at the surface in virtue of the conditions for steady motion, and therefore the boundary equations may be written

$$\left.\begin{aligned} &\psi\cos\alpha+n\left\{\frac{\partial u}{\partial x}\cos\alpha+\frac{\partial u}{\partial y}\cos\beta+\frac{\partial u}{\partial z}\cos\gamma+\frac{\partial u}{\partial x}\cos\alpha+\frac{\partial v}{\partial x}\cos\beta+\frac{\partial w}{\partial x}\cos\gamma\right\} \\ &\qquad\qquad= (v'-g\zeta)\cos\alpha \\ &\psi\cos\beta+n\left\{\frac{\partial v}{\partial x}\cos\alpha+\frac{\partial v}{\partial y}\cos\beta+\frac{\partial v}{\partial z}\cos\gamma+\frac{\partial u}{\partial y}\cos\alpha+\frac{\partial v}{\partial y}\cos\beta+\frac{\partial w}{\partial y}\cos\gamma\right\} \\ &\qquad\qquad= (v'-g\zeta)\cos\beta \\ &\psi\cos\gamma+n\left\{\frac{\partial w}{\partial x}\cos\alpha+\frac{\partial w}{\partial y}\cos\beta+\frac{\partial w}{\partial z}\cos\gamma+\frac{\partial u}{\partial z}\cos\alpha+\frac{\partial v}{\partial z}\cos\beta+\frac{\partial w}{\partial z}\cos\gamma\right\} \\ &\qquad\qquad= (v'-g\zeta)\cos\gamma \end{aligned}\right\} \quad (9),$$

where

$$\zeta = u\cos\alpha+v\cos\beta+w\cos\gamma \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10).$$

The rigorous method of procedure would be first to solve the equation (8) for ψ. Having found ψ we could introduce its value into the right-hand members of (7) and proceed to find solutions of these equations consistent with the boundary conditions (9). Unfortunately the form of equation (8) is such as to make it appear hopeless to carry out this process, and we must have recourse to some method of approximation before we can advance further.

§ 3. *Change of Variables.*

An elastic body, such as that with which we are dealing, will, of course, be capable of an infinite number of independent normal types of vibration. The equations of motion we have found in § 1 are applicable to any one of these types, while the substitution of the solutions in the boundary equations should lead to an equation for the determination of the frequencies. In general, the values of u, v, w will become very small when n is large in such a manner that nu, nv, nw approach finite limits, while the admissible values of λ become large of the order $n^{\frac{1}{2}}$. That type of oscillation with which we are concerned is, however, unique in character in that it continues to exist even when the rigidity is perfect. If, then, in the expressions for u, v, w in terms of n we suppose n to be made infinite, u, v, w should approach finite limits which we will denote by u_0, v_0, w_0.

The quantities u_0, v_0, w_0 denote the displacements of a body which is supposed perfectly rigid. Now the most general small displacement of such a body consists of a translation whose components we may denote by ξ, η, ζ, and a rotation whose components we denote by θ_1, θ_2, θ_3. Thus, the most general values of u_0, v_0, w_0 geometrically possible are

$$\xi - y\theta_3 + z\theta_2,\ \eta - z\theta_1 + x\theta_3,\ \zeta - x\theta_2 + y\theta_1.$$

When, however, we are dealing with the rotation of a rigid body not subject to external disturbing force the quantities ξ, η, ζ, θ_3 will not appear, and we may take

$$u_0 = z\theta_2,\ v_0 = -z\theta_1,\ w_0 = -x\theta_2 + y\theta_1. \quad . \quad . \quad . \quad . \quad . \quad (11).$$

Let us now suppose that

$$u = u_0 + u_1,\ v = v_0 + v_1,\ w = w_0 + w_1,$$

where u_0, v_0, w_0 have the values (11). This is equivalent to supposing that the body is first displaced by rotation as a whole through small angles θ_1, θ_2 about Ox, Oy and is then subjected to elastic distortion, the displacements due to the distortions being u_1, v_1, w_1.

Further, let $\zeta = \zeta_0 + \zeta_1$, $v' = v'_0 + v'_1$, where

$$\left.\begin{aligned}\zeta_0 &= u_0 \cos\alpha + v_0 \cos\beta + w_0 \cos\gamma\\ \zeta_1 &= u_1 \cos\alpha + v_1 \cos\beta + w_1 \cos\gamma\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12),$$

and where v'_0, v'_1 denote the parts of v' due to the harmonic inequalities ζ_0, ζ_1 respectively.

If V_0 denote the non-periodic part of V, $\mathrm{V}_0 + v'_0$ will be the potential at the point

x, y, z due to the attraction of the body when it is rotated without distortion through small angles θ_1, θ_2. If then x_1, y_1, z_1 denote the coordinates of this same point referred to axes obtained by rotating the old axes with the body, on putting in evidence the arguments of the function V_0, we have

$$V_0(x, y, z) + v'_0 = V_0(x_1, y_1, z_1) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13).$$

But the direction cosines of the two sets of axes are given by the scheme

	x	y	z
x_1	1	0	$-\theta_2$
y_1	0	1	θ_1
z_1	θ_2	$-\theta_1$	1

whence $x_1 = x - z\theta_2$, $y_1 = y + z\theta_1$, $z_1 = z + x\theta_2 - y\theta_1$.

Introducing these values in (12) and expanding by TAYLOR's theorem, we find

$$\begin{aligned} V_0(x, y, z) + v'_0 &= V_0(x - z\theta_2,\ y + z\theta_1,\ z + x\theta_2 - y\theta_1) \\ &= V_0(x, y, z) - z\theta_2 \frac{\partial V_0}{\partial x} + z\theta_1 \frac{\partial V_0}{\partial y} + (x\theta_2 - y\theta_1) \frac{\partial V_0}{\partial z} \end{aligned}$$

But from the definition of g we have, at the surface,

$$g \cos \alpha = -\frac{\partial V_0}{\partial x} - \omega^2 x, \quad g \cos \beta = -\frac{\partial V_0}{\partial y} - \omega^2 y, \quad g \cos \gamma = -\frac{\partial V_0}{\partial z}$$

and, therefore,

$$\begin{aligned} v'_0 &= z\theta_2 (g \cos \alpha + \omega^2 x) - z\theta_1 (g \cos \beta + \omega^2 y) - (x\theta_2 - y\theta_1) g \cos \gamma \\ &= g (u_0 \cos \alpha + v_0 \cos \beta + w_0 \cos \gamma) + \omega^2 (\theta_2 xz - \theta_1 yz), \end{aligned}$$

or

$$v'_0 - g\zeta_0 = \omega^2 (\theta_2 xz - \theta_1 yz),$$

whence, finally,

$$v' - g\zeta = \omega^2 (\theta_2 xz - \theta_1 yz) + v'_1 - g\zeta_1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

If now we change our variables from u, v, w to u_1, v_1, w_1, the equations of motion become

$$\left.\begin{aligned} n\nabla^2 u_1 + \lambda^2 u_1 + 2\omega i\lambda v_1 &= -\partial\psi/\partial x - \lambda^2 z\theta_2 + 2\omega i\lambda z\theta_1 \\ n\nabla^2 v_1 + \lambda^2 v_1 - 2\omega i\lambda u_1 &= -\partial\psi/\partial y + \lambda^2 z\theta_1 + 2\omega i\lambda z\theta_2 \\ n\nabla^2 w_1 + \lambda^2 w_1 &= -\partial\psi/\partial z + \lambda^2(x\theta_2 - y\theta_1) \\ \partial u_1/\partial x + \partial v_1/\partial y + \partial w_1/\partial z &= 0 \end{aligned}\right\} \quad . \quad . \quad (15),$$

while in virtue of (14) the boundary equations may be written

$$\left.\begin{aligned} &\psi\cos\alpha + n\left\{\frac{\partial u_1}{\partial x}\cos\alpha + \frac{\partial u_1}{\partial y}\cos\beta + \frac{\partial u_1}{\partial z}\cos\gamma + \frac{\partial u_1}{\partial x}\cos\alpha + \frac{\partial v_1}{\partial x}\cos\beta + \frac{\partial w_1}{\partial x}\cos\gamma\right\} \\ &\qquad - (v'_1 - g\zeta_1)\cos\alpha = \omega^2(\theta_2 xz - \theta_1 yz)\cos\alpha, \\ &\psi\cos\beta + n\left\{\frac{\partial v_1}{\partial x}\cos\alpha + \frac{\partial v_1}{\partial y}\cos\beta + \frac{\partial v_1}{\partial z}\cos\gamma + \frac{\partial u_1}{\partial y}\cos\alpha + \frac{\partial v_1}{\partial y}\cos\beta + \frac{\partial w_1}{\partial y}\cos\gamma\right\} \\ &\qquad - (v'_1 - g\zeta_1)\cos\beta = \omega^2(\theta_2 xz - \theta_1 yz)\cos\beta, \\ &\psi\cos\gamma + n\left\{\frac{\partial w_1}{\partial x}\cos\alpha + \frac{\partial w_1}{\partial y}\cos\beta + \frac{\partial w_1}{\partial z}\cos\gamma + \frac{\partial u_1}{\partial z}\cos\alpha + \frac{\partial v_1}{\partial z}\cos\beta + \frac{\partial w_1}{\partial z}\cos\gamma\right\} \\ &\qquad - (v'_1 - g\zeta_1)\cos\gamma = \omega^2(\theta_2 xz - \theta_1 yz)\cos\gamma. \end{aligned}\right\} (16).$$

§ 4. *Reduction of the Equations when the Body is of the form of a Spheroid of Small Ellipticity.*

The only assumption we have made as yet as to the form of the free surface is that it is a possible figure of equilibrium for a rotating mass of liquid, so that the body may be free from strain when rotating uniformly. The simplest form which can occur and that which presents the greatest interest is the case of a spheroid of revolution of small ellipticity ϵ. We propose for the future to confine ourselves to this case. If we neglect the square of ϵ the angular velocity of rotation is related to ϵ by the equation

$$\epsilon = 15\omega^2/16\pi\rho \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17),$$

where the density ρ is expressed in gravitational units. If then the units of length and time be so chosen that ρ is finite, ω will be a small quantity of the order $\epsilon^{\frac{1}{2}}$.

Take as the equation to the free surface $r = a\{1 + \epsilon T_2\}$ where

$$T_2 = (x^2 + y^2 - 2z^2)/3a^2.$$

Then the direction cosines of the normal to the surface $r = a\ \{1 + \epsilon Q_n\}$, where Q_n is a solid harmonic of order n, are

$$\frac{x}{r} + a\epsilon\left\{\frac{nx}{r^2} Q_n - \frac{\partial Q_n}{\partial x}\right\}, \text{ \&c.,}$$

and thus we have

$$\cos\alpha = \frac{x}{r} + a\epsilon\left\{\frac{2x}{r^2} T_2 - \frac{\partial T_2}{\partial x}\right\},$$

$$\cos\beta = \frac{y}{r} + a\epsilon\left\{\frac{2y}{r^2} T_2 - \frac{\partial T_2}{\partial y}\right\},$$

$$\cos\gamma = \frac{z}{r} + a\epsilon\left\{\frac{2z}{r^2} T_2 - \frac{\partial T_2}{\partial z}\right\},$$

whence

$$\begin{aligned}\zeta_0 &= z\theta_2\cos\alpha - z\theta_1\cos\beta - (\theta_2 x - \theta_1 y)\cos\gamma \\ &= a\epsilon\left[\theta_2\left(x\frac{\partial T_2}{\partial z} - z\frac{\partial T_2}{\partial x}\right) + \theta_1\left(z\frac{\partial T_2}{\partial y} - y\frac{\partial T_2}{\partial z}\right)\right] \\ &= a\epsilon\left[-\theta_2 . \frac{2xz}{a^2} + \theta_1 . \frac{2yz}{a^2}\right],\end{aligned}$$

or

$$\zeta_0 = -\frac{2\epsilon}{a}[\theta_2 xz - \theta_1 yz] \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18).$$

Now observation indicates that in the case of the Earth the oscillation in question differs but slightly in type from the motion of a rigid body, whence we conclude that u_1, v_1, w_1 must be small compared with u, v, w. We propose therefore to make the assumptions, leaving the verification thereof to our subsequent work, that u_1, v_1, w_1, ψ contain the small factor ω^2, while λ is a small quantity of the order ω^3. The latter assumption is justifiable when the body is perfectly rigid, since in this case we have

$$\frac{\lambda}{\omega} = \frac{\mathfrak{C} - \mathfrak{A}}{\mathfrak{A}} = \epsilon = \frac{15\omega^2}{16\pi\rho}.$$

If then, we neglect small quantities of the order ω^4 or ϵ^2, the equations of motion (15) reduce to

$$\left.\begin{aligned} n\nabla^2 u_1 = -\frac{\partial\psi}{\partial x},\quad n\nabla^2 v_1 = -\frac{\partial\psi}{\partial y},\quad n\nabla^2 w_1 = -\frac{\partial\psi}{\partial z}, \\ \frac{\partial u_1}{\partial x} + \frac{\partial v_1}{\partial y} + \frac{\partial w_1}{\partial z} = 0 \end{aligned}\right\} \quad . \quad . \quad . \quad (19).$$

Since all the terms in the boundary equations involve the small factor ω^2, we may replace $\cos\alpha$, $\cos\beta$, $\cos\gamma$, by x/r, y/r, z/r respectively, with errors only of the same order ω^4, and thus the approximate form of the boundary equations is

$$\left.\begin{aligned}
&\psi x + n\left(r\frac{\partial}{\partial r} - 1\right)u_1 + n\frac{\partial}{\partial x}(u_1x + v_1y + w_1z) - (v_1' - g\zeta_1)\,x \\
&\qquad = x\,.\,\omega^2(\theta_2xz - \theta_1yz), \\
&\psi y + n\left(r\frac{\partial}{\partial r} - 1\right)v_1 + n\frac{\partial}{\partial y}(u_1x + v_1y + w_1z) - (v_1' - g\zeta_1)\,y \\
&\qquad = y\,.\,\omega^2(\theta_2xz - \theta_1yz), \\
&\psi z + n\left(r\frac{\partial}{\partial r} - 1\right)w_1 + n\frac{\partial}{\partial z}(u_1x + v_1y + w_1z) - (v_1' - g\zeta_1)\,z \\
&\qquad = z\,.\,\omega^2(\theta_2xz - \theta_1yz).
\end{aligned}\right\} \quad (20).*$$

To the same order we may suppose these equations to hold good at the surface $r = a$, instead of at the surface $r = a\{1 + \epsilon T_2\}$.

§ 5. *Determination of the Elastic Distortions.*

The modified forms (19) of the equations of motion are simply the equations to which we are led in determining the displacements in a strained elastic solid and the solutions of them, when either the displacements or the surface-tractions at the surface of a sphere are given, are well known.† We may readily adapt these solutions so as to satisfy the boundary conditions (20).

Denote for brevity the function $\omega^2(\theta_2xz - \theta_1yz)$ by S_2.

From (19) we obtain at once $\nabla^2\psi = 0$. This equation replaces the more complicated form (8); a particular solution of it is $\psi = AS_2$.

Introducing this value of ψ into the left-hand members of (19), we find

$$\nabla^2 u_1 = -\frac{A}{n}\frac{\partial S_2}{\partial x}, \quad \nabla^2 v_1 = -\frac{A}{n}\frac{\partial S_2}{\partial y}, \quad \nabla^2 w_1 = -\frac{A}{n}\frac{\partial S_2}{\partial z},$$

particular solutions of which are

$$u_1 = -\tfrac{1}{10}\frac{A}{n}r^2\frac{\partial S_2}{\partial x}, \quad v_1 = -\tfrac{1}{10}\frac{A}{n}r^2\frac{\partial S_2}{\partial y}, \quad w_1 = -\tfrac{1}{10}\frac{A}{n}r^2\frac{\partial S_2}{\partial z} \quad . \quad (21).$$

These do not satisfy the last of equations (19), but they make

$$\frac{\partial u_1}{\partial x} + \frac{\partial v_1}{\partial y} + \frac{\partial w_1}{\partial z} = -\tfrac{2}{5}\frac{A}{n}S_2.$$

* For the physical significance of these equations, *vide* § 8 *infra*.

† *Vide* LOVE, 'Elasticity,' vol. 1, chap. 10.

We must therefore add to the particular integrals just found complementary functions which satisfy the equations

$$\left.\begin{array}{l}\nabla^2 u_1 = 0, \quad \nabla^2 v_1 = 0, \quad \nabla^2 w_1 = 0 \\ \dfrac{\partial u_1}{\partial x} + \dfrac{\partial v_1}{\partial y} + \dfrac{\partial w_1}{\partial z} = + \frac{2}{5}\dfrac{A}{n} S_2\end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (22).$$

Now the functions $r^7 \dfrac{\partial}{\partial x}\left(\dfrac{S_2}{r^5}\right)$, $r^7 \dfrac{\partial}{\partial y}\left(\dfrac{S_2}{r^5}\right)$, $r^7 \dfrac{\partial}{\partial z}\left(\dfrac{S_2}{r^5}\right)$ are spherical harmonic functions which remain finite at the origin, and hence, if we take

$$\left.\begin{array}{l} u_1 = B\dfrac{\partial S_2}{\partial x} + Cr^7 \dfrac{\partial}{\partial x}\left(\dfrac{S_2}{r^5}\right), \\ v_1 = B\dfrac{\partial S_2}{\partial y} + Cr^7 \dfrac{\partial}{\partial y}\left(\dfrac{S_2}{r^5}\right), \\ w_1 = B\dfrac{\partial S_2}{\partial z} + Cr^7 \dfrac{\partial}{\partial z}\left(\dfrac{S_2}{r^5}\right), \end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (23)$$

the first three of equations (22) will be satisfied identically, and the fourth will also be satisfied if

$$-3.7.C = \frac{2}{5}\frac{A}{n}, \quad \text{or} \quad C = -\frac{2}{105}\frac{A}{n}.$$

Therefore, adding together the particular integrals (21) and the complementary functions (23), we obtain as a set of solutions of the equations of motion which remain finite at the origin

$$\left.\begin{array}{l} \psi = AS_2, \\ u_1 = -\frac{1}{10}\dfrac{A}{n}r^2\dfrac{\partial S_2}{\partial x} + B\dfrac{\partial S_2}{\partial x} - \frac{2}{105}\dfrac{A}{n}r^7\dfrac{\partial}{\partial x}\left(\dfrac{S_2}{r^5}\right), \\ v_1 = -\frac{1}{10}\dfrac{A}{n}r^2\dfrac{\partial S_2}{\partial y} + B\dfrac{\partial S_2}{\partial y} - \frac{2}{105}\dfrac{A}{n}r^7\dfrac{\partial}{\partial y}\left(\dfrac{S_2}{r^5}\right), \\ w_1 = -\frac{1}{10}\dfrac{A}{n}r^2\dfrac{\partial S_2}{\partial z} + B\dfrac{\partial S_2}{\partial z} - \frac{2}{105}\dfrac{A}{n}r^7\dfrac{\partial}{\partial z}\left(\dfrac{S_2}{r^5}\right), \end{array}\right\} \quad . \quad . \quad (24).$$

We must now prove that these solutions are sufficiently general to satisfy the boundary conditions (20) at the surface of the sphere $r = a$.

From (24) we obtain

$$u_1 x + v_1 y + w_1 z = -\frac{1}{5}\frac{A}{n}r^2 S_2 + 2BS_2 + \frac{2}{35}\frac{A}{n}r^2 S_2 = -\frac{1}{7}\frac{A}{n}r^2 S_2 + 2BS_2 \quad . \quad (25).$$

Since in the small terms which already contain S_2 as a factor we may treat the spheroid as a sphere of radius a, we have

$$\zeta_1 = u_1 \frac{x}{a} + v_1 \frac{y}{a} + w_1 \frac{z}{a} = \frac{1}{a}\left[2B - \tfrac{1}{7}\frac{Aa^2}{n}\right] S_2 \quad . \quad . \quad . \quad . \quad . \quad (26),$$

and therefore

$$v_1{}' = \tfrac{4}{5}\pi\rho a\zeta_1 = \tfrac{4}{5}\pi\rho\left[2B - \tfrac{1}{7}\frac{Aa^2}{n}\right] S_2,$$

whence

$$v_1{}' - g\zeta_1 = (\tfrac{4}{5}\pi\rho a - \tfrac{4}{3}\pi\rho a)\,\zeta_1 = -\tfrac{8}{15}\pi\rho\left[2B - \tfrac{1}{7}\frac{Aa^2}{n}\right] S_2 \quad . \quad . \quad (27).$$

Consider now the different terms of the left-hand members of the boundary equations (20). By means of the formula

$$xS_2 = \frac{r^2}{5}\frac{\partial S_2}{\partial x} - \frac{r^7}{5}\frac{\partial}{\partial x}\left(\frac{S_2}{r^5}\right)$$

we obtain at the boundary

$$x\psi = A\left\{\frac{a^2}{5}\frac{\partial S_2}{\partial x} - \frac{a^7}{5}\frac{\partial}{\partial x}\left(\frac{S_2}{r^5}\right)\right\}.$$

Also we have

$$\left(r\frac{\partial}{\partial r} - 1\right)u_1 = -\tfrac{1}{5}\frac{A}{n}r^2\frac{\partial S_2}{\partial x} - \tfrac{4}{105}\frac{A}{n}r^7\frac{\partial}{\partial x}\left(\frac{S_2}{r^5}\right),$$

and from (25)

$$\begin{aligned}\frac{\partial}{\partial x}(u_1x + v_1y + w_1z) &= -\tfrac{2}{7}\frac{A}{n}xS_2 - \tfrac{1}{7}\frac{A}{n}r^2\frac{\partial S_2}{\partial x} + 2B\frac{\partial S_2}{\partial x} \\ &= -\tfrac{2}{7}\frac{A}{n}\left\{\frac{r^2}{5}\frac{\partial S_2}{\partial x} - \frac{r^7}{5}\frac{\partial}{\partial x}\left(\frac{S_2}{r^5}\right)\right\} - \tfrac{1}{7}\frac{A}{n}r^2\frac{\partial S_2}{\partial x} + 2B\frac{\partial S_2}{\partial x};\end{aligned}$$

therefore, at the surface,

$$\frac{\partial}{\partial x}(u_1x + v_1y + w_1z) = -\tfrac{1}{5}\frac{Aa^2}{n}\frac{\partial S_2}{\partial x} + 2B\frac{\partial S_2}{\partial x} + \tfrac{2}{35}\frac{A}{n}a^7\frac{\partial}{\partial x}\left(\frac{S_2}{r^5}\right).$$

Lastly, from (27),

$$x(v_1{}' - g\zeta_1) = -\tfrac{8}{75}\pi\rho\left[2B - \tfrac{1}{7}\frac{Aa^2}{n}\right]\left[a^2\frac{\partial S_2}{\partial x} - a^7\frac{\partial}{\partial x}\left(\frac{S_2}{r^5}\right)\right]$$

and therefore the first boundary equation becomes

$$\begin{aligned}&\frac{\partial S_2}{\partial x}\left\{A\frac{a^2}{5} - \tfrac{1}{5}Aa^2 - \tfrac{1}{5}Aa^2 + 2nB + \tfrac{8}{75}\pi\rho a^2\left(2B - \tfrac{1}{7}\frac{Aa^2}{n}\right)\right\} \\ &\qquad + a^5\cdot\frac{\partial}{\partial x}\left(\frac{S_2}{r^5}\right)\left\{-A\frac{a^2}{5} - \tfrac{4}{105}Aa^2 + \tfrac{2}{35}Aa^2 - \tfrac{8}{75}\pi\rho a^2\left(2B - \tfrac{1}{7}\frac{Aa^2}{n}\right)\right\} \\ &= \frac{a^2}{5}\frac{\partial S_2}{\partial x} - \frac{a^7}{5}\frac{\partial}{\partial x}\left(\frac{S_2}{r^5}\right).\end{aligned}$$

The remaining boundary equations can be written down by replacing x by y, z respectively. Hence, all three boundary equations will be satisfied provided A, B are subject to the relations

$$-\mathrm{A}a^2 + 10n\mathrm{B} + \tfrac{8}{15}\pi\rho a^2\left(2\mathrm{B} - \tfrac{1}{7}\frac{\mathrm{A}a^2}{n}\right) = a^2,$$

$$\mathrm{A}a^2 - \tfrac{2}{21}\mathrm{A}a^2 + \tfrac{8}{15}\pi\rho a^2\left(2\mathrm{B} - \tfrac{1}{7}\frac{\mathrm{A}a^2}{n}\right) = a^2.$$

Solving these equations, we obtain

$$\mathrm{A} = \frac{21}{19 + \frac{8}{3}\frac{\pi\rho a^2}{n}}, \quad \mathrm{B}n = \frac{4a^2}{19 + \frac{8}{3}\cdot\frac{\pi\rho a^2}{n}},$$

$$2\mathrm{B} - \tfrac{1}{7}\frac{\mathrm{A}a^2}{n} = \frac{5a^2/n}{19 + \frac{8}{3}\frac{\pi\rho a^2}{n}}.$$

Now, let $\epsilon' = 5\omega^2 a^2/38n$, so that ϵ' will denote the ellipticity which would be induced in a sphere of radius a by centrifugal force when distortion is resisted by elasticity alone.* Then the above values may be written

$$\mathrm{A} = \frac{21}{19\,\{1 + \epsilon'/\epsilon\}}, \quad \mathrm{B}\omega^2 = \frac{8\epsilon'/5}{1 + \epsilon'/\epsilon},$$

$$\left(2\mathrm{B} - \tfrac{1}{7}\frac{\mathrm{A}a^2}{n}\right)\omega^2 = \frac{2\epsilon'}{1 + \epsilon'/\epsilon}.$$

Finally from (24)

$$\left.\begin{aligned}
u = u_0 + u_1 &= z\theta_2 + \mathrm{B}\omega^2 z\theta_2 - \tfrac{5}{42}\frac{\mathrm{A}a^2\omega^2}{n}\frac{r^2}{a^2}\theta_2 z + \tfrac{2}{21}\frac{\mathrm{A}a^2\omega^2}{n}\frac{x(\theta_2 xz - \theta_1 yz)}{a^2} \\
&= z\theta_2\left\{1 + \frac{\frac{8}{5}\epsilon' - \frac{r^2}{a^2}\epsilon'}{1 + \epsilon'/\epsilon}\right\} + \tfrac{4}{5}\frac{\epsilon'}{1 + \epsilon'/\epsilon}\frac{x(\theta_2 xz - \theta_1 yz)}{a^2} \\
v = v_0 + v_1 &= -z\theta_1\left\{1 + \frac{\frac{8}{5}\epsilon' - \frac{r^2}{a^2}\epsilon'}{1 + \epsilon'/\epsilon}\right\} + \tfrac{4}{5}\frac{\epsilon'}{1 + \epsilon'/\epsilon}\frac{y(\theta_2 xz - \theta_1 yz)}{a^2} \\
w = w_0 + w_1 &= (-x\theta_2 + y\theta_1)\left\{1 - \frac{\frac{8}{5}\epsilon' - \frac{r^2}{a^2}\epsilon'}{1 + \epsilon'/\epsilon}\right\} + \tfrac{4}{5}\frac{\epsilon'}{1 + \epsilon'/\epsilon}\frac{z(\theta_2 xz - \theta_1 yz)}{a^2}
\end{aligned}\right\} \quad (28),$$

* *Cf.* THOMSON and TAIT, "Natural Philosophy," Part II., § 837.

and from (18), (26)

$$\zeta = \zeta_0 + \zeta_1 = -\frac{2\epsilon}{a}(\theta_2 xz - \theta_1 yz) + \frac{2}{a}\frac{\epsilon'}{(1+\epsilon'/\epsilon)}(\theta_2 xz - \theta_1 yz)$$

$$= -\frac{2}{a}\left\{\epsilon - \frac{\epsilon'}{1+\epsilon'/\epsilon}\right\}(\theta_2 xz - \theta_1 yz),$$

$$= -\frac{2}{a}\left\{\frac{\epsilon}{1+\epsilon'/\epsilon}\right\}(\theta_2 xz - \theta_1 yz) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (29).$$

§ 6. *Determination of the Period.*

The method we have followed hitherto has enabled us to express the displacements at any point of the body by means of two arbitrary constants θ_1, θ_2, but the quantity λ, whose value it is our chief object to determine, has entirely disappeared. We have, in fact, verified that the equations (7), (8), and the boundary conditions (9) are approximately satisfied by the forms (28), while to the same order zero is the approximate value of λ. We require now to have recourse to a method which will enable us to carry our approximations to the value of λ further.

By using the well-known equations of motion of a body of changing form Professor WOODWARD* has shown that the determination of the period may be reduced to the evaluation of the disturbing angular velocities due to the flow of the material relatively to the principal axes of the body. As we have now expressed the displacements, and consequently the velocities, at any point in terms of θ_1, θ_2, which may be taken as the displacements of the body as a whole, we are in a position to calculate these disturbing angular velocities. We might then introduce their values in Professor WOODWARD's equations and proceed to the determination of the period by his method. We propose, however, to make use of equivalent equations which express that the rates of change of angular momentum for the system as a whole about the axes Ox, Oy are zero. It seems somewhat preferable, on account of the additional simplicity of the motion of the axes themselves, to refer to these axes rather than to the moving axes used by WOODWARD, which correspond with our axes Ox_1, Oy_1.

If h_1, h_2 denote the components of angular momentum about Ox, Oy, we have

$$h_1 = \iiint \{\dot{w}y - (\dot{v} + x\omega)\, z\}\, dm,$$

where dm denotes an element of mass, and the integral is taken throughout the volume contained by the displaced surface. Replacing the integral by an integral taken throughout the mean volume and a surface integral, we have

$$h_1 = \iiint \{(\dot{w}y - \dot{v}z) - \omega xz\}\, dm - \rho \iint \zeta\omega xz d\mathrm{S},$$

* 'Astronomical Journal,' xv., No. 345.

or since $\iiint xz\,dm \iint xz\,dm = 0$ and v, w, ζ are each proportional to $e^{i\lambda t}$,

$$h_1 = i\lambda \iiint (wy - vz)\,dm - \rho\omega \iint \zeta xz\,dS,$$

$$\dot{h}_1 = -\lambda^2 \iiint (wy - vz)\,dm - \rho\omega i\lambda \iint \zeta xz\,dS.$$

Similarly

$$h_2 = i\lambda \iiint (uz - wx)\,dm - \rho\omega \iint \zeta yz\,dS,$$

$$\dot{h}_2 = -\lambda^2 \iiint (wy - vz)\,dm - \rho\omega i\lambda \iint \zeta yz\,dS.$$

The equations of angular momentum are

$$\dot{h}_1 - h_2\omega = 0, \qquad \dot{h}_2 + h_1\omega = 0.$$

Replacing h_1, h_2, $\dot{h}_1$, $\dot{h}_2$ by the values we have just found we obtain the following rigorous equations

$$\left.\begin{aligned} -\lambda^2 \iiint (wy - vz)\,dm - \rho\omega i\lambda \iint \zeta xz\,dS & \\ - i\lambda\omega \iiint (uz - wx)\,dm + \rho\omega^2 \iint \zeta xz\,dS = 0, & \\ -\lambda^2 \iiint (uz - wx)\,dm - \rho\omega i\lambda \iint \zeta yz\,dS & \\ + i\lambda\omega \iiint (wy - vz)\,dm - \rho\omega^2 \iint \zeta xz\,dS = 0, & \end{aligned}\right\} \quad . \quad . \quad . \quad (30).$$

Now by using the approximate forms (28) and denoting by $\mathbf{M}$ the mass of the spheroid, we have

$$\begin{aligned} \iiint (wy - vz)\,dm &= \iiint \{ -xy\theta_2 + (y^2 + z^2)\,\theta_1\}\,dm + \iiint (w_1 y - v_1 z)\,dm \\ &= \tfrac{2}{5}\mathrm{M}a^2\theta_1 + \text{terms of order } \epsilon, \\ \iiint (uz - wx)\,dm &= \iiint \{(z^2 + x^2)\,\theta_2 - xy\theta_1\}\,dm + \iiint (u_1 z - w_1 x)\,dm \\ &= \tfrac{2}{5}\mathrm{M}a^2\theta_2 + \text{terms of order } \epsilon. \end{aligned}$$

Also from (29),

$$\iint \zeta yz\,dS = \frac{2\theta_1}{a}\left(\frac{\epsilon}{1 + \epsilon'/\epsilon}\right) \iint y^2 z^2\,dS = \tfrac{8}{15}\pi a^5 \theta_1 \left(\frac{\epsilon}{1 + \epsilon'/\epsilon}\right),$$

$$\iint \zeta xz\,dS = -\tfrac{8}{15}\pi a^5 \theta_2 \frac{\epsilon}{1 + \epsilon'/\epsilon},$$

the errors being of the order ϵ^2.

Hence neglecting terms of order ω^6 only (λ being regarded as of order ω^3) the equations (30) reduce to

$$- i\lambda\omega \,.\, \tfrac{2}{5}\mathrm{M}a^2\theta_2 + \rho\omega^2\theta_1 \,.\, \tfrac{8}{15}\pi a^5 \frac{\epsilon}{1 + \epsilon'/\epsilon} = 0,$$

$$+ i\lambda\omega \,.\, \tfrac{2}{5}\mathrm{M}a^2\theta_1 + \rho\omega^2\theta_2 \,.\, \tfrac{8}{15}\pi a^5 \frac{\epsilon}{1 + \epsilon'/\epsilon} = 0.$$

These equations will be consistent if $\theta_1 = i\theta_2$

$$\lambda = \frac{\tfrac{8}{15}\pi\rho a^5\omega^2 \dfrac{\epsilon}{1 + \epsilon'/\epsilon}}{\tfrac{2}{5}\mathrm{M}a^2\omega} = \omega \frac{\epsilon}{1 + \epsilon'/\epsilon} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (31).$$

This gives the value of λ with errors of the order ω^5.

§ 7. *Numerical Values.*

If we suppose the rigidity of our body to become perfect we should obtain $\epsilon' = 0$, and therefore $\lambda/\omega = \epsilon$.

This is the value we should arrive at if we started by neglecting the elastic distortions. We see now that it is too large and that consequently the effect of elastic deformation is to diminish the frequency or to prolong the period.

The expressions for ϵ, ϵ' are

$$\epsilon = \frac{15\omega^2}{16\pi\rho}, \qquad \epsilon' = \frac{5\omega^2 a^2}{38n}.$$

Taking the sidereal day as 86164 mean solar seconds and using Boys's values* for the mean density of the Earth and the constant of gravitation, viz.: 5·5270 and $6{\cdot}6576 \times 10^{-8}$ we find from the above formula that for a spheroid of the same mean density as the Earth, rotating in a sidereal day,

$$\epsilon = \frac{1}{232}.$$

Again taking the Earth's mean radius as $6{\cdot}371 \times 10^8$ centimetres and $\mathfrak{n} = 8{\cdot}19 \times 10^{11}$,† which is the rigidity of steel, we find

$$n = \mathfrak{n}/\rho = 1{\cdot}482 \times 10^{11},$$

* 'Proc. Roy. Soc.,' 1894, p. 132.

† EVERETT, 'Units and Physical Constants,' pp. 61, 65.

whence

$$\epsilon' = \frac{1}{522},$$

and finally

$$\lambda/\omega = \frac{\epsilon}{1 + \epsilon'/\epsilon} = \frac{1}{335}.$$

We conclude that for a homogeneous spheroid of the same size and mean density as the Earth, the period would be extended from 232 days to 335 days in consequence of elastic distortions, if we suppose the rigidity to be that of steel.

If we take into account the variations in the density of the Earth's strata the problem presented becomes much more complicated. We must replace ϵ by the Precessional Constant, which will no longer be equal to the surface ellipticity. Its value may however be accurately determined by means of data furnished by the Theory of Precession; this value is known to be 1/305. Hence the effective value of ϵ is diminished in the ratio 232 : 305.

As regards the effect of heterogeneity on the value of ϵ', we are, at present, only in a position to make speculations. Professor NEWCOMB points out that in calculating the mean density, greater weight should be given to the density of the superficial layers on account of their greater effective inertia, and hence ϵ' should also be diminished. A reasonable hypothesis seems to be that it is diminished in the same ratio as ϵ. If we make this hypothesis, we find that if the effective rigidity of the Earth were as great as that of steel, the period of the Eulerian nutation would become

$$\frac{305 \times 335}{232} \text{ days} = 440 \text{ days}.$$

This period is slightly in excess of CHANDLER's observed period of 427 days. We therefore conclude that the effective rigidity of the Earth is slightly greater than that of steel.

If we make the same hypothesis as above, with regard to the effects of the variations of density, we may easily calculate what degree of rigidity would be consistent with CHANDLER's observed period. We find for the period of a homogeneous spheroid of the same degree of rigidity

$$\frac{427 \times 232}{305} \text{ days} = 326 \text{ days}.$$

Putting $\lambda/\omega = \frac{1}{326}$, $\epsilon = \frac{1}{232}$ in (31) we obtain

$$1 + \epsilon'/\epsilon = \tfrac{326}{232}$$
$$\epsilon' = \tfrac{94}{232}\epsilon = \tfrac{1}{572},$$

and therefore

$$\mathfrak{n} = \frac{8 \cdot 19 \times 572}{522} \times 10^{11} = 8 \cdot 98 \times 10^{11}.$$

§ 8. *Physical Characteristics of the Motion.*

The height of the waves at the free surface is given by the formula (29), viz. :—

$$\zeta = - \frac{2}{a} \frac{\epsilon}{1 + \epsilon'/\epsilon} (\theta_2 xz - \theta_1 yz).$$

Comparing this with equation (18) we see that ζ may be obtained from ζ_0 by changing θ_1, θ_2 into $\theta_1 \frac{\epsilon}{\epsilon + \epsilon'}$, $\theta_2 \frac{\epsilon}{\epsilon + \epsilon'}$. Thus, to our degree of approximation, the free surface will remain a spheroid of revolution of ellipticity ϵ. The position of the axis of this spheroid may be found by rotating the axis Oz through small angles $\theta_1 \frac{\epsilon}{\epsilon + \epsilon'}$, $\theta_2 \frac{\epsilon}{\epsilon + \epsilon'}$, about O$x$, O$y$ respectively.

Again, from (28) we see that with a high degree of approximation the displacements at the point x, y, z will be given by

$$u = z\theta_2, \quad v = - z\theta_1, \quad w = - x\theta_2 + y\theta_1 \quad . \quad . \quad . \quad . \quad . \quad . \quad (32),$$

in other words, the displacements due to elastic deformation will be negligible compared with the displacements due to the rotation of the body as a whole. We have seen, however, that the elastic distortions will produce an appreciable effect in displacing the axis of figure.

The modified forms (32) indicate that θ_1, θ_2 are the displacements of an axis sensibly fixed in the earth, which we may call the mean axis of figure, while the motion, at any instant, will consist very approximately of a rotation, as a rigid body, with angular velocity ω about an axis whose direction-cosines are $\dot{\theta}_1/\omega$, $\dot{\theta}_2/\omega$, 1. This axis we shall hereafter refer to as the instantaneous axis of rotation. It is, of course, only when we neglect the displacements due to elastic distortion that such an axis exists.

We have found that when $\lambda = \omega \left(\frac{\epsilon^2}{\epsilon + \epsilon'}\right)$, $\theta_1 = i\theta_2$. Taking

$$\theta_1 = \phi e^{i\lambda(t - \tau)}$$

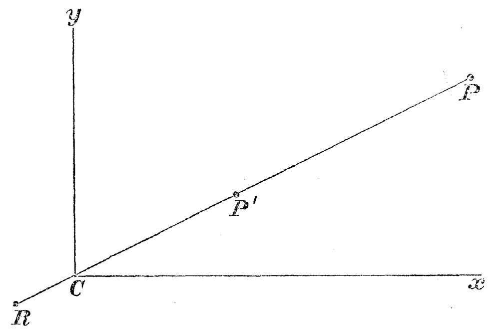
where ϕ, τ are real, we have

$$\theta_2 = - i\phi e^{i\lambda(t - \tau)}.$$

Adding to these the solutions obtained by changing the sign of i, wherever it occurs, we find as the real solution corresponding to the frequencies $\pm\,\omega\,\frac{\epsilon^2}{\epsilon+\epsilon'}$

$$\theta_1 = 2\phi\cos\lambda\,(t-\tau), \qquad \theta_2 = 2\phi\sin\lambda\,(t-\tau),$$

$$\dot{\theta}_1/\omega = -\,2\phi\,\frac{\lambda}{\omega}\sin\lambda\,(t-\tau), \qquad \dot{\theta}_2/\omega = +\,2\phi\,\frac{\lambda}{\omega}\cos\lambda\,(t-\tau).$$



Let C be the point where the axis Oz cuts the surface of the spheroid, and let us take a pair of rectangular axes Cx, Cy, having C as origin and parallel to the original axes Ox, Oy. Let P, P′, R be the points in which the mean axis of figure, the axis of the deformed figure and the instantaneous axis meet the plane Cxy. Taking the radius of the spheroid as unity, the coordinates of P are θ_2, $-\,\theta_1$, or

$$2\phi\sin\lambda\,(t-\tau), \qquad -\,2\phi\cos\lambda\,(t-\tau).$$

The coordinates of P′ are $\theta_2\frac{\epsilon}{\epsilon+\epsilon'}$, $-\,\theta_1\frac{\epsilon}{\epsilon+\epsilon'}$, or

$$2\phi\,\frac{\epsilon}{\epsilon+\epsilon'}\sin\lambda\,(t-\tau), \qquad -\,2\phi\,\frac{\epsilon}{\epsilon+\epsilon'}\cos\lambda\,(t-\tau),$$

and the coordinates of R are $\dot{\theta}_1/\omega$, $\dot{\theta}_2/\omega$, or

$$-\,2\phi\,\frac{\epsilon^2}{\epsilon+\epsilon'}\sin\lambda\,(t-\tau), \qquad +\,2\phi\,\frac{\epsilon^2}{\epsilon+\epsilon'}\cos\lambda\,(t-\tau).$$

We thus see that the points P, P′, R all lie on the same straight line through C, and that this line revolves about C with uniform angular velocity λ relatively to the moving axes Cx, Cy. These axes are themselves rotating with angular velocity ω about C in the same direction, and, hence, the actual angular velocity of the line PP′R about C is $\omega\,\{1+\epsilon^2/(\epsilon+\epsilon')\}$. The distances of R, P, P′ from C remain constant, while since CR is of the order ϵ compared with CP or CP′, which are themselves small quantities, we may suppose the point R to coincide with C. We conclude that

the axis of rotation remains sensibly fixed in direction while the mean axis of figure and the axis of the distorted figure describe cones of revolution about it in a period approximately equal to the period of rotation, these three axes always remaining in a plane.

Let us next consider the motion of the instantaneous axis of rotation relatively to the Earth. Take a new set of axes Ox_1, Oy_1, Oz_1 found by rotating the old set through small angles θ_1, θ_2 about Ox, Oy; these new axes will be sensibly fixed in the Earth. The motion at any instant consists of a rotation whose components about the old axes are $\dot{\theta}_1$, $\dot{\theta}_2$, ω; resolving these rotations about the new axes, we find for the components

$$\dot{\theta}_1 - \omega\theta_2, \quad \dot{\theta}_2 + \omega\theta_1, \quad \omega.$$

Hence the direction-cosines of the axis of rotation relatively to the axes Ox_1, Oy_1, Oz_1 are

$$(\dot{\theta}_1 - \omega\theta_2)/\omega, \quad (\dot{\theta}_2 + \omega\theta_1)/\omega, \quad 1,$$

or

$$-2\phi\left(1 + \frac{\epsilon^2}{\epsilon + \epsilon'}\right)\sin\lambda(t - \tau), \quad +2\phi\left(1 + \frac{\epsilon^2}{\epsilon + \epsilon'}\right)\cos\lambda(t - \tau), \quad 1.$$

Thus, relatively to the Earth, the axis of rotation will describe a cone of revolution about the mean axis of figure with angular velocity λ, the direction of motion in this cone being the same as the direction of the Earth's rotation. This motion would manifest itself by a periodic change in the latitude of places on the Earth's surface, as found by astronomical observations.

The circumstances we have here described are the well-known characteristics of the Eulerian nutation,* with the additional feature that the axis of figure is displaced by centrifugal force towards the axis of rotation. This latter fact has been assumed by NEWCOMB† and other writers‡ as the basis of their work. The law of displacement of the pole of figure assumed by NEWCOMB is, however, not verified. We find

$$\mathrm{PR} : \mathrm{PP'R} = \epsilon + \epsilon' : \epsilon,$$

or

$$\mathrm{PP'} : \mathrm{P'R} = \epsilon' : \epsilon \quad \ldots \ldots \ldots \quad (a),$$

whereas NEWCOMB has taken

$$\mathrm{PP'} : \mathrm{P'R} = \mathrm{E'} : \epsilon \quad \ldots \ldots \ldots \quad (b),$$

where E′ denotes the ellipticity which would be induced in a sphere by centrifugal

* TISSERAND, 'Mécanique Céleste,' vol. 2, p. 494.

† 'Monthly Notices,' March, 1892.

‡ For an account of previous work, *vide* TISSERAND, 'Mécanique Céleste,' vol. 2, chap. xxx.

force, if distortion were resisted by gravitation as well as by elasticity. The quantity ϵ' is the ellipticity which would be induced if elasticity were the only resisting factor.

The reason Professor NEWCOMB's hypothesis is at fault is that the ellipticity ϵ, called by him the "natural" ellipticity of the spheroid, is itself partly maintained by centrifugal force, and that if the rotation were annulled, the spheroid would be distorted so that its ellipticity would no longer be ϵ, but E, say.

The ellipticity induced by rotation about a displaced axis has to be superposed on the ellipticity E about the original axis of rotation, and not on the ellipticity ϵ. Thus, in (b), ϵ should be replaced by E. Now it is obvious that $\epsilon = \mathrm{E} + \mathrm{E}'$, and by a formula given by THOMSON and TAIT*

$$\frac{1}{\mathrm{E}'} = \frac{1}{\epsilon} + \frac{1}{\epsilon'}.$$

Thus

$$\mathrm{E}' = \frac{\epsilon\epsilon'}{\epsilon + \epsilon'} \text{ and } \mathrm{E} = \frac{\epsilon^2}{\epsilon + \epsilon'},$$

and therefore

$$\mathrm{E}' : \mathrm{E} = \epsilon' : \epsilon.$$

With the above correction, then, the laws (a) and (b) become identical.

The spheroid will be distorted not only by centrifugal force about a displaced axis but by the relaxation of centrifugal force about the original axis. The second disturbing factor has been neglected by Professor NEWCOMB.

The disturbing potential will be the difference of the rotation-potentials due to rotation with angular velocity ω about Oz and about Oz_1, that is to

$$\begin{aligned} &\tfrac{1}{2}\omega^2 (x^2 + y^2) - \tfrac{1}{2}\omega^2 (x_1^2 + y_1^2) \\ &= \tfrac{1}{2}\omega^2 (x^2 + y^2) - \tfrac{1}{2}\omega^2 \{(x - z\theta_2)^2 + (y + z\theta_1)^2\} \\ &= \omega^2 (\theta_2 xz - \theta_1 yz). \end{aligned}$$

It is obvious that the equations (19), (20) are the equations for the distortion of an elastic sphere when distorted by forces throughout its mass derivable from this potential function.

Finally the angular velocity of R about P is λ, and therefore the angular velocity of R as viewed from P′ is

$$\lambda \cdot \frac{\mathrm{RP}}{\mathrm{RP}'} = \lambda \frac{\epsilon + \epsilon'}{\epsilon} = \omega\epsilon,$$

which is the third hypothesis made by NEWCOMB.

* 'Natural Philosophy,' Part II., § 840.

§ 9. *Conclusions.*

The existence of the Eulerian nutation, and the fact that it would give rise to a variation in latitude, was first predicted theoretically on the assumption that the Earth could be regarded as a rigid body. Our present work, however, shows that the hypothesis of perfect rigidity, though affording a sufficiently close approximation to the circumstances presented by nature to specify the character of the oscillation, is totally inadequate to lead to a correct determination of the period unless the Earth possesses a very much higher degree of rigidity than is met with in substances which have been subjected to experiment. The only knowledge we have of the amount of the Earth's rigidity arises from the very vague indications furnished by Tidal Theory, and we must therefore have recourse to observation to determine the period with accuracy. This has been effected by Dr. CHANDLER, who, as the result of a discussion of a very large number of observations, has assigned 427 days as the true period. This period is, as the present theory requires, considerably in excess of the Eulerian period of 305 days.

In § 7 we have endeavoured to obtain a numerical estimate of the effective rigidity of the Earth which would be consistent with this observed period, and we have found it to be slightly greater than that of steel, a result which agrees sufficiently closely with the requirements of Tidal Theory. Various causes however combine to render this result liable to a considerable amount of uncertainty. In the first place, our present analysis applies only to a homogeneous spheroid composed of isotropic material, neither of which conditions are fully realized in the case of the Earth. In the second place, there are probably other causes in addition to the elastic deformations of the solid parts of the Earth, which tend to modify the period. In particular we have completely neglected the effects of the mobility of the ocean. According to NEWCOMB these effects will be small compared with the effects of elastic deformation, but different writers have expressed widely different opinions on the subject. Thus WOODWARD* is of opinion that they alone would be sufficient to fully account for the observed extension of the period. NEWCOMB'S view is to some extent confirmed by the smallness of the tide having a 427-day period which has been made the subject of observation by BAKHUYSEN† and A. S. CHRISTIE,‡ and as it has appeared to me to be quite open to question in what manner the results would be affected, I have thought it best not to apply any correction on this account; the results, then, must only be regarded as provisional, pending more complete mathematical investigations on the subject.

In a previous paper§ I have investigated the effects of an internal fluid nucleus,

* 'Astron. Jour.,' No. 345, vol. 15.

† 'Astron. Nach.,' No. 3261.

‡ 'Astron. Jour.,' No. 351.

§ 'Phil. Trans.,' A, 1895.

and found that if the central solid portions of a rigid spheroid were replaced by liquid, the theoretical estimate of the period of oscillation would be diminished. It might appear then that our estimate of the Earth's rigidity would be diminished if we suppose there to be a central fluid nucleus. That this is not so, I think the following considerations will show.

In accordance with the results of the above-mentioned paper the effect of internal fluidity would be to increase the effective value of the quantity we have denoted by ϵ. When the external crust is of considerable thickness, the increase in this quantity is, however, very slight; thus for a crust of about 2,000 miles in thickness we find ϵ is increased in the ratio 305 : 300. Now it seems that if the central solid portions of the Earth were replaced by fluid the increase in the value of ϵ', which denotes the ellipticity due to rotation, would be much more rapid, and that consequently λ/ω would rapidly diminish. We conclude then that the increased effects of the elastic deformations would more than counteract the influence of the reduced effective inertia due to internal fluidity, and that with a given degree of rigidity the period of oscillation would be still further prolonged. The degree of rigidity of the crust necessary to account for a given period would also have to be increased, and as the estimate we have already found is high, the evidence furnished by the latitude-variation still seems opposed to the existence of an internal fluid nucleus.

Finally we may consider the effects of the Earth's viscosity. Unless this be so great that the present work is inapplicable, a circumstance which seems to be quite precluded from the close agreement of our results with observation, the chief effect of internal friction will be to cause the oscillation in question to gradually die out without producing any material change in the period. The dissipative forces arise entirely from the distortion of the parts of the system, and consequently no such forces will occur if the system be absolutely rigid throughout. Now we have seen that the motion consists very approximately of a rotation as a rigid body, and that the elastic distortions are exceedingly minute. Hence we conclude that a very small amount of dissipative force will be called into play, and thus if the motion is once set up, there appears to be no difficulty in accounting for its continuance for a very considerable period, possibly extending over several centuries.